A construction for a class of valuations of the field $k(X_1, \dots, X_d, Y)$ with large value group

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Abstract

Given any algebraically closed field k of characteristic zero and any group G, totally ordered by a suitably chosen ordering, of rational rank less than or equal to d, we construct a valuation of the field $k(X_1, \dots, X_d, Y)$ with value group G. In the case of rational rank equal to d this valuation is induced by a transcendental hypersurface in affine $(d+1)-$space. It is naturally approximated by a sequence of quasi-ordinary hypersurfaces. The value semigroup $\nu(k[X, Y] \setminus 0)$ is the union of the semigroups associated to these quasi-ordinary hypersurfaces.

1 Introduction

Let k be an algebraically closed field of characteristic zero and d an integer. For each commutative group G of rational rank less than (or equal to) d, we construct a zero-dimensional valuation of the field $k(X_1, \dots, X_d, Y)$ whose value group is G. Note that in the case of valuations of the field $k(X_1, \dots, X_d, Y)$ of rational rank equal to $d+1$ we are in the equality case of Abhyankar's inequality ([Bou64]) and the value group has to be $\mathbb{Z}^{d+1}$.

The problem of the existence of the valuations with a given value group and residual extension has solved by "arithmetical" methods, see [Kuh04]. However, our approach is different and more geometric. For example, with this approach question of representing the valuation rings corresponding to these valuations as limits of blowing up algebras of the ring $k[X_1, \dots, X_d, Y]$ seems to be more accessible.

The construction of the valuation is based on generalizing the notion of quasi-ordinary hypersurface singularities ([Lip65], [Lip83]); this is done in Definition 2.1. This generalization gives us a transcendental element $\zeta(X) \in k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]],\ X = (X_1, \dots, X_d)$. As a set $k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$ is the set of formal power series in $X_1, \dots, X_d$ with rational exponents, in which the set of exponents is well-ordered with respect to a total monomial ordering $\preccurlyeq$ which refines the partial ordering $\leq$ on $\mathbb{Q}^d$ (A good ordering, see Definition 3.4). This is in fact a ring (see [Bou64], Chap. 6, Section 3, n° 4, Exemple 6). By a process of truncation of this element $\zeta(X)$, we get the developments $\zeta^{(i)}(X)$. These $\zeta^{(i)}$'s parametrize quasi-ordinary hypersurfaces $f^{(i)}$'s in $\mathbb{A}^{d+1}(k)$ (Definition 2.3). Later, in Section 6, following the ideas of Teissier in [Tei03] and [Tei86], we give a way to compute the $f^{(i)}$'s.

One of the difficulties to construct a valuation with value group in $\mathbb{Q}^d$ is that there is no natural ordering on $\mathbb{Q}^d$. In Section 3, we introduce and study the properties of the good orderings on $\mathbb{Q}^d$.

In the next section using the valuation ring $k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$ we show that there exists an injective morphism $\Theta_\zeta : k[X, Y] \to \mathbb{C}[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$ (see Definition 4.2). With the help of this injection we get the desired valuation ν of the field $k(X, Y)$. We study the properties of this valuation and the semigroup Γ_ζ attached to it. We show that there is a close relation between the semigroup which [GP03] attaches to a quasi-ordinary hypersurface and the semigroup of the valuation Γ_ζ. More precisely, if we denote by Γ_i the semigroups which are attached to the truncated quasi-ordinary hypersurfaces $f^{(i)}$'s, then we have $\Gamma_\zeta = \lim_i \Gamma_i$ of an inductive system $\Gamma_i \stackrel{\times n_{i+1}}{\longrightarrow} \Gamma_{i+1}$ for specific integers n_i. Moreover, we show that given any subgroup of rational rank d there is a transcendental element ζ such that the value group of the valuation attached to this element is G.

In section 5, we show that the $f^{(i)}$'s constitute a sequence of key polynomials in the sense of MacLane ([Mac36]). In order to prove this, we give another way of constructing the valuation ν (Proposition 5.6). This new construction is carried out by a direct introduction of a sequence of valuations ν_i's which approximates the valuation ν. Moreover, the value group of ν_i is equal to the group generated by Γ_i.

In the final section we study an embedding of the spaces Spec$R$, where $R = k[[X]][\zeta(X)]$ and Spec$(\mathbb{C}[X^{\Gamma_\zeta}])$ in an infinite dimensional regular space Spec$k[[X]][U]$, where $U = (U_1, U_2, \ldots)$. We study the ideals defining these embeddings and the relation between them. Moreover, we show that the result of truncating the equations of the embedding Spec$R \hookrightarrow$ Spec$k[[X]][U]$ is a set of equations which gives an embedding of the quasi-ordinary hypersurfaces $f^{(i)} = 0$ in Spec$k[[X]][U]$. Using the constructions of this section and some ideas of [Tei6] and [Tei05], we are able to construct a rational valuation with value group G, for any totally ordered group G of rational rank less than d.

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2 The transcendental hypersurface and its approximations

Generalizing the classical definition of the quasi-ordinary hypersurface singularities (see [Lip88], [Lip65]) we define a transcendental quasi-ordinary hypersurface singularity in the following manner:

Definition 2.1 Fix an element $\zeta(X) = \sum c_\lambda X^\lambda = \sum_{i=1}^{\infty} p_i,\ p_i \in k[X^{\frac{1}{m^{(i)}}}]$, where $X = (X_1, \ldots, X_d)$ and $X^{\frac{1}{m}} = (X_1^{\frac{1}{m}}, \ldots, X_d^{\frac{1}{m}})$. The $m^{(i)}$'s are integers which tend to infinity; they will be described more precisely in Definition 2.4 . We impose the following conditions:

- All the exponents of p_i, i.e., λ's of the monomials of p_i, are ordered with respect to the partial product order $\leq$ on $\mathbb{Q}^d$, with minimum equal to λ_i.
- The partial order on $\mathbb{Q}^d$ induces a total order on the set $\{\lambda_i\}_{i=1}^{\infty}$, i.e., $\lambda_1 < \lambda_2 < \ldots$.
- We define inductively a sequence of subgroups of $\mathbb{Q}^d$ by $Q_0 = \mathbb{Z}^d$, $Q_j = \mathbb{Z}^d + \sum_{\lambda_i < \lambda_{j+1}} \mathbb{Z}\lambda_i$, for $j \in \mathbb{N}$. We impose the condition $\lambda_j \notin Q_{j-1}$.
- If $c_\lambda X^\lambda$ is a term of p_j then $\lambda \in Q_j$.

The above definition is a generalization of [Tei03], subsection 4.4, where a "natural valuation" attached to a "transcendental plane curve", studied through a series of examples from different perspectives: the sequence of point blow ups, the semigroup, the graded valuation ring, ... Moreover, the relations between these approaches studied. In this text we follow the same approach.

Note that if we define $\Lambda = \{\lambda \ : c_\lambda \neq 0\}$ then $\lambda \in p_i \bigcap \Lambda$ iff $\lambda_i \leq \lambda \ngeq \lambda_{i+1}$. We call λ_i's the characteristic exponents of the transcendental hypersurface defined by $Y = \zeta(X)$, see the next proposition. This terminology is justified in Definition 2.3, in which we define for any $i \in \mathbb{N}$, an irreducible quasi-ordinary hypersurface (see [GP03] or [Lip88]) which is parametrized by $X = X, Y = \zeta^{(i)}(X)$ where $\zeta^{(i)}(X)$ is a fractional power series with characteristic exponents $\lambda_1, \ldots, \lambda_i$.

For any good ordering $\preccurlyeq$ we have the inclusions:

$$k[[X]] \subset \widetilde{k[[X]]} = \lim_{N\to\infty} k[[X^{\frac{1}{N}}]] \subset k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]].$$

Proposition 2.2 *The element $\zeta(X)$ is transcendental over the ring $k[X,Y]$. In other words, if $f \in k[X,Y]$ then $f(X,\zeta(X)) \neq 0$.*

Proof. Assume the contrary and let $\zeta(X)$ be the root of an irreducible polynomial $f \in k[X,Y]$. Consider the algebraically closed field $k((X^{\mathbb{Q}^d_{\succcurlyeq 0}}))$, (see [Bou64], Chap. 6, Section 3, n° 4, Exemple 6). We have $\zeta(X) \in k((X^{\mathbb{Q}^d_{\succcurlyeq 0}}))$. In the sequence λ_r of the characteristic exponents the denominators tend to infinity. Therefore, there is an index i such that the denominators of $\lambda_{r,i}$ tend to infinity with r. We can assume that this index is d. Consider the algebraically closed field $k' = k((X^{'\mathbb{Q}^{d-1}_{\succcurlyeq 0}}))$, where $X^{'} = X_1, \dots, X_{d-1}$. We can regard $f(X,Y)$ as a polynomial in the ring $k'[X_d, Y]$ and $\zeta(X)$ as an element of the ring $k'[[X_d^{Q_{\succcurlyeq 0}}]]$. By the Newton-Puiseux theorem all the roots of $f(X,Y)$ are in the ring $\widetilde{k'[[X_d]]}$. It implies that $\zeta(X) \in \widetilde{k'[[X_d]]}$ which is absurd. $\square$

A variant of this proof gives us the following statement: Given any $f \in k[X,Y]$, there does not exist a root $\eta(X) \in k[[X^{Q^d_{\succcurlyeq 0}}]]$ of f, such that the denominators of the terms of η tend to infinity (By denominator of a term $c_\beta X^\beta$ of η we mean: the least natural number n such that $n.\beta \in \mathbb{N}^d$.).

We introduce a sequence of quasi-ordinary hypersurfaces $f^{(i)}$, which approximates the original element $\zeta(X)$.

Definition 2.3 Set $f^{(0)}(X,Y) = Y$, and for any $i \in \mathbb{N}$ define an irreducible quasi-ordinary hypersurface $f^{(i)}(X,Y) \in k[[X]][Y]$ (for the definition of the quasi-ordinary singularities see [Lip83] and for the irreducibility see [GP03]) by the following parametrization:

$$Y = \zeta^{(i)}(X) = \sum_{j=1}^{i} p_j + p^{(i)},$$

where $\frac{p_i + p^{(i)}}{X^{\lambda_i}} \in k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$ and the exponents of the monomials of $p^{(i)}$ are in Q_i, and the first exponent of $p^{(i)}$ is greater than λ_{i+1}.

Definition 2.4 We define for $1 \leq j \leq i \in \mathbb{N} : n_j = [Q_j : Q_{j-1}]$ and $m^{(0)} = 1,\ m^{(i)} = n_1 \dots n_i$. It can be proved that $m^{(i)} = \deg_Y(f^{(i)})$ (see [GP03] or [Lip88]). Moreover, we define the following vectors (originally defined and studied in [GP00]):

$$\gamma_1 = \lambda_1,\ \gamma_j = n_{j-1}\gamma_{j-1} + \lambda_j - \lambda_{j-1},\ j > 1.$$

By $R(f)$, for a quasi-ordinary f, we mean the set of the roots of f in $\widetilde{k[[X]]}$. Following [PP01], we define the notion of the intersection index of two "comparable" quasi-ordinary hypersurfaces.

Definition 2.5 For any two quasi-ordinary hypersurfaces f, g, we say that they are comparable if for any $\eta \in R(f)$ and $\mu \in R(g)$ we have $\eta - \mu = X^\alpha.unit$, where $\alpha \in \mathbb{Q}^d_{\geq 0}$. The intersection index of two such hypersurfaces is defined as follows:

$$(f,g) = \upsilon_X(\mathrm{Res}_Y(f,g)) \in \mathbb{Z}^d.$$

For any two arbitrary root $\eta \in R(f)$ and $\xi \in R(g)$ of two irreducible comparable quasi-ordinary hypersurfaces the coincidence order of η and ξ is by definition the vector $\kappa(\eta,\xi) = \nu_X(\eta - \xi) \in \mathbb{Q}^d_{\geq 0}$. The exponent of contact of such f and g is defined as follows:

$$\kappa(f,g) = max\{\kappa(\eta,\xi), \eta \in R(f), \xi \in R(g)\}.$$

Proposition 2.6 [PP01] *Let g be an irreducible unitary quasi-ordinary hypersurface which is comparable with $f^{(i)}$. We have:*

$$\frac{(f^{(i)},g)}{deg(f^{(i)}).deg(g)} = \frac{\gamma_{i_\kappa}}{n_1 \dots n_{i_\kappa - 1}} + \frac{\kappa - \lambda_{i_\kappa}}{n_1 \dots n_{i_\kappa}}.$$

Here κ is the exponent of contact of $f^{(i)}$ and g. Note that κ is an exponent in the parametrization of $f^{(i)}$, and i_κ is the index of the greatest characteristic exponent λ_j of $f^{(i)}$ such that $\lambda_j \leq \kappa$.

We recall the notion of the semi-roots in our context:

Definition 2.7 We say that $g \in k[[X]][Y]$ is a $j^{th}-$semi-root of $f^{(i)}$, $0 \leq j \leq i$, if the following two conditions are satisfied:
a) $g(0, Y) = Y^{n_1 \dots n_j}$.
b) $g(X, \zeta^{(i)}(X)) = X^{\gamma_{j+1}} \varepsilon_j^{(i)}$, where $\varepsilon_j^{(i)}$ is a unit in $\widetilde{k[[X]]}$.

We have the following lemma (see also [GP00]):

Lemma 2.8 *For any $j \leq i \in \mathbb{N}$, the quasi-ordinary singularity $f^{(j)}$ is a j^{th}-semi-root of $f^{(i)}$.*

Proof. In the case $j = 0$, by definition we have $f^{(0)}(X, Y) = Y$. This gives $f^{(0)}(X, \zeta^{(i)}(X)) = \zeta^{(i)}(X) = X^{\gamma_1}.unit$. For $j > 0$, we use Proposition 2.6. Here $i_\kappa = j+1$, and we have $\frac{(f^{(i)}, f^{(j)})}{\deg(f^{(i)}).\deg(f^{(j)})} = \frac{\gamma_{j+1}}{n_1 \dots n_j}$. We notice that $\deg(f^{(j)}) = n_1 \dots n_j$, which shows that $(f^{(i)}, f^{(j)}) = m^{(i)} \gamma_{j+1}$. □

We need another result (see [GP03] and [PP01]) which allows a $(f^{(0)}, \dots, f^{(i)})-adic$ representation of any element of $k[X, Y]$.

Lemma 2.9 *Given $g \in \mathbb{C}[[X]][Y]$, there exists i_0 such that for $i \geq i_0$, g can be uniquely written as a finite sum $g = \sum c_{l_0 \dots l_i} (f^{(0)})^{l_0} \dots (f^{(i)})^{l_i}$, with $c_{l_0 \dots l_i} \in \mathbb{C}[[X]]$, the $(i+1)-$tuples $(l_0 \dots l_i) \in \mathbb{N}^{i+1}$ verifying $0 \leq l_r \leq n_{r+1} - 1$, for all $r \in \{0, \dots, i\}$.*

Proof. ([PP04]) Make the Euclidean division of g by $f^{(i)}$, by induction we get the $f^{(i)} - adic$ representation of g which is of the form $g = \sum c_{l_i} (f^{(i)})^{l_i}$. Then iterate this process on the coefficients, making at each step the $f^{(j-1)} - adic$ expansions of the coefficients $c_{l_j, \dots, l_i}$. This gives us the claimed *adic* representation. The uniqueness comes from the fact that the $Y-$degrees of the terms $c_{l_0 \dots l_i} (f^{(0)})^{l_0} \dots (f^{(i)})^{l_i}$ are pairwise distinct (see Lemma 7.2 of [PP04]). The only thing which remains to prove is the inequality $0 \leq l_i \leq n_{i+1} - 1$. This is because if i is chosen so large that $m^{(i)} > \deg_Y(g)$, then $f^{(i)}$ (which is of degree $m^{(i)}$) can not appear in the expansion of g, i.e., $l_i = 0$. So, we choose i_0 to be the least i such that $m^{(i)} > \deg_Y(g)$. □

The preceding expansion is called the $(f^{(0)}, \dots, f^{(i)})-adic$, expansion of g. The finite set $\{(l_0 \dots l_i), c_{l_0 \dots l_i} \neq 0\}$ is called the $(f^{(0)}, \dots, f^{(i)}) - adic$ support of g. We set $(f_{[i]}) = (f^{(0)}, \dots, f^{(i)})$ so we can speak of the $(f_{[i]}) - adic$ expansion of an element. We write $c_\ell (f_{[i]})^\ell$ for $c_{l_0 \dots l_i} (f^{(0)})^{l_0} \dots (f^{(i)})^{l_i}$. For a fixed set of functions $\{g_1, \dots, g_n\}$ the next lemma says that for sufficiently large values of i and arbitrary $j \in \mathbb{N}$ the $(f_{[i]}) - adic$ expansion of each g_k is the same as its $(f_{[i+j]}) - adic$ expansion, so in this case for sufficiently large values of i we can speak of $(f_{[\infty]}) - adic$ expansion of g_k's. For example note that the $(f_{[\infty]}) - adic$ expansion of $f^{(i)}$ is itself.

Lemma 2.10 *With the notations of the last lemma, for sufficiently large values of i and any $j \in \mathbb{N}$ the $(f_{[i]}) - adic$ expansion of g and $(f_{[i+j]}) - adic$ expansion of g coincide.*

Proof. For the i_0 chosen in the proof of the last lemma, we have for any $j \geq 0$, $l_{i_0 + j} = 0$. □

Definition 2.11 For any element $\eta \in \widetilde{k[[X]]}$, we define its Newton polyhedron $\mathcal{N}_X(\eta)$ to be the convex hull in $\mathbb{R}^d$ of the set $\mathrm{Supp}_X(\eta) + \mathbb{R}^d_{\geq 0}$, where $\mathrm{Supp}_X(\eta)$ denotes the support of η as a series in the variables X.

The expansion of Lemma 2.10 allows us to compute in an effective way the Newton polyhedron of $g(\zeta)$, where ζ is a root of $f^{(i)} = 0$ (We write $R(f)$ for the set of roots of $f = 0$). This computation is explained by the following two lemmas of [PP01]:

Lemma 2.12 *If $g = \sum c_\ell (f_{[i]})^\ell$, is the $(f_{[i]})-adic$ expansion of $g \in k[[X]][Y]$, then for every $\zeta \in R(f)$, the sets of vertices of the Newton polyhedra $\mathcal{N}_X(c_\ell (f_{[i]})^\ell)$, for varying ℓ, are pairwise disjoint.*

Lemma 2.13 *If $g_1, \dots, g_i \in \widetilde{k[[X]]}$ and the sets of vertices of Newton polyhedra $\mathcal{N}_X(g_1), \dots, \mathcal{N}_X(g_i)$ are pairwise disjoint, then $\mathcal{N}_X(g_1 + \dots + g_i)$ is the convex hull of the union of $\mathcal{N}_X(g_1) \bigcup \dots \bigcup \mathcal{N}_X(g_i)$. In particular, each vertex of $\mathcal{N}_X(g_1 + \dots + g_i)$ is a vertex of one of the polyhedra $\mathcal{N}_X(g_1), \dots, \mathcal{N}_X(g_i)$.*

3 The ordering and the semigroup

Definition 3.1 We associate to $\zeta \in k[[X^{\mathbb{Q}^d_{\geq 0}}]]$, satisfying the conditions of the Definition 2.1, the sequence of the semigroups:

$$\Gamma_i = \mathbb{Z}^d_{\geq 0} + \gamma_1.\mathbb{Z}_{\geq 0} + \ldots + \gamma_i.\mathbb{Z}_{\geq 0}, \ \textit{for } i \in \mathbb{N}.$$

And the semigroup:

$$\Gamma_\zeta = \mathbb{Z}^d_{\geq 0} + \gamma_1.\mathbb{Z}_{\geq 0} + \gamma_2.\mathbb{Z}_{\geq 0} + \ldots.$$

Later, when we attach to the element ζ the valuation ν we will see that:

$$\nu(k[X,Y] \setminus 0) = \Gamma_\zeta.$$

We need the following two lemmas from [GP00]:

Lemma 3.2 *1) The order of the image of γ_j in the group $\frac{Q_j}{Q_{j-1}}$ (see Definition 2.1) is equal to n_j for $j \in \mathbb{N}$.*
2) We have $\gamma_j > n_{j-1}\gamma_{j-1}$, for $j \geq 2$.
3)The vector $n_j\gamma_j$ belongs to the semigroup Γ_{j-1} ($j \in \mathbb{N}$). Moreover, we have a unique relation:

$$n_j\gamma_j = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \ldots + l_{j-1}^{(j)}\gamma_{j-1}$$

such that $0 \leq l_k^{(j)} \leq n_k - 1$, and $\alpha^{(j)} \in \mathbb{Z}^d_{\geq 0}$, for $j \in \mathbb{N}$.

Lemma 3.3 *For any $j \in \mathbb{N}$ the $(f_{[\infty]})-$adic expansion of $(f^{(j-1)})^{n_j}$ is of the following form:*

$$(f^{(j-1)})^{n_j} = c_j f^{(j)} + \sum c^{(j)}_{l_0,\ldots,l_{j-1}} (f^{(0)})^{l_0}(f^{(1)})^{l_1} \ldots (f^{(j-1)})^{l_{j-1}},$$

where $c_j \in k^*$. We have $0 \leq l_r \leq n_{r+1} - 1$, for $r = 0, \ldots, j-1$. The coefficient $c^{(j)}_{l_1^{(j)},\ldots,l_{j-1}^{(j)},0}$ appears, and it is of the form $X^{\alpha^{(j)}}.unit$, where the integers $l_1^{(j)}, \ldots, l_{j-1}^{(j)}$ and the exponent $\alpha^{(j)}$ are given in Lemma 3.2. Moreover, if $X^{\alpha'}$ appears on the coefficient $c^{(j)}_{l_0,\ldots,l_{j-1}}$ then:

$$n_j\gamma_j \leq \alpha' + l_0\gamma_1 + \ldots + l_{j-1}\gamma_j,$$

and equality holds iff $(l_0, \ldots, l_{j-1}) = (l_1^{(j)}, \ldots, l_{j-1}^{(j)}, 0)$.

In order to define the valuation we need to fix a total well-ordering on $\mathbb{Z}^d$ which extends to a total ordering on Γ_ζ. This ordering should verify certain conditions.

Definition 3.4 We say a total ordering $\preccurlyeq$ on $\mathbb{Q}^d$ is a good ordering if:

- It is a monomial ordering on $\mathbb{Q}^d$, i.e., for any $\gamma, \gamma', \gamma" \in \mathbb{Q}^d$ from $\gamma \prec \gamma'$ one has $\gamma + \gamma" \prec \gamma' + \gamma"$.
- It refines the partial ordering $\leq$ on $\mathbb{Q}^d$, i.e., if $u, v \in \mathbb{Q}^d$ and $u < v$ then $u \prec v$

The following proposition shows that every suitably chosen ordering on $\mathbb{Z}^d$ can be expanded to a good ordering on $\mathbb{Q}^d$.

Proposition 3.5 *Every monomial total ordering on $\mathbb{Z}^d$ which refines the partial ordering $\leq$ on $\mathbb{Z}^d$ can be expanded to a good ordering on $\mathbb{Q}^d$.*

Proof. Let $\preccurlyeq$ be such an ordering. Expand this ordering on $\mathbb{Q}^d$ as follows: For $\gamma, \gamma' \in \mathbb{Q}^d : \gamma \prec \gamma'$ iff there exists $n \in \mathbb{N}$ such that $n\gamma, n\gamma' \in \mathbb{Z}^d$ and $n\gamma \prec n\gamma'$. The next lemma shows that we have the following equivalent definition: We have $\gamma \prec \gamma'$ iff for any $n \in \mathbb{N}$ such that $n\gamma, n\gamma' \in \mathbb{Z}^d$ then $n\gamma \prec n\gamma'$. It is clear that $\preccurlyeq$ is a total ordering on $\mathbb{Q}^d$. We show that it is a monomial ordering. Suppose this is not the case. Then there is $\gamma, \gamma', \gamma'' \in \mathbb{Q}^d$ such that $\gamma \prec \gamma'$ but $\gamma + \gamma'' \not\prec \gamma' + \gamma''$ then $\gamma + \gamma'' \succ \gamma' + \gamma''$. By the next lemma and the definition, we can find an $n \in \mathbb{N}$ such that $n\gamma, n\gamma', \ n\gamma'' \in \mathbb{Z}^d$ and $n\gamma + n\gamma'' \succ n\gamma' + n\gamma''$. This gives $\gamma \succ \gamma'$, a contradiction.
This ordering refines the partial ordering on $\mathbb{Q}^d$. Let $\gamma < \gamma'$ and take a natural number n such that $n\gamma, n\gamma' \in \mathbb{Z}^d_{\geq 0}$. By definition of the good ordering $n\gamma \prec n\gamma'$. By the discussion in the first step of the proof we have $\gamma \prec \gamma'$. □

Lemma 3.6 *Let $\preccurlyeq$ be a monomial total ordering on $\mathbb{Z}^d$ which refines the partial ordering $\leq$ on $\mathbb{Z}^d$. For every $a, b \in \mathbb{Z}^d$, if $a \prec b$ then for any $p \in \mathbb{Q}_{\geq 0}$ such that $pa, pb \in \mathbb{Z}^d$ we have $pa \prec pb$.*

Proof. By the monomial ordering property for every $p \in \mathbb{N}$ we have $pa \prec pb$. It suffices to prove the lemma for p^{-1}, where $p \in \mathbb{N}$. If $p^{-1}a \succ p^{-1}b$ then $p.p^{-1}a \succ p.p^{-1}b$ so $a \succ b$, a contradiction. □

Remark 3.7 The ordering introduced in Proposition 3.5, is no longer a well-ordering on $\mathbb{Q}^d$. For example take the set $A = \{u_i = (1, \ldots, 1, \frac{1}{i})\}_{i=1}^{\infty}$. The property that $\preccurlyeq$ refines the partial ordering $\leq$ shows that the set A does not have a smallest element.

Here is a concrete example of a good ordering.

Example 3.8 Consider the $\leq_{d.lex.}$ ordering on $\mathbb{Z}^d$ which is defined as follows:
For any $a, b \in \mathbb{Z}^d$ we have $a <_{d.lex.} b$ iff $(\deg(a) = \sum_{i=1}^{d} a_i < \deg(b)$ or $(\deg(a) = \deg(b)$ and $a <_{lex.} b))$.
This ordering verifies all the conditions of Definition 3.4. It expands to a good ordering, denoted by $\leq_{d.lex.}$ on $\mathbb{Q}^d$.

One way to introduce a monomial ordering $\preccurlyeq$ on a group G is to introduce a subset of the group as the subset of the positive elements, $G_{\succ 0} = \{g \in G : 0 \prec g\}$. For example we have

$$G_{>_{d.lex.}0} = \{\mathbf{u} \in \mathbb{Q}^2 : u_1 + u_2 > 0\} \bigcup \{\mathbf{u} \in \mathbb{Q}^2 : u_1 > 0, u_1 + u_2 = 0\}.$$

Lemma 3.9 *Consider the ordering $\preccurlyeq$ on $\mathbb{Q}^d$. We have:*
1) *It refines the partial ordering $\leq$ iff $\mathbb{Q}^d_{>0} \subset \mathbb{Q}^d_{\succ 0}$.*
2) *It is a total ordering iff for any $\mathbf{u} \in \mathbb{Q}^d$: $\{\mathbf{u}, -\mathbf{u}\} \bigcap \mathbb{Q}^d_{\succ 0} \neq \varnothing$.*
3) *Its restriction on $\mathbb{Z}^d_{\geq 0}$ is a well-ordering iff this restriction refines the partial ordering $\leq$ on $\mathbb{Z}^d_{\geq 0}$.*

proof. The items 1) and 2) are easy to prove. For a proof of 3) we refer to [GP02]. □

As a corollary one can give another characterization of the good orderings.

Corollary 3.10 *The ordering $\preccurlyeq$ on $\mathbb{Q}^d$ is a good ordering if $\mathbb{Q}^d_{>0} \subset \mathbb{Q}^d_{\succ 0}$ and for any $\mathbf{u} \in \mathbb{Q}^d$ we have $\{\mathbf{u}, -\mathbf{u}\} \bigcap \mathbb{Q}^d_{\succ 0} \neq \varnothing$.*

As another corollary we can give another description of the construction given in Proposition 3.5.

Corollary 3.11 *Given a monomial well-ordering $\preccurlyeq$ on $\mathbb{Z}^d$. It has a natural expansion to a good ordering on $\mathbb{Q}^d$, which we denote it with the same notation. We define this expansion with the set of its positive elements: Consider the positive cone in $\mathbb{R}^d$ based on the set of positive elements of $\preccurlyeq$ in $\mathbb{Z}^d$. The set of positive elements will be the intersection of this cone with $\mathbb{Q}^d$. Moreover, this expansion coincides with the expansion defined in Proposition 3.5.*

Definition 3.12 For any two orderings $\preccurlyeq$ and $\preccurlyeq'$ on a group G, we define the set

$$G_+(\preccurlyeq, \preccurlyeq') = (G_{\succ 0} - G_{\succ' 0}) \bigcup (G_{\succ' 0} - G_{\succ 0}).$$

We say the sequence $\{\preccurlyeq_k\}_{k=1}^{\infty}$ of orderings on the group G converges to the ordering $\preccurlyeq$ iff

$$G_+(\preccurlyeq_1, \preccurlyeq) \supset G_+(\preccurlyeq_2, \preccurlyeq) \supset \ldots \text{ and } \bigcap_{k=1}^{\infty} G_+(\preccurlyeq_k, \preccurlyeq) = \varnothing.$$

In this case we write $\lim_{k \to \infty} \preccurlyeq_k = \preccurlyeq$.

Example 3.13 For any $\omega \in \mathbb{R}_{>0}$ define a good ordering $\preccurlyeq_\omega$ on $\mathbb{Q}^2$ by

$$\mathbb{Q}^2_{\succ_\omega 0} = \{\mathbf{u} \in \mathbb{Q}^2 : u_1 + \omega.u_2 > 0\} \bigcup \{\mathbf{u} \in \mathbb{Q}^2 : u_1 + \omega.u_2 = 0, u_1 > 0\}.$$

One can easily prove that this ordering verifies the conditions of the last corollary and it is a good ordering.

Example 3.14 Take a sequence $\{\omega_r\}_{r=1}^{\infty}$ of positive irrational numbers that are increasing and convergent to -1. According to the last example, construct the sequence of orderings $\{\preccurlyeq_{\omega_r}\}_{r=1}^{\infty}$. This is a sequence of good orderings. Then it is easily seen that

$$\lim_{r\to\infty} \preccurlyeq_{\omega_r} = \leq_{d.lex.} .$$

It is interesting to note that $\mathbb{Q}^2$ with the ordering $\preccurlyeq_{\omega_r}$ does not have non-trivial isolated subgroups. In contrary if G is such an isolated subgroup then take $0 \prec_{\omega_r} g \in G$. The group G should contain all the rational points in the section between the line joining the origin to the point g, in the plane, and the line $u_1 + \omega.u_2 = 0$. The group generated by this last set is $\mathbb{Q}^2$. In Example 4.6 we see that $\mathbb{Q}^2$ with ordering $\leq_{d.lex.}$ has a nontrivial isolated subgroup. As a result we have constructed a sequence of orderings on $\mathbb{Q}^2$ with $rank(\mathbb{Q}^2_{\preccurlyeq_{\omega_r}}) = 1$ which converges to the ordering $\leq_{d.lex.}$ with $rank(\mathbb{Q}^2_{\leq_{d.lex.}}) = 2$.

Alternatively, in the above example one could take the ω_r's to be rational numbers and define the same constructions and the same limit. Everything is the same as the argument given in Example 4.6 except that $rank(\mathbb{Q}^2_{\preccurlyeq_{\omega_r}}) = 2$.

4 The valuation and the examples

Given any good ordering $\preccurlyeq$, we define the ring $k[[X^{Q^d_{\succcurlyeq 0}}]]$, which is the ring of power series $z(X) \in k[[X^{Q^d_{\succcurlyeq 0}}]]$, in which the set of exponents are well-ordered with respect to $\preccurlyeq$. This is in fact a valuation ring (see [Bou64], Chap. 6, Section 3, n° 4, Exemple 6). We denote this valuation by ν.

Lemma 4.1 *There is an injective morphism of the rings*

$$\begin{array}{rl} \Theta_\zeta : k[X,Y] & \hookrightarrow k[[X^{Q^d_{\succcurlyeq 0}}]] \\ X & \mapsto X \\ Y & \mapsto \zeta(X). \end{array}$$

Proof. This is clearly a morphism, the injectivity is a result of Proposition 2.2. □

Now, we define the valuation induced by the transcendental element $\zeta(X)$ on the ring $k[X,Y]$, with respect to a good ordering, $\preccurlyeq$, fixed on $\mathbb{Q}^d$:

Definition 4.2 We define a mapping $\nu : k[X,Y] \setminus \{0\} \longrightarrow \mathbb{Q}^d_{\geq 0}$ by:

$$\nu(f) = \nu(\Theta_\zeta(f)).$$

This mapping is a valuation on the ring $k[X,Y]$.

The next proposition shows that this valuation is approximated by the intersection indices of the quasi-ordinary hypersurfaces $f^{(i)}$.

Proposition 4.3 *For any unitary irreducible quasi-ordinary $g \in k[[X]][Y]$, which is comparable with $f^{(i)}$'s, we have:*

$$\nu(g) = \lim_{i\to\infty} \frac{(f^{(i)},\ g)}{deg_Y(f^{(i)})}.$$

Proof. We notice that if i is chosen so large that $\kappa < \lambda_i$ (with the notations of the Proposition 2.6) then for any $j > i$ we have:

$$\frac{(f^{(i)}, g)}{\deg_Y(g).\deg_Y(f^{(i)})} = \frac{(f^{(j)}, g)}{\deg_Y(g).\deg_Y(f^{(j)})}.$$

As a result, the limit is well defined. For the equality, it suffices to note that:

$$\mathcal{N}(g(X,\zeta^{(i)}(X))) = \mathcal{N}(\prod_{r=1}^{m^{(i)}} g(\zeta_r^{(i)})) = \deg_Y(f^{(i)}).\mathcal{N}(g(\zeta^{(i)})) = \mathcal{N}(\mathrm{Res}_Y(f^{(i)}, g)),$$

where $\zeta_r^{(i)}$'s are all the $m^{(i)}$ roots of $f^{(i)} = 0$. $\square$

The following proposition gives an effective way to compute the value $\nu(g)$, for an arbitrary $g \in k[X,Y]$. It also gives essentially another definition of this valuation. We extend the definition of ν to the ring $k[[X]][Y]$ by the same formula.

Proposition 4.4 *We have:*
1) *For any $g \in k[X,Y]$, with the $(f_{[\infty]}) - adic$ expansion $g = \sum c_\ell (f_{[\infty]})^\ell$, the values of the monomials of the $(f_{[\infty]}) - adic$ expansion of g are distinct elements of $\mathbb{Q}^d_{\geq 0}$. Therefore, we have:*

$$\nu(g) = min_\ell\{\nu(c_\ell(f_{[\infty]})^\ell)\}.$$

2) *We have:*

$$\nu(f^{(i)}) = \gamma_{i+1}.$$

3) *We have:*

$$\nu((f^{(j-1)})^{n_j}) = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \ldots + l_{j-1}^{(j)}\gamma_{j-1},$$

where the $l_k^{(j)}$'s and $\alpha^{(j)}$ are defined in the Lemma 3.2. Moreover, there is exactly one term in the $(f_{[\infty]}) - adic$ expansion of $(f^{(j-1)})^{n_j}$ with this value, if ℓ_* is the index of this term then $\ell_* = (l_1^{(j)}, \ldots, l_{j-1}^{(j)}, 0)$.

proof. The first claim is a direct consequence of Lemma 2.12 and the properties of the good orderings. The second one is a consequence of Proposition 4.3. The third one is a consequence of the last step and Lemma 3.3. Alternatively, we can prove the third result directly and as a consequence, yield another proof of Lemma 3.3; We note that by Lemma 2.8, we have $\mathcal{N}((f^{(j-1)})^{n_j}) = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \ldots + l_{j-1}^{(j)}\gamma_{j-1} + \mathbb{R}^d_{\geq 0}$, which gives the first claim of 3). By 1) there is a unique term, say with index ℓ_*, in the $(f_{[\infty]}) - adic$ expansion of $(f^{(j-1)})^{n_j}$ such that $\nu((f^{(j-1)})^{n_j}) = \nu(c_{\ell_*}(f_{[\infty]})^{\ell_*}) = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \ldots + l_{j-1}^{(j)}\gamma_{j-1}$. Using the uniqueness of the representation of the elements of Γ_{j-1}, one can show that ℓ_* is of the claimed form. $\square$

We note that the monomial which appears in the first case of the above proposition is not necessarily a vertex of the Newton polyhedron of $g(\zeta)$.

Corollary 4.5 *The semigroup $\nu(k[X,Y] \setminus 0)$ of the valuation is equal to Γ_ζ. The value group is equal to the subgroup of $\mathbb{Q}^d$ generated by Γ_ζ. We denote this value group by Φ_ζ.*

The next example shows that for suitably chosen ζ the value group will be $\mathbb{Q}^d$. In order to simplify the notations, the example is stated in the case $d = 2$.

Example 4.6 In the set of natural numbers start from $s_1 = 2$ and pick up all the numbers that are power of a prime. Denote by $\{s_i\}_{i=1}^\infty$ the resulting sequence. The first elements are:

$$s_1 = 2, s_2 = 3, s_3 = 4, s_4 = 5, s_5 = 7, s_6 = 8, \ldots.$$

We define:

$$\gamma_1 = (\frac{1}{s_1}, 1), \gamma_2 = (s_2, s_2 + \frac{1}{s_1}),$$

$$\text{and for } i \geq 1 : \begin{cases} \gamma_{2i+1} = (s_2 \ldots s_{2i+1} + \frac{1}{s_{i+1}}, s_2 \ldots s_{2i+1}) \\ \gamma_{2i+2} = (s_2 \ldots s_{2i+2}, s_2 \ldots s_{2i+2} + \frac{1}{s_{i+1}}). \end{cases}$$

One then defines the exponents λ_i's using the inductive formula of Definition 2.4. These λ_i's satisfy the conditions of Definition 2.1: By the construction and the computation of n_i's, which is given in the following, we have $\gamma_j > n_{j-1}\gamma_{j-1}$. This last inequality gives us $\lambda_j > \lambda_{j-1}$. The condition $\lambda_j \notin Q_{j-1}$ is a consequence of the fact that the components of the elements of Q_{j-1} have, as denominators, only $s_1, \ldots, s_{j-1}$. When s_i is a power of the prime p, we have $n_i = p$. As a result $m^{(i)} = \Pi_q q^{\alpha_q}$, where q runs through all the primes less than or equal to s_i and α_q is by definition the greatest power of q such that $q^{\alpha_q} \leq s_i$. By Definition 4.2, the series $\zeta(X) = \sum X^{\lambda_i}$ defines a valuation of $k[X,Y]$.

We see, by induction, that $(\frac{1}{s_i}, 1), (1, \frac{1}{s_i})$ are in the value group Φ_ζ of this valuation. Therefore, by definition of s_i's we have $\Phi_\zeta = \mathbb{Q}^2$. If we give $\mathbb{Q}^2$ the order $\leq_{d.lex.}$, this valuation is of rank two: Define $G = \{(a, -a) : a \in \mathbb{Q}\}$, this is a subgroup of $\mathbb{Q}^2$. It is an isolated subgroup (see [Bou64] for the definition of the isolated subgroups and its relation to the rank of a valuation), since if we take an arbitrary element $0 <_{d.lex.} (a, -a) \in G$ then for any $\mathbf{u} = (u_1, u_2) \in \mathbb{Q}^2$ from $0 <_{d.lex.} \mathbf{u} <_{d.lex.} (a, -a)$ we deduce $deg(\mathbf{u}) = 0$ and then $\mathbf{u} \in G$.

Remark 4.7 Consider the sequence of orderings introduced in Example 3.14. If we denote the semigroups that are attached to the valuations associated by the above example to each of these orderings by $\Gamma_{\zeta, \preccurlyeq_{\omega_r}}$ then as the choice of good ordering does not have any effect on the resulting semigroup, we have $\Gamma_{\zeta, \preccurlyeq_{\omega_r}} = \Gamma_{\zeta, \leq_{d.lex.}}$. Therefore, we have a sequence of orderings which converge to another one. All of these orderings impose the same semigroup but the dimension of the valuation ring for the elements of the sequence is one and the dimension of the valuation ring to which they converge is two.

Example 4.8 We generalize an example of Zariski in [Zar39] and Example 4.22 of [Tei03]. Take $c_1, \ldots, c_d \in \mathbb{N} \bigcup \{\infty\}$ such that at least one of them is ∞ and take d sequence of natural numbers $\{s_j^{(q)}\}_{j=1}^{c_q}$, where $s_j^{(q)} > 1$ (for $q = 1, \ldots, d$), and complete these sequences by setting $s_{c_q+j}^{(q)} = 1$, for $j = 0, \ldots$. Define the following vectors:

$$\gamma_1 = \gamma_0 + \frac{1}{s_1^{(1)}} e_1,$$

Now, for $i \in \mathbb{N}$ set $i = dj + l$, where $j \in \mathbb{N} \bigcup \{0\}$ and $l = 2, \ldots, d+1$ then define:

$$\gamma_i = s_{j+1}^{(l-1)} \gamma_{i-1} + \frac{1}{s_1^{(l)} \ldots s_{j+1}^{(l)}} e_l,$$

where γ_0 is an arbitrary element of $\mathbb{Z}_{\geq 0}^d$ and the e_l's are the transposes of the standard basis of the vector space $\mathbb{Q}^d$. By the definition of the γ_i's it is clear that $n_i = s_{j+1}^{(l)}$. Drop the γ_i's for which $n_i = 1$. As the above example construct the vectors λ_i's. We have $\gamma_i - n_{i-1}\gamma_{i-1} = \frac{1}{s_1^{(l)} \ldots s_{j+1}^{(l)}} e_l > 0$, therefore $\lambda_i > \lambda_{i-1}$, and λ_i is not in the group Q_{i-1} of the Definition 2.4. Consider the element $\zeta = \sum X^{\lambda_i}$, and the valuation attached to it by Definition 4.2. We see, by induction, that $\frac{1}{s_1^{(l)} \ldots s_{j_l}^{(l)}} e_l$ is in the value group of this valuation, Φ_ζ. Therefore, we have:

$$\Phi_\zeta = \{(\frac{p_1}{s_1^{(1)} \ldots s_{j_1}^{(1)}}, \ldots, \frac{p_d}{s_1^{(d)} \ldots s_{j_d}^{(d)}}) : p_1, \ldots, p_d \in \mathbb{Z}, j_1 \leq c_1, \ldots, j_d \leq c_d\}.$$

If we set $s_j^q = j$, for $q = 1, \ldots, d$ and $j \in \mathbb{N}$, the resulting value group is $\Phi_\zeta = \mathbb{Q}^d$.

One may ask whether concerning the value groups the last example is the general situation? More precisely, let ζ be an element which verifies the conditions of Definition 2.1 and consider the valuation induced by it, as in Definition 4.2, with value group Φ_ζ. Does there exist another element $\zeta^{'}$ which comes from the construction of Example 4.8 such that $\Phi_\zeta = \Phi_{\zeta^{'}}$? The answer is no if $d \geq 2$. Here is an example:

Example 4.9 Let $\overline{e} = e_1 + \ldots + e_d$, where e_k's are the standard vectors of the vector space $\mathbb{Q}^d$. For $i \in \mathbb{N}$ we set:

$$\gamma_0 = \overline{e}, \gamma_i = 2\gamma_{i-1} + \frac{1}{2^i}\overline{e},$$

As in the last two examples construct the vectors λ_i. One can show that these vectors verify the conditions of Definition 2.1 (here $n_i = 2$). So, we can consider the element ζ attached to them. We show there is no element $\zeta^{'}$, which comes from a construction as in Example 4.8, such that $\Phi_\zeta = \Phi_{\zeta^{'}}$. In contrary, let $\zeta^{'}$ be such an element and consider the first vector of the construction of $\zeta^{'}$, in Example 4.8, i.e., $\gamma_1' = \gamma_0' + \frac{1}{r} e_1$, where $\gamma_0' \in \mathbb{Z}_{\geq 0}^d$ and $r \in \mathbb{N} \backslash \{1\}$. Then we have $\gamma_1' \in \Phi_\zeta$ which implies that there exists a natural number n and integers $a_1, \ldots, a_n$ and a vector $\mathbf{b} \in \mathbb{Z}^d$ such that: $\sum_{j=1}^n a_j . \gamma_j + \mathbf{b} = \gamma_1'$.

The γ_i's can be written in the form: $\gamma_i = h_i\overline{e} + \frac{l_i}{2^i}\overline{e}$, $h_i, l_i \in \mathbb{N}$, where l_i is an odd number and $l_i < 2^i$. So, the above equation implies: $\frac{1}{r}e_1 - p\overline{e} \in \mathbb{Z}^d$, where $p = \sum_{j=1}^{n} \frac{a_j l_j}{2^j} \in \mathbb{Q}$. When $d > 1$ this implies that $p, p - \frac{1}{r} \in \mathbb{Z}$, which is impossible. In fact, the semigroup Φ_ζ, can be given explicitly as follows:

$$\Phi_\zeta = \{\mathbf{b} + \frac{a_i}{2^i}\overline{e} : \mathbf{b} \in \mathbb{Z}^d, a_i \in \mathbb{Z}, i \in \mathbb{N}\}.$$

For $d = 1$, there will be no contradiction. Because in this case $\overline{e} = e_1$, therefore, $\frac{1}{r}e_1 - p\overline{e} \in \mathbb{Z}^d$ only implies $p - \frac{1}{r} \in \mathbb{Z}$. We can construct the value group which it generates via the construction of Example 4.8. It suffices to set $s_j^{(1)} = 2$, for $j \in \mathbb{N}$.

On the other hand, the following proposition shows that the transcendental elements are general enough to produce any totally ordered group G of rational rank d.

Proposition 4.10 *Suppose G is a totally ordered group of rational rank d with an ordering $\preccurlyeq$ which refines the partial order $\leq$ on G for an arbitrary chosen embedding $G \subseteq \mathbb{Q}^d$ (a good ordering on G). Then there is an element ζ which verifies the conditions of Definition 2.1, such that either $G = \Phi_\zeta$, or for some i, $G = Q_i$ (Definition 2.1).*

Proof. By our assumption on G, we have $\mathbb{Z}^d \subseteq G \subseteq \mathbb{Q}^d$, notice that the inclusion $\mathbb{Z}^d \subseteq \mathbb{Q}^d$ is not necessarily canonical one, however, we do not need the canonicity. Consider a set of generators of G, say $S = \{s_i\}_{i=1}^u$, such that $S \subset \mathbb{Q}^d_{\geq 0}$, where $u \in \mathbb{N}\bigcup\{\infty\}$. Let s'_1 be the first element of S which is not in $G_0 = \mathbb{Z}^d$ and set $\gamma_1 = s'_1, G_1 = G_0 + \mathbb{Z}\gamma_1, n_1 = [G_1 : G_0]$. Assume we have defined the elements $\{\gamma_j, s'_j, n_j, G_j\}_{j=1}^i$. Let s'_{i+1} be the first vector of $S \setminus \{s'_1, \ldots, s'_i\}$ which is not in G_i and set $\gamma_{i+1} = n_i\gamma_i + s'_{i+1}, G_{i+1} = G_i + \mathbb{Z}\gamma_{i+1}, n_{i+1} = [G_{i+1} : G_i]$. Either, this process terminates after finitely many steps, in this case complete the set of γ_i's arbitrarily subject to the conditions of Lemma 3.2, or, it goes on for ever. As in the examples above construct the vectors λ_i. Using the inductive formula of Definition 2.4 and $\gamma_{i+1} - n_i\gamma_i = s'_{i+1}$ we see that: $\lambda_i = s'_1 + \ldots + s'_i$. These vectors verify the conditions of Definition 2.1. Hence they define an element ζ, which is the desired element. $\square$

5 The sequence of key polynomials

In the last section, the construction of the valuation ν on the field $k(X, Y)$ was based on a given valuation, again denoted by ν, on the field $k(X)$. Moreover, this last valuation was induced by fixing a good ordering $\preccurlyeq$, defining the valuation ring $k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$ and considering the valuation induced on the field $k(X)$ from the inclusion $k(X) \subset k((X^{\mathbb{Q}^d_{\succcurlyeq 0}}))$. In this section we explain the relation between the construction of the valuation ν and MacLane's method to expand a given valuation ν on the field k' to the field $k'(Y)$ via a sequence of key polynomials ([Mac36]). In our case k' will be the field $k(X)$. In fact, we show that the quasi-ordinary hypersurfaces $f^{(i)}$'s which attached to the valuation ν are a sequence of key polynomials in MacLane's terminology (Theorem 5.5). In order to prove this result, we give another way of defining the valuations ν_i's (Proposition 5.6) which appear in the MacLane's construction and prove several properties of these valuations including their equivalence with MacLane's construction.

Throughout this section the value group Φ_ω is a sub-group of a totally ordered group G and the ordering on Φ_ω (as value group) is the same as that induced by the ordering on G. Consider an arbitrary valuation ω and a subring R of R_ω. We set $\Gamma = \omega(R \setminus \{0\}) \subset \Phi_{\omega_+} \bigcup\{0\}$; It is the semigroup of (R, ω). For $\phi \in \Phi_\omega$ set:

$$\mathcal{P}_\phi(R) = \{x \in R : \omega(x) \geq \phi\}$$
$$\mathcal{P}^+_\phi(R) = \{x \in R : \omega(x) > \phi\}.$$

The graded algebra associated to (R, ω) is defined by:

$$\mathrm{gr}_\omega R = \bigoplus_{\phi \in \Gamma} \frac{\mathcal{P}_\phi(R)}{\mathcal{P}^+_\phi(R)}.$$

It can be represented (see [Tei03], Proposition 4.1) as a quotient of an infinite dimensional polynomial ring by a binomial ideal, so it is "essentially toric" (see [Tei03], Subsection 4.2).

Definition 5.1 Given a valuation ω on a field k', and given a ring $R \subset R_\omega$, for any $a, b \in R$, we say they are equivalent if their image in $\mathrm{gr}_\omega R$ is the same. In this case we write $a \sim b$. We say b is equivalence-divisible in ω by a if there exists a $c \in R$ such that $b \sim ca$.

Definition 5.2 A key polynomial $\theta(Y) \neq 0$ for a valuation ω of $k'[Y]$ is one which satisfies the following conditions:

- *Irreducibility.* If a product is equivalence-divisible in ω by $\theta(Y)$ then one of the factors is equivalence-divisible by $\theta(Y)$.
- *Minimal degree.* Any non-zero polynomial equivalence-divisible in ω by $\theta(Y)$ has a degree in Y not less than the degree of $\theta(Y)$.
- The leading coefficient of $\theta(Y)$ is 1.

Using such key polynomials MacLane introduces a new valuation based on ω : If ω is a valuation of $k'[Y]$ and $\theta(Y)$ is a key polynomial over ω then choose an arbitrary element $\mu \in G$ such that $\mu > \omega(\theta)$ and set $\omega_1(\theta) = \mu$. For any element $g \in k'[Y]$ with the $\theta - adic$ expansion $g = \sum_i g_i \theta^i$ define:

$$\omega_1(g) = \min_i[\omega(g_i) + i\mu].$$

Theorem 5.3 ([Mac36]) With the notations above, the mapping ω_1 is a valuation on $k'[Y]$. The valuation ω_1 is called an augmented valuation and is denoted by

$$\omega_1 = [\omega, \omega_1(\theta) = \mu].$$

Definition 5.4 ([Mac36]) An ith stage inductive valuation ω_i is any valuation of $k'[Y]$ obtained by a sequence of valuations $\omega_0 = \omega, \omega_1, \ldots, \omega_i$, where for $j = 1, \ldots, i$ we have $\omega_j = [\omega_{j-1}, \omega_j = \mu_j]$. Furthermore, for $j = 2, \ldots, i$, the key polynomials θ_j must satisfy:

- $\theta_1(Y) = Y$
- $\deg\, \theta_j(Y) \geq \deg\, \theta_{j-1}(Y)$.
- $\theta_j(Y) \nsim \theta_{j-1}(Y)$ in ω_{j-1}.

We can symbolize this valuation thus:

$$\omega_i = [\omega_0, \omega_1(\theta_1) = \omega_1, \omega_2(\theta_2) = \mu_2, \ldots, \omega_i(\theta_i) = \mu_i].$$

In the special case that for any $g \in k'[Y]$ there exists some i such that for any $j \geq i$ we have $\omega_j(g) = \omega_i(g)$, one can define the limit augmented valuation:

$$\omega_\infty(g) = \lim_{i \to \infty} \omega_i(g).$$

The relation with the construction of the valuation ν of the last section is as follows:

Theorem 5.5 *Consider the valuation ν of the last section and suppose the transcendental element which is attached to this valuation (Definition 2.1) be ζ and the $f^{(i)}$'s be the quasi-ordinary hypersurfaces attached to it. Then the sequence $\{\theta_i = f^{(i-1)}\}_{i=1}^{\infty}$ is a sequence of key polynomials for the sequence of inductive valuations*

$$\nu_i = [\nu_0 = \nu, \nu_1(\theta_1) = \gamma_1, \nu_2(\theta_2) = \gamma_2, \ldots, \nu_i(\theta_i) = \gamma_i].$$

Moreover, the limit valuation $\lim_{i \to \infty} \nu_i(g)$ exists and is equal to ν. Here ν_0 is a valuation which comes from fixing a good ordering $\preccurlyeq$ on the group $\mathbb{Q}^d$.

We define the valuations ν_i's of Theorem 5.5 in another way which reflects the relation between different *adic* representations and also the relation between the valuations ν_i's and ν. Using the properties of this new definition we are able to prove Theorem 5.5.

Proposition 5.6 *Define the mapping $\nu_i : k[X,Y] \setminus 0 \to \mathbb{Q}^d$ as follows; For any $g \in k[X,Y]$, with the $(f_{[i-1]}) - adic$ expansion $g = \sum c_\ell (f_{[i-1]})^\ell$, set:*

$$\nu_i(g) = min_\ell\{\nu(c_\ell(f_{[i-1]})^\ell)\}.$$

1) *The mapping ν_i defines a valuation.*
2) *For any $j < i$, we have: $\nu_i(f^{(j)}) = \nu(f^{(j)})$.*
3) *For any $g \in k[X,Y]$, we have: $\nu_1(g) \preccurlyeq \nu_2(g) \preccurlyeq \ldots \preccurlyeq \nu(g)$. Moreover, for this g there exists an i such that $\nu_i(g) = \nu(g)$. Therefore, for any $j \geq i$, we have: $\nu_j(g) = \nu(g)$.*
4) *The value semigroup of ν_i is: $\nu_i(k[X,Y] \setminus 0) = \Gamma_i$.*
5) *The valuations ν_i's which are defined in this proposition are equal to the corresponding valuations defined in the Theorem 5.5.*

Proof. For 1), we show that for any $g, h \in k[X,Y] \setminus 0$ we have $\nu_i(g+h) \succcurlyeq \nu_i(g) + \nu_i(h)$ and $\nu_i(gh) = \nu_i(g) + \nu_i(h)$. The first one is a direct consequence of the definition and the uniqueness of the $(f_{[i-1]}) - adic$ representation. For the second one, we show that the monomials in the $(f_{[i-1]}) - adic$ representations of g and h, with minimum value, can not cancel each other in the product $g.h$, through the process of getting the $(f_{[i-1]}) - adic$ representation of $g.h$ from this product. Let $g = \sum_t u_t (f^{(i-1)})^{n_i.t}$ and $h = \sum_t u'_t (f^{(i-1)})^{n_i.t}$ be the unique representations of g and h in $\mathrm{gr}_{\nu_i} k[[X][Y]$, which comes from Lemma 5.9. Now, consider the product $g.h = \sum_{t''} \sum_{\substack{t,t' \\ t+t'=t''}} u_t.u'_{t'}(f^{(i-1)})^{n_i.t''}$. We do the replacements using Lemma 3.3, in each monomials of $g.h$, for those $f^{(j)}$'s that their power is greater than n_j, where $j < i-1$. By Lemma 5.10, such a replacement cannot change the power of $f^{(i-1)}$ of the uniquely generated monomial with minimal value in $\mathrm{gr}_{\nu_i} k[[X][Y]$. Therefore, these replacements for the unique minimum t''_0, which in turn refers to the unique minimums t_0 and t'_0, produces a monomial in the $(f_{[i-1]}) - adic$ representation of $g.h$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$ with value equal to $\nu_i(g) + \nu_i(h)$.
For 2), we note that it is a direct consequence of Proposition 4.3.
For 3), it is sufficient to note that we can write the $(f_{[i+1]}) - adic$ representation of an element from its $(f_{[i]}) - adic$ representation, using the equations given in Lemma 3.3. Moreover, in this process the value of the monomials in the representation can not decrease. As we noted earlier these equations do not change the minimum value.
The two last claims are clear. □

Remark 5.7 The comparison of the propositions 4.4, 4.3, and 5.6 gives us two interpretations of the fact that the valuation ν is the limit of valuations ν_i. The first by associating each ν_i to a specific truncation of the series $\zeta(X)$, the second by associating it to the *adic* expansion in terms of the $f^{(i)}$. The next section unify these interpretations.

Now, we can give a generalization of Proposition 4.10:

Corollary 5.8 *Given any totally ordered subgroup G of rational rank d, ordered by a good ordering, there is an element $\zeta(X)$ which verifies the conditions of Definition 2.1 such that for a unique $i \in \mathbb{N} \bigcup \{\infty\}$ we have $G = \Phi_{\nu_i}$, where ν_i's are those of Theorem 5.5.*

For the following two lemmas we use the notation of Theorem 5.5. Notice that $\theta_{[i]} = f_{[i-1]}$.

Lemma 5.9 *Let $g = \sum_\ell c_\ell (\theta_{[i]})^\ell$ be the $(\theta_{[i]}) - adic$ representation of $g \in k[X,Y]$. Set $\mathrm{in}_{\nu_i}(g) = \sum_{\ell'} c_{\ell'} (\theta_{[i]})^{\ell'}$ which are the monomials of the $(\theta_{[i]}) - adic$ representation of g that have minimum ν_i-value. Then the power of θ_i in these monomials is a power of n_i and for any $t \in \mathbb{N}$ there exists at most one monomial in $\mathrm{in}_{\nu_i}(g)$ such that the power of θ_i for it is $n_i.t$. In the other words, we can write*

$$\mathrm{in}_{\nu_i}(g) = \sum_t u_t \theta_i^{n_i.t},$$

where $t \in \mathbb{N} \bigcup \{0\}$. Here for every t there is a unique ℓ such that $n_i.t = \ell_i$ and $u_t.\theta_i^{n_i.t} = c_\ell.(\theta_{[i]})^\ell$.

Proof. It is sufficient to note that if $\nu_i(c_{\ell_1}(\theta_{[i]})^{\ell_1}) = \nu_i(c_{\ell_2}(\theta_{[i]})^{\ell_2})$ then $n_i \mid \ell_{1,i} - \ell_{2,i}$ and if $\ell_{1,i} = \ell_{2,i}$ then $\ell_1 = \ell_2$. □

Lemma 5.10 *Let $M = c_\ell(\theta_{[i]})^\ell$ be an arbitrary monomial. For an arbitrary $j < i$ with $\ell_j > n_j$, we replace $\theta_j^{n_j}$ by its adic expansion from Lemma 3.3. Let g be the resulting element then we have $\mathrm{in}_{\nu_i}(g) = c_{\ell'}(\theta_{[i]})^{\ell'}$, such that $\ell'_i = \ell_i$.*

Proof. It is sufficient to note that after replacement the monomials which change the power of θ_i have a greater ν_i-value than M. Moreover, there is exactly one unique monomial with minimal ν_i-value which is the same as the ν_i-value of M. □

Proposition 5.11 *With the notations of Theorem 5.5, the element θ_{i+1} is irreducible in $\mathrm{gr}_{\nu_i} k[[X]][Y]$.*

Proof. By Lemma 3.3, we have $c_{i+1}\theta_{i+1} = \theta_i^{n_i} - sX^{\alpha^{(i)}}(\theta_{[i-1]})^{l^{(i)}}$, for some $s \in k$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$. Suppose that $\theta_{i+1} = a.b$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$, for some $a, b \in k[X,Y]$. Then by Lemma 5.9, we have $a = \sum_{t=0}^{P} u_t.\theta_i^{n_i.t}$ and $b = \sum_{t=0}^{Q} u'_t\theta_i^{n_i.t}$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$. From $\nu_i(a) + \nu_i(b) = \nu_i(\theta_{i+1}) = n_i\gamma_i$ we deduce that $P + Q = 1$. Hence, without loose of generality, we can assume that $P = 1$ and $Q = 0$. But then $a.b = u_0u'_0 + u_1u'_0\theta_i^{n_i}$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$. By Lemma 5.10, the element $u_1u'_0$ is a unit in $\mathrm{gr}_{\nu_i} k[[X]][Y]$, therefore, b is a unit in $\mathrm{gr}_{\nu_i} k[[X]][Y]$. □

Proposition 5.12 *If $\theta_{i+1} \mid g$ in $gr_{\nu_i} k[[X]][Y]$ for some $g \in k[X,Y]$ then $\deg_Y(g) \geq \deg_Y(\theta_{i+1})$.*

Proof. We have $g = h\theta_{i+1}$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$ for some $h \in k[X,Y]$. By Lemma 5.10, we can write $g = \sum_{t=0}^{P} u_t.\theta_i^{n_i.t}$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$. Note that $\deg_Y(g) \geq \deg_Y(u_P) + n_i.P.\deg_Y(\theta_i)$. If $\deg_Y(g) < \deg_Y(\theta_{i+1}) = n_i.\deg_Y(\theta_i)$, we have two possibilities: Either, we have $P = 1$ and $u_1 = 1$, which is impossible because by Lemma 5.10, this implies that $h = 1$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$, or, we have $P = 1$; this is also impossible, because by Lemma 5.10, the product $h\theta_{i+1}$ is of the form $\sum_{t=0}^{Q} u'_t\theta_i^{n_i.t}$, such that $Q \geq 1$. □

Proof of Theorem 5.5. By induction, suppose that we have proved ν_i is a valuation. We prove that θ_{i+1} is a key polynomial for ν_i and then by Theorem 5.3 the mapping ν_{i+1} is a valuation. The irreducibility is a result of Proposition 5.11, the minimal degree property is a result of Proposition 5.12. Moreover, the sequence $\{\theta_i\}$ satisfies the conditions of Definition 5.4, hence, it is a sequence of key polynomials. Notice that the condition $\theta_{i+1} \nsim \theta_i$ (in ν_i) is a consequence of the fact that $\nu_i(\theta_{i+1}) = n_i\nu_i(\theta_i) \neq \nu_i(\theta_i)$. □

6 Specialization to the graded valuation ring

Through this section we fix an element $\zeta(X)$ as defined in Definition 2.1 and a sequence of elements $\zeta^{(k)}(X)$ attached to it (Definition 2.3). Following [Tei03], subsection 4.4 and [GP03], in this section we give a geometric interpretation of the construction of the valuation ν and the element $\zeta(X)$ attached to it. Take an infinite sequence of indeterminates $U = (U_1, U_2, \ldots)$. Consider the infinite dimensional space $\mathcal{A} = \mathrm{Spec}(k[[X]][U])$, this will play the role of a regular ambient space. Note that for every element $h \in k[[X]][U]$ there is an $i \in \mathbb{N}$ such that $h \in k[[X]][U_1, \ldots, U_i]$. We embed the variety $\mathcal{S} = \mathrm{Spec}(R)$, $R = k[[X]][\zeta(X)]$, in $\mathcal{A}$ and give natural equations for this embedding in terms of the relations given in Lemma 3.3. Moreover, we give an embedding of the quasi-ordinary hypersurfaces $f^{(r)}(X,Y) = 0$, defined in Definition 2.3, in the ambient space $\mathcal{A}$ such that the equations of this embedding come from truncating the equations of the embedding $\mathcal{S} \hookrightarrow \mathcal{A}$. A specialization of the variety $\mathcal{S}$ to the toric variety $\mathrm{Spec}(\mathrm{gr}_\nu R)$ (see [Tei03], subsection 4.2) will be given via a suitable filtration on the ring $k[[X]][U]$. This filtration is naturally induced from the valuation ν.

The embedding of $\mathcal{S}$ in $\mathcal{A}$ comes from the following morphism:

$$\begin{array}{rl} \Psi : k[[X]][U] & \rightarrow \; R \\ X & \mapsto X \\ U_i & \mapsto f^{(i-1)}(X, \zeta(X)). \end{array}$$

Note that Ψ is surjective, because $U_1 \mapsto f^{(0)}(X, \zeta(X)) = \zeta(X)$.

The valuation ν on $k[[X]][Y]$ (see Definition 4.2) induces a weight on any element of the ring $k[[X]][U]$: For any monomial $X^\beta U^\nu$ we define $\omega(X^\beta U^\sigma) = \nu(\Psi(X^\beta U^\sigma)) = \beta + \sum \sigma_i\gamma_i = \beta + \gamma.\sigma$. For any $\omega \in \Gamma_\zeta$ we define the ideal $\mathcal{I}_\omega$ (res. $\mathcal{I}_\omega^+$) of the ring $k[[X]][Y]$ which contains all the elements with weight greater than or equal to (res. strictly greater than) ω. The sequence of the ideals $\{\mathcal{I}_\omega\}_{\omega\in\Gamma_\zeta}$ is a filtration. Note that the ordering on the index set Γ_ζ of this sequence is the fixed good ordering defined the valuation ν.

Proposition 6.1 *The morphism Ψ induces a surjective morphism of $k[X]-$algebras:*

$$\begin{array}{rl} \mathrm{gr}\Psi : \mathrm{gr}_\omega k[[X]][U] = k[X,U] & \to \mathrm{gr}_\nu R = k[X^{\Gamma_\zeta}] \\ X & \mapsto X \\ U_i & \mapsto \overline{f^{(i-1)}(X,\zeta(X))}. \end{array}$$

Moreover, with the notations of Lemma 3.2, we have $ker(\mathrm{gr}\Psi) =< h_1, h_2, \dots >$, where:

$$\left\{ \begin{array}{lllll} h_1 & := & U_1^{n_1} & - & d_1 X^{\alpha^{(1)}}, \\ h_2 & := & U_2^{n_2} & - & d_2 X^{\alpha^{(2)}} U_1^{l_1^{(2)}}, \\ \dots & \dots & \dots & \dots & \dots \\ h_i & := & U_i^{n_i} & - & d_i X^{\alpha^{(i)}} U_1^{l_1^{(i)}} \dots U_{i-1}^{l_{i-1}^{(i)}}, \\ \dots & \dots & \dots & \dots & \dots \end{array} \right.$$

Proof. In coordinate free terms the morphism $\mathrm{gr}\Psi$ is defined by $\mathrm{gr}\Psi(\overline{a}) = \overline{\Psi(a)}$, for $a \in k[[X]][U]$. The equality $\mathrm{gr}_\omega k[[X]][U] = k[X,U]$ is clear from the definition of the filtration on $k[[X]][U]$ and the equality $\mathrm{gr}_\nu R = k[X^{\Gamma_\zeta}]$ follows from the Proposition 4.4. The proof of the Proposition 38 of [GP03] could be adapted to give a proof of the second part. $\square$

The above proposition shows that $\mathcal{Z}^{\Gamma_\zeta} := \mathrm{Spec}(k[X^{\Gamma_\zeta}])$ is embedded in the infinite dimensional space $\mathcal{A}$. Moreover, the equations defining this embedding are binomial. This is also a general fact, see [Tei03], section 4.

Proposition 6.2 *The kernel of the map $\Psi : k[[X]][U] \to R$ has the following generators:*

$$\left\{ \begin{array}{llllllllll} H_1 & := & U_1^{n_1} & - & d_1 X^{\alpha^{(1)}} & & + & c_1 U_2 & + & r_1(U_1), \\ H_2 & := & U_2^{n_2} & - & d_2 X^{\alpha^{(2)}} U_1^{l_1^{(2)}} & & + & c_2 U_3 & + & r_2(U_1,U_2), \\ \dots & \dots & \dots & \dots & \dots & & \dots & \dots & \dots & \dots \\ H_i & := & U_i^{n_i} & - & d_i X^{\alpha^{(i)}} U_1^{l_1^{(i)}} \dots U_{i-1}^{l_{i-1}^{(i)}} & & + & c_i U_{i+1} & + & r_i(U_1,\dots,U_i), \\ \dots & \dots & \dots & \dots & \dots & & \dots & \dots & \dots & \dots \end{array} \right.$$

for $i \in \mathbb{N}$. The elements c_i are defined in Lemma 3.3 and d_i's are defined in the previous proposition. For any $j \in \mathbb{N}$ the weight of a term $X^\beta U^\nu$ appearing in $r_j(U)$ is strictly greater than $n_j\gamma_j$. The terms appearing in the expansion of $r_j(U)$ are determined explicitly by Lemma 3.3.

Proof. The H_i's are analogous to the equations given in Lemma 3.3. Therefore, $< H_i > \subset \mathrm{Ker}\Psi$. We notice that $in_\omega(H_i) = h_i$, on the other hand, by the last proposition $< h_i >= \mathrm{Ker}(\mathrm{gr}\Psi)$. This gives us $\mathrm{Ker}(\mathrm{gr}\Psi) \subset \mathrm{gr}(\mathrm{Ker}\Psi)$, therefore, $\mathrm{Ker}(\mathrm{gr}\Psi) = \mathrm{gr}(\mathrm{Ker}\Psi)$. Let $g = g(U_1,\dots,U_{i_0}) \in \mathrm{Ker}\Psi$, we show that $g \in < H_i >$. We write g in the form $g = \sum_{(\beta_1,\beta_2)} X^{\beta_1}U^{\beta_2}$. For any i such that $(\beta_2)_i > n_i$ replace $U_i^{n_i}$ with $H_i + X^{\alpha^{(i)}} U_1^{\ell_1^{(i)}} \dots U_{i-1}^{\ell_{i-1}^{(i)}} - c_i U_{i+1} - r_i(U_1,\dots,U_i)$, by Lemma 6.6, this terminates after finitely many steps and we get a representation $g = f(H_1,\dots,H_k) + \sum_{(\beta_1,\beta_2)} X^{\beta_1}U^{\beta_2}$, where f is a polynomial with coefficients in $k[[X]][U]$ and $f(0) = 0$, moreover, $(\beta_2)_i < n_i$. Then we have $\sum_{(\beta_1,\beta_2)} X^{\beta_1}U^{\beta_2} = g - f(H_1,\dots,H_k) \in \mathrm{Ker}\Psi$. If $\sum_{(\beta_1,\beta_2)} X^{\beta_1}U^{\beta_2} = 0$ we are done, otherwise, $\mathrm{in}_\omega(\sum_{(\beta_1,\beta_2)} X^{\beta_1}U^{\beta_2}) \in \mathrm{gr}(\mathrm{Ker}\Psi) = \mathrm{Ker}(\mathrm{gr}\Psi)$, which is impossible because $\mathrm{in}_\omega(\sum_{(\beta_1,\beta_2)} X^{\beta_1}U^{\beta_2}) = X^{\beta_1^*}U^{\beta_2^*}$, for a unique pair (β_1^*,β_2^*), and $\mathrm{gr}\Psi(X^{\beta_1^*}U^{\beta_2^*}) = X^{\gamma.\beta_2^*+\beta_1^*} \neq 0$. $\square$

Remark 6.3 Notice that, unlike what is done in [GP03], it is not possible to arrange the situation so that $d_i = 1$, because we start from a fixed system of semi-roots. Moreover, the equality $\mathrm{in}_\omega(H_i) = h_i$ shows that the ideal defining the embedding $\mathcal{S} \hookrightarrow \mathcal{A}$ specializes through the filtration to the ideal of the embedding $\mathcal{Z}^{\Gamma_\zeta} \hookrightarrow \mathcal{A}$.

Consider a monomial $M = U_1^{q_1} \dots U_j^{q_j} \dots$, and define $V(M) = (q_1, \dots, q_j, \dots)$, $W_2(M) = \frac{q_1}{n_1} + q_2$, $W_{j+1}(M) = \frac{W_j(M)}{n_j} + q_{j+1}$. After one replacement for some term U_j ($q_j \geq n_j$) in M the monomials generated are of the form $M' = U_1^{q_1+m_1u_1} \dots U_{j-1}^{q_{j-1}+m_{j-1}u_1} U_j^{m_ju_1} U_{j+1}^{q_{j+1}+u_2} U_{j+2}^{q_{j+2}} \dots$, such that $m_h < n_h$ ($h \leq j$) and $u_1 + u_2 = [\frac{q_j}{n_j}]$. For $h < j$ we have $\frac{q_h+m_hu_1}{n_h} \leq u_1 - \frac{u_1}{n_h}$.

Lemma 6.4 *With the notations above, for the monomials M' obtained from M after a sequence of replacements for U_j's (for a fixed i and $j \leq i$) we have: $[W_{i+1}(M')] \leq [W_{i+1}(M)]$. The inequality is strict if in at least one of the replacements $u_1 \neq 0$. Moreover, the maximum exponent possible of U_{i+1} in these M''s exists and is less than or equal to $[W_{i+1}(M)]$.*

Proof. It is sufficient to prove it just for one such replacement and use induction. So, if M' is one of the monomials obtained from M by one replacement on U_j ($j \leq i$) and $V(M') = (q'_1, \dots, q'_j, \dots)$ then for some $u_1, u_2 \in \mathbb{Z}_{\geq 0}$ such that $u_1 + u_2 = [\frac{q_j}{n_j}]$, we have

$$W_{i+1}(M') \leq W_{i+1}(M) - \tfrac{q_j}{n_j \dots n_i} + \underbrace{\left(\frac{u_1 - (\frac{u_1}{n_1})}{n_2 \dots n_i}\right) + \dots + \left(\frac{u_1 - (\frac{u_1}{n_j})}{n_{j+1} \dots n_i}\right) + \left(\frac{u_2}{n_{j+1} \dots n_i}\right)}_{\text{A}}.$$

We note that $n_{j+1} \dots n_i.\text{A} \leq u_1 + u_2 - \frac{u_1}{n_1 \dots n_j}$, therefore, $\text{A} \leq \frac{q_j}{n_j \dots n_i} - \frac{u_1}{n_1 \dots n_i}$. This proves the first and the second claim of the lemma. For the last part: suppose it is proved for i, we prove it for $i+1$. Suppose an strategy of the replacements on U_j, for $j \leq i$, generate a maximum power for U_{i+1}. Then every replacement on a U_j can only change the power of U_p, for $p \leq j+1$. Therefore, in the course of the above strategy there is some step where the power of U_i is maximized. By induction this maximum is equal to $[W_i(M)]$. But in this step any replacement on U_j, for $j < i$, can not change the power of U_i, and as a result the power of U_{i+1}. Therefore, in this step without loose of generality we can suppose that the generated monomial is $M'' = U_i^{[W_i(M)]} U_{i+1}^{q_{i+1}}$. Now, by the proof of the first part of the lemma, no matter how the strategy continues the maximum power of U_{i+1} that can be generated is $\frac{[W_i(M)]}{n_i} + q_{i+1}$. □

Corollary 6.5 *The greatest term U_{i_0} that can be generated by the replacements in the monomial M exists and is equal to the largest index j such that $W_j(M) \neq 0$.*

Lemma 6.6 *For any monomial M after finitely many replacements all the monomials M' that are generated are such that $q'_j < n_j$.*

Proof. We use induction on the first index i, such that $q_i \geq n_i$. We prove this index can be shifted, after finitely many replacements, one step to the right and then by the corollary above we are done. So, let $M = M(U_1, \dots, U_i, \dots)$ be a monomial such that $q_j < n_j$ ($j < i$), we use another induction on q_i. We have $W_{i+1}(M) = \frac{q_i}{n_i} + q_{i+1}$. Consider a monomial M' which is obtained by just one replacement from M. If $u_1 = 0$ then for any $j \leq i$ we have $q'_i < q_i$, so, we are done in this case. Otherwise, by Lemma 6.4 we have $[W_{i+1}(M')] = [\frac{W_i(M')}{n_i} + q'_{i+1}] < [W_{i+1}(M)] = [\frac{q_i}{n_i} + q_{i+1}]$, but $q'_{i+1} \geq q_{i+1}$, therefore, $[W_i(M')] < q_i$. So, by induction hypothesis after finitely many replacements in all the monomials M'' that are generated we have $q''_i < n_i$. □

Remark 6.7 Fix a good ordering $\preccurlyeq$. Using some ideas of [Tei6] and [Tei05], we can give for any $d' \geq d+1$ and for any rational group G of rank d a valuation ν' of the field $k(X_1, \dots, X_d, U_1, \dots, U_{t-1})$, where $d' = d+t-1$ and $t \geq 2$, with value group G. Let γ_i's be the generators of the group G which are constructed in the proof of Proposition 4.10 and the relations between γ_i's which are explained in Lemma 3.2. Using the form of the equations introduced in Proposition 6.2, we define a morphism $\Psi' : k[X_1, \dots, X_d][U] \to k[X_1, \dots, X_d, U_1, \dots, U_{t-1}]$ given by $U_{t+i-1} \mapsto U_i^{n_i} - X^{\alpha^{(i)}} U_1^{l_1^{(i)}} \dots U_{i-1}^{l_{i-1}^{(i)}} + r'_i(U_1, \dots, U_i)$, for $i \geq 1$, where $r'_i(U_1, \dots, U_i) \in k[X, U_1, \dots, U_i]$ and they satisfy formally the same conditions of the r_i's of Proposition 4.10 (when we give the weight γ_i to U_i). Then the kernel of this morphism is generated by:

$$\begin{cases} H'_1 & := & U_1^{n_1} & - & X^{\alpha^{(1)}} & & + & r'_1(U_1) & + & U_t \\ H'_2 & := & U_2^{n_2} & - & X^{\alpha^{(2)}}U_1^{l_1^{(2)}} & & + & r'_2(U_1,U_2) & + & U_{t+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots \\ H'_i & := & U_i^{n_i} & - & X^{\alpha^{(i)}}U_1^{l_1^{(i)}}\ldots U_{i-1}^{l_{i-1}^{(i)}} & & + & r'_i(U_1,\ldots,U_i) & + & U_{t+i-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots \end{cases}$$

The construction of the valuation ν' is as follows: We set $\nu'(X_i)=e_i$, where e_i's are the elements of the standard basis of the vector space $\mathbb{Q}^d$, and $\nu'(U_i)=\gamma_i$. We can consider Ψ' as a graded morphism by the grades which come from the $\nu'-$values. For any $g(X,U_1,\ldots,U_{t-1})\in k[X,U_1,\ldots,U_{t-1}]$ we use the H'_i's and the Lemma 6.6 to represent g in the form $g=\sum_{\alpha,\beta}c_{\alpha,\beta}X^\alpha U^\beta(\text{mod}H'_i)$, such that for any β and $j\in\mathbb{N}:\beta_j<n_j$. This representation is unique it is a consequence of the fact that $\text{Ker}\Psi'=<H'_i>$ (see [Tei6]). We say this is the $U-adic$ representation of g subject to the conditions $H'=0$. Then the valuation is defined as:

$$\nu'(g)=min_{\alpha,\beta}\nu'(X^\alpha U^\beta).$$

The proof of Proposition 6.2 shows that this minimum exists and there is only a unique monomial with this minimum exponent. Moreover, for any g,h we have: $\nu'(g+h)\succcurlyeq min\{\nu'(g),\nu'(h)\}$ and $\nu'(gh)=\nu'(g)+\nu'(h)$. The first one is a direct consequence of the definition and the uniqueness of the $U-adic$ representation. The second one is also a direct consequence of uniqueness and the fact that each replacement, using some relation $H'_i=0$, in a monomial does not change the minimum value.

Remark 6.8 The equations $H'_i=0$'s of the last remark can be viewed as a sequence of key polynomials. Transferring the U_{t+j}'s to the other side of the equations we get a set of equations which introduce the U_{t+j}'s as polynomials in $k[X,U_1,\ldots,U_{j+1}]$'s. Using the results of the last section we see that they are a sequence of key polynmials (with respect to the weights γ_i) and there is a sequence of valuations ν_i's attached to this sequence.

We can unify the content of the last remarks and Corollary 5.8 in the following theorem:

Theorem 6.9 *Given any rational group G totally ordered by a good ordering, of rational rank less than or equal to d, there exists a valuation of the field $k(X_1,\ldots,X_d,Y)$ with value group G and with residue field k. This valuation is build from grading a suitably chosen sequence of key polynomials in the ring $k[X,Y]$.*

In the next proposition we give the explicit embedding of the sequence of the quasi-ordinary hypersurfaces $f^{(r)}(X,Y)=0$ in the space $\mathcal{A}$, and the relation between these equations and the equations of the embedding in Proposition 6.2. The equations of the embedding come from the truncation of the equations of Proposition 6.2.

Proposition 6.10 *There exists an embedding of the quasi-ordinary hypersurface $f^{(r)}(X,Y)=0$ (Definition 2.3) in the the space* $\text{Spec}(k[[X]][U_1,\ldots,U_r])$, *in such a way that a set of generators for the ideal of this embedding can be given by a process of truncation of the equations of the embedding $\mathcal{S}\hookrightarrow\mathcal{A}$ which is given in the Proposition 6.2.*

Proof. Consider the embedding:

$$\begin{array}{rl} \Psi_r: k[[X]][U_1,\ldots,U_r] & \to \frac{k[[X]][Y]}{f^{(r)}(X,Y)} \\ X & \mapsto X \\ U_j & \mapsto f^{(j-1)}(X,\zeta^{(r)}(X)). \end{array}$$

Consider the H_i's introduced in Proposition 6.2. Fix a natural number j and truncate H_i's at the jth step in the following sense: Keep $H_1,\ldots,H_{r-1}$ and in the equation of H_r delete the term c_rU_{r+1} and drop all the next H_i's.

Now, the proof of Proposition 6.2 could be repeated to give: The ideal of the embedding Ψ_r is generated by the truncated elements, i.e.,

$$H_1^{(r)}=H_1,\ldots,H_{r-1}^{(r)}=H_{r-1},H_r^{(r)}=H_r-c_rU_{r+1}.$$

The point is that the equations of H_j's, given in the proof of Proposition 6.2, come from the *adic* expansion of the $(f^{(j-1)})^{n_j}$ in Lemma 3.3. These expansions are independent of the parametrizations $\zeta^{(i)}(X)$'s. These equations give us exactly $H_j^{(r)}$'s. For $j = r$ notice that by definition we have $f^{(r)}(X, \zeta^{(r)}(X)) = 0$, hence, the *adic* expansion of $(f^{(r-1)})^{n_r}$ which gives the equation $H_r = 0$ translates to $H_r^{(r)} = 0$. □

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