SEMIGROUPS OF VALUATIONS ON LOCAL RINGS, II

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Let (R, m_R) be a local domain, with quotient field K. Suppose that ν is a valuation of K with valuation ring (V, m_V). Suppose that ν dominates R; that is, $R \subset V$ and $m_V \cap R = m_R$. The value groups Γ of ν which can appear when K is an algebraic function field have been extensively studied and classified, including in the papers MacLane [**?**], MacLane and Schilling [**?**], Zariski and Samuel [33], and Kuhlmann [17]. The most basic fact is that there is an order preserving embedding of Γ into $\mathbf{R}^n$ with the lex order, where n is the dimension of R. The semigroups

$$S^R(\nu) = \{\nu(f) \mid f \in m_R - \{0\}|$$

which can appear when R is a noetherian domain with fraction field K dominated by ν, however, are not well understood, although they are known to encode important information about the ideal theory of R and the geometry and resolution of singularities of Spec$R$.

In Zariski and Samuel's classic book on Commutative Algebra [33], two general facts about semigroups $S^R(\nu)$ of valuations on noetherian local domains are proven (in Appendix 3 to Volume II).

1. For any valuation ν of K which is non negative on R, the semigroup $S^R(\nu)$ is a well ordered subset of the positive part of the value group Γ of ν of ordinal type at most ω^h, where ω is the ordinal type of the well ordered set $\mathbf{N}$, and h is the rank of the valuation.
2. If ν dominates R, the rational rank of ν plus the transcendence degree of V/m_V over R/m_R is less than or equal to the dimension of R.

The second condition is the Abhyankar inequality [**?**].

Prior to this paper, no other general constraints were known on the value semigroup semigroups $S^R(\nu)$. In fact, it was even unknown if the above conditions 1 and 2 characterize value semigroups.

In this paper, we construct an example of a well ordered subsemigroup of $\mathbf{Q}_+$ of ordinal type ω, which is not a value semigroup of a local domain. This shows that the above conditions 1 and 2 do not characterize value semigroups on local domains. We construct this in Corollary **??** by finding a new constraint, Theorem **??**, on a semigroup being a value semigroup of a local domain of dimension n. In Corollary **??**, we give a stronger constraint on regular local rings.

In [**?**], Teissier and the author give some examples showing that some surprising semigroups of rank > 1 can occur as semigroups of valuations on noetherian domains, and raise the general question of finding new constraints on value semigroups and classifying semigroups which occur as value semigroups.

The only semigroups which are realized by a valuation on a one dimensional regular local ring are isomorphic to the natural numbers. The semigroups which are realized by a valuation on a regular local ring of dimension 2 with algebraically closed residue

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field are much more complicated, but are completely classified by Spivakovsky in [29]. A different proof is given by Favre and Jonsson in [**?**], and the theorem is formulated in the context of semigroups by Cutkosky and Teissier [**?**]. However, very little is known in higher dimensions. The classification of semigroups of valuations on regular local rings of dimension two does suggest that there may be constraints on the rate of growth of the number of new generators on semigroups of valuations dominating a noetherian domain. We prove that there is such a constraint, giving a new necessary condition for a semigroup to be a value semigroup. This is accomplished in Theorem **??** and Corollaries **??** and **??**. The constraint is sufficiently strong to allow us to give a very simple example, Corollary **??**, of a well ordered subgroup S of $\mathbf{Q}_+$ of ordinal type ω which is not the semigroup of a valuation dominating an local domain.

1. Asymptotic growth of value semigroups

If R is a local ring, m_R will denote its maximal ideal, and $\ell(N)$ will denote the length of an R module N.

Given a valuation ν which is non negative on R and an element φ of its value group, we will denote by $\mathcal{P}_\varphi(R)$ the ideal $\{x \in R | \nu(x) \geq \varphi\}$ and by $\mathcal{P}^+_\varphi(R)$ the ideal $\{x \in R | \nu(x) > \varphi\}$. When no confusion on the ring is possible we will write $\mathcal{P}_\varphi, \mathcal{P}^+_\varphi$.We note that $\mathcal{P}_\varphi(R)/\mathcal{P}^+_\varphi(R) = 0$ if $\varphi \notin S^R(\nu) \cup \{0\}$.

Suppose that Γ is a totally ordered abelian group, and $a, b \in \Gamma$. We set

$$[a, b] = \{x \in \Gamma \mid a \leq x \leq b\} \text{ and } [a, b[= \{x \in \Gamma \mid a \leq x < b\}$$

Let ν_1 and ν_2 be valuations on the noetherian local domain (R, m), with centers $p_2 \subset_{\neq} p_1$ and such that ν_1 is composed with ν_2 and their ranks differ by one. Let $\lambda \colon \Gamma_1 \to \Gamma_2$ be the corresponding map of value groups. Set $t_1 = \nu_1(p_1), t_2 = \nu_2(p_2)$.

Definition 1.1. Given $\varphi_2 \in \Gamma_2$, denote by $\tilde{\varphi}_2 \in \Gamma_1$ the minimum of $\{\nu_1(f); f \in R,\ \nu_2(f) = \varphi_2\}$. This minimum exists since the semigroup $\nu_1(R \setminus \{0\})$ is well ordered.

Note that $\lambda(\tilde{\varphi}_2) = \varphi_2$ and that $t_1 \in \mathrm{Ker}\lambda$.

We remark that for $\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]$ we have the inclusions:

$$p_1^{y_1} \mathcal{P}_{\varphi_2} \subset \mathcal{P}_{\varphi_1} \subset \mathcal{P}_{\varphi_2},$$

$\mathcal{P}_{\varphi_2} = \mathcal{P}_{\tilde{\varphi}_2}$ and since the valuation $\overline{\nu}_1$ which is the image of ν_1 in R/p_2 is of rank one, the number of elements of $\nu_1(R \setminus \{0\})$ in the interval $[\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]$ is finite (see [33], *loc. cit.*).

Theorem 1.2. *For any function A on R-modules with values in $\mathbf{R}$ which is additive on short exact sequences we have:*

$$\sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1[} A(\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1}) \leq A(M_{\varphi_2}/p_1^{y_1} M_{\varphi_2}), \tag{1}$$

where $M_{\varphi_2} = \mathcal{P}_{\varphi_2}/\mathcal{P}^+_{\varphi_2}$, a finitely generated torsion free R/p_2-module.

Proof. We note that $\mathcal{P}^+_{\varphi_2} \subseteq p_1^{y_1} \mathcal{P}_{\varphi_2}$. The R-module $\mathcal{P}_{\varphi_2}/\mathcal{P}_{\tilde{\varphi}_2 + t_1 y_1}$ is a quotient of $\mathcal{P}_{\varphi_2}/p_1^{y_1}\mathcal{P}_{\varphi_2} = M_{\varphi_2}/p_1^{y_1} M_{\varphi_2}$, and by the additivity of A we have

$$A(\mathcal{P}_{\varphi_2}/\mathcal{P}_{\tilde{\varphi}_2 + t_1 y_1}) = \sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1[} A(\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1}).$$

The result follows, since the value of A can only decrease when passing to a quotient. $\square$

In the special case where ν_1 is a valuation of rank one and ν_2 is the trivial valuation, we have $\Gamma_2 = 0$, $\tilde{\varphi}_2 = 0 \in \Gamma_1$ and, writing φ for φ_1, p for p_1, etc., the inequality reduces to:

$$\sum_{\varphi \in [0,yt[} A(\mathcal{P}_\varphi/\mathcal{P}_\varphi^+) \leq A(R/p_1^y R), \tag{2}$$

Remark The R modules $\mathcal{P}_{\varphi_1}/\mathcal{P}_{\varphi_1}^+$ and $M_{\varphi_2}/p_1^{y_1}M_{\varphi_2}$ have p_1 as their unique minimal prime.

We can take as additive function A on R-modules of dimension $= \dim R/p_1$ the multiplicity e_m with respect to the maximal ideal of R. The inequality (1) then becomes

Corollary 1.3. *We have the inequalities*

$$\sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]} e_m(\mathcal{P}_{\varphi_1}/\mathcal{P}_{\varphi_1}^+) \leq e_m(M_{\varphi_2}/p_1^{y_1}M_{\varphi_2}). \tag{3}$$

It is now possible to evaluate the right hand side thanks to ([B], §7, No.1, Prop 3), which says this: given a finitely generated R-module M of dimension $d \geq 0$, denote by B the set of minimal prime ideals of $\mathrm{Supp}(M)$ such that $\dim(R/p) = d$. Let q be an ideal of R contained in its radical and such that M/qM has finite length. Then we have

$$e_q(M) = \sum_{p \in B} \mathrm{length}_{R_p}(M_p) e_q(R/p).$$

If we apply this to $M_{\varphi_2}/p_1^{y_1}M_{\varphi_2}$ we get:

$$e_m(M_{\varphi_2}/p_1^{y_1}M_{\varphi_2}) = \mathrm{length}_{R_{p_1}}\big((M_{\varphi_2})_{p_1}/m_1^{y_1}(M_{\varphi_2})_{p_1}\big) e_m(R/p_1), \tag{4}$$

where $m_1 = p_1 R_{p_1}$ and the support of the R-module M_{φ_2} is the ideal p_2.
By the main result on Hilbert-Samuel polynomials, in view of the fact that the support of $(M_{\varphi_2})_{p_1}$ is the ideal $p_2 R_{p_1}$, we have an asymptotic estimate:

$$\mathrm{length}_{R_{p_1}}\big((M_{\varphi_2})_{p_1}/m_1^{y_1}(M_{\varphi_2})_{p_1}\big) \asymp \frac{e_{m_1}\big((M_{\varphi_2})_{p_1}\big)}{(\dim(R_{p_1}/p_2R_{p_1})!} y_1^{\dim(R_{p_1}/p_2R_{p_1})}. \tag{5}$$

Let us now consider a valuation ν of rank n.
Let

$$(0) = p_{n+1} \subset p_n \subset p_{n-1} \subset \cdots \subset p_1$$

be the sequence of the centers of valuations ν_i with which a given valuation ν of rank n on a noetherian local domain (R, m) is composed. We assume that the centers are distinct prime ideals.

Let p_0 be a prime ideal of R containing the center p_1 of $\nu = \nu_1$ (p_0 could be equal to p_1). Then

Theorem 1.4. *The sum*

$$\sum_{\varphi_n \in [0, t_n y_n[} \sum_{\varphi_{n-1} \in [\tilde{\varphi}_n, \tilde{\varphi}_n + t_{n-1}y_{n-1}[} \cdots \sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + t_1 y_1[} e_{m_0}((\mathcal{P}_{\varphi_1}/\mathcal{P}_{\varphi_1}^+)_{p_0})$$

is bounded for $y_1, \ldots, y_n$ large enough by a function which behaves asymptotically as

$$\frac{\Pi_{i=0}^n e_{m_i}((R/p_{i+1})_{p_i})}{\Pi_{i=1}^n (\dim(R/p_{i+1})_{p_i})!} \Pi_{i=1}^n y_i^{\dim\ (R/p_{i+1})_{p_i}},$$

where $m_i = p_i R_{p_i}$ for $0 \leq i \leq n$. We take $y_1 \gg y_2 \gg \cdots \gg y_n$ to obtain this asymptotic behavior.

Proof. The proof of this formula is by induction on n. We first prove the formula in the case when $n=1$. We apply formulas (2) and (3) - (5) to the ring R_{p_0}, observe that for $\varphi_1 \in \Gamma_1$,

$$(\mathcal{P}_{\varphi_1})_{p_0} = \{f \in R_{p_0} \mid \nu(f) \geq \varphi_1\},$$

and that $(R_{p_0})_{p_1 R_{p_0}} \cong R_{p_1}$, to obtain

$$\sum_{\varphi_1 \in [0, t_1 y_1[} e_{m_0}((\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1})_{p_0}) \asymp \frac{e_{m_1}(R_{p_1}) e_{m_0}((R/p_1)_{p_0})}{(\dim\, R_{p_1})!} y_1^{\dim\, R_{p_1}},$$

which is the formula for $n=1$.

We now assume that the formula is true for valuations of rank $< n$. We will derive the formula for a rank n valuation ν. We apply the formula to the rank $n-1$ valuation ν_2 which ν is composite with, and the chain of prime ideals

$$(0) = q_n \subset q_{n-1} \subset \cdots \subset q_1 \subset q_0$$

where $q_{n-1} = p_n, \ldots, q_1 = p_2$ are the centers on R of the successive valuations $\nu_n, \ldots, \nu_2$ with which ν_2 is composite, and $q_0 = p_1$, to obtain

(6)

$$\sum_{\varphi_n \in [0, t_n y_n[} \cdots \sum_{\varphi_2 \in [\tilde{\varphi}_3 + t_2 y_2[} e_{m_1}((\mathcal{P}_{\varphi_2}/\mathcal{P}_{\varphi_2^+})_{p_1}) \asymp \frac{\prod_{i=1}^n e_{m_i}((R/p_{i+1})_{p_i})}{\prod_{i=2}^n (\dim\, (R/p_{i+1})_{p_i})!} \prod_{i=2}^n y_i^{\dim\, (R/p_{i+1})_{p_i}}.$$

We now apply formulas (1), and $(3)-(5)$ to the ring R_{p_0} and the valuation $\nu = \nu_1$, to obtain

$$\sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1[} e_{m_0}((\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1})_{p_0}) \asymp \frac{e_{m_1}((\mathcal{P}_{\varphi_2}/\mathcal{P}^+_{\varphi_2})_{p_1}) e_{m_0}((R/p_1)_{p_0})}{(\dim\, (R/p_2)_{p_1})!} y_1^{\dim\, (R/p_2)_{p_1}}. \tag{7}$$

Finally, we sum over (7) and (6) to obtain the desired formula for ν. ☐

2. Amalgamation of valuations

Theorem 2.1. Suppose that $P_i \in K[X,Y]$ are defined for $i \in \mathbf{N}$ by $P_0 = X$ $P_1 = Y$,

$$P_{i+1} = \lambda_i P_i^{m_i} - \lambda_{i,0} M_i - \sum_{\ell'_i} \lambda_{i,\ell'_i} M_{i,\ell'_i}$$

for $i \geq 1$, where $\ell'_i = (\ell'_{i,0}, \ldots, \ell'_{i,i})$, $M_i = \prod_{i'=0}^{i-1} P_{i'}^{\ell_{i,i'}}$, $M_{i,\ell'_i} = \prod_{i'=0}^{i} P_{i'}^{\ell'_{i,i'}}$ with $0 \leq \ell_{i,i'}, \ell_{i,i'} < m_{i'}$ for $1 \leq i' \leq i$, $\ell_{i,0}, \ell'_{i,0} \geq 0$ arbitrary, and $\lambda_i, \lambda_{i,0}, \lambda_{\ell'_i} \in K$ with $\lambda_i, \lambda_{i,0} \neq 0$.

Define $\gamma_0 = 1$ and by induction on i, $\gamma_i = \frac{1}{m_i} \sum_{i'=0}^{i-1} \ell_{i,i'} \gamma_{i'}$. Let $\Gamma_i =< \{\gamma_{i'}\}_{0 \leq i' \leq i} >$ be the group generated by $\gamma_{i'}$ for $0 \leq i' \leq i$. Suppose that

1. $\sum_{i'=0}^{i-1} \ell_{i,i'} \gamma_{i'}$ has order m_i in $\frac{\Gamma_{i-1}}{m_i \Gamma_{i-1}}$ for all $i \geq 1$,
2. $\gamma_{i+1} > m_i \gamma_i$ for all $i \geq 1$, and
3. $\sum_{i'=0}^{i} \ell'_{i,i'} \gamma_{i'} > m_i \gamma_i$ for all ℓ'_i.

Then

1. Suppose that $f \in K[X,Y]$ and ℓ is such that $\deg_y f < \prod_{j=1}^{l-1} m_i$. Then f has a unique finite expansion

$$f = \sum_\alpha a_\alpha \prod_{i=0}^{\ell} P_i^{\alpha_i} \tag{1}$$

with all $\alpha = (\alpha_0, \dots, \alpha_\ell) \in \mathbf{N}^{\ell+1}$, $0 \le \alpha_i < m_i$ for $1 \le i$ and $a_\alpha \in K$,

(2) In the expansion (1), The numbers $\sum_{i=0}^{\ell} \alpha_i \gamma_i$ are distinct for distinct α,

(3) The rule

$$\nu_1(f) = \min_\alpha \sum_{i=0}^{\ell} \alpha_i \gamma_i$$

where the minimum is computed for the expansion (1), determines a nondiscrete rational rank 1 valuation ν_1 of $K(X,Y)$, dominating the local ring $K[X,Y]_{(X,Y)}$.

Theorem 2.2. Suppose that R is a local domain with quotient field K and $\overline{\nu}$ is a valuation of K which dominates R, with value group $\overline{\Gamma}$. Suppose that $\{P_i\}$ is a sequence of polynomials in $K(X,Y)$ satisfying the assumptions of Theorem 2.1. Define by induction on i, $\tau_0 = 0$, and

$$\tau_i = \frac{1}{m_i} \sum_{i'=0}^{i-1} \ell_{i,i'} \tau_{i'} + \overline{\nu}(\lambda_{i,0}) - \overline{\nu}(\lambda_i) \in \overline{\Gamma}_{\mathbf{Q}} = \overline{\Gamma} \otimes_{\mathbf{Z}} \mathbf{Q}.$$

For $f \in K[X,Y]$, define from the expansion (1),

$$\nu(f) = \min_\alpha \{ \sum_{i=0}^{\ell} \alpha_i (\gamma_i, \tau_i) + (0, \mathrm{ord}_z(a(z)) \} \in (\mathbf{Q} \times \overline{\Gamma}_{\mathbf{Q}})_{\mathrm{lex}}.$$

Then ν defines a rank 2 valuation on $K(X,Y)$ which is the composition of ν_1 and $\overline{\nu}$. Further, ν dominates the local ring $A = R[X,Y]_{m_R R[X,Y] + (X,Y)}$, and the center of ν_1 on A is the prime ideal (X,Y).

Theorem 2.3. Suppose that R is a local domain with quotient field K and $\overline{\nu}$ is a valuation of K which dominates R, with value group $\overline{\Gamma}$. Suppose that $\{P_i\}$ is a sequence of polynomials in $K(X,Y)$ satisfying the assumptions of Theorem 2.1 and $\{Q_i\}$ is a sequence of polynomials in $K(U,V)$ satisfying the assumptions of Theorem 2.1, where

$$P_{i+1} = \lambda_i P_i^{m_i} - \lambda_{i,0} M_i - \sum_{\ell_i'} \lambda_{i,\ell_i'} M_{i,\ell_i'}$$

and

$$Q_{i+1} = \mu_i Q_i^{n_i} - \mu_{i,0} N_i - \sum_{\ell_i'} \mu_{i,\ell_i'} N_{i,\ell_i'}$$

Let ν_1 be the valuation on $K(X,Y)$ defined by the $\{P_i\}$, with $\nu_1(P_i) = \gamma_i$, and let ν_2 be the valuation on $K(U,V)$ defined by the $\{Q_i\}$, with $\nu_2(Q_i) = \delta_i$.

Define by induction on i, $\tau_0 = 0$, and

$$\tau_i = \frac{1}{m_i} \sum_{i'=0}^{i-1} \ell_{i,i'} \tau_{i'} + \overline{\nu}(\lambda_{i,0}) - \overline{\nu}(\lambda_i) \in \overline{\Gamma}_{\mathbf{Q}} = \overline{\Gamma} \otimes_{\mathbf{Z}} \mathbf{Q},$$

$\zeta_0 = 0$, and

$$\zeta_i = \frac{1}{n_i} \sum_{i'=0}^{i-1} \ell_{i,i'} \zeta_{i'} + \overline{\nu}(\mu_{i,0}) - \overline{\nu}(\mu_i) \in \overline{\Gamma}_{\mathbf{Q}} = \overline{\Gamma} \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Then

(1) Suppose that $f \in K[X,Y,U,V]$, with $\deg_Y f < \prod_{j=1}^{\ell-1} m_i$ and $\deg_V f < \prod_{j=1}^{m-1} n_i$. Then f has a unique finite expansion

$$f = \sum_{\alpha,\beta} a_{\alpha,\beta} \prod_{i=0}^{\ell} P_i^{\alpha_i} \prod_{i=0}^{m} Q_i^{\beta_i} \tag{2}$$

with all $\alpha = (\alpha_0, \ldots, \alpha_\ell) \in \mathbf{N}^{\ell+1}$, $0 \le \alpha_i < m_i$ for $1 \le i$, all $\beta = (\beta_0, \ldots, \beta_m) \in \mathbf{N}^{m+1}$, $0 \le \beta_i < n_i$ for $1 \le i$, and $a_{\alpha,\beta} \in K$.

(2) Suppose that G is a totally ordered abelian group, and $h_1 : \mathbf{Q} \to G$ and $h_2 : \mathbf{Q} \to G$ are order preserving embeddings. For $f \in K[X,Y,U,V]$, define from the expansion (2),

$$\nu(f) = \min_{\alpha,\beta}\{\sum_{i=0}^{\ell} \alpha_i(h_1(\gamma_i), \tau_i) + \sum_{i=0}^{m} \beta_i(h_2(\delta_i), \zeta_i) + (0, \mathrm{ord}_z(a(z))\} \in (G \times \overline{\Gamma}_{\mathbf{Q}})_{\mathrm{lex}}.$$

Then ν defines a valuation on $K(X,Y)$ which is composite with $\overline{\nu}$. ν has rank 2 or 3, depending on if the rank of $h_1(\mathbf{Q}) + h_2(\mathbf{Q})$ is 1 or 2. Further, ν dominates the local ring $A = R[X,Y,U,V]_{m_R R[X,Y,U,V]+(X,Y,U,V)}$.

3. Wild behavior of the tilde function

Theorem 3.1. Given any decreasing function $f : \mathbf{N} \to \mathbf{Z}$, there exists a valuation ν with value group $(\frac{1}{2^\infty}\mathbf{Z} \times \mathbf{Z})_{\mathrm{lex}}$ which dominates a local ring R of dimension 3, such that for any valuation v equivalent to ν, with value group $(\frac{1}{2^\infty}\mathbf{Z} \times \mathbf{Z})_{\mathrm{lex}}$, we have that for all sufficiently large $n \in \mathbf{N}$, there exists $\lambda \in \frac{1}{2^\infty}\mathbf{Z} \cap [0,n]$ such that $\pi_2(\tilde{\lambda}) < f(n)$.

Theorem 3.2. Given any increasing function $g : \mathbf{N} \to \mathbf{Z}$, there exists a valuation ω with value group $(\frac{1}{2^\infty}\mathbf{Z} \times \mathbf{Z})_{\mathrm{lex}}$ which dominates a local ring R of dimension 3, such that for any valuation w equivalent to ω, with value group $(\frac{1}{2^\infty}\mathbf{Z} \times \mathbf{Z})_{\mathrm{lex}}$, we have that for all sufficiently large $n \in \mathbf{N}$, there exists $\lambda \in \frac{1}{2^\infty}\mathbf{Z} \cap [0,n]$ such that $\pi_2(\tilde{\lambda}) > g(n)$.

Theorem 3.3. Suppose that $f : \mathbf{N} \to \mathbf{Z}$ is a decreasing function and $g : \mathbf{N} \to \mathbf{Z}$ is an increasing function. Then there exists a valuation μ with value group $((\frac{1}{2^\infty}\mathbf{Z} + \frac{1}{2^\infty}\mathbf{Z}\sqrt{2}) \times \mathbf{Z})_{\mathrm{lex}}$ which dominates a local ring A of dimension 5, such that for any valuation m equivalent to μ, with value group $((\frac{1}{2^\infty}\mathbf{Z} + \frac{1}{2^\infty}\mathbf{Z}\sqrt{2}) \times \mathbf{Z})_{\mathrm{lex}}$, we have that for all sufficiently large $n \in \mathbf{N}$, there exists $\lambda_1 \in (\frac{1}{2^\infty}\mathbf{Z} + \frac{1}{2^\infty}\mathbf{Z}\sqrt{2}) \cap [0,n]$ such that $\pi_2(\tilde{\lambda}_1) > f(n)$, and there exists $\lambda_2 \in (\frac{1}{2^\infty}\mathbf{Z} + \frac{1}{2^\infty}\mathbf{Z}\sqrt{2}) \cap [0,n]$ such that $\pi_2(\tilde{\lambda}_2) < g(n)$.

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