Realization certain class of semi-groups as value semi-groups of valuations

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Abstract

Given an ordered semi-group Γ of rational rank r, with a well-ordered minimal system of generators of ordinal type at most ωn, which satisfies a positivity and increasing condition, we construct a valuation centered on the ring of polynomials with $r+n+1$ variables such that the semi-group of the values of the polynomial ring is equal to Γ. The corresponding valuation is constructed using a generalization of Favre and Jonsson's version of MacLane's sequence of key-polynomials [3].

1 Introduction

Recently the interest for studying the structure of the value semi-groups of the valuations centered on a noetherian local-ring has raised (see for example [2]). Several worked examples (e.g., plane branches, irreducible quasi-ordinary hypersurface singularities) suggests that the structure of these semi-groups contain important information on the local uniformization process of the valuation. What type of semi-groups can be realized as the semi-group of values of a noetherian local ring dominated by a valuation ring? Little is known in this respect. We know they are well-ordered of ordinal type $< \omega^h$, for some natural number h ([12], Appendix 3, Proposition 2). Abhyankar's inequality holds between numerical invariants of these valuations (see below). And, such semi-groups have no accumulation point when they are considered as semi-groups of $(\mathbb{R}^n, <_{lex})$ [2].

In this paper we show that given a semi-group Γ of rational rank r, given with a minimal system of generators which is well-ordered of ordinal type at most ωn, $n \in \mathbb{N}$, which satisfies a positivity and increasing condition (Definition 2.2 and Theorem 7.1), there is a polynomial ring $R = k[X_1, \dots, X_d]$ and a valuation ν, which is positive over R, such that the value semi-group of R, $\nu(R \setminus \{0\})$ is equal to Γ. Moreover, we give a bound for d: $d \leq n+r+1$. It seems that this bound is optimal, i.e., in general given such a semi-group Γ, the least number of variables which is needed to realize Γ as the value semi-group of a valuation centered on a polynomial ring is $n+r+1$.

Our basic tool is a generalization of Favre and Jonsson's version of MacLane's sequence of key-polynomials ([3], [6]) for polynomial rings with arbitrary number of variables. The technique of sequences of key-polynomials was first invented by MacLane [6], following ideas of Ostrowski, to produce and describe all the extensions of a discrete rank one valuation ν of a field K to the extension field $L = K(x)$. He attached to any extension, say μ, of the valuation ν, a sequence of polynomials $\phi_i(x)$ of the ring $K[x]$. By induction one can produce any extension μ to L of the valuation ν using valuations constructed by key-polynomials (augmented valuations). In [11], Vaquié generalized MacLane's method to produce all the extensions of any arbitrary valuation of an arbitrary field K to L. He showed that given such an extension of a valuation, there may be many ways to produce such sequences of key-polynomials and augmented valuations. Later Favre and Jonsson showed that in the case of $d = 1$ one can consider a rather simple sequence of toroidal-key-polynomials (SKP), to produce all the pseudo-valuations centered on the ring $k[[X_0, X_1]]$.

In this text, we give a generalization of the sequence of toroidal-key-polynomials of [3] to produce a class of valuations of the field $k((X_0, \dots, X_d))$. Our generalization can not generate all the valuations centered on the polynomial ring. The construction is explicit enough to describe the value semi-group $\nu(k[[X_0, \ldots, X_d]] \setminus \{0\})$. And in addition to realize certain semi-groups as value semi-groups.

Here we recall the basic definitions associated to valuations.

Definition 1.1 Fix a valuation ν.

- The rank $\mathrm{rk}(\nu)$ of ν, is the Krull dimension of the valuation ring R_ν.
- The rational rank of ν, $\mathrm{r.rk}(\nu)$, is the dimension of $\nu(Frac(R_\nu)^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ as a vector space over $\mathbb{Q}$.
- The transcendence degree of ν, $\mathrm{tr.deg}(\nu)$, is the transcendence degree of the extension of F with residue field of ν, $F \subseteq k_\nu := \frac{R_\nu}{\mathfrak{m}_\nu}$.

The principal relation between these numerical invariants is given by Abhyankar's inequalities:

$$\mathrm{rk}(\nu) + \mathrm{tr.deg}(\nu) \leq \mathrm{r.rk}(\nu) + \mathrm{tr.deg}(\nu) \leq \dim R.$$

Moreover, if $\mathrm{r.rk}(\nu) + \mathrm{tr.deg}(\nu) = \dim R$, then value group is isomorphic (as a group) to $\mathbb{Z}^{\mathrm{r.rk}(\nu)}$. When $\mathrm{rk}(\nu) + \mathrm{tr.deg}(\nu) = \dim R$, the value group is isomorphic as an ordered group to $\mathbb{Z}^{\mathrm{rk}(\nu)}$, endowed with the lex. order.

Let R be an integral domain with field of fractions K and let ν be a valuation of K such that its valuation ring R_ν contains R, in this case we say the valuation is centered on the ring R. Let us denote by Φ the totally ordered value group of the valuation ν. Denote by Φ_+ the semigroup of positive elements of Φ and set $\Gamma = \nu(R \setminus \{0\}) \subset \Phi_+ \cup \{0\}$; it is the semigroup of (R, ν); since Γ generates the group Φ, it is cofinal in the ordered set Φ_+.

For $\phi \in \Phi$, set

$$\mathcal{P}_\phi(R) = \{x \in R \mid \nu(x) \geq \phi\}$$
$$\mathcal{P}^+_\phi(R) = \{x \in R \mid \nu(x) > \phi\},$$

where we agree that $0 \in \mathcal{P}_\phi$ for all ϕ, since its value is larger than any ϕ, so that by the properties of valuations the $\mathcal{P}_\phi$ are ideals of R. Note that the intersection $\bigcap_{\phi \in \Phi_+} \mathcal{P}_\phi = (0)$ and that if ϕ is in the negative part Φ_- of Φ, then $\mathcal{P}_\phi(R) = \mathcal{P}^+_\phi(R) = R$.

For $\phi \notin \Gamma$, $\mathcal{P}_\phi(R) = \mathcal{P}^+_\phi(R)$. For each non zero element $x \in R$, there is a unique $\phi \in \Gamma$ such that $x \in \mathcal{P}_\phi \setminus \mathcal{P}^+_\phi$; the image of x in the quotient $(\mathrm{gr}_\nu R)_\phi = \mathcal{P}_\phi / \mathcal{P}^+_\phi$ is the *initial form* $\mathrm{in}_\nu(x)$ of x.

The graded algebra associated with the valuation ν was introduced in ([5],[9]) for the very special case of a plane branch (see [4]), and in [8] in full generality. Later it was extensively used in [10] as a tool to solve the local-uniformization problem. It is

$$\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Gamma} \mathcal{P}_\phi(R) / \mathcal{P}^+_\phi(R).$$

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2 The inductive definition of SKP's

From now on by Φ we mean a totally ordered abelian group of rank $d+1$. Let $\Delta_0 = (0) \subset \cdots \subset \Delta_{d+1} = \Phi$ be its sequence of isolated subgroups (see [12]). We define the sequence of pre-values and the sequence of positively generated values. Attached to a sequence of positively generated values there exists a sequence of key-polynomials (SKP) which are elements of the power series ring $k^{(d)} = k[[X_0, \ldots, X_d]]$[1]. First we need a general lemma on abelian groups.

[1] for any $i \leq d$ we define $k^{(i)} = k[[X_0, \ldots, X_i]]$ and $k_{(i)} = k((X_0, \ldots, X_i))$

Lemma 2.1 *Let Ψ be an abelian group and $\Gamma = \{\gamma_0, \gamma_1, \dots, \gamma_\alpha\}$, α an ordinal number, be a well-ordered sequence of elements of Ψ. For any ordinal $i \le \alpha$ define the subgroups of Ψ, $G_i = (\gamma_j)_{j \le i}$*[2]*, $G_{i^-} = (\gamma_j)_{j<i}$, $n_i = [G_i : G_{i^-}]$, and set $n_0 = \infty$. Then for any $i \le \alpha$ such that $n_i \ne \infty$, we have a unique representation $n_i\gamma_i = \sum_{j<i} m_j\gamma_j$, such that $0 \le m_j < n_j$ when $n_j \ne \infty$, and $m_j \in \mathbb{Z}$ when $n_j = \infty$, and $m_j = 0$ except for a finite number of j. More generally, every element of G_{i^-} has such a unique representation.*

Proof. Let $i \le \alpha$ and $n_i \ne \infty$, we have $n_i\gamma_i \in G_{i^-}$ (by definition of n_i). Thus, there exists a representation $n_i\gamma_i = \sum_{j<i} p_j\gamma_j$, where $p_j \in \mathbb{Z}$, and $p_j = 0$ except for a finite number of j. We define, inductively, a sequence of the elements of the index set α, $A : N' \subset \mathbb{N} \to \{1, \dots, \alpha\}$, as follows:

Let $j_0 < i$ be the greatest ordinal number such that $n_{j_0} \ne \infty$ and $p_{j_0} \ne 0$, the ordinal j_0 exists- as there only a finite numbers of non-zero p_j. Set $A(0) = j_0$. Using Euclidean division, write $p_{j_0} = q_{j_0}n_{j_0} + r_{j_0}$, where $0 \le r_{j_0} < n_{j_0}$. Substituting this for p_{j_0}, and expanding $n_{j_0}\gamma_{j_0}$ in terms of elements of $G_{j_0^-}$, we get $n_i\gamma_i = \sum_{j<j_0} p'_j\gamma_j + r_{j_0}\gamma_{j_0}$, where $p'_j \ne 0$ except for a finite number of j. Now, as before, let $j_1 (< j_0)$ be the first ordinal number such that $n_{j_1} \ne \infty$ and $p'_{j_1} \ne 0$. Set $A(1) = j_1$ and continue as before to obtain $n_i\gamma_i = \sum_{j<j_1} p''_j\gamma_j + r_{j_1}\gamma_{j_1} + r_{j_0}\gamma_{j_0}$, where $0 \le r_j < n_j$. Continue this construction.

Either this construction stops after a finite number of steps, say j_k, then we have $n_i\gamma_i = \sum_{j<i} m_j\gamma_j$, such that $m_j = 0$ except for a finite number of j, and $0 \le m_j < n_j$ when $n_j \ne \infty$. This shows the existence part of the claim in this case. Or, the construction continues for ever, in this case we get a strictly decreasing sequence $A : \mathbb{N} \to \alpha$. But this is impossible: It suffices to note that $A(\mathbb{N})$ is a subset of α without least element, which is impossible (as α is well-ordered). Thus we have proved the existence part of the claim.

For the uniqueness, if we have two such representation $n_i\gamma_i = \sum_{j<i} m_j\gamma_j = \sum_{j<i} m'_j\gamma_j$ then let j_0 be the greatest index such that $m_{j_0} \ne m'_{j_0}$ (as the number of nonzero m_j and m'_j is finite this greatest index exists). Suppose $m_{j_0} > m'_{j_0}$ then $(m_{j_0} - m'_{j_0})\gamma_{j_0} = \sum_{j<j_0}(m'_j - m_j)\gamma_j \in G_{j_0^-}$ which is a contradiction, because $0 \le m_{j_0} - m'_{j_0} < n_{j_0}$. □

Definition 2.2 With the notation of Lemma 2.1, we say the sequence Γ *positively generates* the group Ψ, if for any i we have $m_j \in \mathbb{N}$.

Definition 2.3 A sequence $(\beta_{i,j} \in \Phi)_{i=0\dots d, j=1\dots\tilde{\alpha}_i}$[3], $\tilde{\alpha}_i$ an ordinal number and $\tilde{\alpha}_0 = 1$, is called a *sequence of pre-values* if for any i and j we have

- $\beta_{i,j+1} \succ n_{i,j}\beta_{i,j}$, where $n_{i,j} = \min\{r \in \mathbb{N} :\ r\beta_{i,j} \in (\beta_{i',j'})_{(i',j')<_{lex}(i,j)}\}$
- When j is a limit ordinal then $\beta_{i,j} \succ \beta_{i,j'}$, for any $j' < j$.

Consider the index set $\{(i,j)\}_{i=0\cdots d, j=0\cdots\tilde{\alpha}_i}$, ordered by the *lex.* ordering. As $\tilde{\alpha}_i$ are ordinals, this is a well-ordering. According to Lemma 2.1, when $n_{i,j} \ne \infty$ there exists a unique representation

$$n_{i,j}\beta_{i,j} = \sum_{(i',j') \in S_{i,j} \cup S^c_{i',j'}} m^{(i,j)}_{i',j'}\beta_{i',j'}. \tag{1}$$

where $m^{(i,j)}_{i',j'} = 0$, except for a finite number of $(i',j') <_{lex} (i,j)$, and $S_{i,j} = \{(i',j') \mid (i',j') <_{lex} (i,j),\ m^{(i,j)}_{i',j'} > 0\}$, $S^c_{i,j} = \{(i',j') \mid (i',j') <_{lex} (i,j),\ m^{(i,j)}_{i',j'} < 0\}$. By Lemma 2.1 we have $0 \le m^{(i,j)}_{i',j'} < n_{i',j'}$ if $n_{i',j'} \ne \infty$, and $m^{(i,j)}_{i',j'} \in \mathbb{Z}$ if $n_{i',j'} = \infty$. Thus, if $(i',j') \in S^c_{i,j}$ then $j' = \alpha_{i'}$ and $n_{i',\alpha_{i'}} = \infty$.

Let $\Gamma = (\beta_{i,j} \in \Phi)_{i=0\dots d, j=1\dots\tilde{\alpha}_i}$, ordered by *lex* ordering, be a sequence of pre-values. Let $\Phi_{d,\tilde{\alpha}_d}$ be the group generated by these elements. We say Γ is a *sequence of values* if it positively generates $\phi_{d,\tilde{\alpha}_d}$. This condition is equivalent to $S^c_{i,j} = \emptyset$, for any i and j.

Definition 2.4 (**SKP's**) Given a sequence of values $\Gamma = (\beta_{i,j} \in \Phi)_{i=0\dots d, j=1\dots\tilde{\alpha}_i}$, we attach a sequence of power series $(U_{i,j} \in k^{(d)})_{i=0\dots d, j=1\dots\alpha_i}$, $\alpha_i \le \tilde{\alpha}_i$ to Γ. It is called the *sequence of key-polynomials* of the sequence of values Γ. It is defined by induction on i. For $i = 0$, we set $\alpha_0 = \tilde{\alpha}_0 = 1$ and $U_{0,1} = X_0$. Suppose $U_{i',j'}$ and $\alpha_{i'}$ are defined for $i' < i$. We set $U_{i,1} = X_i$. Suppose $U_{i,j'}$ are defined for $j' < j$. Then we define $U_{i,j}$ as follows

[2]If $a_1, \dots, a_n$ are elements of a group G, by $(a_1, \dots, a_n)$ we denote the subgroup generated by these elements and by $\langle a_1, \dots, a_n\rangle$ the semigroup generated by them.

[3]By $i = 1 \cdots d$ and $j = 1 \cdots \tilde{\alpha}_i$ we mean $i = 1, \dots, d$ and $j = 1, \dots, \tilde{\alpha}_i$.

(P1) If j is not a limit ordinal then

$$U_{i,j} = U_{i,j-1}^{n_{i,j-1}} - \theta_{i,j-1} \prod_{(i',j') \in S_{i,j-1}} U_{i',j'}^{m_{i',j'}^{(i,j-1)}}, \tag{2}$$

where $\theta_{i,j} \in k^*$. This can be written as $U_{i,j} = U_{i,j-1}^{n_{i,j-1}} - \theta_{i,j-1} U^{m^{(i,j-1)}}$.

(P2) If j is a limit ordinal then

$$U_{i,j} = \lim_{j' \to j} U_{i,j'} \in k^{(i-1)}[X_i].$$

In Proposition 2.8 we prove that this limit exists in the ring $k^{(i-1)}[X_i]$. If this limit is equal to zero, then we set $\alpha_i = j$, $\beta_{i,j} = \infty$, and we stop the construction of the key-polynomials at this step, for i. Otherwise, we continue to construct $U_{i,j'}$ for $j' > j$.

If the construction of $U_{i,j}$'s continues for every $j \leq \tilde{\alpha}_i$ then we set $\alpha_i = \tilde{\alpha}_i$.

We denote an SKP by $[U_{i,j}, \beta_{i,j}]_{i=0\ldots d, j=1\ldots\alpha_i}$.

Remark 2.5 The following remarks are in order:

(i) Given any SKP as above, if we consider the data $[U_{i,j}, \beta_{i,j}]_{i=0,1,j=0\cdots\alpha_i}$ then it is a $\Gamma-$SKP for the ring $k[[X_0, X_1]]$ in the sense of [3] for the group $\Gamma = \Phi$.

(ii) The formula of (P1) can be rewritten in the following way.

$$U_{i,j+1} = U_{i,j}^{n_{i,j}} - \theta_{i,j} U_0^{m_0^{(i,j)}} U_1^{m_1^{(i,j)}} \cdots U_{i-1}^{m_{i-1}^{(i,j)}} (U_{i,1}^{m_{i,1}^{(i,j)}} \cdots U_{i,j-1}^{m_{i,j-1}^{(i,j)}}),$$

where $m_{i'}^{(i,j)}$ is equal to $(m_{i',1}^{(i,j)}, \ldots, m_{i',\alpha_{i'}}^{(i,j)})$.

(iii) For a fixed i when α_i is a limit ordinal:

- If there exists an infinite number of j such that $n_{i,j} > 1$ then we have
  - $\deg_{X_i}(U_{i,j}) \to \infty (j \to \alpha_i)$.
  - We have $U_{i,\alpha_i} = \lim_{j \to \alpha_i} U_{i,j} = 0$ (See Lemma 2.12.(ii)).
- Otherwise (we denote this case by writing $U_{i,\alpha_i} \neq 0$), we have
  - $n_{i,j} = 1$, except for a finite number of ordinals j.
  - There is some ordinal j_0 such that $\deg_{X_i}(U_{i,\alpha_i}) = \deg_{X_i}(U_{i,j})$ and $n_{i,j} = 1$, for any $j > j_0$.

(iv) For any limit ordinal $j < \alpha_i$ there are only finitely many $j' < j$ such that $n_{i,j'} > 1$: In contrary, let $j < \alpha$ be the an ordinal such that there are infinitely many $j' < j$ such that $n_{i,j'} > 1$. The argument of the proof of Lemma 2.12.(ii), shows that $U_{i,j} = 0$. Thus, by construction of SKP, we must have $j = \alpha_i$ which is a contradiction.

Definition 2.6 Let $[U_{i,j}, \beta_{i,j}]_{i=0\ldots d, j=1\ldots\alpha_i}$ be an SKP. We define the semigroups $\Gamma_{i,j}$ and the groups $\Phi_{i,j}$, for $i = 0, \ldots, d, j = 1, \ldots, \alpha_i$, as follows:

$$\Gamma_{i,j} = \langle (\beta_0)_1^{\alpha_0}, \ldots, (\beta_{i-1})_1^{\alpha_{i-1}}, \beta_{i,1}, \ldots, \beta_{i,j} \rangle,$$

$$\Phi_{i,j} = (\Gamma_{i,j}),$$

$$\Phi_{i,j}^* = \Phi_{i,j} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Definition 2.7 Consider a power series ring $A = k^{(i)}$. The order of an element $M = \sum_{\mathbf{m}} c_{\mathbf{m}} X^{\mathbf{m}}$ of this ring is $\mathrm{ord}_A(M) = \mathrm{ord}(M) = \min_{\mathbf{m}, c_{\mathbf{m}} \neq 0} \{\sum_{i'=0}^{i} \mathbf{m}_{i'}\}$.

Proposition 2.8 *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$. *Then for any* (i,j) *we have* $U_{i,j} \in k^{(i-1)}[X_i]$.

Proof. The proof is by induction on i and j. For $i = 0$ it is obvious. Suppose it is valid for indices less than i, we prove it for i. When j is not a limit ordinal, formula (P1) represents $U_{i,j}$ as a polynomial in terms of previous U's and the claim is obvious in this case by induction on j.

It remains the case when j is a limit ordinal. We can assume that $j = \alpha_i$ (considering the SKP $[U_{i',j'}, \beta_{i',j'}]_{i'=0\cdots j',j'=1\cdots\alpha'_{i'}}$, where $\alpha'_{i'} = \alpha_{i'}$ for $i' < i$ and $\alpha'_i = j$). We must show that $\lim_{j'\to\alpha_i} U_{i,j'} \in k^{(i-1)}[X_i]$.

If there is infinite number of j such that $n_{i,j} > 1$ then by Lemma 2.12.(ii), we have $U_{i,\alpha_i} = 0 \in k^{(i-1)}[X_i]$. Thus, we can assume $n_{i,j} = 1$, except for a finite number of j. Then by Lemma 2.12.(i), we have $\mathrm{ord}_{k^{(i-1)}[X_i]}(U^{m^{(i,j)}}) \to \infty (j \to \alpha_i)$. By Remark 2.5.(iii), we have $\deg_{X_i}(U^{m^{(i,j)}})$ is limited. Hence $\mathrm{ord}_{k^{(i-1)}}(U^{m^{(i,j)}}) \to \infty (j \to \alpha_i)$. Using this fact and the equality $U_{i,j+1} - U_{i,j} = -\theta_{i,j} U^{m^{(i,j)}}$, for $j \geq j_0$ (where $n_{i,j} = 1$, for $j \geq j_0$), we have $\lim_{j\to\alpha_i} U_{i,j} = U_{i,j_0}^{n_{i,j_0}} - \sum_{j,j_0\leq j<\alpha_i} \theta_{i,j} U^{m^{(i,j)}} \in k^{(i-1)}[X_i]$. □

Remark 2.9 The proof of the proposition shows that for any arbitrary two ordinals $j' < j''$ such that $n_{i,j} = 1$, for $j' < j < j''$, we have $U_{i,j''} = \lim_{j\to j''} U_{i,j} = U_{i,j'}^{n_{i,j'}} - \sum_{j,j'\leq j<j''} \theta_{i,j} U^{m^{(i,j)}}$.

Let $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d,j=1\cdots\alpha_i}$ be an SKP. Fix an $i \leq d$. Consider the abelain ordered group Φ_{i,α_i}. This group is order isomorphic to a subgroup of the ordered group $(\mathbb{R}^n, <_{lex})$, for some n (see [1], Proposition 2.10). Suppose α_i is a limit ordinal. Consider the first index $t \leq d$, such that $\#\{(\beta_{i,j})_t\}_{1\leq j<\alpha_i} = \infty$. The index t is called *the effective component for i* . Notice that this t exists: In contrary, we have $\#\{(\beta_{i,j})_t\}_{1\leq j<\alpha_i, t=1\cdots n} < \infty$. On the other hand, we have $\beta_{i,1} <_{lex} \beta_{i,2} <_{lex} \cdots <_{lex} \beta_{i,\alpha_i}$. But this is impossible when all the components of β_i's come from a finite set. Thus t is well-defined. In [2], it is shown that well-ordered semi-groups of ordinal type $\leq \omega^h$, $h \in \mathbb{N}$, have no accumulation point in $\mathbb{R}^n$, in Euclidean topology. We show that the positively generated semi-groups have a stronger property: The effective component of any sequence of the elements of the semi-group tends to infinity (Lemma 2.11, and Lemma 7.3)

Proposition 2.10 *With the notation of the last paragraph we have:*

(*i*) *There exists $j_{(i)}$, $1 \leq j_{(i)} < \alpha_i$, such that the first $(t-1)$ components of $\beta_{i,j}$ are the same (componentwise), for $j \geq j_{(i)}$, i.e., $(\beta_{i,j})_{t'} = (\beta_{i,j'})_{t'}$, for $j, j' \geq j_{(i)}$ and $t' < t$.*

(*ii*) *For $j > j' > j_{(i)}$ we have $(\beta_{i,j})_t \geq (\beta_{i,j'})_t$.*

(*iii*) *If $U_{i,\alpha_i} = 0$ then:*

(1) $t = \min\{t' |\ 1 \leq t' \leq n,\ \exists j < \alpha_i :\ (\beta_{i,j})_{t'} \neq 0\}$.

(2) $(\beta_{i,j})_{t'} = 0$*, for any* $j < \alpha_i$ *and* $t' < t$.

(3) $(\beta_{i,j})_t \to +\infty (j \to \alpha_i)$.

Proof. The first item is a direct consequence of the definition. For (*ii*), by definition of the SKP's, we have $\beta_{i,j} >_{lex} \beta_{i,j'}$. On the other hand, by (*i*), the first $t-1$ components of $\beta_{i,j}$ and $\beta_{i,j'}$ are the same. Thus $(\beta_{i,j})_t \geq (\beta_{i,j'})_t$.

For (*iii*), set $t_1 = \min\{t' |\ 1 \leq t' \leq n,\ \exists j < \alpha_i :\ (\beta_{i,j})_{t'} \neq 0\}$. By definition of t_1, we have $(\beta_{i,j})_{t'} = 0$, for any $j < \alpha_i$ and $t' < t_1$. So, $t_1 \leq t$. From the definition of the SKP, we deduce that $\beta_{i,j+1} >_{lex} (\prod_{j_0\leq j'\leq j} n_{i,j'})\beta_{i,j_0}$. We choose j_0 such that $(\beta_{i,j_0})_{t_1} \neq 0$ (note that necessarily $(\beta_{i,j_0})_{t_1} > 0$). As $U_{i,\alpha_i} = 0$, there are infinite number of $j > j_0$ such that $n_{i,j} > 1$ $(j \to \alpha_i)$. This shows that $(\beta_{i,j})_{t_1} \to \infty (j \to \alpha_i)$. Thus $t = t_1$. □

Lemma 2.11 *Let $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d,j=1\cdots\alpha_i}$ be an* SKP. *Fix an $i \leq d$ and let t be the effective component for i. Suppose α_i is a limit ordinal then $(\beta_{i,j})_t \to +\infty (j \to \alpha_{i'})$.*

Proof. If $U_{i,\alpha_i} = 0$, then the claim is the content of Proposition 2.10.(*iii*). Assume $U_{i,\alpha_i} \neq 0$. Then, by definition of $U_{i,\alpha_i} \neq 0$, there exists j_0 such that $n_{i,j} = 1$ for $j > j_0$. Notice that in this case there is a finite number of j (in general) such that $n_{i,j} \neq 1$ (by definition of $U_{i,\alpha_i} \neq 0$). And we have $(\beta_{i,j})_t = \sum_{(i',j')\in S_{i,j}} m_{i',j'}^{(i,j)} (\beta_{i',j'})_t$, for $j > j_0$. Define

$$C_i = \{(i', j') \in S_{i,j},\ \max\{j_0, j_{(i)}\} \leq j < \alpha_i,\ (\beta_{i',j'})_t \neq 0\}.$$

If $\#C_i = \infty$ then there exists some $i_0 < i$ and an infinite number of j' such that $(i_0, j') \in C_i$, so we can speak of $j' \to \infty$. As for such (i_0, j') (which are infinite in number) we have $n_{i_0,j'} > 1$, hence α_{i_0} is a limit ordinal and $U_{i_0,\alpha_{i_0}} = 0$. Let t' be the effective component for i_0. By definition of C_i there is at least one j' such that $(\beta_{i_0,j'})_t \neq 0$. But $U_{i_0,\alpha_{i_0}} = 0$, thus by Proposition 2.10.(iii).(2), we have $t' = t$. As $(\beta_{i_0,j'})_t \to \infty (j' \to \infty)$, we have $(\beta_{i,j})_t \to \infty (j \to \alpha_i)$.

If $\#C_i < \infty$ then $(\beta_{i,j})_t$'s are elements of the discrete lattice $L \subset \mathbb{R}$ generated by the finite set of generators $\{(\beta_{i',j'})_t | \ (i', j') \in C_i\}$. Thus, as any bounded region of $\mathbb{R}$ contains only a finite number of elements of the lattice L, the sequence $(\beta_{i,j})_t (j \to \alpha_i)$ can not be contained in any bounded region of $\mathbb{R}$. On the other hand, by Proposition 2.10.(ii), this sequence is increasing, so, it goes to $+\infty$. $\square$

Lemma 2.12 *Consider an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$. *Suppose α_i is a limit ordinal. Then we have the following:*

- (i) *For any $n \in \mathbb{N}$ and $i < d$ there exists ordinal $j_n^{(i)}$ such that* $\mathrm{ord}_{k^{(i-1)}[X_i]}(U^{m^{(i,j)}}) > n$ *for any $j > j_n^{(i)}$.*
- (ii) *If $U_{i,\alpha_i} = 0$ then one can choose the above $j_n^{(i)}$ such that in addition* $\mathrm{ord}_{k^{(i-1)}[X_i]}(U_{i,j}) > n$ *for any $j > j_n^{(i)}$.*

Proof. Suppose both (i) and (ii) are proved for any n' and $i' < i$, and also for $n' \leq n$ and i. We prove them for $n+1$ and i. Suppose t is the effective component for i. For any vector $V \in \mathbb{R}^n$ we define $|V|$ to be its tth component, i.e., $|V| = (V)_t$. Let

$$M^* = \max\{\{|\beta_{i',j'}| : \ (i', j') \in S_{i,j}, \ j' \leq j_{n+1}^{(i')} \text{ when } i' < i, j' \leq j_n^{(i)} \text{ when } i' = i\}.$$

Notice that the cardinality of this set is finite, so M^* is well-defined.

For (i):

By Lemma 2.11, we have $|\beta_{i,j}| \to +\infty (j \to \alpha_i)$. Hence there exists $j_{n+1}^{(i)}$ such that $|\beta_{i,j}| > (n+1)M^*$, for $j \geq j_{n+1}^{(i)}$. The claim is that this number $j_{n+1}^{(i)}$ works. We can assume $j_{(i)} < j_n^{(i)}$ (see Proposition 2.10.(ii)). Suppose $j > j_{n+1}^{(i)}$.

If there exists at least one $(i, j') \in S_{i,j}$ such that $j' \geq j_n^{(i)}$ then we are done. Indeed, if $m_{i,j'}^{(i,j)} > 1$, since $\mathrm{ord}_{k^{(i-1)}[X_i]}(U_{i,j'}) > n$ (by induction hypothesis for (ii), in the case n) then $\mathrm{ord}_{k^{(i-1)}[X_i]}(U^{m^{(i,j)}}) > nm_{i,j'}^{(i,j)} > n+1$. If $m_{i,j'}^{(i,j)} = 1$, since $|\beta_{i,j'}| < |\beta_{i,j}|$ (because $n_{i,j'} > 1$ and $\beta_{i,j} >_{lex} n_{i,j'}\beta_{i,j'}$, and $|.|$ preserves ordering for $j'' > j_{(i)}$), there should be at least one other element $(i'', j'') \in S_{i,j}$. But $\mathrm{ord}_{k^{(i-1)}[X_i]}(U_{i'',j''}) \geq 1$. Therefore, we have $\mathrm{ord}_{k^{(i-1)}[X_i]}(U^{m^{(i,j)}}) > \mathrm{ord}_{k^{(i-1)}[X_i]}(U_{i,j'}) + \mathrm{ord}_{k^{(i-1)}[X_i]}(U_{i'',j''}) > n+1$.

If there exists some $(i', j') \in S_{i,j}$ such that $i' < i$ and $j' > j_{n+1}^{(i')}$ then clearly we are done.

There remains the case that for all $(i', j') \in S_{i,j}$:

- If $i' < i$ then $j' < j_{n+1}^{(i')}$.
- If $i' = i$ then $j' < j_n^{(i)}$.

By definition of M^* and conditions above, we have $|\beta_{i',j'}| < M^*$, for any $(i', j') \in S_{i,j}$. Hence

$$|\beta_{i,j_{n+1}^{(i)}}| \leq |\beta_{i,j}| \leq n_{i,j}|\beta_{i,j}| = \sum_{(i',j') \in S_{i,j}} m_{i',j'}^{(i,j)} |\beta_{i',j'}| < (\sum_{(i',j') \in S_{i,j}} m_{i',j'}^{(i,j)})M^*.$$

Where the first inequality holds because $|.|$ preserves ordering for $j' \geq j_n^{(i)} > j_{(i)}$ (Proposition 2.10.(ii)).

But, by definition of M^*, we have $|\beta_{i,j_{n+1}^{(i)}}| > (n+1)M^*$. Thus $n+1 < \sum_{(i',j') \in S_{i,j}} m_{i',j'}^{(i,j)}$. Finally

$$\mathrm{ord}_{k^{(i-1)}[X_i]}(U^{m^{(i,j)}}) \geq \sum_{(i',j') \in S_{i,j}} m_{i',j'}^{(i,j)} > n+1.$$

For (ii):

As (i) holds for $n+1$ and using induction hypothesis, we can find $j_{n+1}^{(i)}$ such that $\mathrm{ord}(U_{i,j}) > n$, $\mathrm{ord}(U^{m^{(i,j)}}) > n+1$, for $j > j_{n+1}^{(i)}$. If this $j_{n+1}^{(i)}$ does not work for (ii), find the first $j_0 > j_{n+1}^{(i)}$ such that $n_{i,j_0} \neq 1$ (as $U_{i,\alpha_i} = 0$ this j_0 exists) then set $j_{n+1}^{(i)} := j_0$. It is straightforward to check that this new $j_{n+1}^{(i)}$ works also for (ii). $\square$

Example 2.13 Consider the ring $k[X_0, X_1, X_2]$ and the group $\Phi = \mathbb{Z}^3$ with reverse lexicographical order. Consider the valuation ν centered on this ring defined by the SKP $(U_{0,1}, U_{1,1}, (U_{2,j})_{j=1}^{\omega^2})$ and $\beta_{0,1} = (1,0,0), \beta_{1,1} = (0,1,0), \beta_{2,\omega n+j} = (j, n+2, 0)$ for $n \in \mathbb{N}, 0 < j < \omega$ and $\beta_{2,\omega^2} = (0,0,1)$. Here we have the relations

$$U_{2,\omega n+j+1} = U_{2,\omega n+j} - U_{0,1}^{j} U_{1,1}^{n+2}.$$

In this example we have $n_{2,j} = 1$ for any $1 < j < \omega^2$. We see that we can not continue to define U_{2,ω^2+1} the reason is that $(\beta_{2,\omega n})_2 = n+2 \to \infty (n \to \infty)$ and therefore necessarily $\beta_{2,\omega^2} \notin \mathbb{Z}^2 \oplus \{0\}$. Thus, as $\beta_{0,1}, \beta_{1,1} \in \mathbb{Z}^2 \oplus \{0\}$ there does not exist any relation between $\beta_{2,\omega^2}, \beta_{0,1}, \beta_{1,1}$ and we are forced to stop at this step.

Example 2.14 Consider the ring $k[X_0, X_1, X_2]$ and the group $\Phi = \mathbb{Q}$ with the usual order $\leq$. Consider the valuation ν centered on this ring by the SKP $(U_{0,1}, (U_{1,j})_{j=1}^{\omega}, (U_{2,j})_{j=1}^{\omega}, \beta_{i,j})$ which is defined as follows: Let $\{p_i\}_{i=1}^{\infty}$ be a sequence of increasing prime numbers. Define $\beta_{0,1} = 1$, $\beta_{1,1} = \frac{1}{p_1}$, $\beta_{1,j} = m_j + \frac{1}{p_j}$, for $j \geq 2$ where $m_2 = 1$ and $m_{j+1} = p_j m_j + 1$, and $\beta_{2,j} = \beta_{1,j}$, for $j \geq 1$. Then after setting $\theta_{i,j} = 1$, we have $U_{1,j+1} = U_{1,j}^{p_j} - U_{0,1}^{m_{j+1}}$ and $U_{2,j+1} = U_{2,j} - U_{1,j}$.

3 *adic* expansions

Given an SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$. In this section we show that any element f of the power series ring $k^{(d)}$ has a unique expansion in terms of key-polynomials. We give an algorithm for computing this expansion. The algorithm is based on the notion of acceptable vectors $\alpha' \leq \alpha$ attached to the SKP. Any acceptable vector determines an SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha'_i}$. We define the notion of $(U)_{\alpha'} - adic$ expansion and show how one can get $(U)_{\alpha''} - adic$ expansions for $\alpha'' \geq \alpha'$, using $(U)_{\alpha'} - adic$ expansion. In the next section, we use the *adic* expansion of the elements to define a valuation, attached to a given SKP.

Lemma 3.1 *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$. *When $U_{i,j} \neq 0$ it is of the form*

$$U_{i,j} = X_i^{d_{i,j}} + a_{i,j,d_{i,j}-1} X_i^{d_{i,j}-1} + \cdots + a_{i,j,0}$$

where $a_{i,j,j'} \in k^{(i-1)}$, such that the constant term of $a_{i,j,j'}$ is zero. Moreover, when j is not a limit ordinal, we have $d_{i,j} = n_{i,j-1} d_{i,j-1}$ for $1 \leq j < \alpha_i$. If j is a limit ordinal then there exists an ordinal $j_0 < j$, which is not a limit ordinal and for any j' such that $j_0 \leq j' \leq j$, we have $d_{i,j'} = d_{i,j_0} = n_{i,j_0-1} d_{i,j_0-1}$.

Proof. The proofs are all by induction. We prove the last part. By definition of SKP's, it is clear that for any $j' = 1, \ldots, j-1$, we have $m_{i,j'}^{(i,j)} \in S_{i,j}$, so we have $0 \leq m_{i,j'}^{(i,j)} < n_{i,j'}$. By induction we have $n_{i,j'} = d_{i,j'+1}/d_{i,j'}$. Hence $m_{i,j'}^{(i,j)} + 1 \leq d_{i,j'+1}/d_{i,j'}$. So we have

$$\sum_{j'=1}^{j-1} m_{i,j'}^{(i,j)} d_{i,j'} \leq \sum_{j'=1}^{j-1} (\frac{d_{i,j'+1}}{d_{i,j'}} - 1) d_{i,j'} = d_{i,j} - 1 < n_{i,j} d_{i,j}.$$

Hence $\deg_{X_i}(U_{i,j+1}) = n_{i,j} d_{i,j}$. For the last claim we note that when j is a limit ordinal there exists a j_0 such that for any j', $j_0 \leq j' \leq j$, we have $n_{i,j'} = 1$. $\square$

Lemma 3.2 *For any* SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$, *if $U_{i,j} \neq 0$ we have*

$$\deg_{X_i}(U_{i,j}) > \deg_{X_i}(\prod_{j'<j} U_{i,j'}^{p_{i,j'}}),$$

when $0 \leq p_{i,j'} < n_{i,j'}$. In other words $\sum_{j'<j} p_{i,j'} d_{i,j'} < d_{i,j}$. Notice that $p_{i,j'} = 0$, except for a finite number of j'.

Definition 3.3 Fix an SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$. We say that a vector $(\alpha'_0, \ldots, \alpha'_d)$ such that $\alpha'_i \leq \alpha_i$ is an acceptable vector if for any $i = 0, \ldots, d$ and any $j = 0, \ldots, \alpha'{}_i$ and for any $(i', j') \in S_{i,j}$ we have $(i', j') \leq_{lex} (i', \alpha_{i'})$ for $i' < i$, and $(i', j') <_{lex} (i, j)$ when $i' = i$. This means that in the equation (P1) defining $U_{i,j}$ in terms of U's with smaller indices, one needs only indices from α', not necessarily all of α. Notice that an acceptable vector α' determines an SKP, i.e., $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha'_i}$ is an SKP.

Given an SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$ the vector α is an acceptable vector. Moreover, the vector $(1, \ldots, 1) \in \mathbb{N}^d$ is an acceptable vector for any arbitrary SKP.

Definition 3.4 Given any SKP and any acceptable α', one can consider the new SKP defined by this acceptable vector and construct the power series ring $k_{((\alpha',i))} = k[[(U_{i',j'})_{i'\leq i, j'<\alpha'_i, n_{i',j'}\neq 1}, (U_{i',\alpha'_{i'}})_{i'\leq i}]] \subseteq k^{(d)}$. We have $k^{(i)} = k_{((\alpha,i))}$.

Given an SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$ and an acceptable vector $\alpha' = (\alpha'_0, \ldots, \alpha'_d)$, we want to expand an arbitrary element $f \in k^{(d)}$ in terms of U's as an element of the power series ring $k_{((\alpha',d))}$.

Definition 3.5 (*adic* **expansions)** Fix an SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$. Let α' be an acceptable vector for this SKP. For an element $f \in k^{(d)}$ consider the expansion $f = \sum_{I(J)} c_{I(J)} U^{I(J)} \in k_{((\alpha',d))}$, where $I(J) \in \mathbb{N}^1 \times \cdots \times \mathbb{N}^{\alpha'_i} \times \cdots \times \mathbb{N}^{\alpha'_d}$, and $c_{I(J)} \in k$. This expansion is called the $(U)_{\alpha'} - adic$ expansion of f, when for every monomial $U^{I(J)}$ we have $0 \leq I(J)_{i,j} < n_{i,j}$, for any $0 \leq j < \alpha'_i$ and $i = 0, \ldots, d$. Notice that $I(J)_{i,j} = 0$, except for a finite number of j.

Definition 3.6 Fix an SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$ and let α' be an acceptable vector. For any monomial $M(U) = U^{\mathbf{a}} \in k_{((\alpha',d))}$, we define

$$\mathrm{Vdeg}(M) = (\deg_{X_0}(U_0^{\mathbf{a}_0}), \deg_{X_1}(U_1^{\mathbf{a}_1}), \ldots, \deg_{X_d}(U_d^{\mathbf{a}_d})) \in \mathbb{N}^d.$$

Definition 3.7 Fix an SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$ and let α' be an acceptable vector. Let $M(U) = cU^{\mathbf{a}}$ be a monomial of the ring $k_{((\alpha',d))}$ we say that it is a monomial of *adic* form if it satisfies the conditions of monomials of Definition 3.5.

Lemma 3.8 *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$ *and let α' be an acceptable vector. Let $M(U) = cU^{\mathbf{a}} \in k_{((\alpha',d))}$ be a monomial of adic form with respect to this* SKP. *Then* Vdeg(M) *determines the vector* $\mathbf{a}$.

Proof. This is a simple consequence of Lemma 3.2. If we set $n = \deg_{X_i}(U_i^{\mathbf{a}_i})$ then we have $\mathbf{a}_{i,\alpha'_i} = [\frac{n}{d_{i,\alpha'_i}}]$. Suppose by induction we obtained $\mathbf{a}_{i,\alpha'_i}, \ldots, \mathbf{a}_{i,j+1}$ then we have: $\mathbf{a}_{i,j} = [\frac{n - \sum_{j'=j+1}^{\alpha'_i} \mathbf{a}_{i,j'} . d_{i,j'}}{d_{i,j}}]$. Note that if $\mathbf{a}_{i,j} \neq 0$ then for any $j' < j$ such that $d_{i,j} = d_{i,j'}$ we have $\mathbf{a}_{i,j'} = 0$. This shows that in the case of α'_i be of infinite ordinal type also the number of entries of $\mathbf{a}$ computed inductively above, which are nonzero is finite. □

Corollary 3.9 *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$. *Let α' be an acceptable vector. For any two different monomials M, M' of the power series ring $k_{((\alpha',d))}$, we say $M < M'$ if*

$$\mathrm{Vdeg}(M) <_{lex} \mathrm{Vdeg}(M').$$

This is a total well-ordering on the set of monomials of $k_{((\alpha,d))}$ of adic form.

The following proposition shows that the *adic* expansions are well defined elements of the ring $k_{((\alpha,d))}$ and they are unique and it gives an algorithm to compute them.

Proposition 3.10 (Algorithm for getting *adic* **expansions)** *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$. *Let α' and α'' be two acceptable vectors for this* SKP *such that $\alpha' < \alpha''$, with respect to the partial product order of $\mathbb{Z}^{d+1}$. Let $f \in k^{(d)}$ and suppose we know its $(U)_{\alpha'} - adic$ expansion. In order to obtain its $(U)_{\alpha''} - adic$ expansion we do the following:*

Starting from $(U)_{\alpha'} - adic$ expansion of f, for any monomial $M(U)$ in the expansion, and for any $i = 0, \ldots, d$ and $j < \alpha''_i$, do the following replacements, and iterate this process on the resulting expansion as far as possible.

- *If* $n_{i,j+1} > 1$, *replace any occurrence of* $U_{i,j}^{n_{i,j}}$ *in* $M(U)$ *by its equal binomial by* $(P1)$ *of Definition 2.4, namely,*
$$U_{i,j}^{n_{i,j}} = U_{i,j+1} + \theta_{i,j} U^{m^{(i,j)}}.$$
- *If* $n_{i,j+1} = 1$ *then let* $j+1 < j_0 \leq \alpha''_i$ *be the first ordinal such that* $n_{i,j_0} > 1$ *or* $j_0 = \alpha''_i$ *and replace any occurrence of* $U_{i,j}^{n_{i,j}}$ *in* $M(U)$ *by its equal expansion given in Remark 2.9, namely,*
$$U_{i,j}^{n_{i,j}} = U_{i,j_0} + \sum_{j \leq j' < j_0} \theta_{i,j'} U^{m^{(i,j')}}.$$

The resulting expansion is equal to the $(U)_{\alpha''} - adic$ *expansion of the element* f. *Moreover, this expansion is unique.*

Proof. For any element of $k_{((\alpha,d))}$ we define $\mathcal{M}_n$ to be those monomials with ord $= n$. By Lemma 2.12, we know that $\#\mathcal{M}_n$ is finite. We do the replacements of the algorithm (staring from $\alpha' - adic$ expansion of f) in the $n-$th step only on the monomials of $\bigcup_{n' \leq n} \mathcal{M}_{n'}$ of the current expansion. Using Lemma 6.6 of [7], this process terminates after finitely many steps. In this step all the monomials of $\bigcup_{n' \leq n} \mathcal{M}_{n'}$ of the current expansion are of $\alpha'' - adic$ form. Moreover, there exists a number $m(n) < n$, where $m(n) \to \infty (n \to \infty)$, such that in the process of replacements on the monomials of $\bigcup_{n' \leq n} \mathcal{M}_{n'}$ the monomials of $\bigcup_{m' \leq m(n)} \mathcal{M}_{m'}$ does not change (Lemma 2.12). Doing this process as $n \to \infty$ we get an expansion, which satisfies all the properties of $\alpha'' - adic$ expansion. Thus we obtain a $(U)_{\alpha''} - adic$ expansion of f.

Now, we prove that this expansion is unique. Suppose an element $f \in k^{(d)}$ has two different *adic* expansions $f = \sum_{I(J)} c_{I(J)} U^{I(J)} = \sum_{I''(J'')} c''_{I''(J'')} U^{I''(J'')}$. Assume by induction on d the claim is valid for the power series ring $R \otimes_k k^{(d-1)}$, where $R = k[[(U_{d,j})_{j<\alpha_d, n_{d,j} \neq 1}, U_{d,\alpha_d}]]$ is considered as the coefficient ring. Consider f as an element of the ring $R \otimes_k k^{(d-1)}$. The two *adic* expansions of f give two *adic* expansion of $f \in R \otimes_k k^{(d-1)}$ as follows. Setting $U = (U_{(d-1)}, U_d)$ and $I(J) = (I(J)_{(d-1)}, I(J)_d)$ we have

$$\begin{aligned} f &= \textstyle\sum_{I(J)_{(d-1)}} (\sum_{I'(J')_d, I(J)_{(d-1)} = I'(J')_{(d-1)}} c_{I'(J')} U_d^{I'(J')_d}) U_{(d-1)}^{I(J)_{(d-1)}} \\ &= \textstyle\sum_{I''(J'')_{(d-1)}} (\sum_{I'(J')_d, I''(J'')_{(d-1)} = I'(J')_{(d-1)}} c''_{I'(J')} U_d^{I'(J')_d}) U_{(d-1)}^{I''(J'')_{(d-1)}}. \end{aligned}$$

By induction hypothesis, these two *adic* expansions are the same. Suppose M be the least monomial of this expansion, with respect to the ordering of Corollary 3.9, which refers to the indices $I_0(J_0)$ and $I''_0(J''_0)$ (respectively). Then equating the coefficient of M in two *adic* expansions we have

$$g = \sum_{I'(J')_d, I_0(J_0)_{(d-1)} = I'(J')_{(d-1)}} c_{I'(J')} U_d^{I'(J')_d} = \sum_{I'(J')_d, I''_0(J''_0)_{(d-1)} = I'(J')_{(d-1)}} c''_{I'(J')} U_d^{I'(J')_d}.$$

Write $g \mid_{X_0=0,\ldots,X_{d-1}=0} = \sum_{\alpha \in \mathbb{Z}} c_\alpha X_d^\alpha$. Let α_0 be the first α such that $c_\alpha \neq 0$. Then by Lemma 3.1 and 3.8, there is a unique monomial in either of the expansions of g (M and M' respectively) such that $\mathrm{Vdeg}(M) = \mathrm{Vdeg}(M') = \alpha$ (Here $\mathrm{Vdeg}(M) = \deg_{X_d}(M)$). Hence $M = M'$. Thus the least monomials of two expansions of g (with respect to the ordering of Corollary 3.9) are equal. Subtracting this monomial from two representations, and iterating the last procedure we deduce that these two expansions are the same and we are done (A similar argument like the last part works for the initial of the induction $d = 1$). □

Remark 3.11 Given an SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$, and an element $f \in k^{(d)}$ in order to obtain its $(U)_\alpha - adic$ expansion, we can use the algorithm of Proposition 3.10 for the acceptable vectors $\alpha' = (1, \ldots, 1)$ and $\alpha'' = \alpha$. Notice that in this case the $(U)_{\alpha'} - adic$ expansion of every element $f \in k^{(d)}$ is itself.

We also use the notation of $(\alpha') - adic$ expansion. When there is no stress on the special acceptable vector α' or they are clear, we use the simple notation $U_d - adic$ or *adic* expansion.

Remark 3.12 The algorithm of Proposition 3.10, can be applied to obtain the *adic* expansion of an element f starting from any representation of it. For example, suppose two elements f and g are given and we know their *adic* expansions. We can apply the algorithm of Proposition 3.10, to the product of the expansions of f and g to obtain the *adic* expansion of the product fg.

4 Valuations attached to SKP's

In this section we show that to any SKP one can attach a valuation ν of the field $k((X_0,\ldots,X_d))$ centered on the ring $k[[X_0,\ldots,X_d]]$.

Definition 4.1 Let $[U_{i,j},\beta_{i,j}]$ be an SKP. For an acceptable vector α', we define a map

$$\nu_{\alpha'}:\ k^{(d)}\backslash\{0\}\to\Phi$$

by:

- If M is any monomial $M(U)$ with $(U)_{\alpha'}-adic$ expansion $M=c.U^{\mathbf{p}}$, where $c\in k$ then

$$\nu_{\alpha'}(M)=\sum_{i=0}^{d}\sum_{j=0}^{\alpha'_i}p_{i,j}\beta_{i,j}.$$

- If $f\in k^{(d)}$ has the $(U)_{\alpha'}-adic$ expansion $f=\sum_{I(J)}c_{I(J)}U^{I(J)}$ then

$$\nu_{\alpha'}(f)=\min_{I(J)}\{\nu_{\alpha'}(U^{I(J)})\}.$$

For any SKP, we denote the mapping of the definition above by $\nu_\alpha=\mathrm{val}[U_{i,j},\beta_{i,j}]$. We will see that this mapping is a valuation (Theorem 4.7).

Definition 4.2 Let $[U_{i,j},\beta_{i,j}]$ be an SKP and $f\in k^{(d)}$ an arbitrary element and let (α') be an acceptable vector for this SKP. The initial form of f with respect to $\nu_{\alpha'}$ is defined as:

$$\mathrm{in}_{\nu_{\alpha'}}(f)=\sum_{I(J^0)}c_{I(J^0)}U^{I(J^0)},$$

where $f=\sum_{I(J)}c_{I(J)}U^{I(J)}$ is the $(U)_{\alpha'}-adic$ expansion of f and $I(J^0)$ ranges over those indices with minimal $\nu_{\alpha'}-$value.

Definition 4.3 Let $[U_{i,j},\beta_{i,j}]_{i=0\cdots d,j=1\cdots\alpha_i}$ be an SKP and consider the power series ring $k_{((\alpha,d))}$. For any monomial $M(U)=U^{\mathbf{a}}\in k_{((\alpha,d))}$ we define the vector of the powers

$$\mathrm{VP}(M(U))=(\mathbf{a}_{d,\alpha_d},\mathbf{a}_{d-1,\alpha_{d-1}},\ldots,\mathbf{a}_{0,\alpha_0})\in\mathbb{N}^{d+1}.$$

Lemma 4.4 *Fix an* SKP *and suppose that* α' *is an acceptable vector for this* SKP. *Let* $f\in k^{(d)}$ *and suppose* $\mathrm{in}_{\nu_{\alpha'}}(f)=\sum_{I(J)}c_{I(J)}U^{I(J)}$ *then the vector of the powers* $\mathrm{VP}(M)$ *of the monomials* M *of* $\mathrm{in}_{\nu_{\alpha'}}(f)$ *are all different.*

Proof. Let $cU^{I(J)}$ and $c'U^{I'(J')}$ be two monomials of $\mathrm{in}_{\nu_{\alpha'}}(f)$ with equal vector of the powers. We show that for any $j=1,\ldots,\alpha'_d$ the powers of the $U_{d,j}$ in the two monomials are the same. Indeed, let $j'<\alpha'_d$ be the greatest index such that $I(J)_{d,j'}\neq I'(J')_{d,j'}$, note that this maximum index exists. We assume $I(J)_{d,j'}>I'(J')_{d,j'}$. By equating the $\nu_{\alpha'}-$values of the two monomials

$$(I(J)_{d,j'}-I'(J')_{d,j'})\beta_{d,j'}=\sum_{(i'',j'')<_{lex}(d,j')}-(I(J)_{i'',j''}-I'(J')_{i'',j''})\beta_{i'',j''}\in(\beta_{i'',j''})_{(i'',j'')<_{lex}(d,j')},$$

which is clearly a contradiction, because $0<I(J)_{d,j'}-I'(J')_{d,j'}<n_{d,j'}$. Continuing similar argument for $i<d$, we deduce that the two monomials are the same. □

Corollary 4.5 *Fix an* SKP *and suppose* $U_{i,\alpha_i}=0$, *for* $i=1\cdots d$. *Then for any* $0\neq f\in k^{(d)}$ *the initial* $\mathrm{in}_{\nu_\alpha}(f)$ *consists of just one monomial of adic form.*

Lemma 4.6 *Fix an* SKP $[U_{i,j},\beta_{i,j}]$ *and let* α' *be an acceptable vector. For any arbitrary monomial* $M(U)\in k_{((\alpha',d))}$, *where* $M=c.U^{\mathbf{a}}$, *we have:*

(*i*) *The initial form of* M *in its* $(U)_{\alpha'}-adic$ *expansion is just one monomial* $M'=c'U^{\mathbf{a}'}$. *In other words, we have* $\mathrm{in}_{\nu_{\alpha'}}(M)=M'$.

(*ii*) *We have* $\mathbf{a'}_{d,\alpha'_d} = \mathbf{a}_{d,\alpha'_d}$.

(*iii*) *For any two monomials* M *and* M' *of the power series ring* $k_{((\alpha',d))}$ *with equal* $\nu_{\alpha'}-$*values, if* $\mathrm{VP}(M) <_{lex} \mathrm{VP}(M')$ *then* $\mathrm{VP}(\mathrm{in}_{\nu_{\alpha'}}(M)) <_{lex} \mathrm{VP}(\mathrm{in}_{\nu_{\alpha'}}(M'))$.

Proof. For the first claim, let $U_{i,j}$ be a factor of the M with power greater than $n_{i,j}$. Replace $U_{i,j}^{n_{i,j}}$ by its expression from the algorithm for getting *adic* expansion. The claim is that after one such replacement there exists just one monomial with minimal $\nu_{\alpha'}-$value. We prove the claim for the replacements of the first type of algorithm for getting *adic* expansion. For the second type the argument is similar. After a replacement of type one we get two monomials with different $\nu_{\alpha'}-$values:

$$\begin{aligned} M &= \tfrac{M}{U_{i,j}^{n_{i,j}}}(U_{i,j+1} + \theta_{i,j}U^{m^{(i,j)}}) \\ &= \underbrace{c\frac{U^{\mathbf{a}}.U_{i,j+1}}{U_{i,j}^{n_{i,j}}}}_{M_2} + \underbrace{c\theta_{i,j}\frac{U^{\mathbf{a}}U^{m^{(i,j)}}}{U_{i,j}^{n_{i,j}}}}_{M_1}. \end{aligned}$$

Then $\nu_{\alpha'}(M_2) \succ \nu_{\alpha'}(M_1) = \nu_{\alpha'}(M)$. Therefore, we have $\mathrm{in}_{\nu_{\alpha'}}(M) = \mathrm{in}_{\nu_{\alpha'}}(M_1)$. We do the same for M_1. Finally we get a monomial M' whose *adic* expansion is itself, this proves (i).

For the second part we notice the that the proof of the first part shows the following general fact: For the monomial $M(U)$ a replacement on $U_{i,j}^{n_{i,j}}$ can not affect the power of $U_{i',j'}$, for $(i',j') >_{lex.} (i,j)$, of the unique monomial with minimal value of the expansion generated after replacement.

For the last part, suppose $M = U^{\mathbf{a}}$ and $M' = U^{\mathbf{a}'}$. Let $d' \le d$ be the first index such that $\mathbf{a}_{d',\alpha'_{d'}} < \mathbf{a}'_{d',\alpha'_{d'}}$. Then by Lemma 4.4, we have $\mathbf{a}_{i,j} = \mathbf{a}'_{i,j}$ for $i = d'+1,\dots,d$ and $j = 1,\dots,\alpha_i$. Thus the algorithm for getting *adic* expansion for these two monomials for such i and j can be chosen the same. Hence, without loss of generality we can assume that $\mathbf{a}_{i,j} < n_{i,j}$ and $\mathbf{a}'_{i,j} < n_{i,j}$, for $i = d'+1,\dots,d$ and $j < \alpha_i$. Then because $\mathbf{a}_{d',\alpha'_{d'}} < \mathbf{a}'_{d',\alpha'_{d'}}$, by part ($ii$) we are done. □

Theorem 4.7 *Given any* SKP $[U_{i,j},\beta_{i,j}]$, *for any acceptable vector* α', *the mapping* $\nu_{\alpha'}:\ k^{(d)}\backslash\{0\} \to \Phi$ *extends trivially to a* $k-$*valuation of the field* $k((X_0,\dots,X_d))$. *Moreover, for any two acceptable vectors* α' *and* α'' *such that* $\alpha' \le \alpha''$ *and for any* $f \in k^{(d)}$ *we have* $\nu_{\alpha'}(f) \preceq \nu_{\alpha''}(f)$.

Proof. The extension to the field $k((X_0,\dots,X_d))$ is a trivial task. We need only to prove that given any $f,g \in k^{(d)}\backslash\{0\}$ we have $\nu_{\alpha'}(f+g) \succeq \min\{\nu_{\alpha'}(f),\nu_{\alpha'}(g)\}$ and $\nu_{\alpha'}(f.g) = \nu_{\alpha'}(f) + \nu_{\alpha'}(g)$. The first one is a direct consequence of the definition and the uniqueness of the *adic* expansions. For the second equality, let $\mathrm{in}(f) = \sum_{I(J)} c_{I(J)}U^{I(J)}$ and $\mathrm{in}(g) = \sum_{I'(J')} c'_{I'(J')}U^{I'(J')}$. Let $M = c_{I(J_0)}U^{I(J_0)}$ (respectively $M' = c_{I'(J'_0)}U^{I'(J'_0)}$) be the unique (Lemma 4.4) monomial of the expansion of $\mathrm{in}(f)$ (respectively $\mathrm{in}(g)$) with minimal vector of powers, with respect to the *lex.* order. Then by Lemma 4.6, (iii), we see that $\mathrm{in}(M.M') = M''$ is the unique monomial of $\mathrm{in}(f.g)$ with minimal vector of the powers. But $\nu_{\alpha'}(M'') = \nu_{\alpha'}(M) + \nu_{\alpha'}(M') = \nu_{\alpha'}(f) + \nu_{\alpha'}(g)$, by the definition of the mapping $\nu_{\alpha'}$ we have $\nu_{\alpha'}(M'') = \nu_{\alpha'}(f.g)$. For the last part, we note that in the algorithm for getting $\alpha'' - adic$ expansion of an element from its $\alpha' - adic$ expansion every time of substitution we replace a monomial with two new monomials with values equal to or larger than the original monomial. □

Corollary 4.8 *Given an* SKP $[U_{i,j},\beta_{i,j}]_{i=0\cdots d,j=1\cdots\alpha_i}$, *all the* $U_{i,j}$*'s are irreducible elements of the power series ring* $k^{(i-1)}[X_i]$.

Proof. We prove the claim for $U_{d,j}$. Consider the vector (α'), defined by $\alpha'_i = \alpha_i$, for $0 \le i < d$, and $\alpha'_d = j$. This is an acceptable vector. In this proof all the *adic* expansions are $(U)_{\alpha'} - adic$ expansions. Let $U_{d,j}$ be reducible and $U_{d,j} = f.g$, for some non-unit elements $f,g \in k^{(d-1)}[X_d]$. As the $\alpha' - adic$ expansion of $U_{d,j}$ is itself, we have $\mathrm{in}(U_{d,j}) = U_{d,j}$. We can compute this initial in the other way, using initials of f and g. This gives us $U_{d,j} = \mathrm{in}(\mathrm{in}(f).\mathrm{in}(g))$.

On the other hand, $\beta_{d,j} = \nu_{\alpha'}(U_{d,j}) = \nu_{\alpha'}(f) + \nu_{\alpha'}(g)$. Thus the monomials of $\mathrm{in}(f)$ and $\mathrm{in}(g)$) does not have a factor $U_{d,j}$. By Lemma 4.6, (ii), this shows that the monomials $\mathrm{in}(\mathrm{in}(f).\mathrm{in}(g))$ does not have a factor $U_{d,j}$, which is a contradiction. □

Remark 4.9 One should note that in the definition of the SKP's for the ring $k[[X_0,\dots,X_d]]$ the ordering of the variables plays an important role. In other words, changing the coordinates of the rings (even with a permutation) may change totally the system of SKP's attached to the valuation, or even they may not exist. This phenomenon can be seen even in dimension two; For example consider the valuation ν centered on the ring $k[X_0,X_1]$ defined by the SKP, $[(U_{0,1},U_{1,1},U_{1,2},U_{1,3}),(2,3,9,10)]$. Where, we have $U_{1,2}=U_{1,1}^2-U_{0,1}^3,\ U_{1,3}=U_{1,2}-U_{0,1}^3U_{1,1}$. Note that the last two equations are given to us (up to the knowledge of the corresponding θ's) as soon as the sequence of β's $(2,3,9)$ is known. Now, changing the order of the coordinates, we consider the same ring as $k[Y_0,Y_1]$ with $Y_0=X_1,Y_1=X_0$. The same valuation is given by the following SKP's in the new coordinate $\nu=\text{val}[(V_{0,1},V_{1,1},V_{1,2},V_{1,3}),(3,2,9,10)]$. Where the SKP's are as follows.

$$V_{1,2}=V_{1,1}^3-V_{0,1}^2,\ V_{1,3}=V_{1,2}+V_{0,1}^3.$$

The relation between two SKP's is as follows:

$$V_{0,1}=U_{1,1},\ V_{1,1}=U_{0,1},\ V_{1,2}=-U_{1,2}.$$

For $V_{1,3}$ we have:

$$V_{1,3}=V_{1,2}+V_{0,1}^3=-U_{1,2}+U_{1,1}^3=-U_{1,2}+(U_{0,1}^3+U_{1,2})U_{1,1}=-U_{1,3}+U_{1,1}U_{1,2}.$$

As this example shows the explicit relation between the U's and V's is not, in general, trivial.

5 Euclidean expansion and more properties of SKP's

In this section we give another expansion in the ring $k_{(d-1)}[X_d]$, attached to a SKP of the power series ring $k[[X_0,\dots,X_d]]$ $(k_{(i)}:=k((X_0,\dots,X_i)))$. We show that the valuation ν attached to this SKP, can be defined using this new expansion, plus the knowledge of the valuation ν on the field $k_{(d-1)}$. Moreover, we show that the Euclidean expansion can be obtained directly from the *adic* expansion. This is practically interesting, because *adic* expansion is defined only with substitutions while Euclidean expansion is defined using divisions.

Definition 5.1 (**Euclidean expansion**) Fix an SKP $[U_{i,j},\beta_{i,j}]_{i=0\dots,j=1\dots\alpha_i}$. For any $j=1,\dots,\alpha_d$ we define the acceptable vector $\alpha^{(j)}=(\alpha_0,\dots,\alpha_{d-1},j)$. Let $f\in k_{(d-1)}[X_d]$, and consider the expansion $f=\sum_J c_JU_d^J\in k_{(d-1)}[U_d]$ such that $0\le J_{j'}<n_{d,j'}$ for any $0\le j'<j$. This is called the jth Euclidean expansion of f.

Proposition 5.2 (Algorithm for getting Euclidean expansion) *With the notations of Definition 5.1, do the following:*

Consider the greatest index j_0 such that $\deg_{X_d}(f)>d_{d,j_0}$. Divide f by U_{d,j_0} in the ring $k_{(d-1)}[X_d]$ to obtain $f=qU_{d,j_0}+r$, where $q,r\in k_{(d-1)}[X_d]$ and $\deg_{X_d}(r)<d_{d,j_0}$. Iterate the same process for q as far as possible to obtain $f=\sum_t f_tU_{d,j_0}^t$, where $\deg_{X_d}(f_t)<d_{d,j_0}$. Iterate the same procedure for each of the f_t's and the greatest index j', $j'<j_0$, such that $d_{d,j'}<d_{d,j_0}$. Continue as far as possible. This process terminates after finitely many steps. The resulting expansion is equal to the jth *Euclidean expansion of f. Moreover, the Euclidean expansion is unique.*

Proof. As the U_d's which appear in the process are among the elements of the finite set $\{U_{d,j'}/\ n_{d,j'}\neq 1,\text{ and }\deg_{X_d}(f)>d_{d,j'}\}$, the process stops after finitely many steps. We show that the resulting expansion is the jth Euclidean expansion of f. Let U_d^J be a monomial generated in the algorithm above. It is sufficient to show that this monomial is of Euclidean form. Indeed, let j' be the greatest index less than j such that $J_{j'}\ge n_{d,j}$. This means that $\deg_{X_d}(U_{d,1}^{J_1}\cdots U_{d,j'}^{J_{j'}})\ge d_{d,j'+1}$ and we must divide it (in the monomial in the procedure above) by $U_{d,j'+1}$, which is a contradiction.

The uniqueness of Euclidean expansion comes from the fact that (by Lemma 3.8) the $\deg_{X_d}(U_d^J)$ of a monomial of Euclidean form determines the vector J. Therefore there is a unique vector J_0 such that $\deg_{X_d}(U_d^{J_0})=\deg_{X_d}(f)$. This monomial (plus its coefficient)is common in all the possible Euclidean expansions of f. Subtracting this monomial from f, by induction on the degree of f we are done. □

Lemma 5.3 *Fix an* SKP $[U_{i,j},\beta_{i,j}]_{i=0\cdots d,j=1\cdots\alpha_i}$ *and let $f\in k[[X_0,\dots,X_d]]$. The* jth *Euclidean expansion of f can be obtained using the $(\alpha^{(j)})-$adic expansion of it as follows. Let $f=\sum_{I(J)}c_{I(J)}U^{I(J)}$ be the $(\alpha^{(j)})-$ adic expansion of f. Then the Euclidean expansion of f is equal to*

$$\sum_{J'}\Big(\sum_{I(J),I(J)_d=J'}c_{I(J)}\frac{U^{I(J)}}{U_d^{J'}}\Big)U_d^{J'}.$$

Proof. It is clear that the above expansion satisfies all the properties of the jth Euclidean expansion of f. Thus, by uniqueness, it is the Euclidean expansion of f. $\square$

Remark 5.4 Using the above lemma, we extend the notion of Euclidean expansion to the power series ring $k^{(d)}$. An expansion of $f \in k^{(d)}$ of the form $f = \sum_J c_J U_d^J \in k^{(d-1)}[[U_d]]$ which satisfies the conditions of Definition 5.1 is called the Euclidean expansion of f. The above lemma shows that such an expansion can be obtained using *adic* expansion of f. An argument, similar to the proof of Proposition 3.10, shows that this expansion is unique.

Proposition 5.5 *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$ *and let* ν *be the* $k-$*valuation of the field* $k_{(d)}$ *attached to it. Set* $\overline{\nu} = \nu\mid_{k_{(d-1)}}$. *The valuation* ν *(as a valuation of the field* $k_{(d-1)}(X_d)$*) can be defined using the data* $[\overline{\nu}, (U_{d,j})_{j=1}^{\alpha_d}, (\beta_{d,j})_{j=1}^{\alpha_d}]$ *as follows. For any* $f \in k^{(d-1)}[X_d]$ *let* $f = \sum_J f_J U_d^J$ *be its* α_dth *Euclidean expansion then*

$$\nu(f) = \min_J\{\overline{\nu}(f_J) + \beta_d.J\}.$$

Proof. The lemma above shows that the equation of the proposition is just another writing for $\nu(f)$, which is originally the minimum of the values of the monomials in the *adic* expansion of f. $\square$

Remark 5.6 With the notations of the proposition above, if we write $f = \sum_t f_t U_{d,j}^t$, with $\deg_{X_d}(f_t) < d_{d,j}$. Then with a similar argument we have

$$\nu(f) = \min_t\{\nu(f_t) + t\beta_{d,j}\}.$$

Proposition 5.7 *The graded algebra* $\mathrm{gr}_{\nu_\alpha} k_{(d-1)}[X_d]$ *is a Euclidean domain.*

Definition 5.8 Fix an SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$. We consider the set of acceptable vectors $\alpha^{(j)} = (\alpha_0, \ldots, \alpha_{d-1}, j)$, for $j = 1, \ldots, \alpha_d$.

For any $f \in k_{(d-1)}[X_d]$, and any $\alpha^{(j)}$ we define

$$\delta_{\alpha^{(j)}}(f) = \max\{\ell : \ \ell \text{ is power of } U_{d,j} \text{ in the monomials of } \mathrm{in}_{\nu^{\alpha^{(j)}}}(f)\}.$$

Remark 5.9 Let $u \in k^{(d-1)}$ and $f \in k_{(d-1)}[X_d]$ then $\delta_{\alpha^{(j)}}(f) = \delta_{\alpha^{(j)}}(uf)$.

Lemma 5.10 *For any* $f, g \in k_{(d-1)}[X_d]$ *we have*

$$\delta_{\alpha^{(j)}}(f.g) = \delta_{\alpha^{(j)}}(f) + \delta_{\alpha^{(j)}}(g).$$

Proof. First we find $u, v \in k^{(d-1)}$ such that $uf, vg \in k^{(d-1)}[X_d]$, this is always possible. Then by last remark it suffices to prove the lemma for uf and vg, i.e., we can assume $f, g \in k^{(d-1)}[X_d]$. Lemma 4.4 shows that there are unique monomials $f_J U_d^J$ and $g_{J'} U_d^{J'}$ of in(f) and in(g) (respectively) that have maximal $U_{d,j}$ power. Write Euclidean expansion of in($f.g$) using the product in(f).in(g) and algorithm for getting *adic* expansion. We see in(f).in(g) has a unique monomial with $U_{d,j}-$degree equal $\delta_{\alpha^{(j)}}(f) + \delta_{\alpha^{(j)}}(g)$, i.e., $f_J g_{J'} U^J U^{J'}$. Now, Lemma 4.6, (*ii*), shows that after getting *adic* expansion from this product the $U_{d,j}-$powers of the monomials do not change which proves the equality. $\square$

The following lemma is an adaption of the results of [3] in our situation.

Lemma 5.11 *Fix an* SKP $[U_{i,j}, \beta_{i,j}]$, *and let* $\alpha^{(j)}$ *be defined as in Definition 5.8 then*

(i) *For* $f \in k_{(d-1)}[X_d]$, *we have* $\delta_{\alpha^{(j)}}(f) = 0$ *iff* f *is a unit in* $\mathrm{gr}_{\nu_{\alpha^{(j)}}} k_{(d-1)}[X_d]$.

(ii) *If* $f, g \in k_{(d-1)}[X_d]$ *then there exists* $Q, R \in k_{(d-1)}[X_d]$ *such that* $f = Qg + R$ *in* $\mathrm{gr}_{\nu_{\alpha^{(j)}}} k_{(d-1)}[X_d]$ *and* $\delta_{\alpha^{(j)}}(R) < \delta_{\alpha^{(j)}}(g)$.

(iii) *The polynomials* $U_{d,\alpha_j^{(j)}}, U_{d,\alpha_{j+1}^{(j)}}$ *are irreducible in* $\mathrm{gr}_{\nu_{\alpha^{(j)}}} k_{(d-1)}[X_d]$.

(iv) *If* $j' < j$ *then* $U_{d,j'}$ *is a unit in* $\mathrm{gr}_{\nu_{\alpha^{(j)}}} k_{(d-1)}[X_d]$.

(v) *If* $f=\sum_t f_tU_{d,j}^t$, *with* $\deg_{X_d}(f_t)<d_{d,j}$ *and* $\delta_{\alpha^{(j)}}(f)<n_{d,j}$ *then* $f=f_tU_{d,j}^t$ *in* $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$ *for some* $t<n_{d,j}$.

Proof. Throughout the proof we fix the expansion $f=\sum_t f_tU_{d,j}^t$, with $\deg_{X_d}(f_t)<d_{d,j}$.
(i). If $\delta_{\alpha^{(j)}}(f)=0$ then $f=f_0$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$. As $U_{d,j}$ is irreducible and $\deg_{X_d}(f_0)<d_{d,j}$ the polynomial $U_{d,j}$ is prime with f_0. Hence we can find $A,B\in k_{(d-1)}[X_d]$, $\deg_{X_d}(A),\deg_{X_d}(B)<d_{d,j}$ so that $Af_0=1-BU_{d,j}$. Then $\nu_{\alpha^{(j)}}(Af_0)=\nu_{\alpha^{(j)}}(1)\prec\nu_{\alpha^{(j)}}(BU_{d,j})$. Therefore, $Af_0=1$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$. So f_0 and hence f is a unit in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$. Conversely, if f is unit, say $Af=1$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$ for some $A\in k_{(d-1)}[X_d]$ then $\delta_{\alpha^{(j)}}(f)+\delta_{\alpha^{(j)}}(A)=\delta_{\alpha^{(j)}}(1)=0$ so $\delta_{\alpha^{(j)}}(f)=0$.
(ii). Write $g=\sum_t g_tU_{d,j}^t$. It suffices to prove the claim when $g_t=0$ for $t>M:=\delta_{\alpha^{(j)}}(g)$ and using (i) we may assume $g_M=1$. As $\deg_{X_d}(g_t)<d_{d,j}$ for $t\le M$ we have $\deg_{X_d}(g)=Md_{d,j}$. Euclidean division in $k_{(d-1)}[X_d]$ yields $Q,R^1\in k_{(d-1)}[X_d]$ with $\deg_{X_d}(R^1)<\deg_{X_d}(g)$ so that $f=Qg+R^1$. Write $R^1=\sum_t R_tU_{d,j}^t$ and set $N:=\delta_{\alpha^{(j)}}(R^1)$, $R:=\sum_{t\le N}R_tU_{d,j}^t$. Then $f=Qg+R$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$ and

$$\deg_{X_d}(R)=\deg_{X_d}(R_N)+Nd_{d,j}<Md_{d,j}=\deg_{X_d}(f).$$

Hence $N<M$ and we are done.
(iii). We have $\delta_{\alpha^{(j)}}(U_{d,j})=1$ so if $U_{d,j}=fg$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$ then $\delta_{\nu_{\alpha^{(j)}}}(f)=0$ or $\delta_{\alpha^{(j)}}(g)=0$. Hence by (i), f or g is a unit in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$.
For $U_{d,j+1}$, we have $U_{d,j+1}=U_{d,j}^{n_{d,j}}-\theta_{d,j}U^{m^{(d,j)}}$. Let $U_{d,j+1}=fg$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$ with $0<\delta_{\alpha^{(j)}}(f),\delta_{\alpha^{(j)}}<n_{d,j}$. By (v), we can write $f=f_tU_{d,j}^t$, $g=g_{t'}U_{d,j}^{t'}$. Then $U_{d,j+1}=f_tg_{t'}U_{d,j}^{n_{d,j}}$ so $(1-f_tg_{t'})U_{d,j}^{n_{d,j}}=\theta_{d,j}U^{m^{(d,j)}}$. As $U_{d,j}$ is irreducible and $U^{m^{(d,j)}}$ a unit, we have $f_tg_{t'}=1$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$. But then $U^{m^{(d,j)}}=0$ in $\mathrm{gr}_{\nu_{\alpha^{(j)}}}k_{(d-1)}[X_d]$ which is absurd. So we can assume $\delta_{\alpha^{(j)}}(f)=n_{d,j}$ and $\delta_{\alpha^{(j)}}=0$. Hence g is a unit.
(iv). By (i) it suffices to show that $\delta_{\alpha^{(j)}}(U_{d,j'})=0$. If $d_{d,j'}<d_{d,j}$ then this is obvious. If $d_{d,j'}=d_{d,j}$ then $U_{d,j'}=(U_{d,j'}-U_{d,j})+U_{d,j}$ where $\deg_{X_d}(U_{d,j'}-U_{d,j})<d_{d,j}$. Now $\nu_{\alpha^{(j)}}(U_{d,j'})=\beta_{d,j'}<\beta_{d,j}=\nu_{\alpha^{(j)}}(U_{d,j})$, so $\nu_{\alpha^{(j)}}(U_{d,j'}-U_{d,j})<\nu_{\alpha^{(j)}}(U_{d,j})$ and $\delta_{\alpha^{(j)}}(U'_j)=0$.
(v). Suppose $\nu_{\alpha^{(j)}}(f_tU_{d,j}^t)=\nu_{\alpha^{(j)}}(f_{t'}U_{d,j}^{t'})$, where $t\le t'<n_{d,j}$. Then $(t'-t)\beta_{d,j}=\nu_{\alpha^{(j-1)}}(f_t)-\nu_{\alpha^{(j-1)}}(f_{t'})$. Hence $n_{d,j}\mid t'-t$ thus $t'=t$. □

Proof of Proposition 5.7: The item (ii) proves the claim. □

Theorem 5.12 *Fix an* SKP $[U_{i,j},\beta_{i,j}]_{i=0\cdots d,j=1\cdots\alpha_i}$ *and let* ν *be its associated valuation. Consider* $0\neq f\in k[[X_1,\ldots,X_d]]$. *Then initial form of* f *has a unique decomposition of the form:*

(i) *If* $U_{d,\alpha_d}\neq 0,\ n_{d,\alpha_d}=\infty$ *then*

$$f=\tilde{f}U_d^J,\ \text{in }\mathrm{gr}_\nu k_{(d-1)}[X_d],$$

where $\tilde{f}\in k_{(d-1)}$ *and* $0\le J_j<n_{d,j}$, *for* $1\le j<\alpha_d$.

(ii) *If* $U_{d,\alpha_d}\neq 0,\ n_{d,\alpha_d}\neq\infty$ *then*

$$f=p(T)U_d^{\hat{J}},\ \text{in }\mathrm{gr}_\nu k_{(d)},$$

where $p(T)\in k_{(d-1)}[T]$ *and* $0\le J_j<n_{d,j}$, *for* $1\le j\le\alpha_d$, *and* $T=U_{d,\alpha_d}^{n_{d,\alpha_d}}U^{-m^{(d,\alpha_d)}}$. *Moreover, the coefficients of* $p(T)$ *has the same* $\nu-$*value.*

(iii) *If* $U_{d,\alpha_d}=0$ *then*

$$f=\tilde{f}U_d^J,\ \text{in }\mathrm{gr}_\nu k_{(d-1)}[X_d],$$

where $0\le J_j<n_{d,j}$, *for* $1\le j\le\alpha_d$, *and* $J_j=0$, *except for a finite number of* j.

Proof. (i). Suppose $f=\sum_J f_JU_d^J$ is the Euclidean expansion of f (Remark 5.4), where $f_J\in k_{(d-1)}$, and $0\le J_j<n_{d,j}$ for $j<\alpha_d$. We claim that for any two J and J' we have $\nu(f_JU_d^J)\neq\nu(f_{J'}U_d^{J'})$. Indeed, if we have equality, consider the greatest index j_0 such that $J_{j_0}\neq J'_{j_0}$. We have $(J'_{j_0}-J_{j_0})\beta_{d,j_0}=\nu(f_J)-\nu(f_{J'})+\sum j<j_0(J_j-J'_j)\beta_{d,j}$. Then as $j_0<\alpha_d$ (because $n_{d,\alpha_d}=\infty$), we have $n_{d,j_0}\mid J_{j_0}-J'_{j_0}$. Thus $J_{j_0}=J'_{j_0}$ which is absurd.

(ii). We show that any monomial $f_J U_d^J$ of the Euclidean expansion of $\mathrm{in}(f)$ is of the form $\hat{f}_J T^{r_{\alpha_d}} U_d^{\hat{J}}$, in $\mathrm{gr}_\nu k^{(d)}$, for a fixed $\hat{J}$ such that $0 \leq \hat{J}_j < n_{d,j}$, for any j.
Fix J, and make the Euclidean division $J_{\alpha_d} = r_{\alpha_d} n_{d,\alpha_d} + \hat{J}_{\alpha_d}$, $0 \leq \hat{J}_{\alpha_d} < n_{d,\alpha_d}$. And write $f_J U_d^J = \overline{f}_J U_{d,\alpha_d}^{\hat{J}_{\alpha_d}} T^{r_{\alpha_d}} U_d^{\mathbf{a}}$, with $\mathbf{a} := J + r_{\alpha_d}.m_d^{(d,\alpha_d)}$. As

$$U_{d,j}^{n_{d,j}} = \theta_{d,j}(U_{<d-1}^{m_{<d-1}^{(d,j)}}) U_d^{m_d^{(d,j)}}, \text{ in } \mathrm{gr}_\nu k_{(d-1)}[X_d],$$

making the Euclidean division $\mathbf{a}_j = r_j n_{d,j} + \hat{J}_j$, (with $0 \leq \hat{J}_j < n_{d,j}$) for the greatest index j such that $\mathbf{a}_j \neq 0$, we get $\prod_{j' \leq j} U_{d,j'}^{\mathbf{a}_{j'}} = U_{d,j}^{\hat{J}_j} \prod_{j'<j} U_{d,j'}^{\mathbf{a}'_{j'}}$ with $\mathbf{a}'_{j'} \in \mathbb{N}$. We finally get by induction, a representation

$$f_J U_d^J = \hat{f}_J T^{r_{\alpha_d}} U_d^{\hat{J}},$$

where $0 \leq \hat{J}_j < n_{d,j}$, for any j. As $\nu(T) = 0$, with an argument like in the final part of the case (i) one can argue to show that $\hat{J}$ is the same for all J's. Clearly, the coefficients of p has the same $\nu-$value.
(iii). This is similar to (i). □

Corollary 5.13 *Let ν be a valuation as above.*

- (i) *If $U_{d,\alpha_d} \neq 0$, $n_{d,\alpha_d} \neq \infty$ the only irreducible element of $\mathrm{gr}_\nu k_{(d-1)}[X_d]$ is U_{d,α_d}.*
- (ii) *If $U_{d,\alpha_d} \neq 0$, $n_{d,\alpha_d} < \infty$ and moreover we impose the following strong condition: For every two monomial of adic form $U^I, U^J \in k_{(d-1)}$, from $\nu(U^I) = \nu(U^J)$ one gets $U^I = U^J$. Then the irreducible elements of $\mathrm{gr}_\nu k_{(d-1)}[X_d]$ are of the form $U_{d,\alpha_d}^{n_{d,\alpha_d}} - \theta U^{m^{(d,\alpha_d)}}$, for some $\theta \in k$.*
- (iii) *If $U_{d,\alpha_d} = 0$ then $\mathrm{gr}_\nu k_{(d-1)}[X_d]$ is a field.*

Proof. (i). Assume $f \in \mathrm{gr}_\nu k_{(d-1)}[X_d]$ is irreducible. By (i) of the last theorem, $f = \tilde{f} U_d^J$. But $U_{d,j}$ is a unit for $j < \alpha_d$ (by Lemma 5.11, (iv)), so U_{d,α_d} is the only irreducible element in $\mathrm{gr}_\nu k_{(d-1)}[X_d]$ (Lemma 5.11, (iii)).
(ii). We use (ii) of the last theorem. There we construct a polynomial $p(T) \in k_{(d-1)}[T]$. As we are working in the graded ring, we can replace the coefficients of p with their initial, which by assumption is a unique monomial $U^{I_0} \in k_{(d-1)}$. Thus $P(T) = U^{I_0} p'(T)$, where $p'(T) \in k[T]$. Factorize $p'(T) = \prod(T - \theta_l)$, modulo unit factors, we hence get

$$f = U^{I_0} U_{d,\alpha_d}^{\hat{J}_{\alpha_d} - L n_{d,\alpha_d}} \prod_l (U_{d,\alpha_d}^{n_{d,\alpha_d}} - \theta_l U^{m^{(d,\alpha_d)}}),$$

where $L = \deg(p)$. On the other hand Lemma 5.11, (iii), shows that all the elements of the form $U_{d,\alpha_d}^{n_{d,\alpha_d}} - \theta_l U^{m^{(d,\alpha_d)}}$ are irreducible in $\mathrm{gr}_\nu k_{(d-1)}[X_d]$. Thus the decomposition above is the decomposition of f into prime factors in $\mathrm{gr}_\nu k_{(d-1)}[X_d]$.
(iii). It is a result of (iii) of last theorem and Lemma 5.11, (iv). □

Remark 5.14 Consider a valuation ν as above. The strong condition of Corollary 5.13, (ii), is satisfied iff for any $i = 0, \ldots, d-1$ either we have $U_{i,\alpha_i} = 0$ or $U_{i,\alpha_i} \neq 0$ and $n_{i,\alpha_i} = \infty$.

Theorem 5.15 (Homogeneous decomposition) *Let ν be a valuation attached to an* SKP. *Consider the ring $R = k_{((\alpha,d))}$ and the induced valuation on it ν. Every element $f \in R$ has a unique decomposition of the form*

$$f = p(T_{i_1}, \ldots, T_{i_{d_1}}) U^J, \text{ in } \mathrm{gr}_\nu R_\nu,$$

where $d_1 \leq d+1$ and $A = \{i_1, \ldots, i_{d_1}\}$, for any $i \in A$, $n_{i,\alpha_i} \neq \infty$ and $T_i = U_{i,\alpha_i}^{n_{i,\alpha_i}} U^{-m^{(i,\alpha_i)}}$. And $0 \leq J_{i',j} < n_{i',j}$, for $1 \leq j \leq \alpha_{i'}$. And $p(V_1, \ldots, V_{d_1}) \in k[V_1, \ldots, V_{d_1}]$.

Proof. This is a simple induction on Theorem 5.12. □

Theorem 5.16 *Fix an* SKP *$[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$, such that $\alpha_d \geq \omega$. Suppose there exists an infinite sequence of ordinals $s_1 < \cdots < s_\omega = \alpha_d$ such that $n_{d,s_j} > 1$, for any $j < \omega$. Consider the acceptable vectors $\alpha^{(s_j)}$ (see Definition 5.8). For any $f \in k_{(d-1)}[[X_d]]$ there exists $j_* \in \mathbb{N}$ such that for any $j \geq j_*$ we have*

$$\nu_{\alpha^{(s_j)}}(f) = \nu_{\alpha^{(s_{j_*})}}(f).$$

Thus the limit $\lim_{j \to \omega} \nu_{\alpha^{(s_j)}}$ is well-defined and is equal to $\nu_{\alpha^{(s_\omega)}} = \nu$.

Proof. Multiplying f by a suitable factor $u \in k^{(d-1)}$ we can assume $f \in k^{(d)}$. By assumptions, we have $U_{d,\alpha_d} = 0$. Thus by Corollary 4.5, we have $\text{in}_{\nu_\alpha}(f) = c_J U_d^J$, $c_J \in k^{(d-1)}$. Suppose j_* is the maximum index such that $J_{s_{j_*}} \neq 0$. Then by the algorithm of getting *adic* expansion, this j_* works. □

6 SKP-Valuations and numerical invariants

One of the ways to classify the valuations is through their numerical invariants. In this section we show how the arithmetic of the SKP's of an SKP-valuation determines the numerical invariants of the attached valuation on the field $k^{(d)}$.

Proposition 6.1 *Fix an* SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$ *and suppose* ν *be its attached* $k-valuation$. *Let* $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha'_i}$ *be its minimal pseudo-SKP. The valuation* ν *can be defined using the data* $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha'_i}$.

Proof. It is sufficient to note that to define the valuation ν it is sufficient to know the *adic* expansion of elements. Moreover, in the *adic* expansion of an element the $U_{i,j}$'s with $n_{i,j} = 1$ can not appear. Thus the *adic* expansion of every element is defined using only the minimal pseudo-SKP associated to ν. □

The following lemma computes the rank and rational rank and value-semigroup of an SKP valuation in terms of the arithmetic of the SKP.

Lemma 6.2 *Consider a centered* $k-valuation$ *on the ring* $k^{(d)}$ *such that* $\nu = \text{val}[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$. *Let* $\overline{\nu} = \nu \mid_{k_{(d-1)}}$. *It is clear that* $\overline{\nu} = \text{val}[U_{i,j}, \beta_{i,j}]_{i=0\cdots d-1, j=1\cdots\alpha_i}$.

(*i*) *We have* $\text{rk}(\nu) - \text{rk}(\overline{\nu}) \in \{0, 1\}$. *More precisely* $\text{rk}(\nu) = \text{rk}(\overline{\nu}) + 1$ *iff* $\beta_{d,\alpha_d} \notin \Delta$ (Δ *is the smallest isolated subgroup of* Φ *such that* $\Phi^*_{d-1,\alpha_{d-1}} \subset \Delta$), *and* $\text{rk}(\nu) = \text{rk}(\overline{\nu})$ *iff* $\beta_{d,\alpha_d} \in \Delta$.

(*ii*) *We have* $\text{r.rk}(\nu) - \text{r.rk}(\overline{\nu}) \in \{0, 1\}$. *More precisely* $\text{r.rk}(\nu) = \text{r.rk}(\overline{\nu}) + 1$ *iff* $\beta_{d,\alpha_d} \notin \Phi^*_{d-1,\alpha_{d-1}}$, *and* $\text{r.rk}(\nu) = \text{r.rk}(\overline{\nu})$ *iff* $\beta_{d,\alpha_d} \in \Phi^*_{d-1,\alpha_{d-1}}$.

(*iii*) *The semigroup* $\nu(k^{(d)} \setminus \{0\})$ *is equal to* Γ_{d,α_d}.

We define the notion of pseudo-SKP. It allows us to avoid ordinal numbers greater than ω for α_i.

Definition 6.3 For a SKP $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha_i}$ a pseudo-SKP is a subset of U's and β's which comes from dropping an arbitrary number of U's (and associated β's) for which $n_{i,j} = 1$ and also one can not drop U_{i,α_i}'s. To any SKP is attached a minimal pseudo-SKP which is obtained by dropping all $U_{i,j}$ such that $n_{i,j} = 1$. This minimal attached pseudo-SKP is unique. We denote this minimal pseudo-SKP by $[U_{i,j}, \beta_{i,j}]_{i=0\cdots d, j=1\cdots\alpha'_i}$, where $\alpha'_i \leq \omega$ (using the same notation as SKP's).

Theorem 6.4 *Consider a centered* $k-valuation$ *on the ring* $k[[X_0, X_1, X_2]]$, ν, *which is defined by an* SKP, *i.e., let* $\nu = \text{val}[U_{i,j}, \beta_{i,j}]_{i=0,1,2, j=1\cdots\alpha_i}$. *Moreover, we suppose* $\beta_{0,1} \in \Delta_1$. *Then we can compute the numerical invariants of this valuation using the arithmetic of its minimal pseudo*-SKP. *This is summarized in* Table *1*.

Proof. The computation of the rank and the rational-rank is a simple task. The only nontrivial task is the computation of the transcendence degree or the dimension of valuation. It is a direct calculation using Theorem 5.15. For example in the case (I), pick $f, g \in k_{(d-1)}$ with $\nu(f) = \nu(g)$. Then by Theorem 5.15 we have $\text{in}(f) = p(T_1, T_2)U^J$ and $\text{in}(g) = q(T_1, T_2)U^{J'}$. Using the properties of J and J' in the theorem, we see that $J = J'$. Thus $f/g = p(T_1, T_2)/q(T_1, T_2)$. This shows $k_\nu = R_\nu/\mathfrak{m}_\nu = k(T_1, T_2)$. We show that T_1 and T_2 are algebraically independent in k_ν. If T_2 is algebraic over $k(T_1)$, then there is a polynomial $0 \neq p(T) \in k(T_1)[T]$ such that $p = p(T_2) = \sum_i c_i T_2^i = 0$ in k_ν. Regarding T_1 and T_2 as elements of R_ν, we have $p(T_2) = \sum_i c_i T_2^i \in \mathfrak{m}_\nu$. Note that $T_1 = \frac{U_{1,\alpha_1}^{n_{1,\alpha_1}}}{U^{m^\cdot}}$ and $T_2 = \frac{U_{2,\alpha_2}^{n_{2,\alpha_2}}}{U^{m}}$. Multiplying p with a suitable power of $U^{m^\cdot + m}$, say n, we can assume that $U^{n(m^\cdot + m)} p \in k_{((\alpha,2))}$. The condition $p \in \mathfrak{m}_\nu$ implies that the cancelation should occur between initail monomials of monomials of $U^{n(m^\cdot + m)} p$ in the course of getting the *adic* expansion. We show that this is impossible.

	Arithmetic of minimal pseudo-SKP of the valuation ν		rk	r.rk	tr.deg
(I)	$\alpha'_1 < \infty,\ \alpha'_2 < \infty$	$\beta_{i,j} \in \mathbb{Q}\beta_{0,1}$	1	1	2
$(\mathrm{II})_1$	$\alpha'_1 < \infty,\ \alpha'_2 < \infty$	$\beta_{i,j} \in \Delta_1,\ \beta_{1,\alpha'_1} \in \mathbb{Q}\beta_{0,1},\ \beta_{2,\alpha'_2} \in \Delta_1 \backslash \mathbb{Q}\beta_{0,1}$	1	2	1
$(\mathrm{II})_2$	$\alpha'_1 < \infty,\ \alpha'_2 < \infty$	$\beta_{i,j} \in \Delta_1,\ \beta_{1,\alpha'_1} \in \Delta_1 \backslash \mathbb{Q}\beta_{0,1},\ \beta_{2,\alpha'_2} \in \mathbb{Q}\beta_{0,1}$			
$(\mathrm{III})_1$	$\alpha'_1 = \infty,\ \alpha'_2 < \infty$	$\beta_{i,j} \in \mathbb{Q}\beta_{0,1}$	1	1	1
$(\mathrm{III})_2$	$\alpha'_1 < \infty,\ \alpha'_2 = \infty$	$\beta_{i,j} \in \mathbb{Q}\beta_{0,1}$			
(IV)	$\alpha'_1 < \infty, \alpha'_2 < \infty$	$\beta_{1,\alpha'_1} \in \Delta_1 \backslash \mathbb{Q}\beta_{0,1},\ \beta_{2,\alpha'_2} \in \Delta_1 \backslash (\beta_{0,1}, \beta_{1,\alpha'_1}) \otimes \mathbb{Q}$	1	3	0
$(\mathrm{V})_1$	$\alpha'_1 = \infty, \alpha'_2 < \infty$	$\beta_{2,\alpha'_2} \in \Delta_1 \backslash \mathbb{Q}\beta_{0,1}$	1	2	0
$(\mathrm{V})_2$	$\alpha'_1 < \infty, \alpha'_2 = \infty$	$\beta_{1,\alpha'_1} \in \Delta_1 \backslash \mathbb{Q}\beta_{0,1}$			
(VI)	$\alpha'_1 < \infty, \alpha'_2 < \infty$	$\max\{\beta_{i,\alpha'_i}\} \in \Delta_2 \backslash \Delta_1,\ \beta_{1,\alpha'_1} \in (\beta_{0,1}, \beta_{2,\alpha'_2}) \otimes \mathbb{Q}$	2	2	1
$(\mathrm{VII})_1$	$\alpha'_1 < \infty, \alpha'_2 < \infty$	$\beta_{1,\alpha'_1} \in \Delta_2 \backslash \Delta_1,\ \beta_{2,\alpha'_2} \in \Phi \backslash \Delta_2$	3	3	0
$(\mathrm{VII})_2$	$\alpha'_1 < \infty, \alpha'_2 < \infty$	$\beta_{2,\alpha'_2} \in \Delta_2 \backslash \Delta_1,\ \beta_{1,\alpha'_1} \in \Phi \backslash \Delta_2$			
$(\mathrm{VIII})_1$	$\alpha'_1 = \infty, \alpha'_2 < \infty$	$\beta_{2,\alpha'_2} \in \Delta_2 \backslash \Delta_1$	2	2	0
$(\mathrm{VIII})_2$	$\alpha'_1 < \infty, \alpha'_2 = \infty$	$\beta_{1,\alpha'_1} \in \Delta_2 \backslash \Delta_1$			
(IX)	$\alpha'_1 < \infty, \alpha'_2 < \infty$	$\max\{\beta_{i,\alpha'_i}\} \in \Delta_2 \backslash \Delta_1,\ \beta_{1,\alpha'_1} \in \Delta_2 \backslash (\beta_{0,1}, \beta_{2,\alpha'_2}) \otimes \mathbb{Q}$	2	3	0
(X)	$\alpha'_1 = \infty, \alpha'_2 = \infty$		1	1	0

Table 1: Numerical invariants via arithmetic of SKP of the valuation

Write $p = \sum_{i,j} r_{i,j} T_1^i T_2^j,\ r_{i,j} \in k$. Then $U^{n(m^{\cdot}+m)}p = \sum_{i,j} r_{i,j} U^{n_{[i,j]}} U^i_{1,\alpha_1} U^j_{2,\alpha_2}$. By Lemma 4.6.$(ii)$ no cancelation can occur between initial monomials of monomials of $U^{n(m^{\cdot}+m)}p$ with different j's (notice that index $(2, \alpha_2)$ does not occur in $U^{m_{[i,j]}}$). It remains to show that no cancelation can occur for a sum of the form $q_j = \sum_i r_{i,j} U^{n_{[i,j]}} U^i_{1,\alpha_1} U^j_{2,\alpha_2}$. Notice that the power of $U_{2,}$'s, are the same for different monomials of q and the power of U_{1,α_1} are different for any two monomial of q. Now, the proof of Lemma 4.6.(ii) shows that in the course of getting the *adic* expansion of the monomials of q the power of U_{1,α_1} in the initial monomials remain diffrent, for any two monomial of q. Thus no cancelation can occur between the initial monomials of q. □

7 Realization of certain class of semi-groups as value semi-groups of polynomial rings

In this section we give a result on the realization of a semi-group as the semi-group of values which takes a valuation on a polynomial ring.

Theorem 7.1 *Let Γ be a semigroup of an ordered abelian group $(\Psi, \prec)$, given by a minimal system of generators $\{\gamma_j\}_{j \leq \alpha} \subseteq \Phi^+$, $\alpha = \omega n + j^*$, $n, j^* \in \mathbb{N}$ (we denote this by* $\mathrm{o.t}(\Gamma)$*=n). Suppose that Γ is positively generated (Definition 2.2), and $\gamma_{j+1} \succ n_j \gamma_j$ when $n_j \neq \infty$. set $G = (\Gamma)$ and $d =$* $\mathrm{r.rk}(G) + \mathrm{o.t}(\Gamma) + 1$. *Then there exists a valuation ν of the field $k(X_1, \ldots, X_d)$, centered on the polynomial ring $k[X_1, \ldots, X_d]$, such that its value-semigroup is equal to Γ.*

Proof. We give special names for those indices of the γ's that are not rationally independent to the previous ones $\gamma_{s^t_{t'}}$. Introducing a new variable for every $\gamma_{s^t_{t'}+1}$, we construct a set of key-polynomials of this new variable with values equal to γ, up to $\gamma_{s^t_{t'+1}}$. Then, we need to define a new variable. However, the situation differs in the case of limit ordinal: If $s^t_{t'+1}$ is a limit ordinal then the limit key-polynomial which is available is zero and can not take $\gamma_{s^t_{t'+1}}$ as its value. Thus, we are forced to define a new variable for taking value $\gamma_{s^t_{t'+1}}$. The precise definition is as follows.

Set $\{\gamma_{j'}\}_{j' \leq \alpha} = \{\gamma_{s^1_1}\} \cup \{\gamma_{s^t_{t'}+j}\}_{t \in T, t' \in T'_t \cup \{f_{t+1}\}, j \in J_{t,t'}}$, where $T = \{1, \ldots, n+1\}$, $T'_t = \{1, \ldots, f_t\}$ for $t \in T$. $f_t \in \mathbb{N}$, for $t \in T$, $f_0 := 0$, $f_{t+1} := \infty$. $J_{t,t'} = \{1, \ldots, s^t_{t'+1} - s^t_{t'}\}$, for $t \in T$ and $t' < f_t$. For $t \leq n$: $J_{t,f_t} = \{j :\ 1 \leq j < \omega\}$, and for $t = n+1$, $J_{t,f_t} = \{j :\ 1 \leq j \leq j^*\}$, where $\alpha = \omega n + j^*$, $j^* \in \mathbb{N}$ (for the simplicity of notation we assume $f_{n+1} < j^*$). $J_{t,f_{t+1}} = \{0\}$, and $\gamma_{s^t_{f_{t+1}}} = \gamma_{s_1^{t+1}}$, for $t \leq n$. And finally, for $t \in T$ we have $\gamma_{s^t_1} = \gamma_{\omega(t-1)}$ and the finite set $\{\gamma_{s^t_{t'}}\}_{t' \in T'_t}$ are all those γ, $\gamma_{\omega(t-1)} \leq \gamma < \gamma_{\omega(t)}$, with $n_{s^t_{t'}} = \infty$ (see Lemma 2.1, for definition of n).

Then, by Lemma 7.3, we have $\mathrm{r.rk}(G) = f_0 + \ldots + f_{n+1}$.

For $t \in T$ and $t' \in T'_t$ set $i_{t,t'} = f_0 + \ldots + f_{t-1} + t + t'$, and $i_{t,t'} = f_0 + \ldots + f_t + t + 1$, for $t' = f_{t+1}$. Notice that for any t and t' we have $i_{t,t'} > 1$. Thus the total number of i's which has been defined d, is equal to (we give here the formula without assumption $f_{n+1} < j^*$):

$$d = \begin{cases} f_0 + \ldots + f_{n+1} + n & = \text{r.rk}(G) + \text{ot}(\Gamma) \quad : j^* = f_{n+1} \\ f_0 + \ldots + f_{n+1} + n + 1 & = \text{r.rk}(G) + \text{ot}(\Gamma) + 1 \quad : j^* > f_{n+1} \end{cases}$$

Set $\beta_{1,1} = \gamma_{s_1^1}$. It is straightforward to check that the sequence $\{\beta_{1,1}, \beta_{i_{t,t'},j} := \gamma_{s^t_{t'}+j}\}_{i_{t,t'}=2\cdots d, j \in J_{t,t'}}$ is a sequence of values (note the index i starts from 1). The key-polynomials of the SKP attached to this sequence of values are all polynomials of the ring $k[X_0, \ldots, X_{d-1}]$. The valuation ν attached to this SKP has value semi-group Γ. □

Remark 7.2 The following remarks are in order:

- The positivity condition is quite restrictive in general. However, it is well-known that in the case we restrict to the value semi-groups of polynomial rings of two variables, all the value semi-groups are positively generated.
- It seems that the bound d obtained for the number of the variables of the polynomial ring is the best possible. More precisely: Given any semi-group Γ it can not be realized as a value semigroup of a polynomial ring with $< d$ variables.

Lemma 7.3 *With the notation of Theorem 7.1, for any limit ordinal* $\omega(i+1) \leq \alpha$ *we have* $\text{rk}(G_{\omega(i+1)}) = \text{rk}(G_{\omega(i+1)'^-}) + 1$*. In particular,* $n_{\omega(i+1)} = \infty$*.*

Proof. We extend the notion of effective component to this situation. Consider an order embedding $(\Phi, \prec) \subseteq (\mathbb{R}^n, <_{lex})$ such that $\Gamma \subseteq \mathbb{R}^n_{\geq_{les} 0}$. By definition, the effective component for the limit ordinal ωi is the first index $t \leq n$ such that $\#\{(\gamma_j)_t\}_{\omega i \leq j < \omega(i+1)} = \infty$. Like in the case of effective components, one can prove t is well-defined. Note that $(\gamma_j)_{t'} = 0$, for $t' < t$ and $j < \omega(i+1)$. Moreover, one can show that the content of Proposition 2.10.(i) and (ii) hold in this case. Suppose the effective component for ωi is t. Then an argument similar to the proof of Proposition 2.10.(iii), shows that $(\gamma_j)_t \to +\infty (j \to \omega(i+1))$. But $\gamma_{\omega(i+1)} >_{lex} \gamma_{\omega i + j}$, for $j \in \mathbb{N}$. This is possible only if $(\gamma_{\omega(i+1)})_{t'} > 0$, for some $t' < t$. □

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