VALUATION SEMIGROUPS

Let (R, m_R) be a local domain, with quotient field K. Suppose that ν is a valuation of K with valuation ring (V, m_V). Suppose that ν dominates R; that is, $R \subset V$ and $m_V \cap R = m_R$. The possible value groups Γ of ν have been extensively studied and classified, including in the papers MacLane [8], MacLane and Schilling [9], Zariski and Samuel [12], and Kuhlmann [7]. The most basic fact is that there is an order preserving embedding of Γ into $\mathbb{R}^n$ with the lex order, where n is the dimension of R. The semigroup

$$S^R(\nu) = \{\nu(f) \mid f \in m_R - \{0\}|$$

is however not well understood, although it is known to encode important information about the topology, resolution of singularities and ideal theory of R.

In Zariski and Samuel's classic book on Commutative Algebra [12], two general facts about the semigroup $S^R(\nu)$ are proven (in Appendix 3 to Volume II).

1. $S^R(\nu)$ is a well ordered subset of the positive part of the value group Γ of ν of ordinal type at most ω^h, where ω is the ordinal type of the well ordered set $\mathbb{N}$, and h is the rank of the valuation.
2. The rational rank of ν plus the transcendence degree of V/m_V over R/m_R is less than or equal to the dimension of R.

The second condition is the Abhyankar inequality [1].

Prior to this paper, no other general constraints were known on the value semigroup semigroups $S^R(\nu)$. In fact, it was even unknown if the above conditions 1 and 2 characterize value semigroups.

In this paper, we construct an example of a well ordered subsemigroup of $\mathbb{Q}_+$ of ordinal type ω, which is not a value semigroup of a local domain. This shows that the above conditions 1 and 2 do not characterize value semigroups on local domains. We construct this in Corollary 1.4 by finding a new constraint, Theorem 1.1, on a semigroup being a value semigroup of a local domain of dimension n. In Corollary 1.3, we give a stronger constraint on regular local rings.

In [4], Teissier and the author give some examples showing that some surprising semigroups of rank > 1 can occur as semigroups of valuations on noetherian domains, and raise the general question of finding new constraints on value semigroups and classifying semigroups which occur as value semigroups.

The only semigroups which are realized by a valuation on a one dimensional regular local ring are isomorphic to the natural numbers. The semigroups which are realized by a valuation on a regular local ring of dimension 2 with algebraically closed residue field are much more complicated, but are completely classified by Spivakovsky in [10]. A different proof is given by Favre and Jonsson in [5], and the theorem is formulated in the context of semigroups by Cutkosky and Teissier [4]. However, very little is known in higher dimensions. The classification of semigroups of valuations on regular local rings of dimension two does suggest that there may be constraints on the rate of growth of the number of new generators on semigroups of valuations dominating a noetherian domain. We prove that there is such a constraint, giving a new necessary condition for a semigroup to be a value semigroup. This is accomplished in Theorem 1.1 and Corollaries 1.2 and 1.3. The constraint is sufficiently strong to allow us to give a very simple example, Corollary

1.4, of a well ordered subgroup S of $\mathbb{Q}_+$ of ordinal type ω which is not the semigroup of a valuation dominating an local domain.

1. Semigroups of valuations

If R is a local ring, m_R will denote its maximal ideal, and $\ell(N)$ will denote the length of an R module N.

Suppose that Γ is an abelian totally ordered group, and $a, b \in \Gamma$. We set

$$(a,b) = \{x \in \Gamma \mid a < x < b\}.$$

Theorem 1.1. *Suppose that R is a local domain which is dominated by a valuation ν. Let Γ be the value group of ν, $S = \nu(m_R - \{0\})$ be the value semigroup of ν on R, and suppose that s_0 is the smallest element of S. Then*

$$|S \cap (0, ns_0)| < \ell(R/m_R^n)$$

for all $n \in \mathbb{N}$.

Proof. Let $\lambda \in \Gamma$ with $\lambda < ns_0$. Let $f, g \in R$ be such that $f - g \in m_R^n$. Suppose that $\nu(f) \geq \lambda$. Then $g = f + h$ with $h \in m_R^n$, so that

$$\nu(g) \geq \max\{\nu(f), \nu(h)\} \geq \max\{\lambda, ns_0\} = \lambda.$$

Thus

$$H_\lambda = \{[f] \in R/m_R^n \mid \nu(f) \geq \lambda\}$$

is a well defined submodule of R/m_R^n. For $\lambda_1, \lambda_2 \in \Gamma$ with $\lambda_1 < \lambda_2 < ns_0$, we have $H_{\lambda_2} \subset H_{\lambda_1}$. Since $\nu(1) = 0$, we have the inequality of the theorem. $\square$

For a local ring R of dimension d, let

$$P_R(n) = \frac{e(R)}{d!} n^d + \text{ lower order terms}$$

be the Hilbert Samuel polynomial of R, where $e(R)$ is the multiplicity of R. We have that $\ell(R/m_R^n) = P_R(n)$ for $n \gg 0$.

Corollary 1.2. *Suppose that R is a local domain of dimension d which is dominated by a valuation ν. Let $S = \nu(m_R - \{0\})$ be the value semigroup of ν on R, and suppose that s_0 is the smallest element of S. Then*

1. *For $n \gg 0$, $|S \cap (0, ns_0)| < P_R(n)$.*
2. *There exists $c > 0$ such that $|S \cap (0, ns_0)| < cn^d$ for all n.*

Corollary 1.3. *Suppose that R is a regular local ring of dimension d which is dominated by a valuation ν. Let $S = \nu(m_R - \{0\})$ be the value semigroup of ν on R, and suppose that s_0 is the smallest element of S. Then*

$$|S \cap (0, ns_0)| < \binom{d-1+n}{d}$$

for all $n \in \mathbb{N}$.

Corollary 1.4. *There exists a well ordered subsemigroup U of $\mathbb{Q}_+$ such that U has ordinal type ω and $U \neq \nu(m_R - \{0\})$ for any valuation ν dominating a local domain R.*

Proof. Take any subset T of $\mathbb{Q}_+$ such that 1 is the smallest element of S and $n^n \leq |T \cap (0,n)| < \infty$ for all $n \in \mathbb{N}$. For all positive integers r, let

$$rT = \{a_1 + \cdots + a_r \mid a_1, \ldots, a_r \in T\}.$$

Let $U = \omega T = \cup_{r=1}^{\infty} rT$ be the semigroup generated by T. By our constraints, $|U \cap (0,r)| < \infty$ for all $r \in \mathbb{N}$. Thus U is well ordered and has ordinal type ω. By 2 of Corollary 1.2, U cannot be the semigroup of a valuation dominating a local domain. $\square$

2. Asymptotic growth of value semigroups

Suppose that Γ is a totally ordered abelian group of rank 1, and S is a well ordered subsemigroup of ordinal type ω, consisting of positive elements of S. Let s_0 be the smallest element of S. For $n \in \mathbb{N}$, let

$$\varphi(n) = |S \cap (0, ns_0)|.$$

Theorem 1.1 gives a necessary condition for a well ordered semigroup S of rank 1 consisting of positive elements to be the semigroup $\nu(m_R - \{0\})$ of a valuation ν dominating a *fixed* local domain R. Let s_0 be the smallest element of S. We must have

$$\varphi(n) = |S \cap (0, ns_0)| < \ell(R/m_R^n) \tag{1}$$

for all $n \in \mathbb{N}$.

Example 2.1. *The necessary condition (1) is not sufficient. Sufficiency fails for the case of two dimensional regular local rings*

Proof. Let S be the subsemigroup of $\mathbb{Q}_+$ which is generated by 1, 3/2 and 9/4. Then

$$S = \{1, \frac{3}{2}, 2, \frac{9}{4}, \frac{10}{4}, 3, \frac{13}{4}, \frac{14}{4}, \frac{15}{4}, 4, \ldots\}.$$

Define $\varphi(n) = |S \cap (0,n)|$. We have that $\varphi(2) = 2$ and $\varphi(n) = 4n - 7$ for $n \geq 3$. We have

$$\varphi(n) = |S \cap (0,n)| < \frac{(1+n)n}{2} = \binom{1+n}{2} = \ell(R/m_R^n)$$

for all n. Thus S satisfies the bound (1). We will show now that S is not the semigroup of a valuation dominating a regular local ring R of dimension 2. Suppose that there exists a valuation ν dominating a regular local ring R of dimension 2 such that $S = \nu(m_R - \{0\})$. We will derive a contradiction. Let x, y be regular parameters in R. After possibly interchanging x and y, we may assume that $\nu(x) = 1$. There exists $f \in m_R$ such that $\nu(f) = \frac{3}{2}$. Write $f = \alpha_1 x + \alpha_2 y$ with $\alpha_1, \alpha_2 \in R$. We have

$$\min\{\nu(\alpha_1) + \nu(x), \nu(\alpha_2) + \nu(y)\} \leq \nu(f) = \frac{3}{2}$$

with a strict inequality only if $\nu(\alpha_1) + \nu(x) = \nu(\alpha_2) + \nu(y)$. By the restriction that all values are in S, we either have that $\nu(y) = \frac{3}{2}$, or $\nu(y) = 1$ and α_2 is a unit, so that we can make a change of variables, replacing y with f to assume that $\nu(x) = 1$ and $\nu(y) = \frac{3}{2}$. There exists $g \in m_R$ such that $\nu(g) = \frac{9}{4}$. Expand

$$g = \alpha_1 x + \alpha_2 y + \alpha_3 x^2 + \alpha_4 xy + \alpha_5 y^2 + h$$

with $\alpha_1, \ldots, \alpha_5 \in R$ and $h \in m_R^3$, and such that α_i are all either 0 or have nonzero residue in R/m_R. With the convention that $\nu(0) = \infty$, we have

$$\min\{\nu(\alpha_1) + \nu(x), \nu(\alpha_2) + \nu(y), \ldots, \nu(\alpha_5) + 2\nu(y), \nu(h)\} \leq \nu(g) = \frac{9}{4},$$

with a strict inequality only if at least two of the terms have the minimal value. Since all of these values are in S, and $\nu(h) \geq 3$, this is impossible. $\square$

It is worth noting that the necessary condition (1) is sufficient for regular local rings R of dimension 1. In this case the bound is

$$|S \cap (0, ns_0)| \leq \binom{n}{1} - 1 = n - 1$$

for all $n \in \mathbb{N}$. Since $s_0\mathbb{N}_+ \subset S$, we then have that $S = s_0\mathbb{N}_+$. Let μ be the m_R-adic valuation, and let ν be the equivalent valuation $s_0\mu$. Then $S = \nu(m_R - \{0\})$.

In the case of regular local rings R of dimension 1, we see that the bound

$$|S \cap (0, ns_0)| = \ell(R/m_R^n) - 1$$

is achieved for all $n \in \mathbb{N}$. This bound is not achievable for all n however on a regular local ring R of dimension ≥ 2. This can be seen as follows. Suppose that there is a valuation ν dominating a regular local ring R of dimension $d \geq 2$ such that $\varphi(n) = |\nu(m_R - \{0\}) \cap (0, ns_0)| = \ell(R/m_R^n) - 1$ for all n. Then $\varphi(2) = \ell(R/m_R^2) - 1 \geq 2$. Thus there exists a regular system of parameters $x, y, \ldots$ in R such that $\nu(x) = s_0$ and $\nu(y) = s_1$ with $s_0 < s_1 < 2s_0$. There exists $a \in \mathbb{N}$ such that $\frac{a}{a-1}s_0 < s_1$. We have $\nu(y^{a-1}) > as_0$. For $n \geq a$, $y^{a-1}m_R^{n-a}$ generates a submodule G_n of R/m_R^n of length $\binom{d-1+n-a}{d-1}$. Since $\nu(f) > ns_0$ if $f \in y^{a-1}m_R^{n-a}$, we must have that

$$\varphi(n) \leq \ell(R/m_R^n) - 1 - \binom{d-1-a+n}{d-1} < \ell(R/m_R^n) - 1$$

for all $n \geq a$; in fact, $\binom{d-1+n-a}{d-1}$ grows like n^{d-1}, a contradiction.

We can easily achieve growth of $\varphi(n)$ which is asymptotic to n^d on a regular local ring of dimension d. Choose d rationally independent real positive real numbers $\gamma_1, \ldots, \gamma_d$, a regular system of parameters $x_1, \ldots, x_d$ of R and prescribe that $\nu(x_i) = \gamma_i$ for all i. It is an interesting question to determine the exact possible asymptotic behavior of $\varphi(n)$ which is achievable on a local domain. Is it possible to achieve exotic asymptotic growth, such as $n \log n$?

Corollary 1.2 gives a necessary condition for a rank 1 well ordered semigroup S consisting of positive elements to be the value semigroup $\nu(m_R - \{0\})$ of a valuation dominating some local domain R of dimension d. The condition is:

(2) There exists $c > 0$ such that $|S \cap (0, ns_0)| < cn^d$ for all n

where s_0 is the smallest element of S. An interesting question is if (2) is in fact sufficient. (2) is sufficient in the case when $d = 1$, as we now show. Suppose that $S \subset \mathbb{R}_+$ is a semigroup consisting of positive elements which contains a smallest element s_0. Suppose that there exists $c > 0$ such that

$$|S \cap [(0, ns_0)| < cn \tag{3}$$

for all $n \in \mathbb{N}$. By Lemma 2.3 below, we may assume that $S \subset \mathbb{Q}_+$ is finitely generated by some elements $\lambda_1, \ldots, \lambda_r$. There exists $\alpha \in \mathbb{Q}_+$ such that there exists $a_i \in \mathbb{N}_+$ with $\lambda_i = \alpha a_i$ for $1 \leq i \leq r$, and $\gcd(a_1, \ldots, a_r) = 1$. Let $k[t]$ be a polynomial ring over a field k. Let $\nu(f(t)) = \alpha \operatorname{ord}(f(t))$ for $f(t) \in k[t]$. ν is a valuation of $k(t)$. Let R be the one dimensional local domain

$$R = k[t^{a_1}, \ldots, t^{a_r}]_{(t^{a_1}, \ldots, t^{a_r})}.$$

The quotient field of R is $k(t)$, and ν dominates R. We have that $S = \nu(R - m_R)$.

Lemma 2.2. *Suppose that $S \subset \mathbb{R}_+$ is a semigroup consisting of positive elements which contains a smallest element s_0. Suppose that there exists $c > 0$ and $d \in \mathbb{N}_+$ such that*

$$|S \cap (0, ns_0)| < cn^d \tag{4}$$

for all $n \in \mathbb{N}$. Then S is well ordered of ordinal type ω and has rational rank $\leq d$.

Proof. The fact that S is well ordered of ordinal type ω is immediate from (4).

We will prove that the rational rank of S is $\leq d$. After rescaling S by multiplying by $\frac{1}{s_0}$, we may assume that $s_0 = 1$. Suppose that $t \in \mathbb{N}$ and S has rational rank $\geq t$. Then there exist $\gamma_1, \ldots, \gamma_t \in S$ which are rationally independent. Let $b \in \mathbb{N}$ be such that $\max\{\gamma_1, \ldots, \gamma_t\} < b$. For $e \in \mathbb{R}_+$, we have

$$\begin{array}{lll} |S \cap (0, e)| & \geq & |\{a_1\gamma_1 + \cdots + a_t\gamma_t \mid a_1, \ldots, a_t \in \mathbb{N} \text{ and } a_1\gamma_1 + \cdots + a_t\gamma_t < e\}| - 1 \\ & \geq & |\{a_1\gamma_1 + \cdots + a_t\gamma_t \mid a_i \in \mathbb{N} \text{ and } 0 \leq a_i < \frac{e}{tb} \text{ for } 1 \leq i \leq t\}| - 1 \\ & = & |\{(a_1, \ldots, a_t) \in \mathbb{N}^t \mid 0 \leq a_i < \frac{e}{tb} \text{ for } 1 \leq i < t\}| - 1 \end{array}$$

since $\gamma_1, \ldots, \gamma_t$ are rationally independent.

For $a \in \mathbb{N}$ let $n = abt$. Then we see that

$$|S \cap (0, (n+1))| \geq a^t - 1 = \left(\frac{1}{bt}\right)^t n^t - 1.$$

By (4), we see that $t \leq d$. □

Lemma 2.3. *Suppose that $S \subset \mathbb{R}_+$ is a semigroup consisting of positive elements which contains a smallest element s_0. Suppose that there exists $c > 0$ such that*

$$|S \cap (0, ns_0)| < cn \tag{5}$$

for all $n \in \mathbb{N}$. Then S is finitely generated, and the group generated by S is isomorphic to $\mathbb{Z}$.

Proof. By Lemma 2.2, S has rational rank 1, so we may assume that S is contained in $\mathbb{Q}_+$. We may further assume that $s_0 = 1$. Suppose that S is not finitely generated. Then for $e \in \mathbb{N}$, we can find $\lambda_1, \ldots, \lambda_e \in S$ such that $\lambda_i = \frac{a_i}{b_i}$ with $a_i, b_i \in \mathbb{N}_+$, $b_i > 1$ for all i, $\gcd(a_i, b_i) = 1$ for all i and $b_1, \ldots, b_e$ all distinct.

There exist $m_i, n_i \in \mathbb{Z}$ such that $m_i a_i + n_i b_i = 1$ for $1 \leq i \leq e$. Either $m_i > 0$ and $n_i < 0$, or $m_i < 0$ and $n_i > 0$. If $m_i > 0$, then $|m_i|\lambda_i = |n_i| + \frac{1}{b_i}$. If $m_i < 0$, then $|m_i|\lambda_i = |n_i| - \frac{1}{b_i}$. Let $n_0 = \max\{|n_i| + 1 \mid 1 \leq i \leq i\}$. For $n \geq n_0$, we have that $n + \frac{1}{b_i} \in S \cap (n, n+1)$ if $m_i > 0$, and $(n+1) - \frac{1}{b_i} \in S \cap (n, n+1)$ if $m_i < 0$. Thus $|S \cap (n, n+1)| \geq e$ for $n \geq n_0$, which implies that

$$|S \cap (n, n+1)| \geq en - en_0$$

for $n \geq n_0$. For $e > c$, we have a contradiction to (5). □

References

[1] S. Abhyankar, *On the valuations centered in a local domain*, Amer. J. Math. 78 (1956), 321 - 348.
[2] S.D. Cutkosky, *Local factorization and monomialization of morphisms*, Ast/'erisque 260, 1999.
[3] S.D. Cutkosky and L. Ghezzi, *Completions of valuation rings*, Contemp. Math. 386 (2005), 13 -34.
[4] S.D. Cutkosky and B. Teissier, *Semigroups of valuations on local rings*, to appear in Mich. Math. J.
[5] C. Favre and M. Jonsson, *The valuative tree*, Lecture Notes in Math 1853, Springr Verlag, Berlin, Heidelberg, New York, 2004.
[6] W. Heinzer and J. Sally, *Extensions of valuations to the completion of a local domain*, Journal of Pure and Applied Algebra 71 (1991), 175 - 185.

[7] F.-V. Kuhlmann, *Value groups, residue fields, and bad plcaes of algebraic function fields*, Trans. Amer. Math. Soc. 40 (1936), 363 - 395.
[8] S. MacLane, *A construction for absolute values in polynomial rings*, Trans. Amer. Math. Soc. 40 (1936), 363 - 395.
[9] S. MacLane and O. Schilling, *Zero-dimensional branches of rank 1 on algebraic varieties*, Annals of Math. 40 (1939), 507 - 520.
[10] M. Spivakovsky, *Valuations in function fields of surfaces*, Amer. J. Math. 112 (1990), 107 - 156.
[11] B. Teissier, *Valuations, deformations, and toric geometry*, Proceedings of the Saskatoon Conference and Workshop on valuation theory (second volume), F.-V. Kuhlmann, S. Kuhlmann, M. Marshall, editors, Fields Institute Communications, 33, 2003, 361 - 459.
[12] O. Zariski and P. Samuel, *Commutative Algebra Volume II*, D. Van Nostrnad, Princeton, 1960.