Length, Area, Volume, Geometry and Algebra

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Why become a researcher?
French Mathematicians meeting with Bulgarian students.
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Introduction

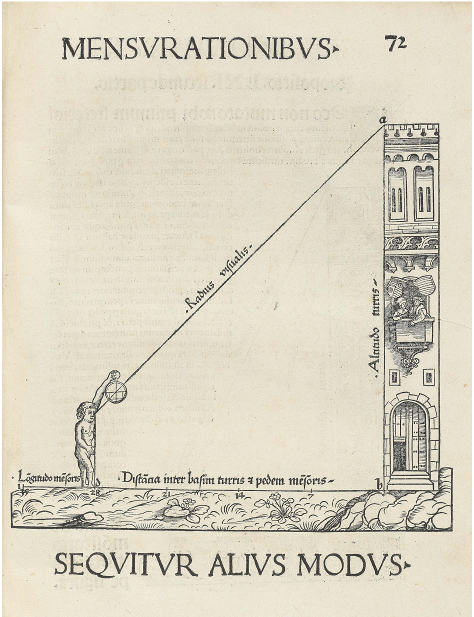
Algebra is generous: she often gives more than is asked of her. (D'Alembert, 1717-1783)

Estimating length is as old as motion of living beings. Measuring length is certainly as old as numbers. The first units of measurement are related to the human body:

- The step
- the cubit (from the elbow to the extremity of the middle finger),
- The foot, handspan, the length of the last part of the thumb (inch), divided in 12 ”lines” and 144 ”points”.

Having well defined units is not sufficient, however, and one must find ways to measure effectively, including objects to which you have no physical access. This is where the theorems of euclidean geometry which relate lengths and angles come into play, added to the fact that length is invariant by translation and rotation, in the plane as in space.

For example you can measure the height of a tower by measuring the angle between the ground and its summit from a point at a given distance from its foot, and using trigonometric functions.


MENSVRATIONIBVS.
72
a
.Radius visualis.
.Altitudo turris.
.Lõgitudo mẽsoris
.Distãtia inter basim turris ⁊ pedem mẽsoris.
b
35
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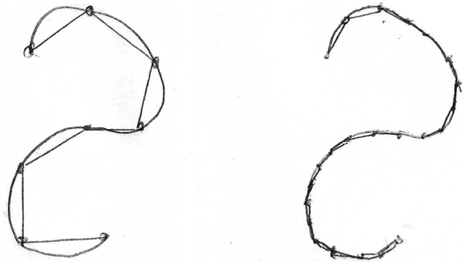
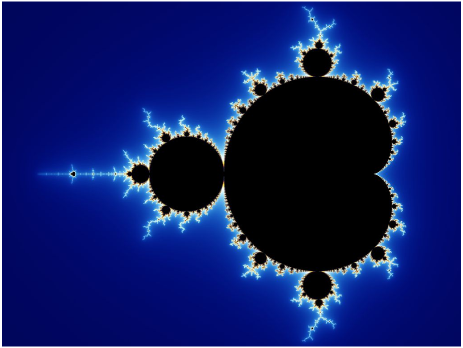
SEQVITVR ALIVS MODVS.

That is why trigonometry is important!

Measuring the length of a plane curve:



Recent results (which are less than 100 years old) allow to give a meaning to the length of extremely complicated curves, whose length cannot be computed in this way (Fractals).



Area

The concept of area is much more subtle, and appeared much later, probably after mankind began cultivating. In ancient Greece, people estimated the size of a field or town by its perimeter, which is the length of its boundary.

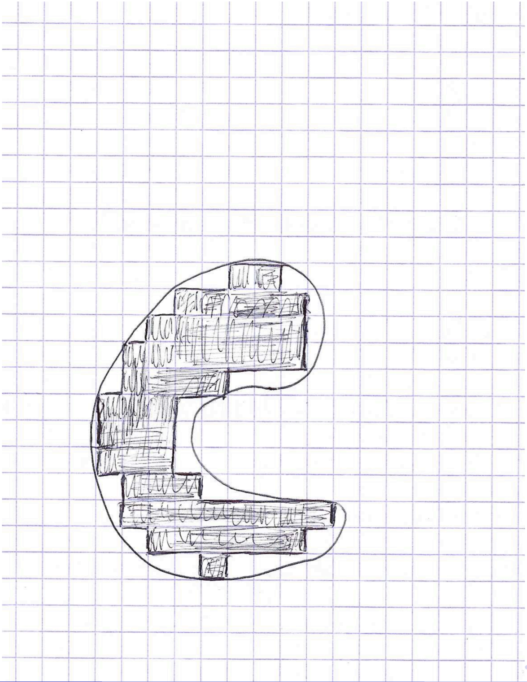
In Homer's Iliad (Ca. 7th century B.C.), the ”measure” of the size of the city of Troy is 10200 steps, its perimeter.

Proclus (411-485) reports court cases of members of Greek communities who, in the first century A.D., decided to divide land equitably according to perimeter and had surprises at the time of harvest!

Fields with the same perimeter may enclose a very small area if they are very elongated or a larger area if they are fairly round.

To estimate the area of a bounded part of the plane, or domain, you fill it with smaller and smaller disjoint squares which approximate its shape better and better and you consider the sum of their areas, which is just the number of squares times the square of their side.

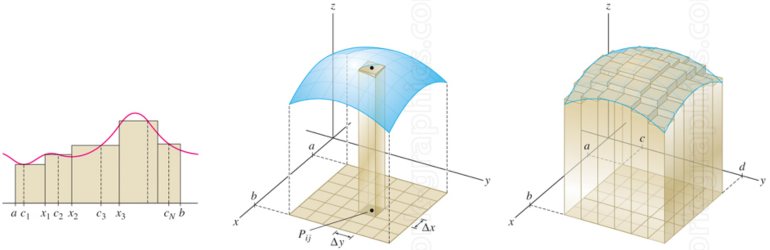
In general the sum will approximate better and better the area of the domain, unless its boundary is extremely complicated as in the fractal we have seen.



Approximating an object of unknown size by objects of known size is a major method!

Volume

Once the notion of area is established, the concept of volume is not such a big step. Replace squares by cubes. The boundary of a solid body is now a surface, which also has an area, which you can again measure by approximating it, in space, by small squares, or small triangles.



(A) In one variable, a Riemann sum approximates the area under the curve by a sum of areas of rectangles.

(B) The volume of the box is $f(P_{ij})\Delta A$ where $\Delta A = \Delta x \Delta y$.

(C) The Riemann sum $S_{N,M}$ is the sum of the volumes of the boxes.

In this lecture, I will present some of the relations between the area of a plane domain and the length of its boundary, the generalizations of this to higher dimension and the characterization of area, volume, etc.

The relation between area and perimeter of a plane domain was a subject of research for Greek mathematicians (5th Century B.C.) and they calculated many examples with polygons (6 centuries before the court cases mentioned by Proclus) and experimentally found an inequality between the perimeter L and the area S of a plane domain:

$$L^2 \geq 4\pi S.$$

This is now called the *isoperimetric inequality*.

The Greeks ”knew” by experience that equality is attained only for the disk, whose perimeter is $2\pi r$ and area πr^2 if it has radius r.

This means that for a given perimeter, the disk is the shape enclosing the largest area.

Why do you think the portholes of most ships or spacecrafts are round? because the light coming in is proportional to the area and the waves come in (or the air seeps out) only through the perimeter if the porthole is closed.

A modern porthole

In his book *Les problèmes des Isopérimètres et des Isépiphanes,* Gauthier-Villars 1929, Tommy Bonnesen writes something like this, about the property of the disk maximizing the area for a given perimeter:
For most people it is a matter of common sense; for the mathematician, on the contrary, it has been the cause of much reflexion. . .

And indeed, a complete proof was given only around 1895, after several attempts in the beginning of the 19th Century.

You could say that the isoperimetric problem had to wait 2.500 years to be solved.

The reason is that the necessary analytic tools were developed only in the 18th and 19th centuries and mathematicians wanted a proof valid for any ”measurable” boundary curve, one which could be meaningfully given a length.

Nowadays there are several proofs of the fact that the disk is the only plane domain maximizing the area for a given perimeter.

In order to present the context of the proof which is most geometric, I need to introduce an apparently quite different topic.

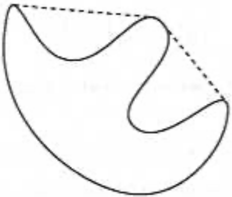
A *convex* domain of the line, the plane or space is a domain which, whenever it contains two points, also contains the segment joining them. The whole line, plane, or space are convex but we are mostly interested in bounded convex subsets.

A bounded convex subset of the line is an interval. An unbounded one which is not the whole line is the set of points which are smaller than, or larger than, a given value.

In the plane or space, examples of unbounded convex domains are the half-planes, or half-spaces, the points which are on one side of a line or a plane, that is, the points where the values of some affine function keep the same sign: points (x, y) such that $u(x, y) = ax + by + c \leq 0$ or $u(x, y) = ax + by + c \geq 0$.

Regions bounded by a plane conic, such as an ellipse, a parabola or half-hyperbola are also bounded or unbounded convex regions. That is because the inside of a cone in 3-space is convex and they are plane sections of it thanks to Apollonius.

Here is an example of a non-convex domain in the plane, together with the smallest convex domain which contains it:

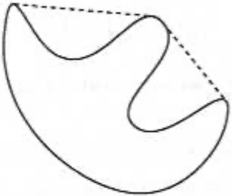


Remark that the intersection of convex domains is a convex domain.

So any bounded domain or solid body has a *convex hull* which is the intersection of the convex domains or bodies which contain it.

It is in fact the intersection of the half-planes or half-spaces which contain it.

In the plane, and only in the plane, the *convex hull* of a domain, which is the intersection of all convex domains containing it, has a smaller perimeter and larger area, so that to prove the optimal character of the disk we can restrict to convex domains. We use the same example as above, but now consider the area and perimeter of the non-convex domain and of the convex one.



The convex one has more area and less perimeter.

It is because bounding lines meet the domain at points in general, and the line itself is a shorter path between these points that following the perimeter.

In higher dimension, one cannot reduce the search for the optimal shape to convex bodies.That is because of the existence of *minimal surfaces* which minimize the area for a given bounding curve, and which are not plane in general.

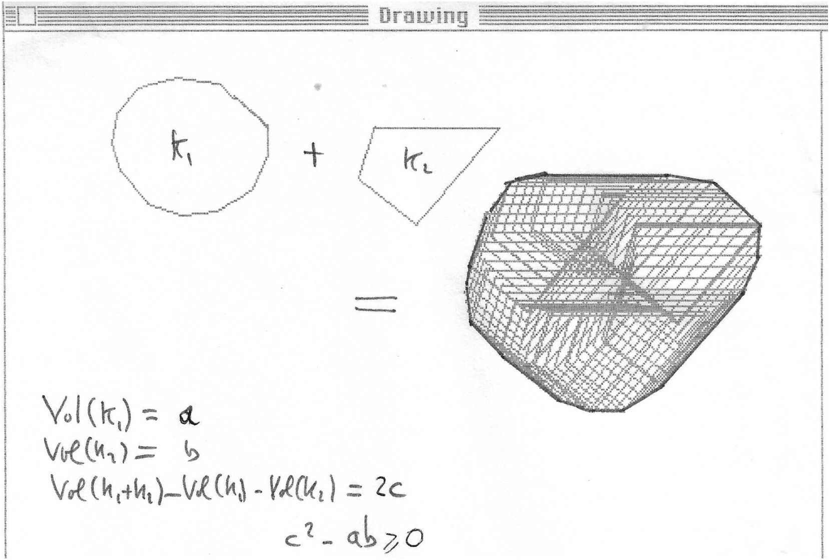
An algebra of convex domains

Given two convex domains in $\mathbf{R}^2$, or convex bodies in $\mathbf{R}^3$ one can define their *Minkowski sum*

$$K_1 + K_2 = \{x + y, x \in K_1, y \in K_2\},$$

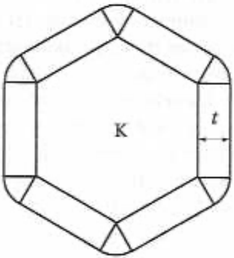
that is, the sum of a point in K_1 and a point in K_2.
You may think of them as vectors; the result, which is a convex domain, is independent of the origin, up to translation.

In all this, we fix an origin at will and multiplication by t is understood as homothety (multiplying all coordinates by t). Everything is independent of the origin, up to translation.


Drawing
K1
+
K2
=
Vol(K1) = a
Vol(K2) = b
Vol(K1+K2) − Vol(K1) − Vol(K2) = 2c
c² − ab ≥ 0

Here is a simpler example, where I add a disk $t\mathbf{D}$ of radius t to the domain bounded by a regular polygon of area S and perimeter L:

$$K + t\mathbf{D} =$$



The area of $K + t\mathbf{D}$ is $S + Lt + \pi t^2$, as you can see by adding the areas of the pieces shown on the figure.

It can be proved that this formula is valid for any convex domain K in the plane :

$$P_K(t) = \mathrm{Area}(K + t\mathbf{D}) = S + Lt + \pi t^2,$$

a polynomial of degree 2 in t, and you can notice that the discriminant of this polynomial is $L^2 - 4\pi S$, so

the isoperimetric inequality is equivalent to the fact that $P_K(t)$ has real roots!

This remains true when the unit disk **D** is replaced by any bounded convex domain and for symmetry we also make a homothety on K:

$$P_{K_1,K_2}(t) = \mathrm{Area}(t_1K_1 + t_2K_2) = v_0t_1^2 + 2v_1t_1t_2 + v_2t^2,$$

with $v_0 = \mathrm{Area}(K_1),\ v_2 = \mathrm{Area}(K_2)$, and $v_1^2 \geq v_0v_2$, or

$$\frac{v_0}{v_1} \leq \frac{v_1}{v_2}.$$

The number v_1 is called the *mixed area* of K_1 and K_2. It is always ≥ 0 and positive unless K_1 and K_2 are parallel segments.

Whenever K_2 is not a disk, the meaning of v_1 is more subtle.

A convex domain with perimeter zero has to be a point while a convex domain can have area zero and positive perimeter if (and only if) it is a segment.

Following work of Bonnesen in 1921, a very satisfactory geometric proof of the extremal character of the disk was developed.
For a plane domain K define the *inradius* $r = R_{int}$ of K as the radius of one of the largest disks contained in K and the *outradius* $R = R_{out}$ as the radius of the smallest disk containing K.

This definition works also for non convex bounded domains.

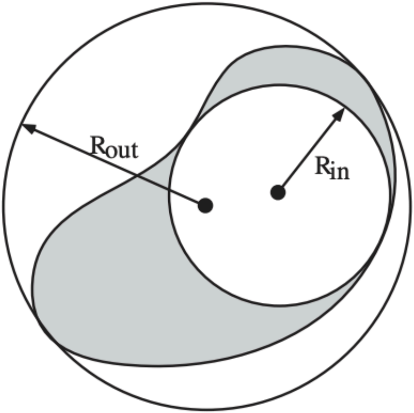


Fig. 2. The disks realizing the circumradius, R_{out}, and inradius, R_{in}, of K.

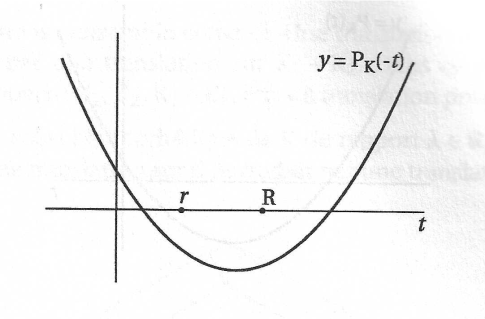
Then we have the inequalities

$$\frac{L - \sqrt{L^2 - 4\pi S}}{2\pi} \leq r \leq R \leq \frac{L + \sqrt{L^2 - 4\pi S}}{2\pi},$$

so that

$$L^2 - 4\pi S \geq \pi^2 (R - r)^2.$$

This immediately implies that $L^2 - 4\pi S = 0$ only for the disk.
The polynomial $P_K(-t) = S - Lt + \pi t^2$ has positive real roots and the inradius and outradius are between them.


y = P_K(-t)
r
R
t

The same is true when you replace the disk by any bounded convex domain but of course you have to adapt the definitions of inradius and outradius.

In the end you have $v_1^2 = v_0 v_2$ only when K_1 and K_2 are homothetic up to translation. Note that K_2 is affected by rotations if it is not the disk.

The polynomial property of volume is also true in space, and in fact in any dimension, for the sum of any two convex bodies. In dimension 3 we get:

$$\mathrm{Vol}(t_1K_1 + t_2K_2) = v_0t_1^3 + 3v_1t_1^2t_2 + 3v_2t_1t_2^2 + v_3t_2^3,$$

The coefficients $v_i = \mathrm{Vol}(K_1^{[3-i]}, K_2^{[i]})$ are called *mixed volumes* for obvious reason, and

$v_0 = \mathrm{Vol}(K_1^{[3]}) = \mathrm{Vol}(K_1)$, $v_3 = \mathrm{Vol}(K_2^{[3]}) = \mathrm{Vol}(K_2)$.

There are two main results, one numerical and one geometric, The numerical one consists of the inequalities: (**Alexandrov-Fenchel inequalities**)

$$\frac{v_0}{v_1} \leq \frac{v_1}{v_2} \leq \frac{v_2}{v_3}.$$

Taking the square of the first ratio, it follows that $(\frac{v_0}{v_1})^2 \leq \frac{v_1}{v_3}$, or

$$v_1^3 \geq v_0^2 v_3$$

.

When K_2 is the unit ball $\mathbf{B}$ in space, the coefficient $3v_1$, which is the derivative at $t = 0$ of the polynomial $P_K(t) = \mathrm{Vol}(K + t\mathbf{B})$, is equal to the area S of the boundary of the body K, so the preceding inequality $v_1^3 \geq v_0^2 v_3$ becomes

$$S^3 \geq 27\mathrm{Vol}(K)^2\mathrm{Vol}(\mathbf{B}),$$

or, since $\mathrm{Vol}(\mathbf{B}) = \frac{4\pi}{3}$,

$$S^3 \geq 36\pi\mathrm{Vol}(K)^2,$$

which is the isoperimetric inequality for bodies in space.
Again, if K is convex, there is equality only if K is a ball.

For the geometric results I will be somewhat more vague.

- Up to multiplication by universal constants, $v_1 = \mathrm{Vol}(K^{[2]}, \mathbf{B}^{[1]})$ is the average area of the orthogonal projections of K to planes and $v_2 = \mathrm{Vol}(K^{[1]}, \mathbf{B}^{[2]})$ is the average length of its projections onto lines.

- The average area of the intersection of K with affine planes and the average length of its intersections with affine lines are also determined by these mixed volumes.

Hadwiger's Theorem

A *real valuation* V on the set of plane convex domains or bodies in space associates to every convex K a non negative real number $V(K)$ such that

$$V(K_1 \cup K_2) = V(K_1) + V(K_2) - V(K_1 \cap K_2)$$

and the valuation of the empty set is 0.

In addition we ask that $V(gK) = V(K)$ where g is any rigid motion in the plane or space (combination of translations and rotations) and that it is continuous with respect to a distance between sets which I shall not define.
It is obvious that the area or volume satisfy this. It is less obvious that the mixed area and mixed volumes with the ball also satisfy it. It is even less obvious that we have:

Hadwigers' Theorem:
A valuation V in the plane or space is a combination with real coefficients of the mixed areas or volumes with the unit disk or ball.

In the plane any valuation is, up to an additive constant, a combination of area and perimeter. In space

$$V(K) = \sum_{i=0}^{3} c_i \mathrm{Vol}(K^{[i]}, \mathbf{B}^{[3-i]}),\ c_i \in \mathbf{R}.$$

In particular the area and the volume are, up to multiplication by a constant, the only valuations such that $V(tK) = t^2 V(K)$ and $V(tK) = t^3 V(K)$ respectively.

Most of what I have shown you, in particular Hadwiger's Theorem, extends to arbitrary dimensions. It even extends to *polyconvexes*, which are finite unions of convex subsets of affine space;

The next few slides are about the polynomial character of the volume of a Minkowski sum.

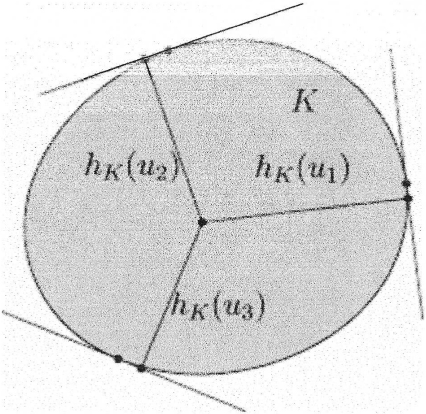
Support function

Let $K \subset \mathbf{R}^2$ be our domain. Consider all affine functions on $\mathbf{R}^2$, say $u\colon \mathbf{R}^2 \to \mathbf{R},\ (x, y) \mapsto u(x, y) = ax + by + c$ with a, b, c in **R**.

Now define a function on the space of linear functions, which has coordinates a, b, c:

$$h_K(u) = \max(u(x), x \in K).$$

Support function: picture



In fact a convex domain is completely determined by its support function in view of the following formula:

$$K = \{x \in \mathbf{R}^2 / u(x) \leq h_K(u) \text{ for all affine functions } u\}.$$

This formula may be hard to process immediately, but if you think about it later you will realize that its meaning is exactly that a convex domain or body is the intersection of the half-spaces which contain it.

We see that

$$h_{K_1+K_2}(u) = h_{K_1}(u) + h_{K_2}(u)$$

and

$$h_{tK}(u) = th_K(u).$$

Idea of the proof of the polynomial character of area

The proof is based on approximating the boundary of K by polygons: take finitely many points on the boundary of K and the intersection of the half-spaces which contain them. Its boundary is a polygon P whose sides S_i have some length ℓ_i. Now S_i is the set of points where some affine function u_i takes its maximum value $h_P(u_i)$, where h_P is the support function of P. The area of P is the sum $\sum_i h_P(u_i)\ell_i$ because the area of a triangle is length of base times height.

We approximate $K_1 + tK_2$ by $P_1 + tP_2$ and using the additivity and homogeneity of the support function show that its area is a polynomial of degree 2 in t. Then we have to take the limit of polynomials *of the same degree*. It is a polynomial of the same degree.

The same proof works in all dimensions.

Replace in this picture the domain by a convex polygon.

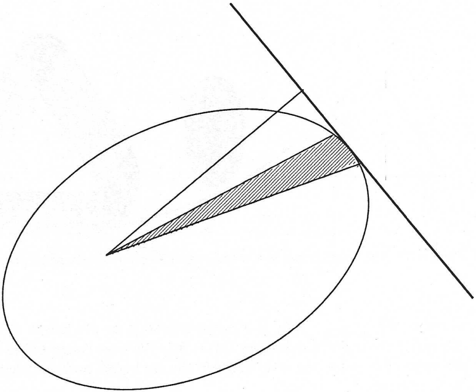


Figure 19: Area of a convex body in terms of the support function

Suppose you need to know the length of a curve like this, but it is in a small area and you do not have the room to uncoil it!

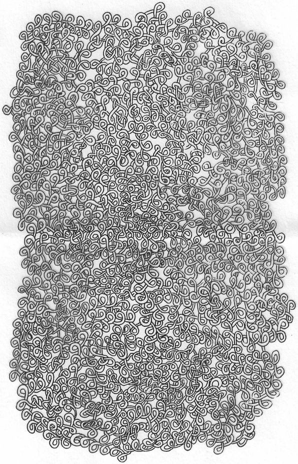


Figure 12: an example of a rather complex curve (thickness has been added for aesthetic reasons).

You are saved by **Crofton's formula**, which tells you that the length of a plane curve C is proportional to the average number of its points of intersection with lines:

$$\mathrm{Length}(C) = a \int n(C \cap \ell) d\ell,$$

where a is a constant which is chosen so that the value of the right hand side, when C is a segment, is the length of the segment, say in meters. If you count the numbers of intersections of C with, say, 30 quite different lines, you get an estimate of the length of the curve.

This is a special case of results which have many applications: estimating the volume of pulmonary vesicles, the area of the lungs, the average volume of the pores in porous media, which is important in geology and oil prospection, and many more.

A page of vocabulary

To understand this the mathematician will think as follows: $d\ell$ is an *invariant measure*, unique up to multiplication by a scalar, on the *space of affine lines in the plane*, the *affine grassmanian*, which is two dimensional. The set of lines which intersect C is *measurable* and the function which to a line associates its number of intersections with C is *integrable* and finally the *integral* is a linear function of the length of the curve. These are notions you meet in many fields of Mathematics. This particular field is called *Integral Geometry* or *Geometric Probability*.

But all this is about things you will learn if you continue to study Mathematics!

THANK YOU FOR YOUR ATTENTION

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