A construction for a class of valuations with large value group on the field $\mathbb{C}(X_1, \ldots, X_d, Y)$

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Abstract

We define a valuation induced by a transcendental hypersurface and a suitably chosen ordering on the group $\mathbb{Q}^d$. It is naturally approximated by a sequence of quasi-ordinary hypersurfaces. The value semigroup $\nu(\mathbb{C}[X, Y] \setminus 0)$ is the union of the semigroups associated to these quasi-ordinary hypersurfaces.

1 Introduction

We construct a class of zero-dimensional valuations with value group being a subgroup of $\mathbb{Q}^d$. The construction is based on generalizing the notion of quasi-ordinary hypersurface singularities ([6]), this is done in Definition 2.1. This generalization gives us a transcendental element $\zeta(X) \in \mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]]$. By a process of truncation of this element, $\zeta(X)$, we get the usual quasi-ordinary hypersurfaces which we denote them by $f^{(i)}$ (Definition 2.3). One of the difficulties to construct a valuation with value groups in $\mathbb{Q}^d$ is that there is no natural ordering on $\mathbb{Q}^d$. In Section 3, we introduce the notion of "good ordering" on $\mathbb{Q}^d$ and study its properties. In the next section we introduce a valuation ring $\mathbb{C}[[X^{\mathbb{Q}^d_{\succ 0}}]]$, which is in fact the valuation ring of the valuation that we will define on the ring $\mathbb{C}[X, Y]$. We show that there is an injective morphism $\Theta_\zeta : \mathbb{C}[X, Y] \to \mathbb{C}[[X^{\mathbb{Q}^d_{\succ 0}}]]$ (see Definiton 4.2). With the help of this injection we get the desired valuation on $\mathbb{C}[X, Y]$. We study the properties of this valuation and the smigroup Γ_ζ attached to it. We show that there is a close relation between the formal semigroup which [3] attaches to a quasi-ordinary hypersurface and the semigroup which comes from the valuation. More precisely, if we denote the formal semigroups which are attached to the truncated quasi-ordinary hypersurfaces, $f^{(i)}$'s, by Γ_i then we have $\Gamma_\zeta = \bigcup \Gamma_i$. In the final section we study an embedding of the spaces Spec$R$, $R = \mathbb{C}[\zeta(X)]$ and Spec$(\mathbb{C}[X^{\Gamma_\zeta}])$ in an infinite dimensional regular space Spec$\mathbb{C}[[X]][U]$, $U = (U_1, U_2, \ldots)$. We study the ideals definig these embeddings and the relation between them.

2 Transcendental hypersurface and its approximation

Generalizing the classical definition of quasi-ordinary hypersurface singularities (see for example [6]) we define a transcendental quasi-ordinary hypersurface singularity in the following manner:

Definition 2.1 *Fix an element* $\zeta(X) = \sum_{i=1}^{\infty} c_\lambda X^\lambda = \sum_{i=1}^{\infty} p_i,\ p_i \in \mathbb{C}[X^{\frac{1}{m^{(i)}}}],\ X = (X_1, \ldots, X_d)$. *Where* $m^{(i)}$*'s are defined in Definition 2.4. We impose the following extra conditions:*

- *All the exponents of* p_i*, i.e.,* λ*'s of the monomials of* p_i*, are ordered with respect to the the partial order* $\leq$ *on* $\mathbb{Q}^d$*, with minimum equal to* λ_i*.*
- *The partial order on* $\mathbb{Q}^d$ *induces a total order on the set* $\{\lambda_i\}_{i=1}^{\infty}$*, i.e.,* $\lambda_1 < \lambda_2 < \ldots$*.*
- *We define the sequence of subgroups of* $\mathbb{Q}^d$*,* $Q_0 = \mathbb{Z}^d$*,* $Q_j = \mathbb{Z}^d + \sum_{\lambda_i < \lambda_{j+1}} \mathbb{Z}\lambda_i$*, for* $j \in \mathbb{N}$*. We impose the condition* $\lambda_j \notin Q_{j-1}$*.*

- *If $c_\lambda X$ is a term of p_j then $\lambda \in Q_j$.*

The above definition is a sort of generalization of [8], Subsection 4.4, where a "natural valuation" attached to a "transcendental plane curve", has been studied through a series of examples from different perspectives: the sequence of point blow ups, the semigroup, the graded valuation ring, ... Moreover, the relations between these approaches has been studied. In this text we follow the same approach.

Note that if we define $\Lambda = \{\lambda \ : c_\lambda \neq 0\}$ then $\lambda \in p_i \bigcap \Lambda$ iff $\lambda_i \leq \lambda \ngeq \lambda_{i+1}$. We call λ_i's the characteristic exponents of the transcendental hypersurface defined by $Y = \zeta(X)$, see the next proposition. This terminology is justified in Definition 2.3, in which we define for any $i \in \mathbb{N}$, an irreducible quasi-ordinary hypersurface (see for example [3] or [6]) with characteristic exponents $\lambda_1, \dots, \lambda_i$.

One can imagine "$\zeta(X)$" as an element of $\mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]]$, which is by definition the set of elements $z(X) \in \mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]]$, in which the set of exponents of each $z(X)$ is well-ordered with respect to a fixed "good ordering " (see Definition 3.4). The properties of "good ordering " shows that $\mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]]$, with natural multiplication and addition, is a ring (see [1], CH 6, Section 3, n° 4, Exemple 6). In this text by $\mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]]$, we mean the ring just explained. We have the inclusions:

$$\mathbb{C}[[X]] \subset \widetilde{\mathbb{C}[[X]]} = \lim_{N\to\infty} \mathbb{C}[[X^{\frac{1}{N}}]] \subset \mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]].$$

Proposition 2.2 *The element $\zeta(X)$ is transcendental on the ring $\mathbb{C}[X,Y]$. In other words, if $f \in \mathbb{C}[X,Y]$ then $f(X,\zeta) \neq 0$.*

Proof. Suppose this is not the case, and there is an element $f \in \mathbb{C}[X,Y]$ such that $f(X,\zeta(X)) = 0$, (We can assume that f is the minimal polynomial of the root $Y = \zeta(X)$ in $\mathbb{C}[X,Y]$). Then we have a parametrization of the algebraic hypersurface $V(f): \ f = 0$, which we denote it by $p: \ \mathbb{C}^d \longrightarrow V(f)$, $X \mapsto (X, \zeta(X)$. The mapping p is surjective, or by duality p^* is injective: Because in contrary, if $0 \neq \overline{g} \in \frac{\mathbb{C}[X,Y]}{(f)}$ and $p^*(\overline{g}) = 0$, or in other words $g(X,\zeta(X)) = 0$, then by the choice of f we have $f \mid g$, a contradiction. The dual map on the coordinate rings is given by

$$\begin{array}{rcl} p^*: \frac{\mathbb{C}[X,Y]}{(f)} & \longrightarrow & \mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]] \\ \overline{X} & \mapsto & X \\ \overline{Y} & \mapsto & \zeta(X). \end{array}$$

By intersection of $V(f)$ with the surface $W = V(I)$, $I =< X_1 - X_2, X_2 - X_3, \dots, X_{d-1} - X_d >$, we get an algebraic curve. The morphism $q : \mathbb{C}^d \cap W \longrightarrow V(f) \cap W$, is a parametrization of this curve. Consider the projection of the ambient space to the space with "$X_1, Y-$coordinates": $\pi : \ \mathbb{A}^{d+1}_{\mathbb{C}} \longrightarrow \mathbb{A}^2_{\mathbb{C}}$. By restriction of π to $V(f) \bigcap W$ we get an algebraic curve C in $\mathbb{A}^2_{\mathbb{C}}$. The parametrization of this curve is the composition of the bottom morphisms of the following diagram:

$$\begin{array}{ccccc} \mathbb{C}^d & \xrightarrow{p} & V(f) & \xrightarrow{\pi|_{V(f)}} & \mathbb{A}^2 \\ \uparrow & & \uparrow & & \uparrow \\ \mathbb{C}^d \cap W & \xrightarrow{q} & V(f) \cap W & \xrightarrow{\pi|_{V(f)\cap W}} & C \end{array}$$

We have the following diagram between the coordinate rings:

$$\begin{array}{ccccc} \mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]] & \xleftarrow{p^*} & \frac{\mathbb{C}[X,Y]}{(f)} & \xleftarrow{\pi|^*_{V(f)}} & \mathbb{C}[X_1, Y] \\ \downarrow & & \downarrow & & \downarrow \\ \frac{\mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]]}{I} & \xleftarrow{q^*} & \frac{\mathbb{C}[X,Y]}{<f,I>} & \xleftarrow{\pi|^*_{V(f)\cap W}} & R_C \end{array}$$

Where by R_C, we mean the coordinate ring of the plane curve C.

Take $t := \overline{X_1}$, as the generator of the ring $\frac{\mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]]}{I}$. We have:

$$\begin{array}{rcl} q^* o\ \pi\ |^*_{V(f)\cap W}\colon\ R_C & \to & \frac{\mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]]}{I} = \mathbb{C}[[t^{\mathbb{Q}_{\geq 0}}]] \\ \overline{X_1} & \mapsto & t \\ \overline{Y} & \mapsto & \zeta(t,\ldots,t). \end{array}$$

By Newton-Puiseux Theorem the algebraic closure of the ring $\mathbb{C}[[X_1]][Y]$ is equal to $\widetilde{\mathbb{C}[[t]]} = \lim_{N\to\infty} \mathbb{C}[[t^{\frac{1}{N}}]] \subset \mathbb{C}[[t^{\mathbb{Q}_{\geq 0}}]]$. The fact that $m^{(i)} \to \infty$, shows that $\zeta(t,\ldots,t) \notin \widetilde{\mathbb{C}[[t]]}$, a contradiction. $\square$

A variant of the proof above gives us the following statement: Given any $f \in \mathbb{C}[X]$, there does not exist a root $\eta(X) \in \mathbb{C}[[X^{Q^d_{\geq 0}}]]$ of f, such that the denominators of terms of η tend to infinity (By denominator of a term $c_\beta X^\beta$ of η we mean: the least natural number n such that $n.\beta \in \mathbb{N}^d$.).

We introduce a sequence of quasi-ordinary hypersurfaces $f^{(i)}$, which in some sense approximates the original element "$\zeta(X)$".

Definition 2.3 *We define for any $i \in \mathbb{N}$, an irreducible quasi-ordinary hypersurface $f^{(i)}(X,Y) \in \mathbb{C}[[X]][Y]$, by the following parametrization:*

$$Y = \zeta^{(i)}(X) = \sum_{j=1}^{i} p_j.$$

Definition 2.4 *We define for $1 \leq j \leq i \in \mathbb{N} : n_j = [Q_j : Q_{j-1}]$ and $m^{(0)} = 1,\ m^{(i)} = n_1 \ldots n_i$. It can be proved that $m^{(i)} = deg_Y(f^{(i)})$ (see [3] or [6]). Moreover, we define the following vectors (originally defined and studied in [4]):*

$$\gamma_1 = \lambda_1,\ \gamma_j = n_{j-1}\gamma_{j-1} + \lambda_j - \lambda_{j-1},\ j > 1.$$

By $R(f)$, for a quasi-ordinary f, we mean the set of roots of f in $\widetilde{\mathbb{C}[[X]]}$. Following [7], we define the notion of intersection index of two "comparable" quasi-ordinary hypersurfaces.

Definition 2.5 *For any two quasi-ordinary hypersurfaces f, g, we say they are comparable if for any $\eta \in R(f)$ and $\mu \in R(g)$ we have $\eta - \mu = X^\alpha.unit$, where $\alpha \in \mathbb{Q}^d_{\geq 0}$. The intersection index of two such hypersurfaces is defined as follows:*

$$(f,g) = v_X(Res_Y(f,g)).$$

Proposition 2.6 *[7] Let g be an irreducible unitary quasi-ordinary hypersurface which is comparable with $f^{(i)}$. We have:*

$$\frac{(f^{(i)},g)}{deg(f^{(i)}).deg(g)} = \frac{\gamma_{i_\kappa}}{n_1\ldots n_{i_\kappa - 1}} + \frac{\kappa - \lambda_{i_\kappa}}{n_1 \ldots n_{i_\kappa}}.$$

Here κ is the exponent of contact of $f^{(i)}, g$. Note that κ is an exponent in parameterization of $f^{(i)}$, and i_κ is the index of the greatest characteristic exponent of $f^{(i)}$ that $\lambda_j \leq \kappa$.

We recall the notion of semi-roots in our context:

Definition 2.7 *Set $X_k^{\frac{1}{m^{(i)}}} = T_k$, for $k = 1\ldots d$, and $\zeta^{(i)}(T) := \zeta^{(i)}(X) \in \mathbb{C}[[T]]$. We say that $g \in \mathbb{C}[[X]][Y]$ is a j^{th}−semi-root of $f^{(i)}$, $j \leq i$, if the following two conditions are satisfied:*
a)$g(0,Y) = Y^{n_1\ldots n_j}$.
b)$g(T_1^{m^{(i)}},\ldots,T_d^{m^{(i)}},\zeta^{(i)}(T)) = T^{m^{(i)}\gamma_{j+1}}\varepsilon_j^{(i)},\ \varepsilon_j^{(i)}$: unit.

We have the following lemma (see also [4]):

Lemma 2.8 *For any $j \leq i \in \mathbb{N}$, the quasi-ordinary singularity $f^{(j)}$ is a j^{th}-semi-root of $f^{(i)}$.*

Proof. We use Proposition 2.6. Here $i_\kappa = j+1$, and we have $\frac{(f^{(i)},f^{(j)})}{deg(f^{(i)}).deg(f^{(j)})} = \frac{\gamma_{j+1}}{n_1\dots n_j}$. We notice that $deg(f^{(j)}) = n_1 \dots n_j$, which shows that $(f^{(i)}, f^{(j)}) = m^{(i)}\gamma_{j+1}$. □

We need another result (see [3] and [7]) which allows a $(f^{(0)}, \dots, f^{(i)}) - adic$ representation of any element of $\mathbb{C}[X, Y]$.

Lemma 2.9 *Given $g \in \mathbb{C}[[X]][Y]$, there exits i_0 such that for $i \geq i_0$, g can be uniquely written as a finite sum $g = \sum c_{l_0\dots l_i}(f^{(0)})^{l_0} \dots (f^{(i)})^{l_i}$, with $c_{l_0\dots l_i} \in \mathbb{C}[[X]]$, the $(i+1)-$tuples $(l_0 \dots l_i) \in \mathbb{N}^{i+1}$ verifying $0 \leq l_k \leq n_{k+1} - 1,\ \forall k \in \{0, \dots, i\}$.*

Proof. This is essentially proved in [7]. The only thing which remains to prove is the inequality $0 \leq l_i \leq n_{i+1} - 1$. Because if i is chosen so large that $m^{(i)} > deg_Y(g)$, then $f^{(i)}$ (which is of degree $m^{(i)}$) can not appear in the expansion of g, i.e., $l_i = 0$. □

The preceding expansion is called the $(f^{(0)}, \dots, f^{(i)})-adic$, expansion of g. The finite set $\{(l_0 \dots l_i)$, $c_{l_0\dots l_i} \neq 0\}$ is called the $(f^{(0)}, \dots, f^{(i)}) - adic$ support of g. We set $(f_{[i]}) = (f^{(0)}, \dots, f^{(i)})$ so we can speak of the $(f_{[i]}) - adic$ expansion of an element. We write $c_\ell(f_{[i]})^\ell$ for $c_{l_0\dots l_i}(f^{(0)})^{l_0} \dots (f^{(i)})^{l_i}$. For a fixed set of functions $\{g_1, \dots, g_n\}$ the next lemma says that for sufficiently large values of i and arbitrary $j \in \mathbb{N}$ the $(f_{[i]}) - adic$ expansion of each g_k is the same as its $(f_{[i+j]}) - adic$ expansion, so in this case for sufficiently large values of i we can speak of $(f_{[\infty]}) - adic$ expansion of g_k's. For example note that the $(f_{[\infty]}) - adic$ expansion of $f^{(i)}$ is itself.

Lemma 2.10 *With the notations of the last lemma, for sufficiently large values of i and any $j \in \mathbb{N}$ the $(f_{[i]}) - adic$ expansion of g and $(f_{[i+j]}) - adic$ expansion of g coincide.*

Proof. For the i chosen in the proof of the last lemma, we have for any $j \geq 0$ $l_{i+j} = 0$.□

This expansion allows us to compute in an effective way the Newton polyhedron of $g(\zeta)$, where ζ is a root of $f^{(i)} = 0$ (We write $R(f)$ for the set of roots of $f = 0$). This computation is explained by the following two lemmas of [7]:

Lemma 2.11 *If $g = \sum c_\ell(f_{[i]})^\ell$, is the $(f_{[i]}) - adic$ expansion of $g \in \mathbb{C}[[X]][Y]$, then for every $\zeta \in R(f)$, the sets of vertices of the Newton polyhedra $\mathcal{N}_X(c_\ell(f_{[i]})^\ell)$, for varing ℓ, are pairwise disjoint.*

Lemma 2.12 *If $g_1, \dots, g_i \in \widetilde{\mathbb{C}[[X]]}$ and the sets of vertices of Newton polyhedra $\mathcal{N}_X(g_1), \dots, \mathcal{N}_X(g_i)$ are pairwise disjoint, then $\mathcal{N}_X(g_1 + \dots + g_i)$ is the convex hull of the union of $\mathcal{N}_X(g_1) \bigcup \dots \bigcup \mathcal{N}_X(g_i)$. In particular, each vertex of $\mathcal{N}_X(g_1 + \dots + g_i)$ is a vertex of one of the polyhedra $\mathcal{N}_X(g_1), \dots, \mathcal{N}_X(g_i)$.*

3 The ordering and semigroup

Definition 3.1 *We associate to $\zeta \in \mathbb{C}[[X^{\mathbb{Q}^d_{\geq 0}}]]$, satisfying the conditions of Definition 2.1, the sequence of semigroups:*

$$\Gamma_i = \mathbb{Z}^d_{\geq 0} + \gamma_1.\mathbb{Z}_{\geq 0} + \dots + \gamma_i.\mathbb{Z}_{\geq 0},\ for\ i \in \mathbb{N}.$$

And the semigroup:

$$\Gamma_\zeta = \mathbb{Z}^d_{\geq 0} + \gamma_1.\mathbb{Z}_{\geq 0} + \gamma_2.\mathbb{Z}_{\geq 0} + \dots.$$

Later, when we attach to the element ζ the valuation ν we will see that:

$$\nu(\mathbb{C}[X, Y] \setminus 0) = \Gamma_\zeta.$$

We need the following two lemmas from [4]:

Lemma 3.2 *1) The order of the image of γ_j in the group $\frac{Q_j}{Q_{j-1}}$ (see Definition 2.1) is equal to n_j for $j \in \mathbb{N}$.*
2) We have $\gamma_j > n_{j-1}\gamma_{j-1}$, for $j \geq 2$.
3) For any vector $u_j \in Q_j$ we have $u_j + n_j\gamma_j \in \gamma_j$.
4)The vector $n_j\gamma_j$ belongs to the semigroup Γ_{j-1} ($j \in \mathbb{N}$). Moreover, we have a unique relation:

$$n_j\gamma_j = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \ldots + l_{j-1}^{(j)}\gamma_{j-1}$$

such that $0 \leq l_k^{(j)} \leq n_k - 1$, and $\alpha^{(j)} \in \mathbb{Z}^d_{\geq 0}$, for $j \in \mathbb{N}$.

Lemma 3.3 *For any $j \in \mathbb{N}$ the $(f_{[\infty]})-adic$ expansion of $(f^{(j-1)})^{n_j}$ is of the following form:*

$$(f^{(j-1)})^{n_j} = c_j f^{(j)} + \sum c^{(j)}_{l_1,\ldots,l_j}(f^{(0)})^{l_1}(f^{(1)})^{l_2}\ldots(f^{(j-1)})^{l_j},$$

where $c_j \in \mathbb{C}^*$. We have $0 \leq l_k \leq n_{k+1} - 1$, for $k = 0, \ldots, j-1$. The coefficient $c^{(j)}_{l_1^{(j)},\ldots,l_{j-1}^{(j)},0}$ appears, and it is of the form $X^{\alpha(j)}.unit$, where the integers $l_1^{(j)}, \ldots, l_{j-1}^{(j)}$ and the exponent $\alpha(j)$ are given in Lemma 2.2. Moreover, if $X^{\alpha'}$ appears on the coefficient $c^{(j)}_{l_1,\ldots,l_j}$ then:

$$n_j\gamma_j \leq \alpha' + l_1\gamma_1 + \ldots + l_j\gamma_j,$$

and equality holds iff $(l_1, \ldots, l_j) = (l_1^{(j)}, \ldots, l_{j-1}^{(j)}, 0)$.

In order to define the valuation we need to fix a total well-ordering on $\mathbb{Z}^d$ which extends to a total ordering on "Γ_ζ". This ordering should verify certain conditions.

Definition 3.4 *We say a total ordering "$\prec$" on $\mathbb{Q}^d$ is a "good ordering" if:*

- *It is a monomial ordering on $\mathbb{Q}^d$, i.e., for any $\gamma, \gamma', \gamma'' \in \mathbb{Q}^d$ from $\gamma \prec \gamma'$ one has $\gamma + \gamma'' \prec \gamma' + \gamma''$.*
- *It refines the partial ordering "$\leq$" on $\mathbb{Q}^d$, i.e., if $u, v \in \mathbb{Q}^d$ and $u < v$ then $u \prec v$*

The following proposition shows that every "suitably choosen" ordering on $\mathbb{Z}^d$ can be expanded in a way to a "good ordering" on $\mathbb{Q}^d$.

Proposition 3.5 *Every global well-ordering on $\mathbb{Z}^d$ which refines the partial ordering "$\geq$" on $\mathbb{Q}^d$ can be expanded to a "good ordering" on $\mathbb{Q}^d$.*

Proof. Let "$\prec$" be such an ordering. Expand this ordering on $\mathbb{Q}^d$ as follows: For $\gamma, \gamma' \in \mathbb{Q}^d$: $\gamma \prec \gamma'$ iff there is $n \in \mathbb{N}$ such that $n\gamma, n\gamma' \in \mathbb{Z}^d$ and $n\gamma \prec n\gamma'$. The next lemma shows that we have the following equivalent definition: We have $\gamma \prec \gamma'$ iff for any $n \in \mathbb{N}$ such that $n\gamma, n\gamma' \in \mathbb{Z}^d$ then $n\gamma \prec n\gamma'$. It is clear that "$\prec$" is a total ordering on $\mathbb{Q}^d$. We show that it is a monomial ordering. Suppose this is not the case. Then there is $\gamma, \gamma', \gamma'' \in \mathbb{Q}^d$ such that $\gamma \prec \gamma'$ but $\gamma + \gamma'' \not\prec \gamma' + \gamma''$ then $\gamma + \gamma'' \succ \gamma' + \gamma''$. By the next lemma and definition, we can find an $n \in \mathbb{N}$ such that $n\gamma, n\gamma', n\gamma'' \in \mathbb{Z}^d$ and $n\gamma + n\gamma'' \succ n\gamma' + n\gamma''$. This gives $\gamma \succ \gamma'$, a contradiction.
This ordering refines the partial ordering on $\mathbb{Q}^d$. Let $\gamma < \gamma'$ and take a natural number n such that $n\gamma, n\gamma' \in \mathbb{Z}^d_{\geq 0}$. By definition of good ordering $n\gamma \prec n\gamma'$. By the discussion in the first step of the proof $\gamma \prec \gamma'$. □

Lemma 3.6 *Let "$\prec$" be a global ordering on $\mathbb{Z}^d$ and $a, b \in \mathbb{Z}^d$. If $a \prec b$ then for any $p \in \mathbb{Q}$ such that $pa, pb \in \mathbb{Z}^d$ we have $pa \prec pb$.*

Proof. By monomial ordering property for every $p \in \mathbb{N}$ we have $pa \prec pb$. It suffices to prove the Theorem for p^{-1}, where $p \in \mathbb{N}$. If $p^{-1}a \succ p^{-1}b$ then $p.p^{-1}a \succ p.p^{-1}b$ so $a \succ b$, a contradiction. □

Remark 3.7 *The ordering introduced in Proposition 3.5, is no longer a well-ordering on* $\mathbb{Q}^d$. *For example take the set* $A = \{u_i = (1, \ldots, 1, \frac{1}{i})\}_{i=1}^{\infty}$. *The property that "*$\prec$*" refines the partial ordering "*$\geq$*", shows that the set* A*, has not an initial element.*

Here is a concrete example of a "good ordering".

Example 3.8 *Consider the "*$<_{d.lex.}$*" ordering on* $\mathbb{Z}^d$ *which is defined as follows:*
For any $a, b \in \mathbb{Z}^d$ *we have* $a <_{d.lex.} b$ *iff* $(deg(a) = \sum_{i=1}^{d} a_i < deg(b)$ *or* $(deg(a) = deg(b)$ *and* $a <_{lex.} b))$.
It could be shown that this ordering verifies all the conditions of the Definition 3.4. It expands to a "good ordering", yet denoted by "$<_{d.lex.}$*" on* $\mathbb{Q}^d$.

One way to introduce a monomial ordering, "$\prec$", on a group, G, is to introduce a subset of the group as the subset of positive elements, $\{0 \prec\} = \{g \in G : 0 \prec g\}$.For example we have

$$\{0 >_{d.lex.}\} = \{\mathbf{u} \in \mathbb{Q}^2 : u_1 + u_2 > 0\} \bigcup \{\mathbf{u} \in \mathbb{Q}^2 : u_1 > 0, u_1 + u_2 = 0\}.$$

Lemma 3.9 *Consider the ordering "*$\prec$*" on* $\mathbb{Q}^d$. *We have:*
1) *It refines the partial ordering "*$\leq$*" iff* $\mathbb{Q}^d_{\geq 0} \subset \{0 \prec\}$.
2) *It is a total ordering iff for any* $\mathbf{u} \in \mathbb{Q}^d$: $\{\mathbf{u}, -\mathbf{u}\} \bigcap \{0 \prec\} \neq \varnothing$.
3) *Its restriction on* $\mathbb{Z}^d_{\geq 0}$ *is a well-ordering iff this restriction refines the partial ordering "*$\leq$*" on* $\mathbb{Z}^d_{\geq 0}$.

proof. The items 1) and 2) are easy to prove. For a proof of 3) we refer to [5]. □

As a corollary one can give another characterization of "good ordering".

Corollary 3.10 *The ordering "*$\prec$*" on* $\mathbb{Q}^d$ *is a "good ordering" if* $\mathbb{Q}^d_{\geq 0} \subset \{0 \prec\}$ *and for any* $\mathbf{u} \in \mathbb{Q}^d$ *we have* $\{\mathbf{u}, -\mathbf{u}\} \bigcap \{0 \prec\} \neq \varnothing$.

As another corollary we can give another description of the construction given in Proposition 3.5.

Corollary 3.11 *Given a monomial well-ordering "*$\prec$*" on* $\mathbb{Z}^d$. *There is a natural expansion of this ordering to a "good ordering" on* $\mathbb{Q}^d$*, which we denote it with the same notation. We define this expansion with the set of its positive elements: Consider the positive cone in* $\mathbb{R}^d$ *based on the set of positive elements of "*$\prec$*" in* $\mathbb{Z}^d$. *The set of positive elements will be the intersection of this cone with* $\mathbb{Q}^d$. *Moreover, this expansion coincides with the expansion defined in Proposition 3.5.*

Definition 3.12 *For any two orderings,* $\prec$ *and* $\prec'$*, we define the set*

$$\Delta_+(\prec, \prec') = (\{0 \prec\} - \{0 \prec'\}) \bigcup (\{0 \prec'\} - \{0 \prec\}).$$

We say the sequnce $\{\prec_k\}_{k=1}^{\infty}$ *of orderings on the group* G *converges to the ordering "*$\prec$*" iff*

$$\Delta_+(\prec_1, \prec) \supset \Delta_+(\prec_2, \prec) \supset \ldots \text{ and } \bigcap_{k=1}^{\infty} \Delta_+(\prec_k, \prec) = \varnothing.$$

In this case we write $\lim_{k \to \infty} \prec_k = \prec$.

Example 3.13 *We introduce for any* $\omega \in \mathbb{R}_{>0}$ *a "good ordering" on* $\mathbb{Q}^2$*, "*$\prec_\omega$*", by*

$$\{0 \prec_\omega\} = \{\mathbf{u} \in \mathbb{Q}^2 : u_1 + \omega.u_2 > 0\} \bigcup \{\mathbf{u} \in \mathbb{Q}^2 : u_1 + \omega.u_2 = 0, u_1 > 0\}.$$

One can easily prove that this ordering verifies the conditions of the last corollary and it is a "good ordering".

Example 3.14 *Take a sequence* $\{\omega_k\}_{k=1}^{\infty}$ *of positive irrational numbers that are increasing and convergent to* -1*. According to the last example, construct the sequence of orderings* $\{\prec_{\omega_k}\}_{k=1}^{\infty}$*. This is a sequence of "good orderings". Then it is easily seen that*

$$\lim_{k\to\infty} \prec_{\omega_k} = >_{d.lex.} .$$

It is interesting to note that $\mathbb{Q}^2$ *with ordering"* $\prec_{\omega_k}$ *" does not have non-trivial isolated subgroups. In contrary if* G *is such an isolated subgroup then take* $0 \prec_{\omega_k} g \in G$*. The group* G *should contain all the rational points in the section between the line joining the origin to the point* g*, in the plane, and the line* $u_1 + \omega.u_2 = 0$*. It could be seen that the group generated by this last set is* $\mathbb{Q}^2$*. In Example 3.7 we will see that* $\mathbb{Q}^2$ *with ordering "*$<_{d.lex.}$*" has a nontrivial isolated subgroup. As a result we have constructed a sequnece of orderings on* $\mathbb{Q}^2$ *with* $rank_{\prec_{\omega_k}}(\mathbb{Q}^2) = 1$ *which converges to the ordering "*$<_{d.lex.}$*" with* $rank_{<_{d.lex.}}(\mathbb{Q}^2) > 1$*.*

Alternatively, in the above example one could take the ω_k's to be rational numbers and define the same constructions and the same limit. All the things are the same with the exception about the rank. The argument given in the Example 4.6 could be repeated to prove that $rank_{\prec_{\omega_k}}(\mathbb{Q}^2) > 1$.

4 The valuation

Given any "good ordering", $\prec$, we can define a ring $\mathbb{C}[[X^{Q^d_{\succcurlyeq 0}}]]$, which is the ring of power series $z(X) \in \mathbb{C}[[X^{Q^d_{\succcurlyeq 0}}]]$, in which the set of exponents are well-ordered with respect to $\prec$. This is in fact a valuation ring (see [1], CH 6, Section 3, n° 4, Exemple 6). We denote this valuation by ν.

Lemma 4.1 *There is an injective morphism of the rings*

$$\Theta_\zeta : \mathbb{C}[X,Y] \hookrightarrow \mathbb{C}[[X^{Q^d_{\succcurlyeq 0}}]]$$
$$Y \mapsto \zeta(X).$$

Proof. This is clearly a morphism, the injectivity is a result of Proposition 2.2. □

Now we define the valuation induced by the transcendental element $\zeta(X)$ on the ring $\mathbb{C}[X,Y]$, with respect to a "good ordering", "$\prec$", fixed on $\mathbb{Q}^d$:

Definition 4.2 *We define a mapping* $\nu : \mathbb{C}[X,Y] \setminus \{0\} \longrightarrow \mathbb{Q}^d_{\geq 0}$ *by:*

$$\nu(f) = \nu(\Theta_\zeta(f)).$$

This mapping is a valuation on the ring $\mathbb{C}[X,Y]$*.*

The following proposition gives an effective way to compute the value $\nu(g)$, for an arbitrary $g \in \mathbb{C}[X,Y]$. It also gives us essentially another definition of this valuation.

Proposition 4.3 *We generalize the definition of* ν *to the set* $\mathbb{C}[[X]][Y]$*, by the same definition. We then have:*
1) *For any* $g \in \mathbb{C}[X,Y]$*, the values of monomials of* $(f_{[\infty]})-$*adic expansion of* g *are distinct elements of* $\mathbb{Q}^d_{\geq 0}$*. Moreover, we have:*

$$\nu(g) = min_\ell\{\nu(c_\ell(f_{[\infty]})^\ell)\}.$$

2) *We have:*

$$\nu(f^{(i)}) = \gamma_{i+1}.$$

3) *We have:*

$$\nu((f^{(j-1)})^{n_j}) = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \ldots + l_{j-1}^{(j)}\gamma_{j-1},$$

where the $l_k^{(j)}$*'s and* $\alpha^{(j)}$ *are defined in Lemma 3.2. Moreover, there is exactly one term in* $(f_{[\infty]})-$*adic expansion of* $(f^{(j-1)})^{n_j}$ *with this value, if* ℓ_* *is the index of this term then* $\ell_* = (l_1^{(j)}, \ldots, l_{j-1}^{(j)}, 0)$*.*

proof. The first claim is a direct consequence of Lemma 2.11 and the properties of good orderings. The second one is a consequence of Definition 2.7 and Lemma 2.8. The third one is a consequence of the last step and Lemma 3.3. Alternatively, we can prove the third result directly and as a consequence, yield another proof of Lemma 3.3; We note that by Lemma 2.8 $\mathcal{N}((f^{(j-1)})^{n_j}) = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \ldots + l_{j-1}^{(j)}\gamma_{j-1} + \mathbb{R}^d_{\geq 0}$, which gives the first claim of 3). By 1) there is a unique term, say with index ℓ_*, in $(f_{[\infty]})-adic$ expansion of $(f^{(j-1)})^{n_j}$ that $\nu((f^{(j-1)})^{n_j}) = \nu(c_{\ell_*}(f_{[\infty]})^{\ell_*}) = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \ldots + l_{j-1}^{(j)}\gamma_{j-1}$. By arguing on uniqueness representation of elements of Γ_{j-1}, one can show that ℓ_* is of the claimed form. □

We note that the monomial which appears in the first case of the above proposition is not neccesarily a vertex of the Newton polyhedron of $g(\zeta)$.

Corollary 4.4 *The semigroup of valuation, $\nu(\mathbb{C}[X,Y] \setminus 0)$, is equal to Γ_ζ. The value group is equal to the subgroup of $\mathbb{Q}^d$ generated by Γ_ζ. We denote this value group by Φ_ζ.*

The next proposition gives yet another feature of this valuation. It shows that, in some sense, this valuation is approximated by intersection index of quasi-ordinary hypersurfaces $f^{(i)}$.

Proposition 4.5 *For any unitary irreducible quasi-ordinary $g \in \mathbb{C}[[X]][Y]$, which is comparable with $f^{(i)}$'s, we have:*

$$\nu(g) = \lim_{i\to\infty} \frac{(f^{(i)},\ g)}{deg_Y(f^{(i)})}.$$

Proof. We notice that if i is chosen so great that $\kappa < \lambda_i$ (with notations of Proposition 1.6) then for any $j > i$ we have:

$$\frac{(f^{(i)}, g)}{deg_Y(g).deg_Y(f^{(i)})} = \frac{(f^{(j)}, g)}{deg_Y(g).deg_Y(f^{(j)})}.$$

As a result, the limit is well defined. For the equality, it suffices to note that (with the notations of Definition 1.7):

$$\mathcal{N}(g(T^{m^{(i)}}, \zeta^{(i)}(T))) = \mathcal{N}(\prod_{k=1}^{m^{(i)}} g(\zeta_k^{(i)})) = deg_Y(f^{(i)}).\mathcal{N}(g(\zeta^{(i)})) = \mathcal{N}(Res_Y(f^{(i)}, g)),$$

where $\zeta_k^{(i)}$'s are all the $m^{(i)}$ roots of $f^{(i)} = 0$. □

The next example shows that for suitably chosen "ζ" the value group will be $\mathbb{Q}^d$. In order to simplify the notations the example is stated in the case $d = 2$.

Example 4.6 *In the set of natural numbers start from $s_1 = 2$ and pick up all the numbers that are power of a prime. The sequence $\{s_i\}_{i=1}^{\infty}$ is the result. The first elements are:*

$$s_1 = 2, s_2 = 3, s_3 = 4, s_4 = 5, s_5 = 7, s_6 = 8, \ldots.$$

We define:

$$\gamma_1 = (\frac{1}{s_1}, 1), \gamma_2 = (s_2, s_2 + \frac{1}{s_1}),$$

$$\text{and for } i \geq 1: \begin{cases} \gamma_{2i+1} = (s_2 \ldots s_{2i+1} + \frac{1}{s_{i+1}}, s_2 \ldots s_{2i+1}) \\ \gamma_{2i+2} = (s_2 \ldots s_{2i+2}, s_2 \ldots s_{2i+2} + \frac{1}{s_{i+1}}). \end{cases}$$

One then defines the exponents λ_i's using the inductive formula of Definition 2.4. These λ_i's satisfy the conditions of Definition 2.1: By the construction and computation of n_i's, which is given in the following, we have $\gamma_j > n_{j-1}\gamma_{j-1}$. This last inequality gives us $\lambda_j > \lambda_{j-1}$. The condition $\lambda_j \notin Q_{j-1}$ is a consequence of the fact that components of elements of Q_{j-1} have, as denominators, only $s_1, \ldots, s_{j-1}$. If s_i's are the powers of a prime p, we have $n_i = p$. As a result $m^{(i)} = \Pi_q q^{\alpha_q}$, where

q runs through all the primes less than (or equal to) s_i and α_q is by definition the greatest power of q such that $q^{\alpha_q} \leq s_i$. By Definition 4.2, the series $\zeta(X) = \sum X^{\lambda_i}$ defines a valuation of $\mathbb{C}[X,Y]$. We see, by induction, that $(\frac{1}{s_i}, 1), (1, \frac{1}{s_i})$ are in the value group of this valuation, we denote this value group by Φ_ζ. Therefore, by definition of s_i's we have $\Phi_\zeta = \mathbb{Q}^2$. This valuation is of rank two; Define $G = \{(a, -a) : a \in \mathbb{Q}\}$. This is a subgroup of $\mathbb{Q}^2$. It is an isolated subgroup: Take an arbitrary element $0 <_{d.lex.} (a, -a) \in G$ then for any $\mathbf{u} = (u_1, u_2) \in \mathbb{Q}^2$ from $0 <_{d.lex.} \mathbf{u} <_{d.lex.} (a, -a)$ we deduce $deg(\mathbf{u}) = 0$ and then $\mathbf{u} \in G$.

Remark 4.7 *Consider the sequence of orderings that introduced in Example 3.14. If we denote the semigroups that are attached to the valuations induced by the above example to each of these orderings by $\Gamma_{\zeta,\prec_{\omega_k}}$ then as the choice of "good ordering" does not have any effect on the resulting semigroup, we have $\Gamma_{\zeta,\prec_{\omega_k}} = \Gamma_{\zeta,>_{d.lex.}}$. Therefore, we have a sequence of orderings which converge to another one. All of these orderings impose the same semigroup but the dimension of the valuation ring for the elements of the sequence is one and the dimension of the valuation ring which they converge to is two.*

Example 4.8 *Take $k_1, \ldots, k_d \in \mathbb{N} \bigcup \{\infty\}$ such that at least one of them is $\infty$and take d sequence of natural numbers $\{s_j^{(q)}\}_{j=1}^{k_q}$, where $s_j^{(q)} > 1$ (for $q = 1, \ldots, d$), and complete these sequences by setting $s_{j+k_q}^{(q)} = 1$, for $j = 0, \ldots$. Define the following vectors:*

$$\gamma_1 = \gamma_0 + \frac{1}{s_1^{(1)}} e_1,$$

$$\gamma_i = s_{j+1}^{(l-1)} \gamma_{i-1} + \frac{1}{s_1^{(l)} \ldots s_{j+1}^{(l)}} e_l,$$

where e_l's are the transpose of the standard basis of the vector space $\mathbb{Q}^d$ and γ_0 is an arbitrary element of _mathbbQ$^d_{\geq 0}$. In the second equation we have $i \in \mathbb{N}$ and it is written as $i = dj + l, j \in \mathbb{N} \bigcup \{0\}, l = 2, \ldots, d+1$. By the definition of the γ_i's, it is clear that $n_i = s_{j+1}^{(l)}$. Drop the γ_i's for which $n_i = 1$. As the above example construct the vectors λ_i's. We have $\gamma_i - n_{i-1}\gamma_{i-1} = \frac{1}{s_1^{(l)} \ldots s_{j+1}^{(l)}} e_l > 0$, therefore $\lambda_i > \lambda_{i-1}$, and λ_i is not in the group Q_{i-1} of Definition 2.4. Consider the element $\zeta = \sum X^{\lambda_i}$, and the valuation attached to it by Definition 4.2. We see, by induction, that $\frac{1}{s_1^{(l)} \ldots s_{j_l}^{(l)}} e_l$ is in the value group of this valuation, Φ_ζ. Therefore, we have:

$$\Phi_\zeta = \{(\frac{p_1}{s_1^{(1)} \ldots s_{j_1}^{(1)}}, \ldots, \frac{p_d}{s_1^{(d)} \ldots s_{j_d}^{(d)}}) : p_1, \ldots, p_d \in \mathbb{Z}, j_1 \leq k_1, \ldots, j_d \leq k_d\}.$$

5 Specialization to the graded valuation ring

Following [8], Subsection 4.4 and [3], in this section we show that the transcendental hypersurface $\mathcal{S} = \mathrm{Spec}(R)$, $R = \mathbb{C}[[X]][\zeta(X)]$, embeds in an infinite dimensional space. We introduce an explicit set of generators for the ideal of this embedding. Moreover, we show that the graded valuation ring $\mathrm{gr}_\nu R$ (which is defined in the following) embeds in this infinite dimensional space with a binomial ideal which comes exactly by considering the "initial forms" of the generators of the first embedding. In fact the situation is general, this is explained in [8], subsection 2.3, see also [2].

Take an infinite sequence of indeterminates $U = (U_1, U_2, \ldots)$. The infinite dimensional ambient space is $\mathcal{A} = \mathrm{Spec}(\mathbb{C}[[X]][U])$. Note that for every element $h \in \mathbb{C}[[X]][U]$ there is an $i \in \mathbb{N}$ such that $h \in \mathbb{C}[[X]][U_1, \ldots, U_i]$. The embedding of $\mathcal{S}$ in $\mathcal{A}$ comes from the following morphism:

$$\begin{aligned} \Psi : \mathbb{C}[[X]][U] &\rightarrow R \\ U_i &\mapsto f^{(i-1)}(X, \zeta(X)). \end{aligned}$$

Note that Ψ is surjective, $U_1 \mapsto Y = \zeta(X)$.

Consider an arbitrary valuation ν and a ring R, $R \subset R_\nu$. We denote the value group of this valuation by Φ. We set $\Gamma = \nu(R \setminus \{0\}) \subset \Phi_+ \bigcup \{0\}$; It is the semigroup of (R, ν). For $\phi \in \Phi$ set:

$$\mathcal{P}_\phi(R) = \{x \in R : \nu(x) \geq \phi\}$$

$$\mathcal{P}_\phi^+(R) = \{x \in R : \nu(x) > \phi\}.$$

The graded algebra associated to (R, ν) is defined by:

$$\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Gamma} \frac{\mathcal{P}_\phi(R)}{\mathcal{P}_\phi^+(R)}.$$

It is a monomial algebra (see [8], Proposition 4.1) and can be represented as a quotient of an infinite dimensional polynomial ring by a binomial ideal, so it is "essentially toric" (see [8], Subsection 4.2).

The valuation ν on $\mathbb{C}[[X]][Y]$ (see Definition **??**) induces a weight on any element of the ring $\mathbb{C}[[X]][U]$: For any monomial $X^\beta U^\nu$ we define $\omega(X^\beta U^\nu) = \nu(\Psi(X^\beta U^\nu)) = \beta + \sum \nu_i \gamma_i$. For any $\omega \in \Gamma_\zeta$ we define the ideal $\mathcal{I}_\omega$ of the ring $\mathbb{C}[[X]][Y]$, which contains all the elements with weight greater than or equal to ω. The sequence of ideals $\{\mathcal{I}_\omega\}_{\omega \in \Gamma_\zeta}$ is a filtration (The ordering on the index set, Γ_ζ, of this sequence is the good ordering fixed to define the valuation ν).

Proposition 5.1 *The morphism Ψ induces a morphism:*

$$\begin{array}{rl} \mathrm{gr}\Psi : \mathrm{gr}_\omega \mathbb{C}[[X]][U] = \mathbb{C}[X, U] & \to \mathrm{gr}_\nu R = \mathbb{C}[X^{\Gamma_\zeta}] \\ U_i & \mapsto X^{\gamma_i}. \end{array}$$

Moreover, with the notations of Lemma 3.2, we have $ker(\mathrm{gr}\Psi) = < h_1, h_2, \dots >$, where $h_i = U_i^{n_i} - d_i X^{\alpha^{(i)}} U_1^{l_1^{(i)}} \dots U_{i-1}^{(i)}$, $d_i \in \mathbb{C}^*$.

Proof. In coordinate free terms the morphism $\mathrm{gr}\Psi$ is defined by $\mathrm{gr}\Psi(\overline{a}) = \overline{\Psi(a)}$, for $a \in \mathbb{C}[[X]][U]$. The equality $\mathrm{gr}_\omega \mathbb{C}[[X]][U] = \mathbb{C}[X, U]$ is clear from the definition of filtration on $\mathbb{C}[[X]][U]$ and the equality $\mathrm{gr}_\nu R = \mathbb{C}[X^{\Gamma_\zeta}]$ comes from Proposition 4.3. The proof of Proposition 38 of [3] could be adapted to give a proof of the second part. □

The above proposition shows that $\mathcal{Z}^{\Gamma_\zeta} := \mathrm{Spec}(\mathbb{C}[X^{\Gamma_\zeta}])$ is embedded in the infinite dimensional space $\mathcal{A}$. Moreover, the equations defining this embedding are binomial. This is also a general fact, see [8], section 4.

Proposition 5.2 *The ideal of the embedding $\mathcal{S} \subset \mathcal{A}$ has the following generators:*

$$\left\{ \begin{array}{llllllllll} H_1 & := & U_1^{n_1} & - & d_1 X^{\alpha^{(1)}} & + & c_1 U_2 & + & r_1(U_1), \\ H_2 & := & U_2^{n_2} & - & d_2 X^{\alpha^{(2)}} U_1^{l_1^{(2)}} & + & c_2 U_3 & + & r_2(U_1, U_2), \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ H_i & := & U_i^{n_i} & - & d_i X^{\alpha^{(i)}} U_1^{l_1^{(i)}} \dots U_{i-1}^{l^{(i)}{}_{i-1}} & + & c_i U_{i+1} & + & r_i(U_1, \dots, U_i), \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array} \right.$$

for $i \in \mathbb{N}$. The elements c_i are defined in Lemma 3.3 and d_i's are those that are defined in the previous proposition. For any $j \in \mathbb{N}$ the weight of a term $X^\beta U^\nu$ appearing in $r_j(U)$ is strictly greater than $n_j \gamma_j$. The terms appearing in the expansion of $r_j(U)$ are determined explicitely by the Lemma 3.3.

Proof. These relations are analogous of the equations intorduced in Lemma 3.3. □

Remark 5.3 *Notice that, unlike [3], it is not possible to arrange the situation such that $d_i = 1$, because we have a pre-fixed system of semi-roots. Moreover, $in_\omega(H_i) = h_i$. In other words, the ideal defining the embedding $\mathcal{S} \subset \mathcal{A}$ specializes through the filtration to the ideal of the embedding $\mathcal{Z}^{\Gamma_\zeta} \subset \mathcal{A}$.*

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