Abstracts

Some aspects of the connection between toric geometry and resolution of singularities

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We know from [2] that normal toric varieties over a field admit (non embedded) resolutions of singularities described by the regular refinements of their fan. The toric *embedded* resolution of singularities for affine toric varieties over an algebraically closed field k was proved in [3] and [5]. The combinatorics works as follows: an affine toric variety $X_0 \subset \mathbf{A}^N(k)$ over k is defined by a prime binomial ideal $I_0 = (u^{m^\ell} - \lambda_\ell u^{n^\ell})_{\ell \in L}$ in $k[u_1, \ldots, u_N]$. The monomial u^m corresponds to a point m in the lattice $M \simeq \mathbf{Z}^N$, and $\lambda_\ell \in k^*$. The vectors $m^\ell - n^\ell \in M$ determine dual hyperplanes H_ℓ in the real vector space $N_{\mathbf{R}}$ generated by the dual lattice $N \simeq \check{\mathbf{Z}}^N$ of M. The intersections with the first quadrant of these hyperplanes determine a fan Σ_0 subdividing the fan whose maximal cone is the first quadrant. The strict transform of X_0 by the corresponding birational map $\pi(\Sigma_0)\colon Z(\Sigma_0) \to \mathbf{A}^N(k)$ of normal toric varieties is the normalization of X_0. The strict transform of X_0 by a birational toric map $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^N(k)$ corresponding to a regular fan Σ subdividing Σ_0 is non singular and transversal to the toric boundary. Such subdivisions provide embedded pseudo[1] resolutions of X_0. The fan Σ can be chosen so as to contains the regular faces of the *weight cone* $\beta = \mathbf{R}^N_{\geq 0} \cap (\bigcap_\ell H_\ell)$, and then $\pi(\Sigma)$ is an embedded resolution.

One may wonder whether such toric maps also (pseudo) resolve the spaces obtained by suitable deformations of the binomial equations. This question comes from the basic observation of [5]: Given a local integral domain R with maximal ideal m and a rational valuation of R corresponding to an inclusion $R \subset R_\nu$ of R in a valuation ring R_ν of its field of fractions, such that $m_\nu \cap R = m$ and $R/m \to R_\nu/m_\nu$ is an isomorphism, we have a faithfully flat specialization of Spec$R$ to the affine toric variety (which may be of infinite embedding dimension) corresponding to the associated graded ring $\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Phi} \mathcal{P}_\phi / \mathcal{P}^+_\phi$ of R with respect to the filtration associated to ν, where $\mathcal{P}_\phi = \{x \in R | \nu(x) \geq \phi\}$, $\mathcal{P}^+_\phi = \{x \in R | \nu(x) > \phi\}$. The fact that ν is a rational valuation implies that $\mathrm{gr}_\nu R$ is a k-algebra and each homogeneous component is a vector space of dimension 1 over k. There is therefore a presentation $\mathrm{gr}_\nu R = k[(U_i)_{i \in I}]/(U^{m^\ell} - \lambda_\ell U^{n^\ell})_{\ell \in L}$ where U^m denotes a monomial, $\lambda_\ell \in k^*$, the sets I and L may be infinite, but countable.

We note that the degrees which actually appear in the graded algebra are the valuations of the elements of R, which form a subsemigroup of the semigroup $\Phi_+ \cup \{0\} = (R_\nu \setminus \{0\})^{mult.}/\{units\}$ of non negative elements of the (totally ordered) value group Φ of ν. In fact $\mathrm{gr}_\nu R$ is isomorphic to the semigroup algebra over k of the semigroup $\Gamma = \nu(R \setminus \{0\})$. If R is noetherian the semigroup Γ is well ordered and therefore has a unique minimal system of generators, indexed by an ordinal,

[1] This means that the restriction over the non singular part is not necessarily an isomorphism.

which is at most ω^h where h is the (archimedian, or real) rank of the value group. By transfinite induction one defines γ_{i+1} as the smallest non zero element of Γ which is not in the semigroup generated by the previous ones.

Let us concentrate on the case where the semigroup Γ is finitely generated and R is a local equicharacteristic and complete noetherian domain with an algebraically closed residue field k. Pick and fix a field of representatives $k \subset R$. Then R appears as an *overweight* deformation of its associated graded ring, in the sense of [6]: there is a continuous and surjective map of k-algebras

$$k[[u_1, \ldots, u_N]] \xrightarrow{\pi} R, \text{ determined by } \ u_i \mapsto \xi_i,$$

for any choice of elements $\xi_i \in R$ whose valuations are the minimal generators of the semigroup Γ or equivalently are such that their initial forms minimally generate the k-algebra $\mathrm{gr}_\nu R$. Giving to u_i the weight $\gamma_i = \nu(\xi_i) \in \Gamma \subset \Phi_+ \cup \{0\}$ determines a weight w on $k[[u_1, \ldots, u_N]]$, with its filtration by weight and the associated graded ring $\mathrm{gr}_w k[[u_1, \ldots, u_N]] \simeq k[U_1, \ldots, U_N]$, now graded by the weight: $\mathrm{deg} U_i = \gamma_i$. Moreover the valuation ideals of R are the images by π of the weight ideals of $k[[u_1, \ldots, u_N]]$ and so the map π induces a surjection of graded k-algebras

$$k[U_1, \ldots, U_N] \xrightarrow{\mathrm{gr}_w \pi} \mathrm{gr}_\nu R, \text{ determined by } \ U_i \mapsto \mathrm{in}_\nu \xi_i,$$

whose kernel is a binomial ideal $(u^{m^\ell} - \lambda_\ell u^{n^\ell})_{\ell \in L}$; it is essentially the presentation of the semigroup algebra of Γ over k which corresponds to an affine toric variety X_0. By flatness the kernel of π is generated by series $F_\ell = u^{m^\ell} - \lambda_\ell u^{n^\ell} + \sum_p c_p^{(\ell)} u^p$ with $c_p^{(\ell)} \in k, \ w(u^p) > w(u^{m^\ell}) = w(u^{n^\ell})$, for $\ell \in L$, a finite set. Let us call X the formal subspace of $\mathbf{A}^N(k)$ defined by the ideal $I = (F_\ell)_{\ell \in L}$; it is an *overweight deformation* of the affine toric variety X_0.

For a regular fan Σ with support the first quadrant of $\check{\mathbf{R}}^N$, the corresponding birational toric map $Z(\Sigma) \to \mathbf{A}^N(k)$ is described in each chart $Z(\sigma)$ corresponding to a maximal cone $\sigma = \langle a^1, \ldots, a^N \rangle$ of Σ, where $a^j \in N$, by

$$\begin{array}{l} u_1 = y_1^{a_1^1} \ldots y_N^{a_1^N} \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ \cdot \quad\quad \cdot \\ u_N = y_1^{a_N^1} \ldots y_N^{a_N^N} \end{array}$$

and the valuation ν of R picks a point in the strict transform of X. A combinatorial argument explained in [8] shows that one can find regular fans Σ subdividing the fan Σ_0 corresponding to the initial binomials of the F_ℓ, and such that for appropriate $\sigma \in \Sigma$ the transforms of the F_ℓ can be written

$$F_\ell \circ \pi(\sigma) = \\ y_1^{\langle a^1, n^\ell \rangle} \ldots y_N^{\langle a^N, n^\ell \rangle} (y_1^{\langle a^1, m^\ell - n^\ell \rangle} \ldots y_N^{\langle a^N, m^\ell - n^\ell \rangle} - \lambda_\ell + \textstyle\sum_p c_p^{(\ell)} y_1^{\langle a^1, p - n^\ell \rangle} \ldots y_N^{\langle a^N, p - n^\ell \rangle}).$$

The point is to find fans for which the inequalities $w(u^p) > w(u^{n^\ell})$ induce inequalities $\langle a^i, p - n^\ell \rangle > 0$. The largest torus-invariant charts of $Z(\Sigma)$ in which the strict transform meets the toric boundary correspond to cones σ of Σ whose intersection with the weight cone β is of maximal dimension $r = \dim R$. The variables y_{i_j}, $1 \leq j \leq r$ corresponding to the vectors $a^{j_i} \in \beta$ do not appear in the transformed binomials $y_1^{\langle a^1, m^\ell - n^\ell \rangle} \ldots y_N^{\langle a^N, m^\ell - n^\ell \rangle} - \lambda_\ell$ and can be taken as local coordinates on the strict transform of X. In fact, *at the point picked by the valuation*, this strict transform is a deformation of the strict transform of X_0 and hence non singular. In summary:

Theorem: *Given a rational valuation ν on a complete equicharacteristic local domain R with an algebraically closed residue field k, if the semigroup of values $\nu(R \setminus \{0\})$ is finitely generated, say by N generators, there is a continuous surjection $k[[u_1, \ldots, u_N]] \xrightarrow{\pi} R$ such that some of the toric modifications of $\mathbf{A}^N(k)$ in the coordinates u_i which resolve the singularities of the toric variety corresponding to $\mathrm{gr}_\nu R$ also produce an embedded local uniformization of the valuation ν on the space $X \subset \mathbf{A}^N(k)$ corresponding to R.*

In the situation of the theorem, by flatness of the deformation, the valuation ν is Abhyankar, which means in this case that the Abhyankar inequality $\dim \mathrm{gr}_\nu R \leq \dim R$ (see [5]) is an equality. Since local uniformization for Abhyankar valuations of algebraic function fields has been proved by Knaf and Kuhlmann in [4], it is natural to ask whether in general the Abhyankar condition implies that the semigroup Γ is finitely generated. An attempt to prove this is in progress. Combined with the theorem above it would have as consequence that the Abhyankar valuations are exactly the quasi-monomial ones, a fact proved by Cutkosky for valuations of rank one using embedded resolution of singularities (see [1], Prop. 2.8).

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