Let ν_1 and ν_2 be valuations on the noetherian local domain (R,m), with centers $p_2 \subset_{\neq} p_1$ and such that ν_1 is composed with ν_2 and their ranks differ by one. Let $\lambda\colon \Gamma_1 \to \Gamma_2$ be the corresponding map of value groups. Set $t_1 = \nu_1(p_1), t_2 = \nu_2(p_2)$.

Given $\varphi_2 \in \Gamma_2$, denote by $\tilde{\varphi}_2 \in \Gamma_1$ the minimum of $\{\nu_1(f); f \in R,\ \nu_2(f) = \varphi_2\}$. Note that $\lambda(\tilde{\varphi}_2) = \varphi_2$ and that $t_1 \in \mathrm{Ker}\lambda$.

We remark that for $\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]$ we have the inclusions

$$p_1^{y_1}\mathcal{P}_{\varphi_2} \subset \mathcal{P}_{\varphi_1} \subset \mathcal{P}_{\varphi_2},$$

$\mathcal{P}_{\varphi_2} = \mathcal{P}_{\tilde{\varphi}_2}$ and since the valuation $\overline{\nu}_1$ which is the image of ν_1 in R/p_2 is of rank one, the number of elements in the interval $[\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]$ is finite.

It follows that for any additive function A on R-modules we have

$$\sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]} A(\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1}) \le A(M_{\varphi_2}/p_1^{y_1} M_{\varphi_2}), \tag{1}$$

where $M_{\varphi_2} = \mathcal{P}_{\varphi_2}/\mathcal{P}^+_{\varphi_2}$, a finitely generated torsion free R/p_2-module.
The R modules $\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1}$ and $M_{\varphi_2}/p_1^{y_1} M_{\varphi_2}$ have p_1 as their unique minimal prime.

In the special case where ν_1 is a valuation of rank one and ν_2 is the trivial valuation, we have $\Gamma_2 = 0$, $\tilde{\varphi}_2 = 0 \in \Gamma_1$ and, writing φ for φ_1, p for p_1, etc., the inequality reduces to:

$$\sum_{\varphi \in [0, yt]} A(\mathcal{P}_{\varphi}/\mathcal{P}^+_{\varphi}) \le A(R/p_1^{y} R), \tag{2}$$

We can take as additive function A on R-modules of dimension $= \dim R/p_1$ the multiplicity e_m with respect to the maximal ideal of R. The inequality (1) then becomes

$$\sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]} e_m(\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1}) \le e_m(M_{\varphi_2}/p_1^{y_1} M_{\varphi_2}). \tag{3}$$

It is now possible to evaluate the right hand side thanks to ([B], §7, No.1, Prop 3), which says this: given a finitely generated R-module M of dimension $d \ge 0$, denote by B the set of minimal prime ideals of $\mathrm{Supp}(M)$ such that $\dim(R/p) = d$. Let q be an ideal of R contained in its radical and such that M/qM has finite length. Then we have

$$e_q(M) = \sum_{p \in B} \mathrm{length}_{R_p}(M_p) e_q(R/p).$$

If we apply this to $M_{\varphi_2}/p_1^{y_1} M_{\varphi_2}$ we get:

$$e_m(M_{\varphi_2}/p_1^{y_1} M_{\varphi_2}) = \mathrm{length}_{R_{p_1}}\big((M_{\varphi_2})_{p_1}/m_1^{y_1}(M_{\varphi_2})_{p_1}\big) e_m(R/p_1), \tag{4}$$

where $m_1 = p_1 R_{p_1}$ and the support of the R-module M_{φ_2} is the ideal p_2.
By the main result on Hilbert-Samuel polynomials, in view of the fact that the support of $(M_{\varphi_2})_{p_1}$ is the ideal $p_2 R_{p_1}$, we have an asymptotic estimate:

$$\mathrm{length}_{R_{p_1}}\big((M_{\varphi_2})_{p_1}/m_1^{y_1}(M_{\varphi_2})_{p_1}\big) \asymp \frac{e_{m_1}\big((M_{\varphi_2})_{p_1}\big)}{(\dim(R_{p_1}/p_2 R_{p_1})!} y_1^{\dim(R_{p_1}/p_2 R_{p_1})}. \tag{5}$$

Let

$$(0) = p_{n+1} \subset p_n \subset p_{n-1} \subset \cdots \subset p_1$$

be the sequence of the centers of valuations ν_i with which a given valuation ν of rank n on a noetherian local domain (R,m) is composed. We assume that the centers are distinct prime ideals.

Let p_0 be a prime ideal of R containing the center p_1 of $\nu = \nu_1$ (p_0 could be equal to p_1). Then the sum

$$\sum_{\varphi_n \in [0, t_n y_n]} \sum_{\varphi_{n-1} \in [\tilde{\varphi}_n, \tilde{\varphi}_n + t_{n-1} y_{n-1}]} \cdots \sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + t_1 y_1]} e_{m_0}((\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1})_{p_0})$$

is bounded for $y_1, \ldots, y_n$ large enough by a function which behaves asymptotically as

$$\frac{\Pi_{i=0}^n e_{m_i}((R/p_{i+1})_{p_i})}{\Pi_{i=1}^n (\dim (R/p_{i+1})_{p_i})!} \Pi_{i=1}^n y_i^{\dim\ (R/p_{i+1})_{p_i}},$$

where $m_i = p_i R_{p_i}$ for $0 \leq i \leq n$. We take $y_1 \gg y_2 \gg \cdots \gg y_n$ to obtain this asymptotic behavior.

The proof of this formula is by induction on n. We first prove the formula in the case when $n = 1$. We apply formulas (2) and (3) - (5) to the ring R_{p_0}, observe that for $\varphi_1 \in \Gamma_1$,

$$(\mathcal{P}_{\varphi_1})_{p_0} = \{f \in R_{p_0} \mid \nu(f) \geq \varphi_1\},$$

and that $(R_{p_0})_{p_1 R_{p_0}} \cong R_{p_1}$, to obtain

$$\sum_{\varphi_1 \in [0, t_1 y_1]} e_{m_0}((\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1})_{p_0}) \asymp \frac{e_{m_1}(R_{p_1}) e_{m_0}((R/p_1)_{p_0})}{(\dim\ R_{p_1})!} y_1^{\dim\ R_{p_1}},$$

which is the formula for $n = 1$.

We now assume that the formula is true for valuations of rank $< n$. We will derive the formula for a rank n valuation ν. We apply the formula to the rank $n-1$ valuation ν_2 which ν is composite with, and the chain of prime ideals

$$(0) = q_n \subset q_{n-1} \subset \cdots \subset q_1 \subset q_0$$

where $q_{n-1} = p_n, \ldots, q_1 = p_2$ are the centers on R of the successive valuations $\nu_n, \ldots, \nu_2$ with which ν_2 is composite, and $q_0 = p_1$, to obtain

(6)

$$\sum_{\varphi_n \in [0, t_n y_n]} \cdots \sum_{\varphi_2 \in [\tilde{\varphi}_3 + t_2 y_2]} e_{m_1}((\mathcal{P}_{\varphi_2}/\mathcal{P}_{\varphi_2^+})_{p_1}) \asymp \frac{\prod_{i=1}^n e_{m_i}((R/p_{i+1})_{p_i})}{\prod_{i=2}^n (\dim\ (R/p_{i+1})_{p_i})!} \prod_{i=2}^n y_i^{\dim\ (R/p_{i+1})_{p_i}}.$$

We now apply formulas (1), and $(3) - (5)$ to the ring R_{p_0} and the valuation $\nu = \nu_1$, to obtain

$$\sum_{\varphi_1 \in [\tilde{\varphi}_2, \tilde{\varphi}_2 + y_1 t_1]} e_{m_0}((\mathcal{P}_{\varphi_1}/\mathcal{P}^+_{\varphi_1})_{p_0}) \asymp \frac{e_{m_1}((\mathcal{P}_{\varphi_2}/\mathcal{P}^+_{\varphi_2})_{p_1}) e_{m_0}((R/p_1)_{p_0})}{(\dim\ (R/p_2)_{p_1})!} y_1^{\dim\ (R/p_2)_{p_1}}. \tag{7}$$

Finally, we sum over (7) and (6) to obtain the desired formula for ν.