Let ν_1 and ν_2 be valuations on the noetherian local ring (R, m), with centers $p_2 \subset_{\neq} p_1$ and such that ν_1 is composed with ν_2 and their ranks differ by one. Let $\lambda\colon \Gamma_1 \to \Gamma_2$ be the corresponding map of value groups. Set $t_1 = \nu_1(p_1), t_2 = \nu_2(p_2)$.

Given $\varphi_2 \in \Gamma_2$, denote by $\tilde{\varphi}_2 \in \Gamma_1$ the minimum of $\{\nu_1(f); f \in R,\ \nu_2(f) = \varphi_2\}$. Note that $\lambda(\tilde{\varphi}_2) = \varphi_2$ and that $t_1 \in \mathrm{Ker}\lambda$. Since the ranks of the valuations differ by one, the ordered group $\mathrm{Ker}\lambda$ is of rank one, and we identify it with a subgroup of the ordered group $\mathbf{R}$.

We remark that for $\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]$ we have the inclusions

$$p_1^{y_1}\mathcal{P}_{\varphi_2} \subset \mathcal{P}_{\varphi_1} \subset \mathcal{P}_{\varphi_2},$$

and since the valuation $\overline{\nu}_1$ which is the image of ν_1 in R/p_2 is of rank one, the number of elements in the interval $[\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]$ is finite.

The elements of the group Γ_i will be denoted by φ_i or γ_i, and it is understood that $\mathcal{P}_{\varphi_i} = \{x \in R; \nu_i(x) \geq \varphi_i\}$.

Lemma 0.1. *For any additive function A on R-modules we have*

$$(1) \qquad \sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]} A(\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1}) \leq A(M_{\varphi_2}/p_1^{y_1} M_{\varphi_2}),$$

where $M_{\varphi_2} = \mathcal{P}_{\varphi_2}/\mathcal{P}^+_{\varphi_2}$, which is a finitely generated torsion free R/p_2-module.

Indeed, by additivity the left hand side is equal to $A(\frac{M_{\varphi_2}}{\mathcal{P}_{\varphi_2}/\mathcal{P}_{\tilde{\varphi}_2 + t y_1}})$ which is a quotient of $M_{\varphi_2}/p_1^{y_1} M_{\varphi_2}$. □

In the special case where ν_1 is a valuation of rank one and ν_2 is the trivial valuation, we have $\Gamma_2 = 0$, $\tilde{\varphi}_2 = 0 \in \Gamma_1$ and the inequality reduces to:

$$(2) \qquad \sum_{\varphi \in [0, y_1 t_1]} A(\mathcal{P}_{\varphi}/\mathcal{P}^+_{\varphi}) \leq A(R/p_1^{y_1} R),$$

We can take as additive function A on R-modules of dimension $\leq \mathrm{dim}R/p_1$ the multiplicity in dimension $\mathrm{dim}R/p_1$ with respect to the maximal ideal. The inequality (1) then becomes

$$(3) \qquad \sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]} e_m(\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1}) \leq e_m(M_{\varphi_2}/p_1^{y_1} M_{\varphi_2}),$$

It is possible to evaluate the right hand side thanks to the:

Proposition 0.2. (See ([B], §7, No.1, Prop 3)*Given a finitely generated R-module M of dimension $d \geq 0$, denote by B the set of minimal prime ideals of* $\mathrm{Supp}(M)$ *such that* $\dim(R/p) = d$. *Let q be an ideal of R contained in its radical and such that M/qM has finite length. Then we have*

$$e_q(M) = \sum_{p \in B} \mathrm{length}(M_p) e_q(R/p).$$

□

We shall also use the following

Corollary 0.3. *Let (R, m) be a noetherian local ring and let M be a finitely generated module whose annihilator is a prime ideal p. Let q be a prime ideal of R containing p. Then we have the inequality*

$$e_{qR_q}(M_q) \leq e_m(M).$$

By Proposition 0.2, we have the following two equalities:

$$e_{qR_q}(M_q) = \text{length}_{(R_q)_{pR_q}}((M_q)_{pR_q}) e_{qR_q}(R_q/pR_q) \text{ and } e_m(M) = \text{length}_{R_p}(M_p) e_m(R/p).$$

Now it suffices to use the fact that $(R_q)_{pR_q} = R_p$, $(R/p)_{q/p} = R_q/pR_q$ and $(M_q)_{pR_q} = M_p$, and the semicontinuity of mutiplicity (see [L]) which tells us that $e_{qR_q}(R_q/pR_q) \leq e_m(R/p)$. ☐

If we apply Proposition 0.2 to $M_{\varphi_2}/p_1^{y_1}M_{\varphi_2}$ we get:

$$e_m(M_{\varphi_2}/p_1^{y_1}M_{\varphi_2}) = \text{length}_{R_{p_1}}\big((M_{\varphi_2})_{p_1}/m_1^{y_1}(M_{\varphi_2})_{p_1}\big) e_m(R/p_1),$$

where $m_1 = p_1R_{p_1}$ and the support of the R-module M_{φ_2} is the ideal p_2.
By the main result on Hilbert-Samuel polynomials, in view of the fact that the support of $M_{\varphi_2})_{p_1}$ is the ideal $p_2R_{p_1}$, we have an asymptotic estimate:

$$\text{length}_{R_{p_1}}\big((M_{\varphi_2})_{p_1}/m_1^{y_1}(M_{\varphi_2})_{p_1}\big) \asymp \frac{e_{m_1}\big((M_{\varphi_2})_{p_1})\big)}{(\dim(R_{p_1}/p_2R_{p_1})!} y_1^{\dim(R_{p_1}/p_2R_{p_1})}$$

Let

$$(0) = p_{n+1} \subset p_n \subset p_{n-1} \subset \cdots \subset p_1 \subset m$$

be the sequence of the centers of valuations ν_i with which a given valuation ν of rank n is composed.

Let

$$\Gamma \xrightarrow{\lambda_1} \Gamma_1 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_n} \Gamma_n \longrightarrow 0$$

be the corresponding sequence of morphisms of value groups. For each $\varphi_{j+1} \in \nu_{j+1}(R \setminus \{0\}) \subset \Gamma_{j+1}$, define $\tilde{\varphi}_{j+1} = \min\{\nu_j(f); f \in R, \nu_{j+1}(f) = \varphi_{j+1}\}$, and remark the equality of valuation ideals $\mathcal{P}_{\varphi_{j+1}} = \mathcal{P}_{\tilde{\varphi}_{j+1}}$ with respect to the two valuations ν_{j+1} and ν_j.

Define also $t_j = \nu_j(p_j)$ and remark that it is in the subgroup of rank one $\text{Ker}\lambda_{j+1}$ of Γ_j, so that we may identify it with a real number. We assume chosen from the beginning an identification of all the $\text{Ker}\lambda_j$ with subgroups of $\mathbf{R}$.

Theorem 0.4. *For each $j \in \{1, \ldots, n\}$ and every $\varepsilon > 0$ the sum*

$$\sum_{\varphi_n \in [0, t_n y_n]} \sum_{\varphi_{n-1} \in [\tilde{\varphi}_n, \tilde{\varphi}_n + t_{n-1}y_{n-1}]} \cdots \sum_{\varphi_j \in [\tilde{\varphi}_{j+1}, \tilde{\varphi}_{j+1} + t_j y_j]} e_m(\mathcal{P}_{\varphi_j}/\mathcal{P}^+_{\varphi_j})$$

is bounded for $y_j, \ldots, y_n$ large enough by

$$(1+\varepsilon)\frac{\prod_{i=j-1}^n e_{m_i}(R_{p_i}/p_{i+1}R_{p_i})}{\prod_{i=j}^n (\dim R_{p_i}/p_{i+1}R_{p_i})!} \prod_{i=j}^n y_i^{\dim R_{p_i}/p_{i+1}R_{p_i}},$$

where $m_i = p_iR_{p_i}$.

The proof is by descending induction on j. The first step of the induction concerns the valuation ν_n, which is of rank one and is given by considering the special case of Lemma 0.1 where the valuation is of height one, and taking as additive function the multiplicity in dimension $\dim R/p_1$. Applying Proposition 0.2 to the right hand side of the inequality gives:

$$\sum_{\varphi \in [0, y_1 t_1]} A(\mathcal{P}_\varphi/\mathcal{P}^+_\varphi) \leq H_{R_{p_1}}(y_1) e_m(R/p_1),$$

where $H_{R_{p_1}}(y_1)$ is the Hilbert-Samuel function of the local ring R_{p_1} with respect to its maximal ideal $m_1 = p_1R_{p_1}$. It is for large y_1 a polynomial in y_1 whose highest degree term is $\frac{e_{m_1}(R_{p_1})}{(\dim R_{p_1})!} y_1^{\dim R_{p_1}}$, which gives the result in this case.

Define now $t_j = \nu_j(p_j) \in \Gamma_j$ for each j and remark that it is in the kernel of the map $\Gamma_j \to \Gamma_{j+1}$, which is of rank one. We can therefore think of it as a real number.
We are going to use the following

Lemma 0.5. *For each j we have $p_j^{y_j}\mathcal{P}_{\varphi_{j-1}} \subset \mathcal{P}_{\tilde{\varphi}_{j-1}+t_jy_j}$ and the interval $[\tilde{\varphi}_{j-1}, \tilde{\varphi}_{j-1}+t_jy_j]$ is finite.*

This follows directly from [ZS], Vol. II, appendix 3, Lemma 4. □

Assume that the result is known for j and prove it for $j-1$. Let us apply the result above to

In view of the lemma, we have the inequality

$$\sum_{\varphi_j \in [\tilde{\varphi}_{j+1}, \tilde{\varphi}_{j+1}+t_jy_j]} e_m(\mathcal{P}_{\varphi_j}/\mathcal{P}^+_{\varphi_j}) \leq e_m(M_{\varphi_{j+1}}/p_j^{y_j}M_{\varphi_{j+1}}).$$

Now we apply Proposition 0.2 and the properties of the Hilbert-Samuel polynomial of the R_{p_j}-module $(M_{\varphi_{j+1}})_{p_j}$, whose annihilator is $p_{j+1}R_{p_j}$, to obtain an asymptotic estimate

$$e_m(M_{\varphi_{j+1}}/p_j^{y_j}M_{\varphi_{j+1}}) \asymp \frac{e_{m_j}((M_{\varphi_{j+1}})_{p_j})}{(\mathrm{dim}R_{p_j}/p_{j+1}R_{p_j})!} y_j^{\mathrm{dim}R_{p_j}/p_{j+1}R_{p_j}},$$

so that for each $\varphi_{j+1} \in \Gamma_{j+1}$ we have for any $\varepsilon > 0$ for y_j sufficiently large (in a way which depends on ε and φ_j) an inequality

$$\sum_{\varphi_j \in [\tilde{\varphi}_{j+1}, \tilde{\varphi}_{j+1}+t_jy_j]} e_m(\mathcal{P}_{\varphi_j}/\mathcal{P}^+_{\varphi_j}) \leq (1+\varepsilon)\frac{e_{m_j}\big((M_{\varphi_{j+1}})_{p_j}\big)}{(\mathrm{dim}R_{p_j}/p_{j+1}R_{p_j})!} y_j^{\mathrm{dim}R_{p_j}/p_{j+1}R_{p_j}}.$$

Now we can use Corollary 0.3 to see that

$$e_{m_j}\big((M_{\varphi_{j+1}})_{p_j}\big) \leq e_m\big(M_{\varphi_{j+1}}\big) = e_m(\mathcal{P}_{\varphi_j}/\mathcal{P}_{\varphi_{j+1}}).$$

Summing up over the $\varphi_{j+1} \in [\tilde{\varphi}_{j+2}, \tilde{\varphi}_{j+2} + t_{j+1}y_{j+1}]$, which are finite in number, then over the $\varphi_{j+2} \in [\tilde{\varphi}_{j+3}, \tilde{\varphi}_{j+3} + t_{j+2}y_{j+2}]$ and so on, we obtain the statement. □
Remark If the centers of ν_j and ν_{j+1} coincide, the variable y_j does not appear in the bounding expression; this corresponds to the fact that there are only finitely many elements of $\nu_j(R \setminus \{0\})$ in the union of all intervals $[\tilde{\varphi}_{j+1}, \tilde{\varphi}_{j+1} + t_jy_j]$ (see [ZS], *loc.cit.*).

Corollary 0.6. *For each $\varepsilon > 0$, the number of elements of the semigroup $\nu(R \setminus \{0\})$ contained in the union of all the intervals appearing in the theorem is bounded, for y_j large enough, by*

$$(1+\varepsilon)\frac{\prod_{i=0}^n e_{m_i}(R_{p_i}/p_{i+1}R_{p_i})}{\prod_{i=1}^n(\mathrm{dim}R_{p_i}/p_{i+1}R_{p_i})!}\prod_{i=1}^n y_i^{\mathrm{dim}R_{p_i}/p_{i+1}R_{p_i}},$$

with the convention that $p_0 = m$.

Indeed, for the valuation ν itself, the multiplicity of $\mathcal{P}_\varphi/\mathcal{P}^+_\varphi$ is equal to its length, because it is a finite dimensional vector space over $k = R/m$, and it is nonzero if and only if φ is in $\nu(R \setminus \{0\})$. □

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