Abstracts

Toric geometry and the Semple-Nash modification

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(joint work with Pedro Daniel González Pérez)

The Semple-Nash modification of an algebraic variety over a field of characteristic zero is a birational map $\nu\colon NX \to X$ such that $\nu^*\Omega^1_X$ has a locally free quotient of rank $d = \dim X$ and which is minimal for this property. It is unique up to unique X-isomorphism. Geometrically the set-theoretic fiber $|\nu^{-1}(x)|$ is the set of limit positions at x of tangent spaces to X at non singular points tending to x. In [6] Semple asks whether iterating this operation leads to a non singular model of X. The same question was asked by John Nash in the late 1960's. Nobile proved in [5] that the map ν is an isomorphism if and only if X is non singular, so that to answer positively Semple's question it suffices to prove that for some k we have $N^{k+1}X = N^kX$. The case of dimension one is easily settled positively, and Gérard Gonzalez Sprinberg proved in [4] that iterating the operation of *normalized* Semple-Nash modification (i.e., at each step, the modification followed by normalization) eventually resolves singularities of 2-dimensional normal toric varieties. This eventually led to the best result on this problem to this day, which is due to Spivakovsky (see [7]): iterating the normalized Semple-Nash modification eventually resolves singularities of normal surfaces over an algebraically closed field.

Our basic idea is that in higher dimensions it should be easier in the toric case to deal with the Semple-Nash modification without normalization. Gonzalez Sprinberg showed that the Semple-Nash modification, in characteristic zero, is isomorphic with the blowing up of the *logarithmic jacobian ideal*, which is an equivariant sheaf of ideals defined as follows:

Let k be a field and let Γ be a finitely generated subgroup of a lattice $M \simeq \mathbf{Z}^d$. Denote by $(\gamma_i)_{i\in F}$ a set of generators. The logarithmic jacobian ideal is the monomial ideal of the semigroup algebra $k[t^\Gamma]$ generated by the elements $t^{\gamma_{i_1}+\cdots+\gamma_{i_d}}$ for all $(i_1, \ldots, i_d) \subset F$ such that $\gamma_{i_1} \wedge \ldots \wedge \gamma_{i_d} \neq 0$ in $M_{\mathbf{R}}$.

Since we wanted to describe precisely blowing-ups (and not only normalized blowing-ups) and modifications we wrote down the basic theory of toric varieties without any assumption of normality or of projectivity. However Sumihiro's theorem (see [8]) on the existence of a covering by invariant affine varieties of a variety on which a torus of the same dimension acts with a dense orbit fails without the assumption of normality, and we have to set the existence of such a covering as part of the definition of a toric variety. Then an abstract toric variety has a combinatorial description: it corresponds to certain semigroups in the convex duals of the cones of a fan, which satisfy a natural gluing-up condition. This generalizes the definition of [2] which concerns toric varieties equivariantly embedded in projective space.

Given a rational strictly convex cone $\sigma \in N_{\mathbf{R}}$ and a finitely generated semigroup Γ, such that $\mathbf{Z}\Gamma = M$ and $\mathbf{R}_{\geq 0}\Gamma = \check{\sigma} \subset M_{\mathbf{R}}$ one has the affine toric variety $\mathrm{Spec}k[t^{\Gamma}]$ with its torus $T^M = \mathrm{Spec}k[t^M]$. For a face $\tau < \sigma$, the lattice $M(\tau;\Gamma)$ spanned by $\Gamma \cap \tau^{\perp}$ is in general a sublattice of finite index in $M(\tau) = M \cap \tau^{\perp}$. This corresponds to the fact that the normalization map is equivariant and its restriction to an orbit of the normalization $\mathrm{Spec}k[t^{\check{\sigma}\cap M}]$ may not be one to one. The variety $T^{\Gamma\cap\tau^{\perp}} = \mathrm{Spec}k[t^{\Gamma\cap\tau^{\perp}}]$ with torus $T^{M(\tau;\Gamma)}$ is an orbit closure in $\mathrm{Spec}k[t^{\Gamma}]$.

The semigroup $\Gamma_\tau = \Gamma + M(\tau;\Gamma)$ is again a finitely generated semigroup, generating the group M and with the property that the cone $\mathbf{R}_{\geq 0}\Gamma_\tau$ which it generates in $M_{\mathbf{R}}$ is equal to $\check{\tau}$.

A toric variety is a triple $(N, \Sigma, \mathbf{\Gamma})$ consisting of a lattice N, a fan Σ in $N_{\mathbf{R}}$ and a family of subsemigroups of the dual lattice M of N,

$$\mathbf{\Gamma} = (\Gamma_\sigma)_{\sigma\in\Sigma}$$

such that:
- The group $\mathbf{Z}\Gamma_\sigma$ generated by Γ_σ is equal to M for all $\sigma \in \Sigma$.
- The cone $\mathbf{R}_{\geq 0}\Gamma_\sigma$ generated by Γ_σ is equal to $\check{\sigma}$ for all $\sigma \in \Sigma$.
- Far all $\sigma \in \Sigma$ and $\tau < \sigma$, we have $\Gamma_\tau = \Gamma_\sigma + M(\tau;\Gamma_\sigma)$.

The toric variety $T_{\Sigma}^{\mathbf{\Gamma}} = \bigcup_{\sigma\in\Sigma} T^{\Gamma_\sigma}$ is then obtained by gluing up the affine toric varieties T^{Γ_σ} as in the normal case.

Now if we define an abstract toric variety as an irreducible separated algebraic variety X over k equipped with an algebraic action of a torus $T^M \subset X$ of the same dimension extending the action of the torus on itself by translation and covered by finitely many open affine invariant subsets, then we have:

Theorem 1. (See [3]) *An abstract toric variety is equivariantly isomorphic to a variety* $T_{\Sigma}^{\mathbf{\Gamma}}$.

In fact, with the natural definitions of toric morphisms, this sets up an equivalence of categories, see [3] for details.
The logarithmic jacobian ideals sheafify into an equivariant sheaf of ideals on $T_{\Sigma}^{\mathbf{\Gamma}}$.

Given a monomial ideal $I = (t^{n_1}, \ldots, t^{n_s})$ in $k[t^{\Gamma}]$, it determines a piecewise linear function $\mathrm{ord}_I \colon \sigma \to \mathbf{R}$ by $\nu \mapsto \min_{m\in I}\langle \nu, m\rangle$. The cones of linearity σ_i correspond to the vertices $(m_1, \ldots, m_k)$ in $\check{\sigma}$ of the Newton polyhedron of I and they are the cones of maximal dimension of a fan Σ_I with support $|\sigma|$. If to each cone $\sigma_i \subset \sigma$ we attach the semigroup $\Gamma_i = \Gamma + \langle (n_j - m_i)_{j=1,\ldots,s}\rangle \subset \check{\sigma}_i \cap M$ and attach to the faces of the σ_i the semigroups corresponding to the gluing up rules explained above, we see that we have a triple $(N, \Sigma_I, \mathbf{\Gamma}_I)$ corresponding to a toric variety $T_{\Sigma_I}^{\mathbf{\Gamma}_I}$ endowed with an equivariant proper map to T^{Γ}. It is the blowing-up of the ideal I. This construction globalizes into the blowing up of a sheaf of equivariant ideals on a toric variety, and we can apply it to the sheaf of logarithmic jacobian ideals to obtain an analogue of the Semple-Nash modification, which is defined over any field.

The analogue in the toric case, in any characteristic, of Nobile's result is an easy combinatorial lemma: the blowing up of the logarithmic jacobian ideal is an isomorphism if and only if the toric variety $T_{\Sigma}^{\mathbf{\Gamma}}$ is non singular.

So iterating the logarithmic jacobian blowing up of a toric variety $T_\Sigma^{\mathbf{\Gamma}}$ produces a sequence of refinements $\Sigma^{(j)}$ of the fan Σ, with attached systems of semigroups $\mathbf{\Gamma}^{(j)}$. The goal is to prove that they stabilize: the refinement stops only when the space is non singular, as we have seen.

Ewald and Ishida have introduced in [1] the analogue in toric geometry of the Zariski-Riemann manifold in algebraic geometry: it is the space $ZR(M)$ of (additive) preorderings $\leq$ on M in the following sense:

- $\forall\ m, n \in M,\ m \leq n$ or $n \leq m$.
- $m \leq n$ and $n \leq p$ imply $m \leq p$.
- $m \leq n$ implies $m + p \leq n + p,\ \forall p \in M$,

endowed with the topology defined by the basis of open sets

$$\mathcal{U}(\theta) = \{\nu \in ZR(M)/\check{\theta} \subset L(\nu)\},$$

where θ is a rational convex cone of $N_{\mathbf{R}}$ and $L(\nu)$ is the semigroup of elements of M which are $\geq_\nu 0$, or non negative for the preorder ν.

With this topology, the space $ZR(M)$ is quasi compact, and it behaves like a space of valuations. In particular the preorders defined by vectors ν of the dual lattice N of M, as $m \leq_\nu n \iff \langle m, \nu\rangle \leq \langle m, \nu\rangle$ correspond to divisorial valuations. A preorder $\nu \in ZR(M)$ picks a cone in each refinement $\Sigma^{(j)}$ of Σ, and to prove that the sequence of toric varieties $(N, \Sigma^{(j)}, \mathbf{\Gamma}^{(j)})$ stabilizes, it suffices to check that for every preorder it stabilizes along the sequence of cones picked by that preorder. Then we have the following partial result:

Theorem 2. (See [3]) *The sequence $(N, \Sigma^{(j)}, \mathbf{\Gamma}^{(j)})$ stabilizes for any order that is lexicographic with respect to some basis of N. In particular it stabilizes for the "divisorial" preorders associated to vectors of N.*

References

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