INTRODUCTION TO CURVE SINGULARITIES

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INTRODUCTION

My purpose in the course was to provide a simple introduction to the resolution of singularities of curves in analytic Geometry and to the basic invariants of these singularities. However, the course is not elementary or self contained in the usual sense that it uses only simple notions and results. On the contrary, taking advantage of the existence nowadays of excellent treatises of complex analytic Geometry, I freely refer to them for some results, for example on normalization. My hope is to entice the reader to study them carefully by showing some applications of some of the material they contain in the fairly intuitive context of singularities of curves.

The study of singularities of functions begins with that of analytic functions of one variable, say $f(x)$. The local study consists in the assertion that in a neighbourhood of a zero of order (also called multiplicity) k of a formal or convergent power series $f(x)$, which we may assume to be the origin after a translation on the variable x, one may write $f(x) = x^k g(x)$ with $g(0) \neq 0$, and by a change of variable $x_1 = xg(x)^{\frac{1}{k}}$ which is well defined near 0 since $g(0) \neq 0$, we get the function $f(x') = x'^k$; the point is that the lowest exponent of the series $f(x)$ determines completely the local behaviour of the function $f(x)$ up to an analytic isomorphism. Over an algebraically closed field, say the field $\mathbf{C}$ of complex numbers, the global study in the case of polynomials functions of one variable consists on one hand in the assertion that the total number of roots counted with multiplicity is the degree of the polynomial and on the other hand on the residue theorem.

We will begin the study of functions of two variables in the neighbourhood of a zero. First let us consider the formal situation, that is the case where we deal with formal power series in two variables without constant term. It means that we do not worry about the convergence of the series which we construct. It is then of course awkward to speak the "mappings" defined by formal power series, since its value may be undefined but it is customary to do this, with a cautionary "formal" in front of everything, or an "oid" after. Thus a formal power series in two variables $f(x, y) \in \mathbf{C}[[x, y]]$ represents a formal mapping $(\mathbf{C}^2, 0) \to (\mathbf{C}, 0)$ and the quotient ring $\mathbf{C}[[x, y]]/(f)$ is known as an *algebroid plane curve* although if the series $f(x, y)$ is divergent, there is no germ of curve in $(\mathbf{C}^2, 0)$ defined by $f(x, y) = 0$ and such that this ring is the ring of formal functions on this germ. In any case, the first goal is to try and *parametrize* such a curve, that is, find all possible different pairs of power series $x(t), y(t) \in \mathbf{C}[[t]]$ without constant term such that $f(x(t), y(t)) = 0$. This is the geometric problem of presenting any (formal) germ of curve in $\mathbf{C}^2$ as the union of the images of (formal) maps $(\mathbf{C}, 0) \to (\mathbf{C}^2, 0)$. Let me first describe an apparently more

"analytic" problem, which will turn out to be equivalent, and its solution by Newton, who used a favourite method of analysts: successive approximations.

1. NEWTON'S STUDY OF PLANE CURVE SINGULARITIES

A general reference for this paragraph is [W]. Let $f(x,y) \in \mathbf{C}[[x,y]]$ be a formal power series without constant term. We seek series $y(x)$ without constant term such that $f(x,y(x)) = 0$.

Let us first eliminate a marginal case;

If $f(0,y) = 0$, it means that $f(x,y)$ is divisible by some power of x; let a be the maximum power of x dividing $f(x,y)$, and let us set $f(x,y) = x^a f'(x,y)$. Geometrically, the equality $f(0,y) = 0$ means that the curve $f(x,y) = 0$ contains the y-axis, and the equality above means that this axis should be counted a times in the curve. This component may be parametrized by $x = 0, y = t$ and we are left with the problem of parametrizing the rest of the curve, which is defined by $f'(x,y) = 0$. We now have $f'(0,y) \neq 0$, and we may thus reduce to the case $f(0,y) \neq 0$. From now on we shall assume that $f(0,y) \neq 0$. We may then write, since $f(0,y)$ is a formal power series in y, $f(0,y) = y^k g(y)$, with $g(0) \neq 0$.

The proof of the existence of parametrizations proceeds by induction on the integer k. If $k = 1$, we have $\frac{\partial f}{\partial y}(0,0) \neq 0$, and by the implicit function theorem there exists a unique formal power series $y(x) \in \mathbf{C}[[x]]$ such that $y(0) = 0$ and $f(x,y(x)) = 0$. We now assume that $k > 1$.

Considering series $f(x,y)$ of the form $y^p - x^q$ with $p,q > 1$ and $(p,q) = 1$ shows that one cannot hope to find series in powers of x. Newton's idea is to seek solutions which are *fractional power series* in x, that is, he seeks series in $x^{\frac{1}{m}}$ for some integer m, say $\phi(x^{\frac{1}{m}}) \in \mathbf{C}[[x^{\frac{1}{m}}]]$ such that $f(x,\phi(x^{\frac{1}{m}})) = 0$.
More precisely he seeks solutions of the form:

$$y = x^\nu(c_0 + \phi_0(x^{\frac{1}{m}})) \qquad \text{with } c_0 \neq 0, \quad \nu \in \mathbf{Q}_+, \quad \phi_0 \text{ without constant term.}$$

If we write

$$f(x,y) = \sum_{i,j \in \mathbf{N}} a_{i,j} x^i y^j \qquad \text{with } a_{0,0} = 0$$

and substitute, we get

$$\sum_{i,j} a_{i,j} x^{i+\nu j}(c_0 + \phi_0(x^{\frac{1}{m}}))^j$$

and we seek ν, $c_0 \neq 0$ and a series $\phi_0(x^{\frac{1}{m}})$ such that this series is zero. In particular, its lowest order terms in x must be zero. Since ϕ_0 has no constant term, if we denote by μ the minimum value of $i + \nu j$ for (i,j) such that $a_{i,j} \neq 0$, we have

$$\sum_{i,j} a_{i,j} x^{i+\nu j}(c_0 + \phi_0(x^{\frac{1}{m}}))^j = x^\mu \sum_{i+\nu j = \mu} a_{i,j} c_0^j + x^\mu h(x^{\frac{1}{m}}) \text{ where } h \text{ has no constant term.}$$

So c_0 must satisfy

$$\sum_{i+\nu j = \mu} a_{i,j} c_0^j = 0$$

For this equation to have a non-zero root in $\mathbf{C}$, it is necessary and sufficient that the sum has more than one term.

Let us consider in the (i,j)-plane the set of points (i,j) such that $a_{i,j} \neq 0$. It is a subset $\mathcal{N}(f)$ of the first quadrant $\mathbf{R}_0^2 = \{(i,j)\ /i \geq 0,\ j \geq 0\}$, called the Newton cloud of the series $f(x,y)$. Any two subset A and B of $\mathbf{R}^d$ can be added coordinate-wise, to give the Minkowski sum $A + B = \{a + b, a \in A,\ b \in B\}$ of A and B. Let us consider the subset $\mathcal{N}_+(f) = \mathcal{N}(f) + \mathbf{R}_0^2$ of $\mathbf{R}_0^2$; its boundary is a sort of staircase with possibly infinite vertical or horizontal parts. The Newton polygon $\mathcal{P}(f)$ of $f(x,y)$ is defined as the boundary of the convex hull of $\mathcal{N}_+(f)$. It is a broken line with infinite horizontal and vertical sides, possibly different from the coordinate axis.

Note that it is a natural generalization to the case of two variables of the construction of the "lowest exponent" for a function of one variable.

The following is a picture of a Newton polygon in the case where the infinite sides do coincide with the coordinate axis, or equivalently where the area bounded by the polygon is finite.

Recall that the convex hull is by definition the intersection of the half-spaces containing $\mathcal{N}_+(f)$, so that the number $\mu = \min_{a_{i,j} \neq 0}\{i + \nu j\}$ is the minimal abscissa of the intersection points with the i-axis of the lines with slope $\frac{-1}{\nu}$ meeting $\mathcal{N}_+(f)$. Let us denote by L_ν the line which gives this minimum; an example in drawn on the picture. So the polynomial

$$\sum_{i+\nu j=\mu} a_{i,j} c_0^j$$

corresponds to the sum of the terms $a_{i,j} x^i y^j$ such that (i,j) lies on the intersection of the line L_ν with the Newton polygon.

A necessary and sufficient condition for this polynomial to have more than one term is that $\frac{-1}{\nu}$ is the slope of one of the sides of the Newton polygon. For simplicity of notation,

let us call ν the *inclination* of the line of slope $\frac{-1}{\nu}$. Let us denote by $\frac{l}{h}$ the inclination of the "first side" of the Newton polygon of f, that is, the side with the smallest inclination. Let c_0 be a non zero root of the corresponding equation, and let us make the change of variables

$$\begin{aligned} x =& x_1^h \\ y =& x_1^l(c_0 + y_1) \end{aligned}$$

The substitution in $f(x, y)$ gives

$$f(x_1^h, x_1^\ell(c_0 + y_1)) = \sum a_{i,j} x_1^{hi+\ell j}(c_0 + y_1)^j \ .$$

By definition of μ, for each $a_{i,j} \neq 0$, we have $hi + \ell j \geq \mu h$, so we may factor the series above as

$$x_1^{\mu h} f_1(x_1, y_1) \text{ , where } \ f_1(x_1, y_1) = \sum a_{i,j} x_1^{hi+\ell j-\mu h}(c_0 + y_1)^j \ .$$

We remark that

$$f_1(0, y_1) = \sum_{i+\nu j=\mu} a_{i,j}(c_0 + y_1)^j \ ,$$

and since $a_{0,k} \neq 0$ by definition of k, *the order in* y_1 *of* $f_1(0, y_1)$ *is* $\leq k$.

Since c_0 has been chosen as a root of the polynomial $\sum_{i+\nu j=\mu} a_{i,j} c_0^j$, this order is ≥ 1. We remark that

The order in y_1 *of* $f_1(0, y_1)$ *is equal to* k *if and only if* c_0 *is a root of multiplicity* k *of the polynomial* $\sum_{i+\nu j=\mu} a_{i,j} T^j = 0$

But then we must have an equality

$$\sum_{i+\nu j=\mu} a_{i,j} T^j = a_{0,k}(T - c_0)^k$$

which implies by the binomial formula and since $\mathbf{C}$ is a field of characteristic zero, that the term in T^{k-1} is not zero; this is possible only if ν is an integer and then the equality above shows that the "first side of the Newton polygon" meets the horizontal axis at the point $(\nu k, 0)$, which corresponds to the monomial $x^{\nu k}$, which has the non zero coefficient $(-1)^k a_{0,k} c_0^k$, so it is actually the only finite side of the Newton polygon of $f(x, y)$, which means that we may write in this case

$$f(x, y) = a_{0,k}(y - c_0 x^\nu)^k + \sum_{i+\nu j>\mu} a_{i,j} x^i y^j \quad \text{with } \nu \in \mathbf{N}, \ \mu = \nu k.$$

Making the change of variables

$$\begin{aligned} x =& x_1 \\ y =& y_1 + c_0 x_1^\nu \end{aligned}$$

the series $f(x, y)$ becomes

$$f'(x, y) = a_{0,k} y_1^k + \sum_{i+\nu j>\mu} a_{i,j} x_1^i (y_1 + c_0 x_1^\nu)^j$$

The monomials which appear are of the form $x_1^{i+\nu l}y_1^{j-l}$, so that they all satisfy $i+\nu l+\nu(j-l)=i+\nu j>\nu k$. This means that *if the order of $f_1(0,y_1)$ is k, the Newton polygon of $f_1(x_1,y_1)$ still contains the point $(0,k)$ and the inclination ν_1 of its first side is strictly greater than ν .*

The proof now proceeds as follows, :

a) If the order in y_1 of $f_1(0,y_1)$ is less than k, by the induction hypothesis, there exist an integer m_1 and a series $\phi_1(x_1^{\frac{1}{m_1}})\in\mathbf{C}[[x_1^{\frac{1}{m_1}}]]$ such that

$$f_1(x_1,\phi_1(x_1^{\frac{1}{m_1}}))=0$$

By the definition of f_1, this implies that

$$f(x_1^h,x_1^\ell(c_0+\phi_1(x_1^{\frac{1}{m_1}}))=0$$

If we set $m=m_1h$ and $\phi(x^{\frac{1}{m}})=x^{\frac{l}{h}}(c_0+\phi_1(x^{\frac{1}{m}}))\in\mathbf{C}[[x^{\frac{1}{m}}]]$, we have $f(x,\phi(x^{\frac{1}{m}}))=0$ and the result in this case.

b) If the order in y_1 of $f(0,y_1)$ is still equal to k, we saw that ν is an integer and the inclination of the first side of the Newton polygon of the function $f_1(x_1,y_1)$ obtained from $f(x,y)$ as above is strictly greater than ν.

We now set $\nu_0=\nu\in\mathbf{N}$ and repeat the same analysis for f_1, defining a function $f_2(x_2,y_2)$. If again the order of $f_2(0,y_2)$ is k, the slope of the first side of the Newton polygon of $f_1(x_1,y_1)$ is an integer $\nu_1>\nu_0$ and after the change of variables $x=x_2, y=y_2+c_0x_2^{\nu_0}+c_1x^{\nu_1}$ the slope of the Newton polygon has become greater than ν_1.

There are two possibilities;

– either after a finite number of such steps we get a function $f_p(x_p,y_p)$ such that $f(0,y_p)$ is of order $<k$, and by the induction hypothesis we have a series $\phi_p(x^{\frac{1}{m_p}})\in\mathbf{C}[[x^{\frac{1}{m_p}}]]$ such that $f_p(x,\phi_p(x^{\frac{1}{m_p}}))=0$, and so a series

$$y=c_0x^{\nu_0}+c_1x^{\nu_1}+\cdots+c_{p-1}x^{\nu_{p-1}}+\phi_p(x^{\frac{1}{m_p}})$$

such that $f(x,y(x))=0$;

Or the order remains indefinitely equal to k and we have an infinite increasing sequence of integers

$$\nu_0\ <\nu_1\ <\ldots<\nu_p<\ \ldots$$

and a formal power series

$$\phi_\infty(x)=c_0x^{\nu_0}+c_1x^{\nu_1}+\cdots+c_px^{\nu_p}+\cdots\in\mathbf{C}[[x]]$$

such that the Newton polygon of the function $f_\infty(x_\infty,y_\infty)$ obtained from $f(x,y)$ by the change of variables $x=x_\infty$, $y=y_\infty+\phi_\infty(x)$ has a Newton polygon containing the point $(0,k)$ and with inclination 0. This means that $f_\infty(x_\infty,y_\infty)$ is divisible by y_∞^k, so we may write

$$f_\infty(x_\infty,y_\infty)=y_\infty^kg(x_\infty,y_\infty)$$

This implies that the order of $g(0, y_\infty)$ is zero, so $g(0,0) \neq 0$. Geometrically, our curve is the non singular curve $y = \phi_\infty(x)$ counted k times. Indeed, for each integer p, we have

$$f(x, c_0 x^{\nu_0} + c_1 x^{\nu_1} + \cdots + c_p x^{\nu_p}) = x^{\nu_0+\nu_1+\cdots+\nu_p} f_p(x, 0),$$

so that by Taylor's expansion theorem, $f(x, \phi(x)) = 0$. This completes in the formal case the proof of the existence of a fractional power series such that $f(x, y(x)) = 0$.

In order to describe all the solutions of the equation $f(x, y) = 0$, it is convenient to develop a little more the formalism of the Newton polygon. Let $\mathcal{P}$ and $\mathcal{P}'$ be two Newton polygons; we can define their sum $\mathcal{P} + \mathcal{P}'$ as the boundary of the convex hull of the Minkowski sum of the convex domains in $\mathbf{R}^2_+$ bounded by $\mathcal{P}$ and $\mathcal{P}'$ respectively. It is easy to verify that the following equality holds for $f,\ f' \in \mathbf{C}[[x, y]]$

$$\mathcal{P}(ff') = \mathcal{P}(f) + \mathcal{P}(f').$$

Any Newton polygon has a *length* and an *height* which are the length of the horizontal and vertical projections of its finite part, respectively.

We say that a Newton polygon is *elementary* if it has only one finite side. If it bounds a finite area, it is then uniquely determined by its length and height. We use the following notation for such an elementary Newton polygon.

We also need a little more algebra, beginning with the following fundamental theorem: One says that a holomorphic function $f(x_1, \ldots, x_n, y)$ defined on a neighborhood of 0 in $\mathbf{C}^n \times \mathbf{C}$ is y-regular (of order k) if $f(0, y)$ has a zero of finite order k at $0 \in \{0\} \times \mathbf{C}$. Geometrically this means that if we consider the germ of hypersurface $(W, 0) \subset \mathbf{C}^n \times \mathbf{C}$ defined by $f(x_1, \ldots, x_n, y) = 0$ and the first projection $p\colon W \to \mathbf{C}^n$, then for a small enough representative, if W is not empty (i.e $k \geq 1$), the fiber $p^{-1}(0)$ is the single point 0. In other words, the fiber is a finite subset of $\{0\} \times \mathbf{C}$. The general idea of the avatars of the Weierstrass preparation theorem is that finiteness of the fiber over one point x in an analytic map implies finiteness of the fibers above points sufficiently close to x.

Weierstrass preparation Theorem(see [K], [Ł] p.110).-*If $f(x_1, \ldots, x_n, y)$ is regular of order k in y, there exist a unique polynomial of the form*

$$P(x_1, \ldots, x_n, y) = y^k + a_1(x_1, \ldots, x_n) y^{k-1} + \cdots + a_k(x_1, \ldots, x_n)$$

with $a_i \in \mathbf{C}\{x_1, \dots, x_n\}$ and a convergent series $u(x_1, \dots, x_n)$ with $u(0) \neq 0$, i.e invertible in $\mathbf{C}\{x_1, \dots, x_n\}$ such that we have the equality of convergent series

$$f(x_1, \dots, x_n, y) = u(x_1, \dots, x_n, y)P(x_1, \dots, x_n, y).$$

The polynomial P is said to be *distinguished* in y, or to be a Weierstrass polynomial.

If we start with any power series f, we have the same result but in the ring of formal power series.
It can be shown that, given a function f, for almost every choice of coordinates in $C^n \times \mathbf{C}$, the function f is distinguished with respect to the last coordinate.

It follows from the Weierstrass preparation theorem that provided we have chosen coordinates such that $f(0, y) \neq 0$, say $f(0, y) = a_q y^q + \cdots$ with $a_q \neq 0$, it is equivalent to seek solutions of $f(x, y) = 0$ and of $P(x, y) = 0$, where $P(x, y)$ is the Weierstrass polynomial

$$u^{-1}(x, y)f(x, y) = y^q + a_1(x)y^{q-1} + \cdots + a_q(x) = 0 \quad \text{with } a_i(x) \in \mathbf{C}[[x]]$$

Now from an algebraic point of view, we must consider the field of fractions $\mathbf{C}((x))$ of the integral domain $\mathbf{C}[[x]]$; the irreducible polynomial $T^m - x \in \mathbf{C}((x))[T]$ defines an algebraic extension of degree m of $\mathbf{C}((x))$, denoted by $\mathbf{C}((x^{\frac{1}{m}}))$ which is a Galois extension with Galois group equal to the group μ_m of m-th roots of unity in $\mathbf{C}$. The action of μ_m is exactly the change in determination of $x^{\frac{1}{m}}$, determined by $x^{\frac{1}{m}} \mapsto \omega x^{\frac{1}{m}}$ for $\omega \in \mu_m$. A series of the form $y = \sum a_i x^{\frac{i}{m}}$ such that the greatest common divisor of m and all the exponents i which effectively appear is 1 gives m different series as ω runs through μ_m.

Suppose now that our function f is an irreducible element of $\mathbf{C}[[x, y]]$, and that the order in y of $f(0, y)$ is $k < \infty$. Then the construction described above provides a series $y(x^{\frac{1}{m}})$ with $m \leq k$ such that $f(x, y(x^{\frac{1}{m}})) = 0$. The product

$$\Pi_{\omega \in \mu_m}(y - y(\omega x^{\frac{1}{m}}))$$

is a polynomial $Q(x, y) \in \mathbf{C}[[x]][y]$ which, by the algorithm of division of polynomials in $\mathbf{C}((x))[y]$, divides $P(x, y)$; the rest of the division of P by Q is a polynomial of degree $< m - 1$ with m different roots; it is zero.
We have therefore $Q(x, y) = P(x, y)$ and $m = k$ in this case.
We remark that the expansions $y(\omega x^{\frac{1}{m}})$ all have the same initial exponent $\frac{l}{h}$, and by the definition of $Q(x, y)$, only monomials $x^i y^j$ with $\frac{i}{\ell} + \frac{j}{h} \geq \frac{\mu}{\ell}$ appear, and the monomial x^h actually appears. So we have verified:

Proposition.- *The Newton polygon of an irreducible series is elementary, and of the form $\{\frac{p}{k}\}$, where k is the order of $f(0, y)$.*

Now it is known that rings such as $k[[x, y]]$, where k is a field, or $\mathbf{C}\{x, y\}$ are *factorial*; each element has a decomposition $f = f_1^{a_1} \dots f_r^{a_r}$ where f_i is irreducible, which means that it cannot be factored again as a product $f_i = gh$ in a non trivial way, that is, without g or h being an invertible element in $k[[x, y]]$, (= a series with a non zero constant term).

My aim now is to prove the following

Theorem.- *a) Let k be an algebraically closed field of characteristic zero, and let $f \in k[[x,y]]$ be a power series without constant term such that $f(0,y) \neq 0$. Consider the decomposition $f = uf_1^{a_1} \dots f_r^{a_r}$ of f into irreducible Weierstrass polynomials $f_i^{a_i}$, with a factor u which is invertible in $k[[x,y]]$. For each index i, $1 \leq i \leq r$, there are power series without constant term $x_i(t), y_i(t) \in k[[t]]$ such that $f(x_i(t), y_i(t)) \equiv 0$; we may choose $x_i(t) = t^{m_i}$ where m_i is the degree of the Weierstrass polynomial f_i, and $y_i(t)$ is then uniquely determined. Moreover if we then write $y_i(t) = c_i t^{l_i} + \dots$ with $c_i \in k^*$, then the Newton polygon of f in the coordinates (x,y) is the sum*

$$\mathcal{N}(f) = \sum_1^r \{\frac{a_i m_i}{a_i l_i}\}.$$

Here we have to allow the case where for some i, $y_i(t) \equiv 0$, that is $l_i = \infty$.
b) If $k = \mathbf{C}$ and $f \in \mathbf{C}\{x,y\}$ is a convergent power series, the series $x_i(t)$ and $y_i(t)$ are also convergent.
Remark: if we do not assume $f(0,y) \neq 0$, a similar result holds, but we may no longer apply Weiertrass' theorem and we have to allow expansions of the form $x = 0$, $y = t$ and the corresponding Newton polygons appears as summands in $\mathcal{N}(f)$.

The geometric interpretation of this result is that if we take any reduced analytic plane curve $f = uf_1 \dots f_r$ with f_i irreducible, i.e all $a_i = 1$, the curve defined by $f = 0$ is a sufficiently small neighborhood of the origin is the analytic image of a representative of a complex-analytic map-germ

$$\bigsqcup_{i=1}^r (\mathbf{C}, 0)_i \longrightarrow (\mathbf{C}^2, 0).$$

Conversely, given two power series $x(t), y(t) \in k[[t]]$ without constant term, one may *eliminate* t between them to produce an equation $f(x,y) = 0$ with the property that $f(x(t), y(t) = 0$. Indeed, by using the "natural" elimination process (see[T1]) we may do this in such a way that eliminating t beween $x(t^q), y(t^q)$ produces the equation $f^q(x,y)$, so that we may even represent parametrically a non-reduced equation.

There are several ways to prove this theorem; one is to prove the convergence first, either directly by providing bounds for the coefficient of the series produced by Newton's method, which works but is inelegant, or by considering the analytic curve $f(t^m, y) = 0$, and proving that it is a ramified analytic covering of the t-axis; it is also the union of m non singular curves, so each of them is analytic, and this proves the convergence of the series. (see [Ł], II.6).

These proofs give no basis for generalizations to higher dimension, so I chose to present a geometric method of constructing the analytic map

$$\bigsqcup_{i=1}^r (\mathbf{C}, 0)_i \longrightarrow (\mathbf{C}^2, 0).$$

This method was perfected by Hironaka and is the basis for his method of resolution in all dimensions over a field of characteristic zero.

2. RESOLUTION BY BLOWING UPS

Let us consider the projective space $\mathbf{P}^n(\mathbf{C})$ as the space of lines through the origin in $\mathbf{C}^{n+1}$. If we choose coordinates $x_0, \ldots, x_n$ on $\mathbf{C}^{n+1}$ the projective space is covered by affine charts U_i, the points of which correspond to the lines contained in the open set $x_i \neq 0$. It is customary to take homogeneous coordinates $(u_0 : \cdots : u_n)$ on the projective space, corresponding to the lines given parametrically by $x_i = u_i t$, or by the equations $x_i u_j - x_j u_i = 0$, where it is enough to take the n equations for which $j = i+1$ and $i < n$. The term "homogeneous coordinates " means that for any $\lambda \in k*$ the coordinates $(u_0 : \cdots : u_n)$ and $(\lambda u_0 : \cdots : \lambda u_n)$ define the same point.

Now consider the subvariety Z of the product space $\mathbf{C}^{n+1} \times \mathbf{P}^n$ defined by these n equations. It is a nonsingular algebraic variety of dimension $n+1$ and the first projection induces an algebraic morphism $B_0 \colon Z \to \mathbf{C}^{n+1}$.

The fiber $B_0^{-1}(0)$ is the entire projective space $\mathbf{P}^n(k)$ since when all x_i are zero, all the equations between the u_j vanish,while the fiber $B_0^{-1}(x)$ for a point $x \neq 0$ consists of a unique point because then the coordinates x_i determine uniquely the ratios of the u_j which means a point of $\mathbf{P}^n(k)$. "Blowing up a point "replaces the observer at the point by what he sees", because the observer essentially sees a projective space (in fact a sphere, but this is just a metaphor).

A basic properties of blowing up is that it separates lines: in fact consider the algebraic map $\delta \colon \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{P}^n$ which to a point outside the origin associates the line joining the origin to this point. Of course we cannot extend the definition of this map through the origin; The graph of δ however, is an algebraic subvariety of $(\mathbf{C}^{n+1} \setminus \{0\}) \times \mathbf{P}^n$, and we may take the closure (for the strong topology if $k = \mathbf{C}$, or for the Zariski topology) of this graph. It is a good exercise to check that this closure coincides with Z as defined above. A point of $B_0^{-1}(0)$ is precisely a direction of line, so the map $\delta \circ B_0$ can be defined there as the map which to this point associates the direction: in Z we have separated all the lines meeting at the origin.

Let us consider in more detail the case $n = 1$. Then Z is a surface covered by two affine charts corresponding to the charts of the projective space: for convenience of notation set $u_0 = u,\ u_1 = v,\ x_0 = x,\ x_1 = y$ so that Z is defined by $vx - uy = 0$. On the open set U of Z where $u \neq 0$ we may taxe as coordinates $x_1 = x,\ y_1 = \frac{v}{u}$ and then the map induced by B_0 on U is described in these coordinates by

$$\begin{aligned} x \circ B_0 =& x_1 \\ y \circ B_0 =& x_1 y_1 \end{aligned}$$

and similarly on the open set V defined by $v \neq 0$, we take as coordinates $x_1 = \frac{u}{v},\ y_1 = y$ and the map B_0 is described by

$$\begin{aligned} x \circ B_0 =& x_1 y_1 \\ y \circ B_0 =& y_1 \end{aligned}$$

Remark that in the first chart the projective space $B_0^{-1}(0)$ is defined by $x_1 = 0$ and in the second by $y_1 = 0$ (remember that they are coordinates on two distinct charts, and on the intersection of the two charts they define the same subspace). It is a crucial property of blowing up that it transforms the blown-up subspace (here the origin) into a subspace defined locally by one equation (called a divisor); it is a good exercise to check that this is the case in any dimension. The space $B_0^{-1}(0)$ is called the *exceptionnal divisor*. We are now able to study the effect on a function $f(x, y)$ (formal or convergent) of its composition with the map B_0. Consider the expansion of f as a sum of homogeneous polynomials

$$f(x, y) = f_m(x, y) + f_{m+1}(x, y) + \cdots + f_{m+k}(x, y) + \cdots ,$$

where f_j is homogeneous of degree j. In the chart U, we may write

$$f \circ B_0 = f(x_1, x_1y_1) = x_1^m\big(f_m(1, y_1) + x_1f_{m+1}(1, y_1) + \cdots + x_1^kf_{m+k}(1, y_1) + \cdots\big)$$

and there is a similar expansion in the other chart. Now if we look at the zero set of $f \circ B_0$ we see that in each chart it contains the exceptionnal divisor counted m times. If we remove this exceptionnal divisor as many times as possible, i.e divide $f \circ B_0$ by x_1^m in the first chart and by y_1^m in the second, we obtain the equation of a curve on the surface Z, either formal or defined near $B_0^{-1}(0)$, which no longer contains the exceptionnal divisor. This curve is called the *strict transform* of the original curve. We also say that the equation obtained in this way is the strict transform of f). In the first chart it is $x_1^{-m}f(x_1, x_1y_1)$, and in the second $y_1^{-m}f(x_1y_1, y_1)$.

By construction, the strict transform meets the exceptionnal divisor only in finitely many points; let us determine them: in the first chart they are given by $f_m(1, y_1) = 0$ and in the second, by $f_m(1, y_1) = 0$. By construction of the projective space the points we seek are therefore the points in the projective line defined by the homogeneous equation $f_m(u, v) = 0$. The homogeneous polynomial f_m of lowest degree appearing in $f(x, y)$ is called the *initial form* and $f_m(x, y) = 0$ is a union of m lines (counted with multiplicity) called the *tangent cone* of f at the point 0. So we see that the strict transform of f meets the exceptionnal divisor at the points in this projective space corresponding to the lines which are in the tangent cone at 0 of our curve.

In particular, if our curve has two components with tangent cones meeting only at the origin, their strict transforms are disjoint. Consider for example $f(x, y) = (y^2-x^3)(y^3-x^2)$.

In order to analyze in more detail what goes on, we have to *assume that k is algebraically closed, which we will do from now on* , and introduce the concept of *intersection number* of two curves at a point. The simplest definition (but not the most useful for computations) is the following:
Let $f, h \in k[[x, y]]$ be series without constant term and without common irreducible factor. Let (f, h) denote the ideal generated by f and h in $k[[x, y]]$. Then the dimension

$$\dim k[[x, y]]/(f, h)$$

is finite and is by definition the intersection number of the two curves at 0. If $k = \mathbf{C}$ and f, h are in $\mathbf{C}\{x, y\}$, then the dimension above is also

$$\dim \mathbf{C}\{x, y\}/(f, h)$$

where now (f,h) is the ideal generated in $\mathbf{C}\{x,y\}$.

To prove the finiteness we first remark that it is sufficient to prove it after replacing k by its algebraic closure and then we may use the Hilbert nullstellensatz which tells us that since $f=0,\ h=0$ meet only at the origin, the ideal (f,h) contains a power of the maximal ideal $\mathbf{m}=(x,y)$ say $\mathbf{m}^N$. This implies the finiteness since $k[[x,y]]/(f,h)$ is then a quotient vector space of $k[[x,y]]/\mathbf{m}^N$ and also shows that we may without changing the ideal assume that f,h are polynomials of degree $<N$, so that for example if f,h are convergent power series the vector spaces $\mathbf{C}\{x,y\}/(f,h)$ and $\mathbf{C}[[x,y]]/(f,h)$ are equal.

The definition of intersection multiplicity at the point 0, of the two curves $f=0,\ h=0$, say in the analytic case is then

$$\big(f,h\big)_0 = \dim \mathbf{C}\{x,y\}/(f,h).$$

Note that we use large parentheses for the intersection number, small ones for the ideal generated by f,g.

In any case this definition of the intersection multiplicity has the advantage to suggest the following intuitive interpretation :
Consider a 1-parameter deformation of one of the two functions, say $f+\epsilon$; it is possible to show that if f,h converge in a nice neighborhood U of 0, for small enough ϵ, then the two curves $h=0, f+\epsilon=0$ meet in U transversally at points which are non-singular on each. Moreover, these points tend to 0 as ϵ tends to 0, and the number of these points is $\dim\mathbf{C}\{x,y\}/(f,h)$. So this number may be thought of as the number of ordinary intersections (i.e transverse intersection of non-singular curves) which are concentrated at 0.

There is another way to present this intersection number, which is very useful for computations:
Suppose that $h(x,y)=uh_1^{e_1}\dots h_r^{e_r}$ with $u(0)\neq 0$. For each i, $1\leq i\leq r$, let us parametrize the curve $h_i(x,y)=0$ by $x(t_i),y(t_i)$. Now substitute these power series in $f(x,y)$; we get a series in t_i, the order of which we denote by I_i. Then we have

$$I_i = \dim\mathbf{C}\{x,y\}/(f,h_i)\ ,$$

and

$$\big(f,h\big)_0 = \sum_1^r e_i I_i.$$

Remark: Given a germ of curve $f=0$, where $f=f_m+f_{m+2}+\cdots$, its multiplicity at the origin may be defined as the smallest degree m of a monomial appearing in the series f. A better definition is to say that the multiplicity is the intersection number $\big(f,\ell\big)_0$ for a sufficiently general linear form ℓ. In fact, we have

$$m \leq \big(f,\ell\big)_0$$

with equality if and only if the line $\ell(x,y)=0$ is not in the tangent cone defined by $f_m(x,y)=0$.

Indeed, we may parametrize the line $\ell=0$ by $x=\alpha t,\ y=\beta t$; then we substitute in f:

$$f(\alpha t, \beta t) = f_m(\alpha, \beta)t^m + f_{m+1}(\alpha, \beta)t^{m+1} + \cdots$$

is of order $\geq m$, and of order m exactly if and only if $f_m(\alpha, \beta) \neq 0$.

It is convenient, given a curve $f(x, y)$ and a point z in the plane, to define the *multiplicity of f at z* as follows: take coordinates $(x', y,)$ centered at z, which means that they vanish at z; if $z = (a, b)$ we may take $x' = x - a, y' = y - b$. Then expand f in those coordinates (of course we assume that z is in the domain of convergence of f).

We get $f'(x', y') = f(a + x', b + y')$. Then we compute the lowest degree terms appearing in the expansion of f' and denote this by $m_z(f)$ or, if X is the curve $f(x, y) = 0$, by $m_z(X)$. We see that $m_z(f) = 0$ unless $f(z) = 0$, and that if ℓ is a line through z, we have $m_z(X) \leq \big(X, \ell\big)_z$ with equality except if ℓ is in the tangent cone of X at z.

Let us apply this, in our blowing up as described above, to the line $x_1 = 0$ (the exceptionnal divisor) and the strict transform $f_1(x_1, y_1) = 0$, at a point x' with coordinates $x_1 = 0, y_1 = t_1)$ where $f_m(1, t_1) = 0$ i.e a point of intersection of the strict transform with the exceptionnal divisor. We have

$$f_1 = f_m(1, y_1) + x_1 f_{m+1}(1, y_1) + \cdots$$

and if we denote by $e_{x'}$ the multiplicity of t_1 as a root of the polynomial $f_m(1, Y)$, it follows from what we saw above that we have

$$e_{x'} \geq m_{x'}(f_1)$$

with equality unless the curve $f_1(x_1, y_1) = 0$ is tangent to the exceptionnal divisor at the point x', in the sense that the tangent at x' to the exceptionnal divisor is in the tangent cone of $f_1 = 0$ at the point x'. Since the multiplicity of f_1 is zero at points where $f_m(1, y_1)$ does not vanish, we see that if we look at all the points x' in the blown up surface Z which are mapped to our origin by the projection $Z \to \mathbf{C}^2$, which we denote by $x' \to 0$, we have

$$\sum_{x' \to 0} m_{x'}(f_1) \leq \sum_{x' \to 0} e_{x'} = m \ ,$$

so that in particular, if *there is a point x' of the strict transform X' of X which is mapped to* 0 *and is of multiplicity m on $f_1 = 0$, then it is the only point of X' mapped to* 0 *and X' is transversal to the exceptionnal divisor at x'.* This fact and its generalizations play a crucial role in Hironaka's proof of the resolution of singularities.

In order to show that the situation which we have just described cannot persist indefinitely in a sequence of blowing ups, we have to use the intersection number in another manner, according to Hironaka:

Given a germ of a plane curve (X, x) with r branches $(X_i, x)_{1 \leq i \leq r}$ and a nonsingular curve W through the point x, define the *contact exponent* of W with X at x as follows:

$$\delta_x(W, X) = \min_{i=1}^{r} \left(\frac{\big(X_i, W\big)_x}{m_x(X_i)} \right)$$

and the *contact exponent* of X at x as follows

$$\delta_x(X) = \max_W \delta_x(W, X),$$

where W runs through the set of germs at x of non-singular curves.

Lemma.-*Let $f(x, y) = 0$ be an equation for X. If the coordinates (x, y) are chosen in such a way that $x = 0$ is not tangent to X at x and W is defined by $y = 0$, the rational number $\delta_x(W, X)$ is the inclination of the first side of the Newton polygon of $f(x, y)$.*

By definition of $\delta_x(W, X)$ is enough to prove that for an irreducible f, the inclination of the only side of it Newton polygon is $\frac{\left(X,W\right)_x}{m_x(X)}$, but if we parametrize X by $x = t^m,\ y = t^q + \cdots$, we find that the transversality condition implies $m \leq q$, and we have $\left(X, W\right)_x = q$; the result follows.

Lemma.- *Assume that W is the curve $y = 0$ and that $f(x, y)$ is in Weierstrass form, i.e.,*

$$f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x) \qquad a_i(x) \in \mathbf{C}\{x\},$$

then the inclination of the first side of its Newton polygon is

$$\delta_x(W, X) = \min_{1 \leq i \leq n-1} \frac{\nu_0(a_i)}{i}.$$

Here as usual $\nu_0(a(x))$ denotes the order of vanishing at the origin of the series $a(x)$.

Indeed, the point $(0, n)$ is a vertex of the first side of the Newton polygon, and the lemma is just the observation that if we write $a_i(x) = \alpha_i x^{c_i} + \cdots$, the other vertices of the Newton polygon are among the points $(c_i, n-i)$, which follows directly from the definition.

A nonsingular curve W such that $\delta_x(W, X) = \delta_x(X)$ is said to have *maximal contact* at x. non singular curves with maximal contact are the nonsingular curves which it is hardest to separate from X by a succession of blowing ups (in the sense of separating strict transforms), and so when they eventually separate, something nice should happen; indeed once they separate, there is no point of multiplicity $m_x(X)$ in the iterated strict transform mapping to x. As one says, "the multiplicity has dropped". Hironaka's approach to resolution uses the existence of varieties with maximal contact to build an induction on the dimension.

The next step is to prove the existence of curves with maximal contact.

Assume that a non singular curve W defined by $y = 0$ does not have maximal contact with X at x. We way assume that the curve $x = 0$ is transversal to $f(x, y) = 0$, which means that $f(0, y) = a_{0,m}y^m + \cdots$, where m is the multiplicity of f at 0. By a change of variable $y = (a_{0,m})^{\frac{1}{m}} y'$, which does not change the contacts, we may assume that $a_{0,m} = 1$. To say that $\delta_x(W, X) < \delta_x(X)$ means that there is a series $A(x)$ such that the contact of the curve $f(x, y) = 0$ with $y - A(x) = 0$ is greater than its contact with $y = 0$. By a change of the coordinate x which does not affect the contacts, we may assume that $A(x) = \xi x^d$

for some integer d and $\xi \in \mathbf{C}^*$. Let us now compute the power series expansion in the coordinates $x' = x, y' = y - A(x)$;

$$f'(x,y') = \sum_{\frac{i}{\delta}+j\geq m} a_{i,j}x^i(y'+\xi x^d)^j = \sum_{\frac{k}{\delta'}+\ell\geq m} a'_{k,\ell}x^k y'^\ell.$$

By expanding the powers of $y' + \xi x^d$ we get, for each (i,j), and $k \leq j$ the inequality $\frac{i+kd}{\delta} + j - k \geq m$ but we know that $\frac{i}{\delta} + j \geq m$. From this follows the inequality $d \geq \delta$. Isolating the terms which lie on the first side of the Newton polygon, we get:

$$(*) \sum_{\frac{i}{\delta}+j=m} a_{i,j}x^i y^j + \sum_{\frac{i}{\delta}+j>m} a_{i,j}x^i y^j = \sum_{\frac{k}{\delta'}+\ell\geq m} a'_{k,\ell}x^k y'^\ell,$$

and the slope of the first side of the Newton polygon of the right-hand side is $\delta' > \delta$. Let us first assume that $\delta = 1$. Remark that all the terms $x^k y'^\ell$ with $\frac{k}{\delta'} + \ell \geq m$ except y'^m are in the ideal $(x,y')^{m+1}$. Therefore we must have the equality

$$\sum_{\frac{i}{\delta}+j=m} a_{i,j}x^i y^{=} y'^m \text{ mod.}(x,y)^{m+1}$$

so that the left hand side is the m-th power of $y - \xi x^d$. This implies that $d = 1 = \delta$ since the left hand side is homogeneous.

If $\delta > 1$ we follow the same method. Since we know that $d \geq \delta$, it is easy to check that the ideal of $k[[x,y]]$ generated by the monomials $x^k y'^\ell$, $\frac{k}{\delta'} + \ell \geq m$, $k \neq 0$ is contained in the ideal I generated by the monomials $x^i y^j$, $\frac{i}{\delta} + j > m$. Looking at the equation $(*)$ modulo I gives us

$$\sum_{\frac{i}{\delta}+j=m} a_{i,j}x^i y^j = y'^m \text{ mod.}I$$

which again by homogeneity shows that $d = \delta$ and the sum on the left hand side is $(y - \xi x^d)^m$.

Note that this argument also works if $\delta_x(X) = \infty$. So there are two possibilities:

1) We have $\delta_x(W,X) < \delta_x(X)$; in this case the sum of the terms of $f(x,y)$ lying on the first side of the Newton polygon is of the form $(y - \xi x^d)^m$.
2) The sum of the terms of $f(x,y)$ lying on the first side of the Newton polygon is not of the form $(y - \xi x^d)^m$.
In the first case, as we have seen, $d = \delta_x(W,X)$. We make the change of variables $x' = x$; $y' = y - \xi x^d$ and in the new coordinates x', y', if W' is the curve $y' = 0$, we have $\delta_x(W',X) > \delta_x(W,X)$. This follows easily from the computation we have just made; an effect of the change of variables is that all the terms lying on the first side of the Newton polygon, of inclination d, are transformed into the single term monomial y'^m. So the inclination of the new Newton polygon has to be $> d$; but we know this inclination to

be $\delta_x(W', X)$. If we have not reached $\delta_x(X)$, we continue the same procedure, and after possibly infinitely many steps, i.e. after a change of variables of the form

$$x' = x \ ; \ y' = y - \xi_1 x^{d_1} - \xi_2 x^{d_2} - \cdots - \xi_r x^{d_r} - \cdots$$

we reach the stage where the sum of terms on the first side of the Newton polygon is not a $m-th$ power, so $\delta_x(W_s, X) = \delta_x(X)$, with s possibly infinite. Since the denominators of the $\delta_x(W, X)$'s are bounded, the series is infinite only in the case where $\delta_x(X) = \infty$. At least formally this series converges, since we have $d_1 > d_2 > \cdots > d_r > \ldots$, but we can omit the proof of convergence if we work in $\mathbf{C}\{x, y\}$ since the equality $\delta_x(X) = \infty$ means that in some coordinates $f(x, y)$ is of the form $u(x, y)y^m$ where u is an invertible element in $k[[x, y]]$; indeed for any other case, we see from the definition that $\delta_x(X) < \infty$. But the Weierstrass preparation theorem tells us that if such a presentation exists with formal power series, it also exists with convergent power series, so that the series defining our final coordinates converges.

So in all cases, we can find a nonsingular curve W which has *maximal contact* with X at x, i.e. such that $\delta_x(W, X) = \delta_x(X)$.

Remark that all the discussion above is valid on a germ of a non singular surface, since it is analytically isomorphic to the plane. The definition of the blowing up is independant of the choice of coordinates, and makes sense on any nonsingular surface.

The next step is to study the behavior of the contact under blowing up of the origin. I will leave the proof of this as an exercise, since it is a direct application of what we have just seen and the definition of blowing up:

Theorem(Hironaka).- *Let m be an integer, let $f(x, y) = 0$ define a germ of a plane curve, $(X, 0) \subset (\mathbf{C}, 0)$ of multiplicity m and let $(W, 0) \subset (\mathbf{C}, 0)$ be a non singular curve with maximal contact with X at 0. If, after blowing up the point 0 by the map $B_0 \colon Z \to \mathbf{C}^2$, there is a point $x' \in X'$ of multiplicity m in the strict transform $X' \subset Z$ of X, then*
1) The point x' is the only point of X' mapped to 0 by B_0,
2) The strict transform W' of W by B_0 contains the point x', and W' has maximal contact with X' at x',
3) We have the equality $\delta_{x'}(W', X') = \delta_x(W, X) - 1$.

Corollary.- *The maximal length of a sequence of infinitely near points of multiplicity m on the strict transforms of X, each mapping to its predecessor in successive blowing ups*

$$\cdots Z^{(r)} \to Z^{(r-1)} \to \cdots \to Z^{(2)} \to Z^{(1)} \to \mathbf{C}^2$$

is equal to the integral part $[\delta_x(X)]$.

This suffices to show that unless the curve is of the form $y^m = 0$, the multiplicity of its strict transform in the sequence of blowing ups obtained by blowing up at each step the points of maximal multiplicity drops after a finite number of steps. By induction on the multiplicity, this proves the resolution of the singularity of X at 0 by a finite number of blowing ups of points on non singular surfaces.

We should remark that the map $X' \to X$ of the strict transform of X to X is defined by itself, without any reference to an embedding $(X, 0) \subset (\mathbf{C}^2, 0)$ (see [K]).

We have proved a local result, but if now we consider any algebraic or analytic curve, it has finitely many singular points, and the local resolution processes at each point are independant, so we have:

Theorem.-*Given an algebraic or analytic plane curve X there exists a finite sequence of point blowing ups such that in the composed map $X' \to X$ the curve X' has no singularities.*

Actually we can get, by the same method, a better result, known as *embedded resolution* and originally due to Max Nœther, as follows:

Theorem.- *Given a curve X on a non singular surface S, there exists a finite sequence of blowing ups of points*

$$S^{(r)} \to \cdots \to S^{(1)} \to S$$

such that if we denote by $\pi\colon S^{(r)} = S' \to S$ their compositum, then the inverse image of the singular points of X (the exceptionnal divisor) is a union of non singular curves meeting transversally on the non singular surface S', and the strict transform X' of X by π is a non singular curve meeting transversally these curves.

In analytic terms, if $f(x,y) = 0$ is a local equation for X in S, then $f \circ \pi$ is, at every point x' of S', of the form $(f \circ \pi)_{x'} = u^a v^b$ for suitable local coordinates of S' at x'. Of course a and b will be zero unless we have $x' \in \pi^{-1}(X)$.

The induced map $\pi\colon X' \to X$ is a resolution of singularities of X. If we fix a singular point $x \in X$, let r be the number of analytically irreducible components of the germ (X, x). The number of points in $\pi^{-1}(x)$ is equal to r and for a small enough representative X_x of the germ (X, x), the part $\pi^{-1}(X_x)$ of X' lying over X_x consists of r non singular curves $\mathbf{D}_i$, each marked with one of the points of $\pi^{-1}(x)$. The image by π of each of these non singular curves $\mathbf{D}_i$ is one of the irreducible components of X_x.

If we choose for each non singular curve $\mathbf{D}_i$ a coordinate t_i vanishing at the only point z_i of $\mathbf{D}_i$ lying over x, then $\mathbf{D}_i$ is described parametrically, in local coordinates (u, v) on S' centered at z_i, by *convergent* power series $u(t_i), v(t_i)$, because of the implicit function theorem. Since the map $\pi\colon S' \to S$ is a composition of algebraic maps, $x \circ \pi$ and $y \circ \pi$ are at worst convergent power series in (u, v), so when we restrict them to $\mathbf{D}_i$, we get convergent power series in t_i. This shows that each branch of our curve has a convergent parametrization, and from this we deduce that the formal parametrization constructed by Newton's method converges.

Note that this convergence argument works equally well with the first resolution theorem. The new fact in the resolution result above with respect to the resolution theorem is the transversality of the strict transform with the exceptionnal divisor, which is not part of the resolution theorem as we have stated it above. The proof of this improvement is not difficult: it amounts to resolving singularities, by a sequence of points blowing up, of the union of the strict transform and the exceptional divisor of the map which resolves the singularities of X.
As an example, given an integer $m > 1$, after one blowing up the strict transform of a curve with equation $y^m - x^{m+1} = 0$, but is not transversal to the exceptionnal divisor.

It is the first example of a fundamental fact of analytic or algebraic geometry:*you can make spaces (in fact, their strict transforms) transversal by well chosen sequences of blowing ups.*

3. RESOLUTION OF SPACE CURVES

1. Integral dependance

To prove a resolution theorem for space curves, one meets the difficulty that their equations may be complicated (for example to define a curve in $\mathbf{C}^n$ one may need more that $n-1$ equations; those for which $n-1$ equations suffice are called *complete intersections* , and also that rather different looking sets of equations may generate the same ideal in $\mathbf{C}\{x_1, \ldots, x_n\}$ and therefore define the same curve. In the proofs above we have used constructions which depend heavily on the equation. Moreover, even to show that a germ of a complex curve in $\mathbf{C}^d$ has a finite number of irreducible components, which are analytic germs, is not completely trivial (see [L], II.5). There are two possibilities: we can conceptualize and abstract the proof for plane curves to make it less dependent on the equation, or try to reduce to the plane curve case. As it happens, the two methods are not so different, at least for one of the ways of abstracting the ideas.
To reduce to the plane curve case, the natural idea is to project the space curve X to a plane curve X_1. One can then show that a resolution of X_1 has to map to X, and that this map is a resolution of the singularities of X!.

The key idea is that of *normalization.* The Italian geometers called *normal* a projective variety $Z \in \mathbf{P}^n$ having the property that any map $Z' \to Z$ presenting Z as a "general" projection by a linear map $\mathbf{P}^{n'} \setminus L \to \mathbf{P}^n$ of an algebraic variety $Z' \subset \mathbf{P}^{n'}$ had to be an isomorphism. A typical non normal surface in $\mathbf{P}^3$ is therefore a general projection of a non singular surface in $\mathbf{P}^4$; such a projection has a curve of double points, on which are finitely many more complicated singular points, the "pinch points". Here the meaning of "general" has to be made precise;
The variety Z is normal if any map $\pi\colon Z' \to Z$ which
a) is finite-to-one and
b) induces an isomorphisme $Z' \setminus \pi^{-1}(U) \to U$ over the complement U of a closed algebraic or analytic subset of Z is an isomorphism.

The resolution theorem we saw above shows that a singular curve in $\mathbf{P}^2$ cannot be normal.

The concept of normalization was "localized" and transfigured into a concept of commutative algebra, as follows: Recall that the total ring of quotients of a ring A is the ring of equivalence classes of couples (a, b) of elements of A, where b is not a zero divisor in A with addition $(a, b)+(a', b') = (ab'+ba', bb')$ and componentwise multiplication, the equivalence being $(a, b) \equiv (a', b')$ when $ab' - ba' = 0$. The map $a \mapsto (a, 1)$ induces an injection of A in F and we indentify A with its image in F. If A is an integral domain, F is its field of fractions.

Definition Let A be a commutative ring without nilpotent elements, and let F be its total ring of quotients.
Definition.- An element $h \in F$ is *integral* over A if it satisfies an equation

$$h^k + a_1 h^{k-1} + \cdots + a_k = 0 \ \text{ with } a_i \in A.$$

Example.- Consider the germ of plane curve X in $\mathbf{C}^2$ defined by the equation $y^p - x^q = 0$. The quotient $\mathcal{O}$ of the ring $\mathbf{C}\{x, y\}$ by the ideal generated by $y^p - x^q$ is the ring of germs of analytic functions on the germ X (the restrictions to X of two analytic functions on $\mathbf{C}^2$ coincide if and only if their difference is in the ideal). The ring $\mathcal{O}$ is an integral domain; let K be its field of fractions. If we keep the notations x, y, etc.. for the restictions to X of functions on $\mathbf{C}^2$, we have $\frac{y}{x} \in F$. I claim that if $p \leq q$, it is integral over $\mathcal{O}$; indeed, we have the relation

$$(\frac{y}{x})^p - x^{q-p} = 0 \ .$$

We can remark that the function $\frac{y}{x}$ is defined and analytic on the strict transform of X by the blowing up of the origin for any sufficiently small representative of the germ X. We remark also that the condition $p \leq q$ is equivalent to saying that the meromorphic function $\frac{y}{x}$ remains bounded on X for any small representative.

Proposition.- *Given a ring A without nilpotent elements, let F be its total ring of fractions; the set of elements of F integral over A is a ring for the operations induced by those of F.*

This ring is called the normalization of A (or the integral closure of A in F) and often denoted by $\overline{A}$. Of course we have $A \subset \overline{A}$; a ring such that $A = \overline{A}$ is said to be integrally closed. Is is not difficult to check that $\overline{A}$ is integrally closed.

If A is nœtherian and integrally closed, any injective map $A \to B$ to a subring B of the total ring of fractions of A which makes B into a finite A-module is an isomorphism; this is the translation of the original definition of normality. To prove it, check that if h is an element of B, the powers of h cannot all be linearly independant over A, so h satisfies an integral dependance relation, and if A is normal, it is in A!

An important theorem is that if A is an analytic algebra, i.e a quotient of a convergent power series ring by some ideal, then $\overline{A}$ is a finite sum of integrally closed analytic algebras, and moreover that the injection $A \to \overline{A}$ makes $\overline{A}$ into a *finitely generated* A-module. Taking a common denominator (in F) for a finite set of generators of the A-module $\overline{A}$, we see that the ("conductor") ideal $\mathcal{C} = \{d \in \overline{A},\ d.\overline{A} \subset A\}$ is not zero.
Another important fact is that if the analytic algebra of germs of functions on a curve at a point is normal, the point is non singular on the curve, and the analytic algebra is isomorphic to a convergent power series ring in one variable $\mathbf{C}\{t\}$. ([Ł], VI.3, Thm.2)

So this abstract idea, normalization, provides us with a proof of the resolution of singularities of space curves: given $(C, 0) \in (\mathbf{C}^d, 0)$, the normalization $\mathcal{O} \to \overline{\mathcal{O}}$ of the (reduced) analytic algebra of germs of functions on C is an analytic algebra which is a product $\Pi_{i=1}^r \mathbf{C}\{t_i\}$ of a finite number of convergent power series rings in one variable. If $x_1, \ldots, x_d$ generate the maximal ideal of $\mathcal{O}$, we get r d-uples of convergent power series

expansions $x_j(t_i)$, which are our Newton series in this case. They geometrically correspond to a map

$$\bigsqcup_{i=1}^{r}(\mathbf{C},0)_i \to (C,0)$$

which is our resolution of singularities. However, normalization is geometrically subtle in general, and the finiteness of normalization is a subtle theorem; in addition, we may seek a more geometric proof, as follows

We now turn to the definition of plane projections of a space curve.
Let $(C,0) \in (\mathbf{C}^d,0)$ be a germ of a (reduced) space curve defined by an ideal $I \subset \mathbf{C}\{x_1,\dots,x_d\}$. Let us choose a linear projection $p\colon \mathbf{C}^d \to \mathbf{C}^2$. Let M denote the space of all such projections; think of it as a set of $d \times 2$ matrices of rank 2. We endow M with the topology (complex or Zariski) induced by that of the space of matrices. We wish to consider only the projections such that $p|C\colon C \to p(C)$ is finite to one. If that is not the case, the kernel of p, which is a linear subspace of codimension 2 of $\mathbf{C}^d$, contains one of the irreducible components of the curve C; the intersection is analytic, so it is either of dimension 0 or 1. By looking at the equations of C, it is not too difficult to check that the projections which do not contain a component of C form a dense open set of M. The fact that they are those which induce a finite map $C \to p(C)$ is a consequence of the Weierstrass preparation theorem.
Assume now that the map $C \to p(C)$ is finite. Again by the Weiertrass theorem, it means that the map of analytic algebras $\mathbf{C}\{x,y\} \to \mathcal{O}$ defined by $f \mapsto (f \circ p)|C$ makes $\mathcal{O}$ a $\mathbf{C}\{x,y\}$-module of finite type. Since $\mathbf{C}\{x,y\}$ is nœtherian, it means we have a presentation by an exact sequence of $\mathbf{C}\{x,y\}$-modules:

$$\mathbf{C}\{x,y\}^q \to \mathbf{C}\{x,y\}^p \to \mathcal{O} \to 0$$

An argument detailed in [T1] shows that since C is of dimension 1, we must have $q = p$, so the first map is described by a square matrix with entries in $\mathbf{C}\{x,y\}$. Let $\phi(x,y)$ be the determinant of that matrix. This determinant is, up to an invertible factor, independant of the choice of the presentation. Then the image $p(C)$ is the plane curve with equation $\phi(x,y) = 0$.

On the other hand, let us say that a plane projection $p\colon \mathbf{C}^d \to \mathbf{C}^2$ is *general* for C if it has the following property:
For any sequence of couples of points $(a_i,b_i) \in (C \setminus \{0\}) \times (C \setminus \{0\})$ tending to 0, the limit direction of the secant line $\overline{a_i,b_i}$ (for any subsequence) is *not* contained in the kernel of p.

We will see in the next paragraph that all general projections of a given space curve are topologically indistinguishable as curves in $\mathbf{C}^2$. In [T2] it is shown that if p is general for C, then the inclusion of the ring $\mathcal{O}_1$ of the image $X_1 = p(X)$ as defined above into the ring $\mathcal{O}$ (induced by the composition of functions with p) induces an isomorphism of the total rings of fractions of these two rings, and because $\mathcal{O}$ is a finite $\mathcal{O}_1$-module, every element of $\mathcal{O}$ is integral over $\mathcal{O}_1$, as we saw above. Therefore $\mathcal{O}$ is contained in the normalization $\overline{\mathcal{O}_1}$ of $\mathcal{O}_1$.

Therefore $\overline{\mathcal{O}_1}$ is also the normalization of $\mathcal{O}$, and it is a finite $\mathcal{O}$-module for general reasons (it suffices to know that the integral closure of $\mathcal{O}_1$ is a finite $\mathcal{O}_1$-module). Now we can use the universal property of blowing ups: in $\overline{\mathcal{O}_1}$ all ideals become principal and generated by a non zero divisor in each $\mathbf{C}\{t_i\}$. By the universal property of blowing up ([Ł], VII.5) if we blow up the origin in $\mathcal{O}$, the resulting algebra is still contained in $\overline{\mathcal{O}_1}$, and as we repeat blowing up points, we get an increasing sequence of $\mathcal{O}$-algebras contained in $\overline{\mathcal{O}_1}$, all having the same total ring of fractions. Since $\overline{\mathcal{O}_1}$ is a finite $\mathcal{O}$-module, this sequence stabilizes after finitely many steps. We have to show that this limit algebra is $\overline{\mathcal{O}_1}$. But if this were not the case, the maximal ideal of one of the component local algebras would not be principal, so we could blow it up and get a strictly bigger algebra, contradicting the stability.

In conclusion, we have shown that any space curve singularity can also be desingularized by a finite sequence of point blowing ups.

One can also prove embedded resolution for space curves; it is not much more difficult than in the plane curve case. I refer to the lecture of J. Castellanos for facts about space curves.

4. PUISEUX CHARACTERISTIC EXPONENTS

Let

$$f(x,y) = 0 \quad \text{with } f(x,y) \in \mathbf{C}\{x,y\}$$

be an equation for a branch $(X,0) \subset (\mathbf{C}^2,0)$, which means that the series f is an irreducible element of $\mathbf{C}\{x,y\}$.

As we saw, we may assume thanks to the Weierstrass preparation theorem that f is of the form

$$f(x,y) = y^m + a_1(x)y^{m-1} + \cdots + a_m(x)$$

where m is the intersection multiplicity at 0 of the branch C with the axis $x = 0$. We saw also that we may assume that m is equal to the multiplicity $n = m_0(X)$ of the curve X at the origin.

For this last paragraph we return to the traditional notation; n will be the multiplicity of the curve, which we have hitherto denoted by m, and the curve will be denoted by X instead of C.

As we saw, the branch X can be parametrized near 0 as follows

$$\begin{aligned} x(t) &= t^n \\ y(t) &= a_m t^m + a_{m+1}t^{m+1} + \cdots + a_j t^j + \cdots \qquad \text{with } m \geq n \end{aligned}$$

Let us now consider the following grouping of the terms of the series $y(t)$: set $\beta_0 = n$ and let β_1 be the smallest exponent appearing in $y(t)$ which is not divisible by β_0. If no such

exponent exists, it means that y is a power series in x, so that our branch is analytically isomorphic to $\mathbf{C}$, hence non singular. Let us suppose that this is not the case, and set $e_1 = (n, \beta_1)$, the greatest common divisor of these two integers. Now define β_2 as the smallest exponent appearing in $y(t)$ which is not divisible by e_1. Define $e_2 = (e_1, \beta_2)$; we have $e_2 < e_1$, and we continue in this manner. Having defined $e_i = (e_{i-1}, \beta_i)$, we define β_{i+1} as the smallest exponent appearing in $y(t)$ which is not divisible by e_i. Since the sequence of integers

$$n > e_1 > e_2 > \cdots > e_i > \cdots$$

is strictly decreasing, there is an integer g such that $e_g = 1$. At this point, we have structured our parametric representation as follows:

$$\begin{aligned}
x(t) &= t^n \\
y(t) &= a_n t^n + a_{2n} t^{2n} + \cdots + a_{kn} t^{kn} + a_{\beta_1} t^{\beta_1} + a_{\beta_1+e_1} t^{\beta_1+e_1} + \cdots + a_{\beta_1+k_1e_1} t^{\beta_1+k_1e_1} \\
&\quad + a_{\beta_2} t^{\beta_2} + a_{\beta_2+e_2} t^{\beta_2+e_2} + \cdots + a_{\beta_q} t^{\beta_q} + a_{\beta_q+e_{q-1}} t^{\beta_q+e_{q-1}} + \cdots \\
&\quad + a_{\beta_g} t^{\beta_g} + a_{\beta_g+1} t^{\beta_g+1} + \cdots
\end{aligned}$$

where by construction the coefficients of the $t^{\beta_i}; i \geq 1$ are not zero. Let us define integers n_i and m_i by the equalities

$$\begin{aligned}
e_{i-1} &= n_i e_i \\
\beta_i &= m_i e_i \qquad\qquad \text{for } 1 \leq i \leq g
\end{aligned}$$

and note that we may rewrite the expansion of y into powers of t as an expansion of y into fractional powers of x as follows:

$$\begin{aligned}
y =& a_n x + a_{2n} x^2 + \cdots + a_{kn} x^k + a_{\beta_1} x^{\frac{m_1}{n_1}} + a_{\beta_1+e_1} x^{\frac{m_1+1}{n_1}} + \cdots + a_{\beta_1+k_1e_1} x^{\frac{m_1+k_1}{n_1}} \\
&+ a_{\beta_2} x^{\frac{m_2}{n_1n_2}} + a_{\beta_2+e_2} x^{\frac{m_2+1}{n_1n_2}} + \cdots + a_{\beta_q} x^{\frac{m_q}{n_1n_2\cdots n_q}} + a_{\beta_q+e_{q-1}} x^{\frac{m_q+1}{n_1n_2\cdots n_q}} + \cdots \\
&+ a_{\beta_g} x^{\frac{m_g}{n_1n_2\cdots n_g}} + a_{\beta_g+1} x^{\frac{m_g+1}{n_1n_2\cdots n_g}} + \cdots
\end{aligned}$$

The set of pairs of coprime integers (m_i, n_i) are sometimes also called the Puiseux characteristic pairs. Their datum is obviously equivalent to that of the characteristic exponents β_i. The sequence of integers $B(X) = (\beta_0, \beta_1, \ldots, \beta_g)$, where $\beta_0 = n$, may be characterized algebraically as follows: let μ_n denote the group of n-th roots of unity. For $\omega \in \mu_n$ let us compute the order in t of the series $y(t) - y(\omega t)$. If we write $\omega = e^{\frac{2\pi i k}{n}}$, we have

$$y(\omega t) = a_n \omega^n t^n + \cdots + a_{\beta_1} \omega^{\beta_1} t^{\beta_1} + \cdots$$

and we see that multiplying t by ω does not affect the terms in t^{jn}. The term in t^{β_1} is unchanged if and only if $\omega^{\beta_1} = 1$, that is $\frac{k\beta_1}{n}$ is an integer, i.e; $k\beta_1 = ln$ or $km_1 = ln_1$ with the notations introduced above. Since n_1 and m_1 are coprime, this means that k is a multiple of n_1, which is equivalent to saying that ω belongs to the subgroup $\mu_{\frac{n}{n_1}}$ of μ_n consisting of $\frac{n}{n_1} = n_2 \cdots n_g$-th roots of unity. If this is the case, then the coefficients of

all the terms of the form $t^{\beta_1+je_1}$ in the Puiseux expansion are also unchanged when t is multiplied by ω, and the first term which may change is $a_{\beta_2}t^{\beta_2}$. An argument similar to the previous one shows that if $\omega \in \mu_{\frac{n}{n_1}}$, then $\omega^{\beta_2} = 1$ if and only if $\omega \in \mu_{\frac{n}{n_1 n_2}}$, and so on.

Finally, if we denote by v the order in t of an element of $\mathbf{C}\{t\}$, we see that

$$v(y(t) - y(\omega t)) = \beta_i \text{ if and only if } \omega \in \mu_{\frac{n}{n_1\cdots n_{i-1}}} \setminus \mu_{\frac{n}{n_1\cdots n_i}} \text{ for } 1 \leq i \leq g$$

This provides an algebraic characterization, and a sequence of cyclic subextensions

$$\mathbf{C}\{x\} \subset \mathbf{C}\{x^{\frac{1}{n_1}}\} \subset \mathbf{C}\{x^{\frac{1}{n_1 n_2}}\} \subset \cdots \subset \mathbf{C}\{x^{\frac{1}{n_1 n_2 \cdots n_i}}\} \subset \cdots \subset \mathbf{C}\{x^{\frac{1}{n}}\}$$

corresponding to the nested subgroups $\mu_{\frac{n}{n_1\cdots n_i}}$ of th group μ_n.

This shows that the sequence $(\beta_0, \beta_1, \ldots \beta_g)$ depends only upon the ring inclusion $\mathbf{C}\{x\} \subset \mathcal{O}_{X,0}$.

We shall see later in a different way that this sequence does not depend upon the choice of coordinates (x, y) in which we write the Puiseux expansion *as long as the curve* $x = 0$ *is transversal to* X.If this is not the case, one still obtains other characteristic exponents, which are related to the transversal ones by the *inversion formula* which I leave as an exercise.

For example consider the curve with equation $y^3 - x^2 = 0$.

I refer to the lectures of Lê for the Burau-Zariski topological interpretation of the characteristic sequence $(\beta_0, \beta_1, \ldots \beta_g)$ as a characteristic of the iterated torus knot that one obtains upon intersecting the branch X with a sufficiently small sphere in $\mathbf{C}^2$ centered at the origin.

Given a germ of a reduced plane curve X, it has a decomposition $X = \bigcup_{i=1}^r X_i$ into branches; each branch has its characteristic sequence $B(X_i)$, and as numerical characters of X, we have also the intersection numbers $(X_i, X_j)_0$ of distinct branches at 0.

If we remember that these intersection numbers are equal to the linking numbers in $\mathbf{S}^3$ of the knots corresponding to X_i and X_j and are therefore topological characters of the link $X \cap \mathbf{S}^3_\epsilon$, since Milnor proved (see Lê's lectures) that the curve X is homeomorphic to the cone with vertex 0 drawn on this link, we expect that the collection of the characteristic sequences of the branches and their intersection numbers may be a topological invariant of the curve X.

Let us define the local topological type of a germ of subspace of $\mathbf{C}^N$ as follows:

Definition.- Two subspaces X_1 and X_2 of $\mathbf{C}^N$ are topologically equivalent at 0 if there exist neighbourhoods U and V of 0 in $\mathbf{C}^N$ and an homeomorphism $\psi: U \to V$ such that $\psi(X_1 \cap U) = X_2 \cap V$. Two germs at 0 of subspaces are topologically equivalent if they have representatives which are topologically equivalent at 0.

Theorem (Zariski, Lejeune-Jalabert).- *Two germs of plane curves* $X = \cup_{i\in I} X_i$ *and* $X' = \cup_{i\in I'} X'_i$ *are topologically equivalent if and only if there exists a bijection* $\phi: I \to I'$ *between their branches which preserves characteristics and intersection numbers, that is, satisfies*

$$B(X'_{\phi(i)}) = B(X_i) \text{ for } i \in I, \quad (X'_{\phi(i)}, X'_{\phi(j)})_0 = (X_i, X_j)_0 \text{ for } i \neq j.$$

Topological equivalence is less strict a relation than analytic (or even $\mathcal{C}^1$) equivalence.

Let X_1 and X_2 each consist of four lines through the origin in $\mathbf{C}^2$. According to the previous theorem, these two germs are topologically equivalent. However, if there was a germ et 0 of a $\mathcal{C}^1$ (and in particular analytic) isomorphism of $\mathbf{C}^2$ to itself, sending X_1 to X_2, its tangent linear map at 0 would have to send X_1 onto X_2. But two quadruplets of lines through 0 are linearly equivalent if and only if they have the same cross-ratio. If the slopes of the lines of X_1 are a_1, b_1, c_1, d_1, and similarly for X_2, the cross ratios are

$$(\frac{a_1 - a_3}{a_1 - a_4})(\frac{a_2 - a_4}{a_2 - a_3})$$

and the numbers obtained by permutation. It is therefore easy to find examples where X_1 and X_2 are not $\mathcal{C}^1$-equivalent.

In particular, in an analytic family of curves such as the surface in $\mathbf{C}^3$ with equation

$$(y - x)(y + x)(y - 2x)(y + tx) = 0$$

for small values of t, the fibers are all analytically inequivalent but topologically equivalent.

Theorem.- *Given two reduced germs of plane curves* $(X, 0) \subset (\mathbf{C}^2, 0)$ *and* $(X', 0) \subset (\mathbf{C}^2, 0)$ *the following conditions are equivalent:*
1) X *and* X' *are topologically equivalent,*
2) There exists an integer d*, a germ of curve* $(C, 0) \subset (\mathbf{C}^d, 0)$ *and two linear projections* $p, p' \colon \mathbf{C}^d \to \mathbf{C}^2$*, both general for* C *at* 0*, and such that* $p(C) = X, \quad p'(C) = X'$*,*
3) There exists a one-parameter family of germs of plane curves that is a germ along $\{0\} \times U$ *of a surface in* $\mathbf{C}^2 \times U$*, where* U *is a disk in* $\mathbf{C}$*, say with equation* $f(x, y, u) = 0$ *and* $v, v' \in U$ *such that the germs of plane curve* $f(x, y, v) = 0, \ f(x, y, v') = 0$ *are isomorphic to* $X, \ X'$ *respectively and all the germs* $f(x, y, t) = 0$ *have the same topological type for* $t \in U$*.*

In fact, the theory of Lipschitz saturation, summarized in [T2], shows that, given the topological type of a germ of plane curve X, there exists a space curve $X^s \subset \mathbf{C}^N$, unique up to isomorphism, such that the germs of plane curves having the same topological type as are exactly the images of X^s by a linear projection which is generic for X^s.

The δ invariant of a plane curve singularity.

Let $\mathcal{O}$ be the analytic algebra of a germ of curve $(X, 0)$, plane or not, and let $\overline{\mathcal{O}}$ be its normalization. Since it is an $\mathcal{O}$-module of finite type with the same total ring of quotients, a version of the Hilbert Nullstellensatz shows that the quotient *vector space* over $\mathbf{C}$ is finite dimensional. So we may define an invariant to measure how far $\mathcal{O}$ is from being integrally closed, i.e regular:

$$\delta_X = \dim_{\mathbf{C}} \frac{\overline{\mathcal{O}}}{\mathcal{O}}$$

In the case of plane curves, this invariant has a geometrical interpretation, (see [T3]) which I will describe only in the case of a branch, for simplicity:

Let $t^n, y(t)$ be a parametrization of our branch X. Consider the product of the normalization of X with itself, with coordinates $(t't')$ and the two curves in $(\mathbf{C}^2, 0) = (\overline{X} \times \overline{X}, 0)$ defined by

$$\frac{t^n - t'^n}{t - t'} = 0 \quad \frac{y(t) - y(t')}{t - t'} = 0$$

The intersection number of these two curves at the origin is equal to $2\delta_X$; if now we perturb slightly the parametrization of X by $t^n + v\alpha t, y(t) + v\beta t$ with two "general " complex numbers $\alpha,\ \beta$, we can see that the two curves now have deformed equations and for small v they now meet transversally in $2\delta_X$ points in $\overline{X} \times \overline{X}$. this means that the curve defined parametrically by $t^n + v\alpha t, y(t) + v\beta t$ has δ_X ordinary double points (two branches meeting transversally), which tend to 0 as v tends to 0. So we can view δ_X as the number of ordinary double points which have coalesced to form the singularity of X at the origin. Of course, for an ordinary double point $\delta = 1$.

5. THE SEMIGROUP OF A BRANCH

There is another natural object associated to the inclusion $\mathcal{O} \to \overline{\mathcal{O}}$; again I will decribe it only in the case of a branch.

Let $\mathcal{O}$ be the analytic algebra of a germ of analytically irreducible curve X, and let $\overline{\mathcal{O}}$ be its normalization; we have an injection $\mathcal{O} \to \overline{\mathcal{O}}$ which makes $\overline{\mathcal{O}}$ an $\mathcal{O}$-module of finite type and $\overline{\mathcal{O}}$ is a subalgebra of the fraction field of $\mathcal{O}$. Since $\overline{\mathcal{O}}$ is isomorphic to $\mathbf{C}\{t\}$, the order in t of the series defines a mapping $\nu\colon \mathbf{C}\{t\} \setminus 0 \to \mathbf{N}$ which satisfies
i) $\nu(a(t)b(t)) = \nu(a(t)) + \nu(b(t))$ and
ii) $\nu(a(t) + b(t)) \geq \min(\nu(a(t)), \nu(b(t)))$ with equality if $\nu(a(t)) \neq \nu(b(t))$;
in other words, ν is a *valuation* of the ring $\mathbf{C}\{t\}$.

We consider the valuations of the elements of the subring $\mathcal{O}$, i.e the image Γ of $\mathcal{O} \setminus \{0\}$ by ν; in view of i), it is a semigroup contained in $\mathbf{N}$. The fact that $\overline{\mathcal{O}}$ is a finite $\mathcal{O}$-module implies that $\mathbf{N} \setminus \Gamma$ is finite, and in fact we have for the δ invariant of C the equality

$$\delta_X = \#(\mathbf{N} \setminus \Gamma)$$

Now we seek a minimal set of generators of Γ as a semigroup:
Let $\overline{\beta_0}$ be the smallest non zero element in Γ, let $\overline{\beta_1}$ be the smallest element of Γ which is nor a multiple of $\overline{\beta_0}$, let $\overline{\beta_2}$ be the smallest element of Γ which is not a combination with non negative integral coefficients of $\overline{\beta_0}$ and $\overline{\beta_1}$, i.e is not in the semigroup $\langle \overline{\beta_0}, \overline{\beta_1} \rangle$, and so on. Finally, since $\mathbf{N} \setminus \Gamma$ is finite, we find in this way a minimal set of generators:

$$\Gamma = \langle \overline{\beta_0}, \overline{\beta_1}, \ldots, \overline{\beta_g} \rangle$$

This set is uniquely determined by the semigroup Γ, and of course determines it.
By a theorem of Apery and Zariski, if $(X, 0)$ is a plane branch, the datum of these generators, or of the semigroup, is equivalent to the datum of the Puiseux characteristic of $(X, 0)$, or of its topological type.

Let us take the notations introduced for the Puiseux pairs; it is easy to check that if we set $\beta_0 = n$, the multiplicity, then $\overline{\beta_0} = \beta_0 = n, \overline{\beta_1} = \beta_1$. After that is becomes more complicated. Zariski ([Z], Th. 3.9) proved the following formula for $q = 2, \dots, g$:

$$\overline{\beta_q} = (n_1 - 1)n_2 \dots n_{q-1}\beta_1 + (n_2 - 1)n_3 \dots n_{q-1}\beta_3 + \dots + (n_{q-1} - 1)\beta_{q-1} + \beta_q,$$

which can be summarized in the following recursive formula:

$$\overline{\beta_q} = n_{q-1}\overline{\beta_{q-1}} - \beta_{q-1} + \beta_q$$

It follows easily from this that the datum of the semigroup is equivalent to the datum of the multiplicity n and the Puiseux exponents β_i of the curve.

The semigroups coming from plane branches are characterized among all semigroups of analytically irreducible germs of curves by the following two properties:

1) $n_i\overline{\beta_i} \in \langle \overline{\beta_0}, \dots, \overline{\beta_{i-1}} \rangle$

2) $n_i\overline{\beta_i} < \overline{\beta_{i+1}}$

That the semigroups of plane branches have these properties follows from the induction formula and the inequalities $\beta_i < \beta_{i+1}$. The converse can be proved by the construction outlined below (see [Z], appendix).

Conversely, given a semigroup Γ in $\mathbf{N}$ with finite complement, we can associate to it an analytic (in fact algebraic) curve, called the *monomial curve* associated to Γ. If $\Gamma = \langle \overline{\beta_0}, \overline{\beta_1}, \dots, \overline{\beta_g} \rangle$, the monomial curve C^Γ is described parametrically by

$$\begin{aligned} u_0 &= t^{\overline{\beta_0}} \\ u_1 &= t^{\overline{\beta_1}} \\ &\vdots \\ u_g &= t^{\overline{\beta_g}} \end{aligned}$$

On the other hand, the relations 1) above mean that there exist natural numbers $\ell_i^{(j)}$ such that we have

$$\begin{aligned} n_1\overline{\beta_1} &= \ell_0^{(1)}\overline{\beta_0} \\ n_2\overline{\beta_2} &= \ell_0^{(2)}\overline{\beta_0} + \ell_1^{(2)}\overline{\beta_1} \\ &\vdots \\ n_j\overline{\beta_j} &= \ell_0^{(j)}\overline{\beta_0} + \dots + \ell_{j-1}^{(j)}\overline{\beta_{j-1}} \\ &\vdots \\ n_g\overline{\beta_g} &= \ell_0^{(g)}\overline{\beta_0} + \dots + \ell_{g-1}^{(j)}\overline{\beta_{g-1}} \end{aligned}$$

These relations translate into equations for the curve $C^\Gamma \subset \mathbf{C}^{g+1}$; since $u_i = t^{\overline{\beta_i}}$, our curve satisfies the g equations

$$u_j^{n_i} - u_0^{\ell_0^{(j)}} u_1^{\ell_1^{(j)}} \dots u_{j-1}^{\ell_{j-1}^{(j)}} = 0, \quad 1 \le j \le g,$$

and it can be shown that they actually define $C^\Gamma \subset \mathbf{C}^{g+1}$, so that if Γ is the semigroup of a plane branch, C^Γ is a complete intersection.

Remark that if we give to u_i the weight $\overline{\beta_i}$, the i-th equation is homogeneous of degree $n_i\overline{\beta_i}$.

The connection between a plane curve X having semigroup Γ and the monomial curve is much more precise and interesting than the formal relation we have just seen; by small deformations of the monomial curve one obtains all the branches with the same semigroup. In fact the best way to understand all branches with semigroup Γ is to consider the not necessarily plane curve C^Γ (C^Γ is plane if and only if C has only one characteristic exponent).

By definition of Γ, there are elements $\xi_q \in \mathcal{O}$ with $\nu(\xi_q) = \overline{\beta_q}$. We can write these elements in $\mathbf{C}\{t\}$ as

$$\xi_q = t^{\overline{\beta_q}} + \sum_{j > \overline{\beta_q}} \gamma_{q,j} t^j.$$

Let us consider the one-parameter family of parametrizations

$$\begin{aligned} u_0 =& t^m \\ u_1 =& t^{\overline{\beta_1}} + \sum_{j > \overline{\beta_1}} v^{j - \overline{\beta_1}} \gamma_{1,j} t^j \\ & \cdot \\ & \cdot \\ u_g =& t^{\overline{\beta_g}} + \sum_{j > \overline{\beta_g}} v^{j - \overline{\beta_g}} \gamma_{g,j} t^j \end{aligned}$$

The reader can check that for $v \neq 0$, the curve thus described is isomorphic to our original curve C. (hint: make the change of parameter $t = vt'$ and the change of coordinates $u_j = v^{-\overline{\beta_j}} v'_j$, and remember the definition of the ξ_j). For $v = 0$, we have the parametric description of the monomial curve.

So we have in fact described a map

$$\mathbf{C} \times \mathbf{C} \to \mathbf{C}^{g+1} \times \mathbf{C}$$

which induces the identity on the second factors (with coordinate v). The image of this map is a surface, which is the total space of a deformation of the monomial curve, all of its fibers except the one for $v = 0$ being isomorphic to our plane curve C.

So the monomial curve is a specialization, in this family, of our plane curve. In this specialization the multiplicity and the semigroup remain constant; in a rather precise sense

it is an equisingular specialization, or one may say that *the plane curve is an equisingular deformation of the monomial curve with the same semigroup.*

The same phenomenon can be also observed in the language of equations rather than parametrizations. Let us consider a one parameter family of equations for curves in $\mathbf{C}^{g+1}$, of the form

$$\begin{aligned}
&u_1^{n_1} - u_0^{\ell_0^{(1)}} - vu_2 = 0\\
&u_2^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}} - vu_3 = 0\\
&.\\
&.\\
&u_{g-1}^{n_{g-1}} - u_0^{\ell_0^{(g-1)}} u_1^{\ell_1^{(g-1)}} \dots u_{g-2}^{\ell_{g-2}^{(g-1)}} - vu_g = 0\\
&u_g^{n_g} - u_0^{\ell_0^{(g)}} u_1^{\ell_1^{(g)}} \dots u_{g-1}^{\ell_{g-1}^{(g)}} = 0
\end{aligned}$$

For $v = o$ we get the equations of the monomial curve, and for $v \neq 0$ we get a curve which has semigroup Γ; this is a general heuristic principle of equisingularity: we have added to each equation of the monomial curve, homogeneous of degree $n_i\overline{\beta_i}$, a perturbation of degree $\overline{\beta_{i+1}} > n_i\overline{\beta_i}$, and this should not change the equisingularity class (the perturbation is "small" compared to the equation).

Notice that for each fixed $v \neq 0$ the curve described by the above equations is a plane curve: for simplicity take $v = 1$; then use the first equation to compute $u_2 = u_1^{n_1} - u_0^{\ell_0^{(1)}}$, substitute this in the next equation, and use this to compute u_3 as a function of u_0, u_1, and so on. Finally the last equation gives us the equation of a plane curve of the form

$$\left(\cdots\left((u_1^{n_1} - u_0^{\ell_0^{(1)}})^{n_2} - u_0^{\ell_0^{(2)}} u_1^{\ell_1^{(2)}}\right)^{n_3} - \cdots\right)^{n_g} - u_0^{\ell_0^{(g)}} u_1^{\ell_1^{(g)}} (u_1^{n_1} - u_0^{\ell_0^{(1)}})^{\ell_2^{(g)}} \cdots = 0$$

The first consequence is that we can produce explicitely the equation of a plane curve with given characteristic exponents: compute the semigroup and its generators, and then write the equation above.

A more important fact is that one can show (see [Z], appendix) that any plane curve with a given semigroup appears up to isomorphism as a fiber in a deformation depending on a finite number of parameters: it is a deformation of the monomial curve obtained by adding to the j-th equation a polynomial in the u_i's of order $> n_j\overline{\beta_j}$, and these polynomials can in principle be explicitely computed.

Finally, all the plane branches with the same semigroup have "the same" process of resolution of singularities: you have to blow up points according to the same rules, the multiplicities of the strict transforms are the same, and so on. So the resolution of the plane curve described above shows the structure of the resolutions of all the curves with the same semigroup. First you resolve the curve $u_1^{n_1} - u_0^{\ell_0^{(1)}} = 0$; when its strict transform is non singular (after a number of blowing ups which depends on the continued fraction expansion of the ratio $\frac{\ell_0^{(1)}}{n_1}$, you take it as a coordinate axis: then you have one parenthesis

less in the equation above (the point is that the form of the equation does not change), and you proceed like this. After g such steps the branch is resolved.

So the deformation to the monomial curve also explains to us how to resolve the singularities, and it is perhaps the best description. Can we generalize it to higher dimensions?

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