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Extending valuations of local domains to complete local domains without changing the value group.

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Abstract

Let (R, m, k) be an excellent local noetherian domain with field of fractions K. Let

$$\nu : K^* \twoheadrightarrow \Gamma$$

be a valuation centered at R and let R_ν be the corresponding valuation ring of K, dominating R. Denote by $\widehat{R}$ the m-adic completion of R. In the applications of valuation theory to commutative algebra and the study of singularities, one is often induced to replace R by its m-adic completion $\widehat{R}$ and ν by a suitable extension $\widehat{\nu}_-$ to $\frac{\widehat{R}}{P}$ for a suitably chosen prime ideal P, such that

$$P \cap R = (0).$$

In a previous article we gave a systematic description of all such extensions $\widehat{\nu}_-$ and defined the notion of tight extensions that are of particular interest for applications (see Herrera, Olalla, Spivakovsky and Teissier, *Extending a valuation centered in a local domain to its formal completion*, Proc. London Math. Soc. (3) 105 (2012) 571–621). If $\widehat{\nu}_-$ is a tight extension then its graded algebra is birational to that of ν (the converse is not known and might not be true). In particular, the value group of $\widehat{\nu}_-$ is Γ. The existence of tight extensions was conjectured by the last author (see Teissier, *Valuations, deformations, and toric geometry*, Fields Institute Communications, **33**, 2003, 361-459). In the present paper we give a proof of Teissier's conjecture. An intended application of this result is an important step in two recent approaches to local uniformization in positive characteristic.

1. *Introduction*

All the rings in this paper will be commutative with 1.

Let (R, m, k) be a local noetherian domain with field of fractions K and R_ν a valuation ring, birationally dominating R. Let $\nu : K^* \twoheadrightarrow \Gamma$ be the corresponding valuation of K, centered at R. Here Γ is the value group; it is totally ordered and abelian. Let $\widehat{R}$ denote the m-adic completion of R. In the applications of valuation theory to commutative algebra and the study of singularities, one is often induced to replace R by its m-adic completion $\widehat{R}$ and ν by a suitable extension $\widehat{\nu}_-$ to $\frac{\widehat{R}}{P}$ for a suitably chosen prime ideal P, such that $P \cap R = (0)$. In the paper [**9**] we gave, assuming that R is excellent, a systematic description of all such extensions $\widehat{\nu}_-$ and among them identified the class of tight extensions. The existence of tight extensions (see Conjecture 1 of [**9**]) is the restatement in the framework of [**9**] of a conjecture of the last author going back to 2003 (see [**16**], *proposition* 5.19).

The main result of the present paper a proof of Teissier's conjecture:
Given an excellent local domain R and a valuation ν of R that is strictly positive on its maximal ideal m, there exists a prime ideal P of the m-adic completion $\widehat{R}$

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such that $P\bigcap R=(0)$ and an extension $\hat\nu_-$ of ν to $\frac{\widehat R}{P}$ such that the natural inclusion

$$\mathrm{gr}_\nu R\subset \mathrm{gr}_{\hat\nu_-}\widehat R/P \tag{1.1}$$

of the associated graded rings is scalewise-birational (see the definitions below)**. In particular, the valuations ν and $\hat\nu_-$ have the same value group.**

The only assumption about R we ever used in [**9**] is a weaker and more natural condition than excellence: namely, we require the completion homomorphism $R\to\widehat R$ to be regular. Local rings R having this property are called G-rings or quasi-excellent.

One specific application we have in mind has to do with the approaches to proving the Local Uniformization Theorem in arbitrary characteristic which are described in [**15**] and [**16**]).

These approaches to local uniformization both make important use, in different ways, of the structure of the graded ring $\mathrm{gr}_\nu R$ associated to the filtration of an excellent local domain R defined by the valuation ν. Extending ν to a valuation $\hat\nu_-$ of a complete local domain $\widehat R/P$ in such a manner that the corresponding extension $\mathrm{gr}_\nu R\subset \mathrm{gr}_{\hat\nu_-}\widehat R/P$ of graded rings is scalewise-birational allows one to have access to the many advantages of completeness without losing the algebraicity of the centers of blowing-up (in [**15**]) or of the embeddings into appropriate larger affine spaces where a birational toric map will provide local uniformization (in [**16**]).

Let r denote the (real) rank of ν. Let $(0)=\Delta_r\subsetneqq\Delta_{r-1}\subsetneqq\dots\subsetneqq\Delta_0=\Gamma$ be the isolated subgroups of Γ and $P_0=(0)\subsetneqq P_1\subseteq\dots\subseteq P_r=m$ the prime valuation ideals of R, which need not, in general, be distinct. Assuming that R is excellent, in [**9**] we canonically associated to ν a chain of $2r+2$ prime ideals

$$H_0\subset H_1\subset\dots\subset H_{2r}=H_{2r+1}=m\widehat R,$$

satisfying $H_{2\ell}\cap R=H_{2\ell+1}\cap R=P_\ell$ and such that $H_{2\ell}$ is a minimal prime of $P_\ell\widehat R$ for $0\le\ell\le r$. We called H_i the ***i*-th implicit prime ideal** of $\widehat R$, associated to R and ν. The role of these ideals is explained at the beginning of section 2.1 below.
The ideals H_i behave well under local blowing ups along ν (that is, birational local homomorphisms $R\to R'$ such that ν is centered in R'). This means that given a local blowing up $R\to R'$ along ν, the i-th implicit prime ideal H'_i of $\widehat{R'}$ has the property that $H'_i\cap\widehat R=H_i$. In this situation we say that the collection $\{H'_i\}$ forms a **tree** (see subsection 1.1).

For a prime ideal P in a ring R, $\kappa(P)$ will denote the residue field $\frac{R_P}{PR_P}$.

Let $(0)=\mathbf{m}_0\subsetneqq\mathbf{m}_1\subsetneqq\dots\subsetneqq\mathbf{m}_{r-1}\subsetneqq\mathbf{m}_r=\mathbf{m}_\nu$ be the prime ideals of the valuation ring R_ν. By definitions, our valuation ν is a composition of r rank one valuations, $\nu=\nu_1\circ\nu_2\dots\circ\nu_r$, where ν_ℓ is a valuation of the field $\kappa(\mathbf{m}_{\ell-1})$, centered at $\frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}(R_\nu)_{\mathbf{m}_\ell}}$. In particular, ν_ℓ defines a valuation on $\frac{R}{P_{\ell-1}}$ with value group $\frac{\Delta_{\ell-1}}{\Delta_\ell}$ by restriction (see [**21**], Chapter VI, §10, p. 43 for the definition of composition of valuations; more information is given below in subsection 1.1, where we interpret each $\mathbf{m}_\ell$ as the limit of a tree of ideals). For each integer $\ell\in\{1,\dots,r\}$, let $\mu_\ell:=\nu_\ell\circ\nu_{\ell+1}\circ\dots\circ\nu_r$; it is the residual valuation induced by ν on $\frac{R}{P_{\ell-1}}$.

1.1. *Trees.*

We consider birational ν-*extensions* $R\to R'$ of local rings, that is, injective homomorphisms such that R' is an R-algebra essentially of finite type with field of fractions K, dominated by R_ν (in particular, we have $m'\cap R=m$). Such extensions form a direct system $\{R'\}$ with

$$\lim_{\overrightarrow{R'}}R'=R_\nu. \tag{1.2}$$

We will consider many direct systems of rings and of ideals indexed by $\{R'\}$; direct limits will always be taken with respect to the direct system $\{R'\}$.

DEFINITION 1. A **tree** of R'-algebras is a direct system $\{S'\}$ of rings, indexed by the directed set $\{R'\}$, where S' is an R'-algebra. Note that the tree maps are not required to be injective or birational. A morphism $\{S'\} \to \{T'\}$ of trees is the datum of a map of R'-algebras $S' \to T'$ for each R' commuting with the tree morphisms $S' \to S''$ and $T' \to T''$ for each map $R' \to R''$.

DEFINITION 2. Let $\{S'\}$ be a tree of R'-algebras. For each S', let I' be an ideal of S'. We say that $\{I'\}$ is a **tree of ideals** if for every arrow $b_{S'S''}\colon S' \to S''$ in our direct system, we have $b^{-1}_{S'S''}I'' = I'$. We have the obvious notion of inclusion of trees of ideals. In particular, we may speak about chains of trees of ideals.

EXAMPLES. For each non-negative element $\beta \in \Gamma$, the valuation ideals $\mathcal{P}'_\beta \subset R'$ of value β form a tree of ideals of $\{R'\}$. Similarly, the i-th prime valuation ideals $P'_i \subset R'$ form a tree. If $rk\ \nu = r$, the prime valuation ideals P'_i give rise to a chain

$$(0) = P'_0 \subsetneqq P'_1 \subseteq \cdots \subseteq P'_r = m' \tag{1.3}$$

of trees of prime ideals of $\{R'\}$.

In [**9**] we proved that the implicit prime ideals H'_i form a tree of ideals of $\widehat{R}$.

We showed that specifying an extension $\widehat{\nu}_-$ of ν as above is equivalent to specifying a chain

$$\tilde{H}'_0 \subset \tilde{H}'_1 \subset \cdots \subset \tilde{H}'_{2r} = m'\widehat{R}' \tag{1.4}$$

of trees of prime valuation ideals of $\widehat{R'}$ such that $H'_\ell \subset \tilde{H}'_\ell$ for all $\ell \in \{0, \dots, 2r\}$, and valuations $\widehat{\nu}_1, \widehat{\nu}_2, \dots, \widehat{\nu}_{2r}$, where $\widehat{\nu}_i$ is a valuation of the field $k_{\widehat{\nu}_{i-1}}$ (the residue field of the valuation ring $R_{\widehat{\nu}_{i-1}}$), arbitrary when i is odd and satisfying certain conditions, coming from the valuation $\nu_{\frac{i}{2}}$, when i is even.

Recall that for each $\beta \in \Gamma$ one associates to β the valuation ideals

$$\begin{aligned} \mathcal{P}_\beta(R) = &\quad \{x \in R/\nu(x) \geq \beta\} \\ \mathcal{P}^+_\beta(R) = &\quad \{x \in R/\nu(x) > \beta\} \end{aligned}$$

of R (with the convention that $\nu(0) = \infty$, an element larger than any element of Γ) and the graded ring

$$\mathrm{gr}_\nu R = \bigoplus_{\beta \in \Gamma_+} \frac{\mathcal{P}_\beta(R)}{\mathcal{P}^+_\beta(R)}.$$

Note that if the ring R is noetherian, the semigroup $\Phi \subset \Gamma_{\geq 0}$ of values of ν on $R \setminus \{0\}$ is well ordered. If $R \to R'$ is a birational extension of local domains dominated by R_ν, we will write $\mathcal{P}'_\beta$ for $\mathcal{P}_\beta(R')$.

NOTATION. Let $\mathcal{T} = \{R'\}$, the tree defined above. For a ring $R' \in \mathcal{T}$, we shall denote by $\mathcal{T}(R')$ the subtree of $\mathcal{T}$ consisting of all the ν-extensions R'' of R'.

Let

$$\Gamma \hookrightarrow \widehat{\Gamma} \tag{1.5}$$

be an extension of ordered groups of the same rank. Let

$$(0) = \widehat{\Delta}_r \subsetneqq \widehat{\Delta}_{r-1} \subsetneqq \cdots \subsetneqq \widehat{\Delta}_0 = \widehat{\Gamma}$$

be the isolated subgroups of $\widehat{\Gamma}$, so that the inclusion (1.5) induces inclusions

$$\Delta_\ell \hookrightarrow \widehat{\Delta}_\ell \quad \text{and} \tag{1.6}$$

$$\frac{\Delta_\ell}{\Delta_{\ell+1}} \hookrightarrow \frac{\widehat{\Delta}_\ell}{\widehat{\Delta}_{\ell+1}}. \tag{1.7}$$

Let $G \hookrightarrow \widehat{G}$ be an extension of graded algebras without zero divisors, such that G is graded by Γ_+ and $\widehat{G}$ by $\widehat{\Gamma}_+$. The graded algebra G is endowed with a natural valuation with value group Γ and similarly for $\widehat{G}$ and $\widehat{\Gamma}$. Both of these natural valuations will be denoted by *ord*.

Definition 3. We say that the extension $G \hookrightarrow \widehat{G}$ is **scalewise-birational** if $G_0 = \widehat{G}_0$ and for all homogeneous elements $x \in \widehat{G}$ and $\ell \in \{1, \dots, r-1\}$ such that $ord\ x \in \widehat{\Delta}_\ell$ there exists $y \in G$ such that $ord\ y \in \Delta_\ell$ and $xy \in G$.

Consider a tree of prime ideals H' of $\widehat{R'}$ with $H' \cap R' = (0)$ and a valuation $\widehat{\nu}_-$, centered at $\lim\limits_{\to} \frac{\widehat{R'}}{H'}$.

Definition 4. We say that $\hat{\nu}_-$ is a **scalewise-birational extension of** ν if for each R' the inclusion

$$\mathrm{gr}_\nu R' \subset \mathrm{gr}_{\hat{\nu}_-} \widehat{R'}/H' \tag{1.8}$$

is scalewise-birational.

Of course, if $\hat{\nu}_-$ is a scalewise-birational extension of ν then the inclusion (1.8) is birational in the usual sense (that is, induces equality of fraction fields) which, in turn, implies that $\widehat{\Gamma} = \Gamma$.

The main result of the present paper is:

Theorem 1.1. *There exists a tree of prime ideals $\tilde{H}'_0$ of $\widehat{R'}$ with $\tilde{H}'_0 \cap R' = (0)$ and a scalewise-birational extension $\widehat{\nu}_-$, centered at $\lim\limits_{\to} \frac{\widehat{R'}}{\tilde{H}'_0}$.*

The results of [**2**] and the example given in Remark 5.20, 4) of [**16**] shows that the morphism of associated graded rings is not an isomorphism in general.

Acknowledgement. We acknowledge earlier work in this area by W. Heinzer and J. Sally [**7**] (which deals with the case dim $R = 2$) as well as by the school of S. D. Cutkosky [**3**, **4**] that helped inspire some of the authors to think about the subject.

2. *A review of some of the main definitions and results from [7]*

2.1. *Definition and first properties of implicit prime ideals*

Let $0 \leq \ell \leq r$. We recall the definition of one of the main objects in the theory, the *implicit prime ideals* of ν in $\widehat{R}$. Broadly speaking, the odd-numbered implicit prime ideals represent the fact that elements of $\widehat{R}$ can be limits of sequences of elements of R of ever increasing valuation, tending to infinity at least for one of the valuations with which ν is composed, while even-numbered implicit ideals represent the fact that while some quotients $\frac{R}{P_\ell}$ may not be analytically irreducible, the valuation ν "chooses" one of the irreducible components of the completion. These representations must then be "stabilized" with respect to the tree $\mathcal{T}$ in the way shown by equation (2.1) below.

The $(2\ell+1)$-st **implicit prime ideal** $H_{2\ell+1} \subset \widehat{R}$ is defined by

$$H_{2\ell+1} = \bigcap_{\beta\in\Delta_\ell} \left(\left(\lim_{\overrightarrow{R'}} \mathcal{P}'_\beta \widehat{R'}\right) \bigcap \widehat{R}\right), \tag{2.1}$$

where R' ranges over $\mathcal{T}$. As usual, we think of (2.1) as a tree equation: if we replace R by any other $R'' \in \mathcal{T}$ in (2.1), it defines the corresponding ideal $H''_{2\ell+1} \subset \widehat{R''}$.

Proposition 2.1. *([**9**], Proposition 3.1) We have $H'_{2\ell+1} \cap R' = P'_\ell$.*

The implicit ideal $H_{2\ell}$ is defined as the unique minimal prime ideal of $P_\ell\widehat{R}$ contained in $H_{2\ell+1}$.

Here the uniqueness follows from the fact that R being excellent (or a G-ring), the ring $\widehat{R} \otimes_R \kappa(P_\ell)$ is regular, hence a domain. Thus the only minimal prime of $\widehat{R} \otimes_R \kappa(P_\ell)$ is (0) and hence the unique minimal prime of $P_\ell\widehat{R}$ in $\widehat{R}$ is the kernel of the natural map $\widehat{R} \to \widehat{R} \otimes_R \kappa(P_\ell)$.

In ([**9**], Theorem 8.1) we prove the primality of the implicit ideals.

2.2. *A classification of extensions of ν to $\widehat{R}$.*

One is naturally led to consider the more general problem of extending ν not only to rings of the form $\frac{\widehat{R}}{P}$ but also to the ring $\lim\limits_{\overrightarrow{R'}} \frac{\widehat{R'}}{P'}$, where P' is a tree of prime ideals of $\widehat{R'}$, such that $P' \cap R' = (0)$.

In §5 of [**9**], we gave a systematic description of all the possible extensions $\widehat{\nu}_-$ of ν to $\lim\limits_{\overrightarrow{R'}} \frac{\widehat{R'}}{P'\widehat{R'}}$, as compositions of $2r$ valuations,

$$\widehat{\nu}_- = \widehat{\nu}_1 \circ \cdots \circ \widehat{\nu}_{2r}, \tag{2.2}$$

satisfying certain conditions. We associate to each extension $\widehat{\nu}_-$ of ν a chain

$$\tilde{H}'_0 \subset \tilde{H}'_1 \subset \cdots \subset \tilde{H}'_{2r} = m'\widehat{R'} \tag{2.3}$$

of prime $\widehat{\nu}_-$-ideals, corresponding to the decomposition (2.2), satisfying

$$\tilde{H}'_{2\ell} \cap R' = \tilde{H}'_{2\ell+1} \cap R' = P'_\ell. \tag{2.4}$$

.

By definitions, for $1 \le i \le 2r$, $\widehat{\nu}_i$ is a valuation of the field $k_{\widehat{\nu}_{i-1}}$.

Proposition 2.2. *([**9**], Proposition 5.3) We have*

$$H'_i \subset \tilde{H}'_i \qquad \textit{for all } i \in \{0, \dots, 2r\}. \tag{2.5}$$

One of the main theorems of [**9**] (Theorem 5.4) says that specifying the valuation $\widehat{\nu}_-$ is equivalent to specifying the following data:

(1) A chain of trees (2.3) of prime ideals of $\widehat{R'}$ satisfying (2.5) and some additional conditions.

(2) For each i, $1 \le i \le 2r$, a valuation $\widehat{\nu}_i$ of $k_{\widehat{\nu}_{i-1}}$ (where $\widehat{\nu}_0$ is taken to be the trivial valuation by convention), whose restriction to $\lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_{i-1})$ is centered at the local ring $\lim\limits_{\overrightarrow{R'}} \frac{\widehat{R'}_{\tilde{H}'_i}}{\tilde{H}'_{i-1}\widehat{R'}_{\tilde{H}'_i}}$.

The data $\{\widehat{\nu}_i\}_{1\le i\le 2r}$ is subject to the following additional condition: if $i = 2\ell$ is even then $rk\ \widehat{\nu}_i = 1$ and $\widehat{\nu}_i$ is an extension of ν_i to $k_{\widehat{\nu}_{i-1}}$ (which is naturally an extension of $k_{\nu_{i-1}}$).

In particular, such extensions $\widehat{\nu}_-$ always exist, and usually there are plenty of them.

2.3. *Tight extensions and scalewise birationality*

In §6 of [**9**] we define and study, among other things, a class of extensions $\widehat{\nu}_-$ which are particularly useful for the applications called the **tight** extensions (see [**9**, Definition 6.1]). One of the properties of tight extensions is the fact that

$$\tilde{H}'_{2\ell} = \tilde{H}'_{2\ell+1} \quad \text{for } 1 \leq \ell \leq r. \tag{2.6}$$

PROPOSITION 2.3. *([**9**], Proposition 6.7) The extension $\widehat{\nu}_-$ is tight if and only if for each R' in our direct system the natural graded algebra extension $\mathrm{gr}_\nu R' \to \mathrm{gr}_{\widehat{\nu}_-}\widehat{R'}$ is scalewise-birational.*

REMARK 1. Proposition 2.3 allows us to rephrase Conjecture 1 of [**9**] as follows: the valuation ν admits at least one tight extension $\widehat{\nu}_-$.

The only material from [**9**] used in the proof of the main theorem of the present paper is the definition (2.1) of implicit ideals, their primality (Theorem 8.1) and the equality

$$H'_{2\ell+1} \cap R' = P'_\ell$$

(Proposition 3.1), but not the somewhat technical Theorem 5.4.

3. *A proof of the existence of a scalewise-birational extension $\hat{\nu}_-$*

The purpose of this section is to prove Theorem 1.1 by constructing a valuation $\widehat{\nu}_-$ whose associated graded algebra is a scalewise-birational extension of $\mathrm{gr}_\nu R$.

The construction consists of describing the trees of ideals $\tilde{H}'_i$, $0 \leq i \leq 2r$ and, for each i, a valuations $\widehat{\nu}_i$ of the residue field $k_{\widehat{\nu}_{i-1}}$, such that $\widehat{\nu}_- = \widehat{\nu}_1 \circ \dots \widehat{\nu}_{2r}$. More precisely, for $\ell \in \{1, \dots, r\}$, we will construct, recursively in the descending order of ℓ, a tree $\tilde{H}'_{2\ell-1}$ of prime ideals of $\widehat{R'}$, $R' \in \mathcal{T}$, such that $\tilde{H}'_{2\ell-1} \cap R' = P'_{\ell-1}$, and a scalewise-birational extension $\widehat{\mu}_{2\ell}$ of μ_ℓ to $\lim\limits_{\overrightarrow{R' \in \mathcal{T}}} \frac{\widehat{R'}}{\tilde{H}'_{2\ell-1}}$. The valuation $\widehat{\mu}_2$ will be the desired scalewise-birational extension $\widehat{\nu}_-$ of $\mu_1 = \nu$.

We will have $\tilde{H}'_{2\ell} = \tilde{H}'_{2\ell+1}$.

Let $\widehat{\mathbf{R}} := \lim\limits_{\overrightarrow{R' \in \mathcal{T}}} \widehat{R'}$ and $\mathbf{H}_i = \lim\limits_{\overrightarrow{R' \in \mathcal{T}}} \tilde{H}'_i$. Then $\hat{\nu}_{2\ell+1}$ will be the trivial valuation of the field $\frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell+1}}}{\mathbf{H}_{2\ell}\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell+1}}}$.

We define $\tilde{H}'_{2r} = \tilde{H}'_{2r+1} = m'\widehat{R'}$ and let $\widehat{\mu}_{2r+2} = \widehat{\nu}_{2r+2}$ be the trivial valuation of the residue field k'.

Next, assume that $\ell \in \{1, \dots, r\}$, that the tree $\left\{\tilde{H}'_{2\ell+1}\right\} \supset \left\{H'_{2\ell+1}\right\}$ of prime ideals of $\widehat{R'}$ and a scalewise-birational extension $\widehat{\mu}_{2\ell+2}$ of $\mu_{\ell+1}$ to $\lim\limits_{\overrightarrow{R'}} \frac{\widehat{R'}}{\tilde{H}'_{2\ell+1}}$ are already constructed for $R' \in \mathcal{T}$ and that

$$\tilde{H}'_{2\ell+1} \cap R' = P'_\ell.$$

Define $\tilde{H}'_{2\ell} := \tilde{H}'_{2\ell+1}$ and let $\widehat{\nu}_{2\ell+1}$ be the trivial valuation of the field $\frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell+1}}}{\mathbf{H}_{2\ell}\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell+1}}}$.

It remains to construct the ideals $\tilde{H}'_{2\ell-1} \subset \widehat{R'}$ and a scalewise-birational extension $\widehat{\mu}_{2\ell}$ of μ_ℓ to $\frac{\widehat{R'}}{\tilde{H}'_{2\ell-1}}$ for R' in $\mathcal{T}$.

Consider the set $\mathcal{I}$ of all the trees of prime ideals $\{H'\}$ of $\left\{\widehat{R}'\right\}$ contained in $\tilde{H}'_{2\ell}$ and satisfying the condition

$$H' \cap R' = P'_{\ell-1}. \tag{3.1}$$

The collection $\mathcal{I}$ is not empty since the tree $\{H'_{2\ell-1}\}$ belongs to it by Proposition 2.1 (Proposition 3.1 of [**9**]). Let $\left\{\tilde{H}'_{2\ell-1}\right\}$ be a maximal (in the sense of inclusion) element of $\mathcal{I}$. Here by "maximal" we mean that there is no tree $\{I'\}$ in $\mathcal{I}$ containing $\left\{\tilde{H}'_{2\ell-1}\right\}$ such that $\left\{\tilde{H}'_{2\ell-1}\right\} \subsetneqq I'$ for some I' in $\mathcal{I}$ and hence for all I' that are sufficiently far out in the tree. Such a maximal tree $\left\{\tilde{H}'_{2\ell-1}\right\}$ exists by Zorn's Lemma.

By the induction assumption, the valuation $\widehat{\mu}_{2\ell+2}$ is already defined on $\frac{\widehat{\mathbf{R}}}{\mathbf{H_{2\ell+1}}}$ and, for all $R' \in \mathcal{T}$, the inclusion $\mathrm{gr}_{\mu_{\ell+1}} \frac{R'}{P'_\ell} \subset \mathrm{gr}_{\widehat{\mu}_{2\ell+2}} \frac{\widehat{R'}}{\tilde{H}'_{2\ell+1}}$ of graded algebras is scalewise-birational.

It remains to describe the valuation $\widehat{\mu}_{2\ell}$ centered in $\frac{\widehat{R'}}{\tilde{H}'_{2\ell-1}}$, $R' \in \mathcal{T}$. We will first define $\widehat{\nu}_{2\ell}$ and then define $\widehat{\mu}_{2\ell}$ as $\widehat{\nu}_{2\ell} \circ \widehat{\mu}_{2\ell+2}$.

Lemma 3.1. *Let μ be a rank one valuation, centered in a local noetherian domain (A, M). Let*

$$\Psi = \mu(A \setminus \{0\}).$$

Then Ψ contains no infinite strictly increasing bounded sequences.

Proof. For an element $\beta \in \Psi$, let $\mathcal{P}_\beta := \mathcal{P}_\beta(A)$ denote the μ-ideal of A of value β. An infinite ascending sequence $\alpha_1 < \alpha_2 < \dots$ in Ψ, bounded above by an element $\beta \in \Psi$, would give rise to an infinite descending chain of ideals in $\frac{A}{\mathcal{P}_\beta}$. Thus it is sufficient to prove that $\frac{A}{\mathcal{P}_\beta}$ has finite length.

Let $\delta := \mu(M)(= \min(\Psi \setminus \{0\}))$. Since $\mathrm{rk}\ \mu = 1$, there exists $n \in \mathbf{N}$ such that $\beta \le n\delta$. Then $M^n \subset \mathcal{P}_\beta$, so there is a surjective homomorphism $\frac{A}{M^n} \twoheadrightarrow \frac{A}{\mathcal{P}_\beta}$. Thus $\frac{A}{\mathcal{P}_\beta}$ has finite length, as desired. $\square$

Remark 2. See [**5**, Theorem 3.2] for a more general result about the absence of accumulation points in semigroups.

Corollary 3.2. *The semigroup $\nu\left(\frac{R'_{P'_\ell}}{P'_{\ell-1}} \setminus \{0\}\right)$ is isomorphic to $\mathbf{N}$ as an ordered set.*

Proof. Since ν_ℓ is a rank 1 valuation, centered in the local noetherian domain $\frac{R'_{P'_\ell}}{P'_{\ell-1}}$, the semigroup $\nu\left(\frac{R'_{P'_\ell}}{P'_{\ell-1}} \setminus \{0\}\right)$ contains no infinite strictly increasing bounded sequences. The corollary now follows from the fact that $\nu\left(\frac{R'_{P'_\ell}}{P'_{\ell-1}} \setminus \{0\}\right)$ is well ordered. $\square$

LEMMA 3.3. *Fix a ring R' in $\mathcal{T}$ and a non-zero element $x \in \frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R'}_{\tilde{H}'_{2\ell}}}$. Then*

$$(x)\bigcap \frac{R'_{P'_\ell}}{P'_{\ell-1}} \neq (0).$$

Proof. Let $Q_1,\dots,Q_s$ be the set of minimal primes of (x) in $\frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R'}_{\tilde{H}'_{2\ell}}}$. It is sufficient to prove that

$$Q_i \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}} \neq (0) \quad \text{for all} \quad i \in \{1,\dots,s\}. \tag{3.2}$$

Indeed, once (3.2) is established, let y_i be a non-zero element of $Q_i \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}}$. Then there exists $N \in \mathbf{N}$ such that $0 \neq \left(\prod_{i=1}^{s} y_i\right)^N \in (x)\bigcap \frac{R'_{P'_\ell}}{P'_{\ell-1}}$, as desired.

To prove (3.2), we argue by contradiction. Assume that (x) has a minimal prime Q' in $\frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R'}_{\tilde{H}'_{2\ell}}}$ satisfying

$$Q' \cap \frac{R'_{P'_\ell}}{P'_{\ell-1}} = (0).$$

For every ν-extension $R' \to R''$, there exists a minimal prime Q'' of (x) in $\frac{\widehat{R''}_{\tilde{H}''_{2\ell}}}{\tilde{H}''_{2\ell-1}\widehat{R''}_{\tilde{H}''_{2\ell}}}$ lying over Q' and satisfying

$$Q'' \cap \frac{R''_{P''_\ell}}{P''_{\ell-1}} = (0). \tag{3.3}$$

By Zorn's lemma, there exists a tree $\{P''\}$ of prime ideals of $\left\{\frac{\widehat{R''}_{\tilde{H}''_{2\ell}}}{\tilde{H}''_{2\ell-1}\widehat{R''}_{\tilde{H}''_{2\ell}}}\right\}$ satisfying (3.3). Taking the preimage of $\{P''\}$ in $\left\{\widehat{R''}\right\}$ produces a tree $\{H''\}$ satisfying (3.1) and properly containing the tree $\left\{\tilde{H}''_{2\ell-1}\right\}$, contradicting the maximality of $\left\{\tilde{H}''_{2\ell-1}\right\}$. This completes the proof of the lemma. $\square$

For a positive element $\bar{\beta} \in \frac{\Delta_{\ell-1}}{\Delta_\ell}$ denote by $\widehat{\mathcal{P}}'_{\bar{\beta},\ell}$ the ideal

$$\widehat{\mathcal{P}}'_{\bar{\beta},\ell} = \frac{\left(\mathcal{P}'_{\bar{\beta}} + \tilde{H}'_{2\ell-1}\right)\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R'}_{\tilde{H}'_{2\ell}}}. \tag{3.4}$$

Since by the definition of $\tilde{H}'_{2\ell-1}$ we have $\bigcap_{\bar{\beta}} \widehat{\mathcal{P}}'_{\bar{\beta},\ell} = (0)$, we can define the pseudo-valuation $\widehat{\nu}_{2\ell}$ centered at $\frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R'}_{\tilde{H}'_{2\ell}}}$ by

$$\widehat{\nu}_{2\ell}(x) := \max\left\{ \bar{\beta} \in \left(\frac{\Delta_{\ell-1}}{\Delta_\ell}\right)_+ \;\middle|\; x \in \widehat{\mathcal{P}}'_{\bar{\beta},\ell} \right\}. \tag{3.5}$$

LEMMA 3.4. *We have*

$$\widehat{\mathcal{P}}'_{\bar{\beta}}\bigcap \frac{R'_{P'_\ell}}{P'_{\ell-1}} = \mathcal{P}'_{\bar{\beta}}. \tag{3.6}$$

Proof. The inclusion $\supset$ is clear from definitions. Let us prove the opposite inclusion.

By Zorn's lemma, there exist extensions $\tilde{\nu}$ of ν_ℓ to valuations centered in the ring $\frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R'}_{\tilde{H}'_{2\ell}}}$. Fix one such extension $\tilde{\nu}$ and let $\tilde{\mathcal{P}}'_{\bar{\beta}}$ denote the greatest $\tilde{\nu}$-ideal of $\frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R'}_{\tilde{H}'_{2\ell}}}$ among those of value at least $\bar{\beta}$. We have

$$\widehat{\mathcal{P}}'_{\bar{\beta},\ell} \subset \tilde{\mathcal{P}}'_{\bar{\beta}}, \tag{3.7}$$

so $\widehat{\mathcal{P}}'_{\bar{\beta},\ell} \bigcap \frac{R'_{P'_\ell}}{P'_{\ell-1}} \subset \tilde{\mathcal{P}}'_{\bar{\beta}} \bigcap \frac{R'_{P'_\ell}}{P'_{\ell-1}} = \mathcal{P}'_{\bar{\beta}}$, which proves the desired inclusion. □

COROLLARY 3.5. *The restriction of the pseudo-valuation $\widehat{\nu}_{2\ell}$ to $\frac{R'_{P'_\ell}}{P'_{\ell-1}}$ coincides with ν_ℓ.*

REMARK 3. The inclusion (3.7) is equivalent to saying that for every $x \in \frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R'}_{\tilde{H}'_{2\ell}}}$ we have $\tilde{\nu}(x) \le \widehat{\nu}_{2\ell}(x)$. The next lemma shows that this inequality is, in fact, an equality.

Keep the notation of the above remark. Let $\bar{\beta} = \widehat{\nu}_{2\ell}(x)$.

LEMMA 3.6. *We have*

$$\tilde{\nu}(x) = \bar{\beta}. \tag{3.8}$$

Proof. Take a non-zero element $y \in (x) \bigcap \frac{R'_{P'_\ell}}{P'_{\ell-1}}$ (such an element exists by Lemma 3.3). Let $u \in \frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R'}_{\tilde{H}'_{2\ell}}}$ be such that $y = ux$. Since $\widehat{\nu}_{2\ell}$ is a pseudo-valuation extending ν_ℓ, we have

$$\nu_\ell(y) = \widehat{\nu}_{2\ell}(y) = \widehat{\nu}_{2\ell}(xu) \ge \bar{\beta} + \widehat{\nu}_{2\ell}(u) \ge \tilde{\nu}(x) + \tilde{\nu}(u) = \tilde{\nu}(y) = \nu_\ell(y), \tag{3.9}$$

where the first inequaliity is given by the ultrametric triangle rule and the second follows from Remark 3, applied separately to x and u. Thus all the inequalities in (3.9) are equalities and the result follows. □

COROLLARY 3.7.

(i) *The pseudo-valuation $\widehat{\nu}_{2\ell}$ is, in fact, a valuation.*

(ii) *The valuation $\widehat{\nu}_{2\ell}$ is the unique valuation extending ν_ℓ to $\frac{\widehat{R'}_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R'}_{\tilde{H}'_{2\ell}}}$.*

REMARK 4. Since the construction above is valid for all R'' in $\mathcal{T}$, $\widehat{\nu}_{2\ell}$ extends naturally to a valuation $\widehat{\nu}_{2\ell} : \frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell+1}} \setminus \{0\} \to \frac{\Delta_{\ell-1}}{\Delta_\ell}$ (which we still denote by $\widehat{\nu}_{2\ell}$). The valuation $\widehat{\nu}_{2\ell}$ is centered in the local ring $\frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell+1}}$. Moreover, $\widehat{\nu}_{2\ell}$ is the unique valuation extending ν_ℓ with this property.

Let

$$\mathbf{P}_{\bar{\beta}} := \lim_{\substack{\longrightarrow \\ R' \in \mathcal{T}}} \frac{\mathcal{P}'_{\bar{\beta}} R'_{P'_\ell}}{P'_{\ell-1} R'_{P'_\ell}},$$

$$\widehat{\mathbf{P}}_{\bar\beta,\ell} := \lim_{\overrightarrow{R'\in\mathcal{T}}} \widehat{\mathcal{P}'}_{\bar\beta,\ell},$$

and similarly for $\mathbf{P}^+_{\bar\beta}$ and $\widehat{\mathbf{P}}^+_{\bar\beta,\ell}$. Recall that

$$\mathbf{m}_\ell = \lim_{\overrightarrow{R'\in\mathcal{T}}} P'_\ell.$$

By definitions (see (3.4)), for all $\bar\beta \in \left(\frac{\Delta_{\ell-1}}{\Delta_\ell}\right)_+$, we have a natural surjective homomorphism

$$\lambda_{\bar\beta} : \frac{\mathcal{P}'_{\bar\beta}}{\mathcal{P}'^+_{\bar\beta}} \otimes_{\kappa(P'_\ell)} \kappa\left(\tilde H'_{2\ell}\right) \longrightarrow \frac{\widehat{\mathcal{P}'}_{\bar\beta,\ell}}{\widehat{\mathcal{P}'}^+_{\bar\beta,\ell}} \tag{3.10}$$

of $\kappa\left(\tilde H'_{2\ell}\right)$-vector spaces. Passing to direct limits as R' runs over the tree $\mathcal{T}$, we obtain a natural surjective homomorphism

$$\lambda_{\bar\beta} : \frac{\mathbf{P}_{\bar\beta}}{\mathbf{P}^+_{\bar\beta}} \otimes_{\kappa(\mathbf{P}_\ell)} \kappa(\mathbf{H}_{2\ell}) \longrightarrow \frac{\widehat{\mathbf{P}}_{\bar\beta,\ell}}{\widehat{\mathbf{P}}^+_{\bar\beta,\ell}} \tag{3.11}$$

of $\kappa\left(\mathbf{H}_{2\ell}\right)$-vector spaces.

REMARK 5.
(i) We have

$$\frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}(R_\nu)_{\mathbf{m}_\ell}} = R_{\nu_\ell}.$$

The ideal $\mathbf{P}_{\bar\beta} \subset \frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}(R_\nu)_{\mathbf{m}_\ell}}$ is the ν_ℓ-ideal of value $\bar\beta$; in particular, $\mathbf{P}_{\bar\beta}$ is principal, generated by any element $x \in \frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}(R_\nu)_{\mathbf{m}_\ell}}$ such that $\nu_\ell(x) = \bar\beta$.

(ii) By definitions (see (3.4)), we have

$$\widehat{\mathbf{P}}_{\bar\beta,\ell} = \mathbf{P}_{\bar\beta}\frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell-1}\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}. \tag{3.12}$$

Hence the ideal $\widehat{\mathbf{P}}_{\bar\beta,\ell}$ is also principal.

By Remark 5, both $\kappa(\mathbf{H}_{2\ell})$-vector spaces in (3.11) are one-dimensional.

COROLLARY 3.8. *The map (3.11) (and hence also* (3.10)*) is an isomorphism.*

COROLLARY 3.9. *The graded algebra*

$$gr_{\nu_\ell}\frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}(R_\nu)_{\mathbf{m}_\ell}} \otimes_{\kappa(\mathbf{m}_\ell)} \kappa(\mathbf{H}_{2\ell}) \cong \bigoplus_{\bar\beta\in\left(\frac{\Delta_{\ell-1}}{\Delta_\ell}\right)_+} \frac{\widehat{\mathbf{P}}_{\bar\beta,\ell}}{\widehat{\mathbf{P}}^+_{\bar\beta,\ell}} = gr_{\widehat{\nu}_{2\ell}}\frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell-1}\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}$$

is an integral domain.

The extension $\widehat{\mu}_{2\ell}$ of μ_ℓ to $\frac{\widehat{\mathbf{R}}}{\mathbf{H}_{2\ell-1}}$ is defined by $\widehat{\mu}_{2\ell} = \widehat{\nu}_{2\ell} \circ \widehat{\mu}_{2\ell+2}$. This completes the definition of $\widehat{\mu}_{2\ell}$, $\ell \in \{1,\dots,r\}$, by descending recursion on ℓ.

Lemma 3.10. *The natural inclusion*

$$\mathrm{gr}_{\mu_\ell}\frac{R}{P_{\ell-1}} \hookrightarrow \mathrm{gr}_{\widehat{\mu}_{2\ell}}\frac{\widehat{R}}{\tilde{H}_{2\ell-1}} \tag{3.13}$$

of graded algebras is scalewise-birational.

Proof. In the proof that follows we will keep in mind the following commutative diagram

$$\begin{tikzcd} & \mathrm{gr}_{\mu_\ell}\frac{R'}{P'_{\ell-1}} \arrow[r, hook] & \mathrm{gr}_{\widehat{\mu}_{2\ell}}\frac{\widehat{R'}}{\tilde{H}'_{2\ell-1}} \\ \mathrm{gr}_{\mu_{\ell+1}}\frac{R}{P_\ell} \arrow[r, hook] & \mathrm{gr}_{\mu_{\ell+1}}\frac{R'}{P'_\ell} \arrow[r, hook] \arrow[u, hook] & \mathrm{gr}_{\widehat{\mu}_{2\ell+2}}\frac{\widehat{R'}}{\tilde{H}'_{2\ell}} \arrow[u, hook] \end{tikzcd} \tag{3.14}$$

of natural inclusions, valid for all R' in $\mathcal{T}$. Here the lower row is a composition of two birational injections of graded algebras, where the second inclusion is scalewise-birational by the induction hypothesis, and the top row is an inclusion of graded algebras whose scalewise birationality we want to prove.

Take a homogeneous element $\bar{x} \in \mathrm{gr}_{\widehat{\mu}_{2\ell}}\frac{\widehat{R}}{\tilde{H}_{2\ell-1}}$ and let x be a representative of $\bar{x}$ in $\frac{\widehat{R}}{\tilde{H}_{2\ell-1}}$. Let $\bar{\beta} = \widehat{\mu}_{2\ell}(x)$. Take an element $y \in \mathbf{P}_{\bar{\beta}}$ and $u \in \frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell-1}\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}} \setminus \mathbf{H}_{2\ell}\frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell-1}\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}$ such that $x = yu^{-1}$ (the existence of y and u follows from Remark 5 (ii)).

Let R' be such that $y \in \frac{R'_{P_\ell}}{P'_{\ell-1}R'_{P_\ell}}$ and $u \in \frac{\widehat{R}'_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R}'_{\tilde{H}'_{2\ell}}} \setminus \tilde{H}'_{2\ell}\frac{\widehat{R}'_{\tilde{H}'_{2\ell}}}{\tilde{H}'_{2\ell-1}\widehat{R}'_{\tilde{H}'_{2\ell}}}$. Write $y = \frac{y_1}{y_2}$ with $y_1, y_2 \in \frac{R}{P_{\ell-1}}$. Write u as $u = \frac{w}{s}$, where

$$w, s \in \frac{\widehat{R}'}{\tilde{H}'_{2\ell-1}} \setminus \frac{\tilde{H}'_{2\ell}}{\tilde{H}'_{2\ell-1}} \tag{3.15}$$

for some R' in $\mathcal{T}$.

Let $\bar{\ }$ denote the operation of taking the natural image of an element of $\frac{\widehat{\mathbf{R}}}{\mathbf{H}_{2\ell-1}}$ in $\mathrm{gr}_{\widehat{\mu}_{2\ell}}\frac{\widehat{\mathbf{R}}}{\mathbf{H}_{2\ell-1}}$, its initial form with respect to the filtration defined by $\widehat{\mu}_{2\ell}$. The inclusion (3.15) implies that $\bar{w}, \bar{s} \in \mathrm{gr}_{\widehat{\mu}_{2\ell+2}}\frac{\widehat{R'}}{\tilde{H}'_{2\ell}}$. In view of the birationality of the second row of diagram (3.14), there exist $\bar{w}_1, \bar{w}_2, \bar{s}_1, \bar{s}_2 \in \mathrm{gr}_{\mu_{\ell+1}}\frac{R}{P_\ell}$ satisfying $\bar{w} = \frac{\bar{w}_1}{\bar{w}_2}$ and $\bar{s} = \frac{\bar{s}_1}{\bar{s}_2}$.

Putting together all of the above, we obtain

$$\bar{x}(\bar{y}_2\bar{w}_1\bar{s}_2) \in \mathrm{gr}_{\mu_\ell}\frac{R}{P_{\ell-1}}$$

and $\bar{y}_2\bar{w}_1\bar{s}_2 \in \mathrm{gr}_{\mu_\ell}\frac{R}{P_{\ell-1}}$, the latter containment implying that

$$ord(\bar{y}_2\bar{w}_1\bar{s}_2) \in \Delta_{\ell-1}.$$

This completes the proof of the lemma. □

We put $\hat{\nu}_- = \widehat{\mu}_2 = \hat{\nu}_2 \circ \cdots \circ \hat{\nu}_{2r}$, where for each ℓ the valuation $\hat{\nu}_{2\ell}$ is centered in the local ring $\frac{\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}{\mathbf{H}_{2\ell-1}\widehat{\mathbf{R}}_{\mathbf{H}_{2\ell}}}$.

This completes the construction of the valuation $\hat{\nu}_-$.

The scalewise birationality of the extension $\hat{\nu}_-$ of ν to the ring $\frac{\widehat{R}}{\tilde{H}_1}$ is given by Lemma 3.10, applied with $\ell = 1$. This completes the proof of Theorem 1.1. □

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