

Example.

Let $R = k[x, y]_{(x, y)}$. Let ν be a valuation of the quotient field of R defined by its generating sequence $\{P_i\}_{i \geq 0}$ as follows

$$\begin{aligned} P_0 &= x, & \nu(P_0) &= 1 \\ P_1 &= y, & \nu(P_1) &= 10 + \frac{1}{2} \\ P_2 &= P_1^2 - x^{21}, & \nu(P_2) &= 10^2 + \frac{1}{2^2} \\ P_3 &= P_2^2 - x^{190} P_1, & \nu(P_3) &= 10^4 + \frac{1}{2^3} \\ P_{i+1} &= P_i^2 - x^{\alpha(i)} P_{i-1}, & \nu(P_{i+1}) &= 10^{2^i} + \frac{1}{2^{i+1}}, & \alpha(i) &= 2 \cdot 10^{2^{i-1}} - 10^{2^{i-2}}. \end{aligned}$$

Denote by S the semigroup $\nu(m_R)$. Then for all $d \geq 10$, $|S \cap [d, d+1]| \leq 2 \log_{10} d$, and $|S \cap [1, 10]| = 9$. Thus $|S \cap [1, d+1]| \leq 2d \log_{10} d$, when $d \geq 10$.

Let k be an integer such that $4 \cdot 10^{2^{k-1}} \leq d \leq 4 \cdot 10^{2^k}$. Then $|S \cap [e, e+1]| \geq 2^k$ for all $e \geq d/2$. Thus $|S \cap [1, d+1]| > 2^k d/2 > (\log_{10} d - 1)d/2 \geq \frac{d}{4} \log_{10} d$, when $d \geq 100$.

Thus for $d \gg 0$

$$\frac{d}{4} \log_{10} d < |S \cap [1, d+1]| < 2d \log_{10} d.$$

More precisely, $f(d) = |S \cap [1, d+1]|$ is a piecewise linear function. For example, if k is such that $2 \cdot 10^{2^{k-1}} \leq d \leq 10^{2^k}$, then $f(d) = 2^k(d - 2 \cdot 10^{2^{k-1}} + 1) + C$, where $C = f(2 \cdot 10^{2^{k-1}} - 1) = |S \cap [1, 2 \cdot 10^{2^{k-1}}]|$.