A construction for a class of valuations of the field $k(X_1, \ldots, X_d, Y)$ with large value group

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Abstract

Given any algebraically closed field k of characteristic zero and any abelian group G, of rational rank less than or equal to d, totally ordered by a suitably chosen ordering, we construct a valuation of the field $k(X_1, \ldots, X_d, Y)$ with value group G. In the case of rational rank equal to d this valuation is induced by a fractional power series parametrization of a transcendental hypersurface in affine $(d+1)-$space which is naturally approximated by a sequence of quasi-ordinary hypersurfaces. The value semigroup $\nu(k[X, Y] \setminus 0)$ is the union of the semigroups associated to these quasi-ordinary hypersurfaces.

1 Introduction

Let k be an algebraically closed field of characteristic zero and d an integer. For each commutative group G of rational rank less than or equal to d, we construct a zero-dimensional $k-$valuation of the field $k(X_1, \ldots, X_d, Y)$ whose value group is G. Note that in the case of valuations of the field $k(X_1, \ldots, X_d, Y)$ of rational rank equal to $d+1$ we are in the equality case of Abhyankar's inequality ([**?**]) and the value group has to be $\mathbb{Z}^{d+1}$.

The problem of the existence of the valuations with a given value group and residual extension has been solved by "arithmetical" methods, see [**?**]. However, our approach is different and more geometric. For example, with this approach the question of representing the valuation rings corresponding to these valuations as inductive limits of localizations of blowing-up algebras of the $k-$algebra $k[X_1, \ldots, X_d, Y]$ seems to be more accessible.

The construction of the valuation is based on generalizing the notion of quasi-ordinary hypersurface singularities ([**?**], [**?**]); this is done in Definition 2.1 in the next section. This generalization allows us to describe elements $\zeta(X) \in k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$, $X = (X_1, \ldots, X_d)$ which are transcendental over $k(X)$. As a set $k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$ is the set of formal power series in $X_1, \ldots, X_d$ with rational exponents, in which the set of exponents is well-ordered with respect to a total monomial ordering $\preccurlyeq$ which refines the partial ordering $\leq$ on $\mathbb{Q}^d$ (A good ordering, see Definition 3.4). This is in fact a valuation ring (see [**?**], Chap. 6, Section 3, n° 4, Exemple 6). The generalized quasi-ordinary series $\zeta(X)$ can be used to define injective morphisms of k-algebras $\Theta_\zeta : k[X, Y] \to k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$ (see Definition 4.1). With the help of such injections we get valuations $\nu = \nu_\zeta$ of the field $k(X, Y)$.

By a process of truncation of this element $\zeta(X)$ according to its "critical exponents" (see Definition 2.1) we get expansions $\zeta^{(i)}(X)$ which parametrize quasi-ordinary hypersurfaces $f^{(i)} = 0$ in $\mathbb{A}^{d+1}(k)$ (Definition 2.3). We relate the semigroups Γ_i attached ([**?**]) to the irreducible quasi-ordinary hypersurfaces $f^{(i)} = 0$ to the semigroup $\Gamma_\zeta = \nu_\zeta(k[X, Y] \setminus \{0\})$ attached to $\zeta(X)$. More precisely, if we denote by Γ_i the semigroup associated to $f^{(i)} = 0$, then we have $\Gamma_\zeta = \varinjlim \Gamma_i$ for an inductive system

$\Gamma_i \xrightarrow{\times n_{i+1}} \Gamma_{i+1}$ with integers n_i determined by the exponents of $\zeta(X)$. Later, in Section 6, following the ideas of Teissier in [?] and [?], we give a way to compute the $f^{(i)}$.

One of the difficulties to construct a valuation with value group in $\mathbb{Q}^d$ is that there is no natural ordering on $\mathbb{Q}^d$. In Section 3, we introduce and study the properties of the good orderings on $\mathbb{Q}^d$.

In section 4 we also show that given any abelian group of rational rank d there is a transcendental element $\zeta(X)$ such that the value group of the valuation attached to this element is G. To illustrate the method we give example of constructions of valuations with large value groups, such as $\mathbb{Q}^d$ itself.

In section 5, we show that the $f^{(i)}$ constitute a sequence of key polynomials in the sense of MacLane ([?]) for the extension to $k(X,Y)$ of the restriction of ν to $k(X)$. In order to prove this, we give another way of constructing the valuation ν (Proposition 5.6). This new construction is carried out by a direct introduction of a sequence of valuations ν_i which approximates the valuation ν. Moreover, the value group of ν_i is equal to the group generated by Γ_i. Let In the final section we study an embedding of the spaces Spec$R$, where $R = k[[X]][\zeta(X)]$ and Spec$(k[X^{\Gamma_\zeta}])$ in an infinite dimensional regular space Spec$k[[X]][U]$, where $U = (U_1, U_2, \ldots)$. We study the ideals defining these embeddings and the relation between them. Moreover, we show that the result of truncating the equations of the embedding Spec$R \hookrightarrow$ Spec$k[[X]][U]$ is a set of equations which gives an embedding of the quasi-ordinary hypersurfaces $f^{(i)} = 0$ in Spec$k[[X]][U]$. Using the constructions of this section and some ideas of [?] and [?], we are able to construct a rational valuation with value group G for any totally ordered group G of rational rank less than d.

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2 The transcendental hypersurface and its approximations

Generalizing the classical definition of the quasi-ordinary hypersurface singularities (see [?], [?]) we define a transcendental quasi-ordinary hypersurface singularity in the following manner:

Definition 2.1 Fix an element $\zeta(X) = \sum c_\lambda X^\lambda = \sum_{i=1}^{\infty} p_i,\ p_i \in k[X^{\frac{1}{m^{(i)}}}]$, where $X = (X_1, \ldots, X_d)$ and $X^{\frac{1}{m}} = (X_1^{\frac{1}{m}}, \ldots, X_d^{\frac{1}{m}})$. The $m^{(i)}$ are integers which tend to infinity with i; they will be described more precisely in Definition 2.4 . We impose the following conditions:

- All the exponents of p_i, i.e., the λ's of the monomials of p_i, are ordered with respect to the partial product order $\leq$ on $\mathbb{Q}^d$, with minimum equal to λ_i.
- The partial order on $\mathbb{Q}^d$ induces a total order on the set $\{\lambda_i\}_{i=1}^{\infty}$, i.e., $\lambda_1 < \lambda_2 < \ldots$.
- We define inductively a sequence of subgroups of $\mathbb{Q}^d$ by $Q_0 = \mathbb{Z}^d$, $Q_j = \mathbb{Z}^d + \sum_{\lambda_i < \lambda_{j+1}} \mathbb{Z}\lambda_i$, for $j \in \mathbb{N}$. We impose the condition $\lambda_j \notin Q_{j-1}$.
- If $c_\lambda X^\lambda$ is a term of p_j then $\lambda \in Q_j$.

The λ_i are called characteristic exponents of the series.

The above definition is a generalization of [?], subsection 4.4, where a "natural valuation" attached to a

"transcendental plane curve", studied through a series of examples from different perspectives: the sequence of point blow ups, the semigroup, the graded valuation ring, ... Moreover, the relations between these approaches studied. In this text we follow the same approach.

Note that if we define $\Lambda = \{\lambda\ : c_\lambda \neq 0\}$ then $\lambda \in \Lambda$ is the exponent of a term of p_i iff $\lambda_i \leq \lambda \ngeq \lambda_{i+1}$. We call the λ_i the *characteristic exponents* of the transcendental hypersurface defined by $Y = \zeta(X)$, see the next proposition. This terminology is justified in Definition 2.3, in which we define for any $i \in \mathbb{N}$, an irreducible quasi-ordinary hypersurface (see [?] or [?]) which is parametrized by $X = X, Y = \zeta^{(i)}(X)$ where $\zeta^{(i)}(X)$ is a fractional power series with characteristic exponents $\lambda_1, \ldots, \lambda_i$.

For any good ordering $\preccurlyeq$ we have the inclusions:

$$k[[X]] \subset \widetilde{k[[X]]} = \bigcup_N k[[X^{\frac{1}{N}}]] \subset k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]].$$

Proposition 2.2 *The element $\zeta(X)$ is transcendental over the ring $k[X,Y]$. In other words, the morphism of $k-$algebras*

$$\begin{array}{rrcl} \Theta_\zeta : & k[X,Y] & \to & K[X,\zeta(X)] \\ & X & \mapsto & X \\ & Y & \mapsto & \zeta(X) \end{array}$$

is injective.

Proof. Assume the contrary and let $\zeta(X)$ be the root of an irreducible polynomial $f \in k[X,Y]$. Consider the algebraically closed field $k((X^{\mathbb{Q}^d_{\succcurlyeq 0}}))$, (see [?], Chap. 6, Section 3, n° 4, Exemple 6). We have $\zeta(X) \in k((X^{\mathbb{Q}^d_{\succcurlyeq 0}}))$. In the sequence λ_r of the characteristic exponents the denominators tend to infinity. Therefore, there is an index i such that the denominators of $\lambda_{r,i}$ tend to infinity with r. We can assume that this index is d. Consider the algebraically closed field $k' = k((X'^{\mathbb{Q}^{d-1}_{\succcurlyeq 0}}))$, where $X' = (X_1, \ldots, X_{d-1})$. We can regard $f(X,Y)$ as a polynomial in the ring $k'[X_d, Y]$ and $\zeta(X)$ as an element of the ring $k'[[X_d^{Q_{\succcurlyeq 0}}]]$. By the Newton-Puiseux theorem all the roots of $f(X,Y)$ are in the ring $\widetilde{k'[[X_d]]}$. It implies that $\zeta(X) \in \widetilde{k'[[X_d]]}$ which is absurd. $\square$

A variant of this proof gives us the following statement: Given any $f \in k[X,Y]$, there does not exist a root $\eta(X) \in k[[X^{Q^d_{\succcurlyeq 0}}]]$ of f, such that the denominators of the terms of η tend to infinity (By denominator of a term $c_\beta X^\beta$ of η we mean: the least natural number n such that $n.\beta \in \mathbb{N}^d$.).

We introduce a sequence of quasi-ordinary hypersurfaces $f^{(i)}$, which approximates the original element $\zeta(X)$.

Definition 2.3 Set $f^{(0)}(X,Y) = Y$, and for any $i \in \mathbb{N}$ define an irreducible quasi-ordinary hypersurface $f^{(i)}(X,Y) \in k[[X]][Y]$ (for the definition of the quasi-ordinary singularities see [?] and for the irreducibility see [?]) by the following parametrization:

$$Y = \zeta^{(i)}(X) = \sum_{j=1}^{i} p_j + p^{(i)},$$

where $\frac{p_i + p^{(i)}}{X^{\lambda_i}} \in k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$, the exponents of the monomials of $p^{(i)}$ are in Q_i, and the first exponent of $p^{(i)}$ is greater than λ_{i+1}.

Definition 2.4 We define for $1 \le j \le i \in \mathbb{N} : n_j = [Q_j : Q_{j-1}]$ and $m^{(0)} = 1,\ m^{(i)} = n_1 \ldots n_i$. It can be proved that $m^{(i)} = \deg_Y(f^{(i)})$ (see [?] or [?]). Moreover, we define the following vectors (originally defined and studied in [?]):

$$\gamma_1 = \lambda_1,\ \gamma_j = n_{j-1}\gamma_{j-1} + \lambda_j - \lambda_{j-1},\ j > 1.$$

By $R(f)$, for a quasi-ordinary f, we mean the set of the roots of f in $\widetilde{k[[X]]}$. Following [?], we define the notion of the intersection index of two "comparable" quasi-ordinary hypersurfaces.

Definition 2.5 For any two quasi-ordinary hypersurfaces f, g, we say that they are comparable if for any $\eta \in R(f)$ and $\mu \in R(g)$ we have $\eta - \mu = X^\alpha.unit$, where $\alpha \in \mathbb{Q}^d_{\geq 0}$. The intersection index of two such hypersurfaces is defined as follows:

$$(f,g) = \upsilon_X(\mathrm{Res}_Y(f,g)) \in \mathbb{Z}^d.$$

For any two arbitrary root $\eta \in R(f)$ and $\xi \in R(g)$ of two irreducible comparable quasi-ordinary hypersurfaces the coincidence order of η and ξ is by definition the vector $\kappa(\eta,\xi) = \nu_X(\eta - \xi) \in \mathbb{Q}^d_{\geq 0}$. The exponent of contact of such f and g is defined as follows:

$$\kappa(f,g) = max\{\kappa(\eta,\xi), \eta \in R(f), \xi \in R(g)\}.$$

Proposition 2.6 [?] *Let g be an irreducible unitary quasi-ordinary hypersurface which is comparable with $f^{(i)}$. We have:*

$$\frac{(f^{(i)},g)}{deg(f^{(i)}).deg(g)}=\frac{\gamma_{i_\kappa}}{n_1\dots n_{i_\kappa-1}}+\frac{\kappa-\lambda_{i_\kappa}}{n_1\dots n_{i_\kappa}}.$$

Here κ is the exponent of contact of $f^{(i)}$ and g. Note that κ is an exponent in the parametrization of $f^{(i)}$, and i_κ is the index of the greatest characteristic exponent λ_j of $f^{(i)}$ such that $\lambda_j\leq\kappa$.

We recall the notion of the semi-roots in our context:

Definition 2.7 We say that $g\in k[[X]][Y]$ is a $j^{th}-$semi-root of $f^{(i)}$, $0\leq j\leq i$, if the following two conditions are satisfied:
a) $g(0,Y)=Y^{n_1\dots n_j}$.
b) $g(X,\zeta^{(i)}(X))=X^{\gamma_{j+1}}\varepsilon_j^{(i)}$, where $\varepsilon_j^{(i)}$ is a unit in $\widetilde{k[[X]]}$.

We have the following lemma (see also [?]):

Lemma 2.8 *For any $j\leq i\in\mathbb{N}$, the quasi-ordinary singularity $f^{(j)}$ is a j^{th}-semi-root of $f^{(i)}$.*

Proof. By construction, the quasi-ordinary hypersurfaces $f^{(i)},f^{(j)}$ are comparable. In the case $j=0$, by definition we have $f^{(0)}(X,Y)=Y$. This gives $f^{(0)}(X,\zeta^{(i)}(X))=\zeta^{(i)}(X)=X^{\gamma_1}.unit$. For $j>0$, we use Proposition 2.6. Here $i_\kappa=j+1$, and we have $\frac{(f^{(i)},f^{(j)})}{\deg(f^{(i)}).\deg(f^{(j)})}=\frac{\gamma_{j+1}}{n_1\dots n_j}$. We notice that $\deg(f^{(j)})=n_1\dots n_j$, which shows that $(f^{(i)},f^{(j)})=m^{(i)}\gamma_{j+1}$, so that the order in X of $f^{(j)}(X,\zeta^{(i)}(X))$ must be γ_{j+1}, which gives the result. □

We need another result (see [?] and [?]) which allows a $(f^{(0)},\dots,f^{(i)})-adic$ representation of any element of $k[X,Y]$.

Lemma 2.9 *Given $g\in k[[X]][Y]$, there exists i_0 such that for $i\geq i_0$, g can be uniquely written as a finite sum $g=\sum c_{l_0\dots l_i}(f^{(0)})^{l_0}\dots(f^{(i)})^{l_i}$, with $c_{l_0\dots l_i}\in\mathbb{C}[[X]]$, the $(i+1)-$tuples $(l_0\dots l_i)\in\mathbb{N}^{i+1}$ verifying $0\leq l_r\leq n_{r+1}-1$, for all $r\in\{0,\dots,i\}$.*

Proof. ([?]) Make the Euclidean division of g by $f^{(i)}$, by induction we get the $f^{(i)}-adic$ representation of g which is of the form $g=\sum c_{l_i}(f^{(i)})^{l_i}$. Then iterate this process on the coefficients, making at each step the $f^{(j-1)}-adic$ expansions of the coefficients $c_{l_j,\dots,l_i}$. This gives us the claimed *adic* representation. The uniqueness comes from the fact that the $Y-$degrees of the terms $c_{l_0\dots l_i}(f^{(0)})^{l_0}\dots(f^{(i)})^{l_i}$ are pairwise distinct (see Lemma 7.2 of [?]). The only thing which remains to prove is the inequality $0\leq l_i\leq n_{i+1}-1$. This is because if i is chosen so large that $m^{(i)}>\deg_Y(g)$, then $f^{(i)}$ (which is of degree $m^{(i)}$) can not appear in the expansion of g, i.e., $l_i=0$. So, we choose i_0 to be the least i such that $m^{(i)}>\deg_Y(g)$. □

The preceding expansion is called the $(f^{(0)},\dots,f^{(i)})-adic$ expansion of g. The finite set $\{(l_0\dots l_i),\ c_{l_0\dots l_i}\neq 0\}$ is called the $(f^{(0)},\dots,f^{(i)})-adic$ support of g. We set $(f_{[i]})=(f^{(0)},\dots,f^{(i)})$ so we can speak of the $(f_{[i]})-adic$ expansion of an element. We write $c_\ell(f_{[i]})^\ell$ for $c_{l_0\dots l_i}(f^{(0)})^{l_0}\dots(f^{(i)})^{l_i}$. For a fixed set of functions $\{g_1,\dots,g_n\}$ the next lemma says that for sufficiently large values of i and arbitrary $j\in\mathbb{N}$ the $(f_{[i]})-adic$ expansion of each g_k is the same as its $(f_{[i+j]})-adic$ expansion, so in this case for sufficiently large values of i we can speak of the $(f_{[\infty]})-adic$ expansion of the g_k. For example note that the $(f_{[\infty]})-adic$ expansion of $f^{(i)}$ is itself.

Lemma 2.10 *With the notations of the last lemma, given $g\in k[[X]][Y]$, for sufficiently large values of i and any $j\in\mathbb{N}$ the $(f_{[i+j]})-adic$ expansion of g and its $(f_{[i]})-adic$ expansion coincide.*

Proof. For the i_0 chosen in the proof of the last lemma, we have for any $j\geq 0$, $l_{i_0+j}=0$. □

Definition 2.11 For any element $\eta\in\widetilde{k[[X]]}$, the Newton polyhedron $\mathcal{N}_X(\eta)$ is the boundary of the convex hull in $\mathbb{R}^d$ of the set $\mathrm{Supp}_X(\eta)+\mathbb{R}^d_{\geq 0}$, where $\mathrm{Supp}_X(\eta)$ denotes the support of η as a series in the variables X.

The expansion of Lemma 2.10 allows us to compute in an effective way the Newton polyhedron of $g(\zeta)$, where ζ is a root of $f^{(i)} = 0$ (We write $R(f)$ for the set of roots of $f = 0$). This computation is explained by the following two lemmas of [**?**]:

Lemma 2.12 *If $g = \sum c_\ell (f_{[i]})^\ell$, is the $(f_{[i]})-$adic expansion of $g \in k[[X]][Y]$, then for every $\zeta \in R(f)$, the sets of vertices of the Newton polyhedra $\mathcal{N}_X(c_\ell (f_{[i]})^\ell)$, for varying ℓ, are pairwise disjoint.*

Lemma 2.13 *If $g_1, \dots, g_i \in \widetilde{k[[X]]}$ and the sets of vertices of Newton polyhedra $\mathcal{N}_X(g_1), \dots, \mathcal{N}_X(g_i)$ are pairwise disjoint, then $\mathcal{N}_X(g_1 + \dots + g_i)$ is the convex hull of the union of $\mathcal{N}_X(g_1) \bigcup \dots \bigcup \mathcal{N}_X(g_i)$. In particular, each vertex of $\mathcal{N}_X(g_1 + \dots + g_i)$ is a vertex of one of the polyhedra $\mathcal{N}_X(g_1), \dots, \mathcal{N}_X(g_i)$.*

3 The ordering and the semigroup

Definition 3.1 We associate to the series $\zeta \in k[[X^{\mathbb{Q}^d_{\geq 0}}]]$, satisfying the conditions of the Definition 2.1, the sequence of the semigroups:

$$\Gamma_i = \mathbb{Z}^d_{\geq 0} + \gamma_1.\mathbb{Z}_{\geq 0} + \dots + \gamma_i.\mathbb{Z}_{\geq 0}, \ for\ i \in \mathbb{N}.$$

And the semigroup:

$$\Gamma_\zeta = \mathbb{Z}^d_{\geq 0} + \gamma_1.\mathbb{Z}_{\geq 0} + \gamma_2.\mathbb{Z}_{\geq 0} + \dots.$$

Later, when we attach to the element ζ the valuation ν we will see that:

$$\nu(k[X, Y] \setminus 0) = \Gamma_\zeta.$$

We need the following two lemmas from [**?**]:

Lemma 3.2 *1) The order of the image of γ_j in the group $\frac{Q_j}{Q_{j-1}}$ (see Definition 2.1) is equal to $n_j > 1$ for $j \in \mathbb{N}$.*
2) We have $\gamma_j > n_{j-1}\gamma_{j-1}$, for $j \geq 2$.
3)The vector $n_j \gamma_j$ belongs to the semigroup Γ_{j-1} ($j \in \mathbb{N}$). Moreover, we have a unique relation:

$$n_j \gamma_j = \alpha^{(j)} + l_1^{(j)} \gamma_1 + \dots + l_{j-1}^{(j)} \gamma_{j-1}$$

such that $0 \leq l_k^{(j)} \leq n_k - 1$, and $\alpha^{(j)} \in \mathbb{Z}^d_{\geq 0}$, for $j \in \mathbb{N}$.

Lemma 3.3 *For any $j \in \mathbb{N}$ the $(f_{[\infty]}) - adic$ expansion of $(f^{(j-1)})^{n_j}$ is of the following form:*

$$(f^{(j-1)})^{n_j} = c_j f^{(j)} + \sum c^{(j)}_{l_0, \dots, l_{j-1}} (f^{(0)})^{l_0} (f^{(1)})^{l_1} \dots (f^{(j-1)})^{l_{j-1}},$$

where $c_j \in k^*$. We have $0 \leq l_r \leq n_{r+1} - 1$, for $r = 0, \dots, j-1$. The coefficient $c^{(j)}_{l_1^{(j)}, \dots, l_{j-1}^{(j)}, 0}$ appears, and it is of the form $X^{\alpha^{(j)}}.unit$, where the integers $l_1^{(j)}, \dots, l_{j-1}^{(j)}$ and the exponent $\alpha^{(j)}$ are given in Lemma 3.2. Moreover, if $X^{\alpha'}$ appears on the coefficient $c^{(j)}_{l_0, \dots, l_{j-1}}$ then:

$$n_j \gamma_j \leq \alpha' + l_0 \gamma_1 + \dots + l_{j-1} \gamma_j,$$

and equality holds iff $(l_0, \dots, l_{j-1}) = (l_1^{(j)}, \dots, l_{j-1}^{(j)}, 0)$.

From now on (unless otherwise specified) by G we mean a totally ordered abelian group of rational rank d with an ordering $\preccurlyeq$ (a group is ordered by the relation $\preccurlyeq$ if from $x \preccurlyeq y$ and $z \preccurlyeq t$ one deduces that $x + z \preccurlyeq y + t$). We remark that an ordered group is torsion-free so that by tensorization with $\mathbb{Q}$ any such group G can be considered as a subgroup of $\mathbb{Q}^d$. Fix a copy of $\mathbb{Z}^d \subseteq G$. The ordering $\preccurlyeq$ defines an ordering on $\mathbb{Z}^d$ with respect to the inclusion also denoted by $\preccurlyeq$.

Definition 3.4 With the notation of the paragraph above, we say the ordering $\preccurlyeq$ on G is a good ordering, with respect to the inclusion $\mathbb{Z}^d \subseteq G$ if it refines the partial ordering $\leq$ on $\mathbb{Z}^d$, i.e., if $u, v \in \mathbb{Z}^d$ and $u < v$ then $u \prec v$.

In the case $G = \mathbb{Q}^d$ by a good ordering we mean a good ordering with respect to the standard inclusion $\mathbb{Z}^d \subset \mathbb{Q}^d$.

Notice that tensoring the relation $\mathbb{Z}^d \subseteq G$ by $\mathbb{Q}$ shows that the inclusion $\mathbb{Z}^d \subseteq G$ induces a natural embedding of G in $\mathbb{Q}^d$, so $\mathbb{Z}^d \subseteq G \subseteq \mathbb{Q}^d$. Later we will show that there is a natural extension of the good ordering of G to this $\mathbb{Q}^d$. The next lemma shows that every totally ordered group can be considered as a group with a good ordering.

Lemma 3.5 *Given any totally ordered group $(G, \preccurlyeq)$ there exists a suitable inclusion $\mathbb{Z}^d \subseteq G$ such that $\preccurlyeq$ is a good ordering with respect to this inclusion.*

Proof. Let $(0) = \Delta_0 \subset \Delta_1 \subset \ldots \Delta_d = G$ be the sequence of isolated subgroups. For any $i = 1, \ldots, d$ choose $a_i \in \Delta_i \setminus \Delta_{i-1}$ such that $0 \preccurlyeq a_i$. The embedding $\mathbb{Z}^d \to G$ defined by $e_i \mapsto a_i$ (where the e_i are the elements of the standard basis of $\mathbb{Z}^d$) is what we need. It is indeed an embedding: otherwise there exists a relation of the form $n_i a_i = \sum_{j=1}^{i-1} n_j a_j$, where $n_j \in \mathbb{Z}$. This shows that $n_i a_i \in \Delta_{i-1}$ and therefore $a_i \in \Delta_{i-1}$ which is a contradiction. $\square$

The following proposition shows that every good ordering of G can be extended to a good ordering of $\mathbb{Q}^d$.

Proposition 3.6 *Consider a good ordering $\preccurlyeq$ of G with respect to $\mathbb{Z}^d \subseteq G$. This ordering can be extended to a good ordering of $\mathbb{Q}^d$ with respect to the natural embedding $\mathbb{Z}^d \subseteq G \subseteq \mathbb{Q}^d$.*

Proof. Let $\preccurlyeq$ be such an ordering. extend this ordering on $\mathbb{Q}^d$ as follows: For $\gamma, \gamma' \in \mathbb{Q}^d : \gamma \prec \gamma'$ iff there exists $n \in \mathbb{N}$ such that $n\gamma, n\gamma' \in \mathbb{Z}^d$ and $n\gamma \prec n\gamma'$. The next lemma shows that we have the following equivalent definition: We have $\gamma \prec \gamma'$ iff for any $n \in \mathbb{N}$ such that $n\gamma, n\gamma' \in \mathbb{Z}^d$ then $n\gamma \prec n\gamma'$. It is clear that $\preccurlyeq$ is a total ordering on $\mathbb{Q}^d$ as a set. We show that it is a total ordering as a group. Suppose this is not the case. Then there is $\gamma, \gamma', \gamma'', \gamma''' \in \mathbb{Q}^d$ such that $\gamma \prec \gamma'$ and $\gamma'' \prec \gamma'''$ but $\gamma + \gamma'' \not\prec \gamma' + \gamma'''$ then $\gamma + \gamma'' \succ \gamma' + \gamma'''$. By the next lemma and the definition, we can find an $n \in \mathbb{N}$ such that $n\gamma, n\gamma',\ n\gamma'', n\gamma''' \in \mathbb{Z}^d$ and $n\gamma + n\gamma'' \succ n\gamma' + n\gamma'''$. This is a contradiction. Because $n\gamma \prec n\gamma'$ and $n\gamma'' \prec n\gamma'''$ and because $\preccurlyeq$ is an ordering on the group $\mathbb{Z}^d$ we deduce that $n\gamma + n\gamma'' \prec n\gamma' + n\gamma'''$. This ordering refines the partial ordering on $\mathbb{Q}^d$. Let $\gamma < \gamma'$ and take a natural number n such that $n\gamma, n\gamma' \in \mathbb{Z}^d_{\geq 0}$. By definition of the good ordering $n\gamma \prec n\gamma'$. By the discussion in the first step of the proof we have $\gamma \prec \gamma'$. $\square$

Lemma 3.7 *Let $\preccurlyeq$ be a total ordering on $\mathbb{Z}^d$ which refines the partial ordering $\leq$ on $\mathbb{Z}^d$. For every $a, b \in \mathbb{Z}^d$, if $a \prec b$ then for any $p \in \mathbb{Q}_{\geq 0}$ such that $pa, pb \in \mathbb{Z}^d$ we have $pa \prec pb$.*

Proof. By the ordering property for every $p \in \mathbb{N}$ we have $pa \prec pb$. It suffices to prove the lemma for p^{-1}, where $p \in \mathbb{N}$. If $p^{-1}a \succ p^{-1}b$ then $p.p^{-1}a \succ p.p^{-1}b$ so $a \succ b$, a contradiction. $\square$

Remark 3.8 The ordering introduced in Proposition 3.6, is no longer a well-ordering on $\mathbb{Q}^d$. For example take the set $A = \{u_i = (1, \ldots, 1, \frac{1}{i})\}_{i=1}^{\infty}$. The property that $\preccurlyeq$ refines the partial ordering $\leq$ shows that the set A does not have a smallest element.

Here is a concrete example of a good ordering.

Example 3.9 Consider the $\leq_{d.lex.}$ ordering on $\mathbb{Z}^d$ which is defined as follows:
For any $a, b \in \mathbb{Z}^d$ we have $a <_{d.lex.} b$ iff $(\deg(a) = \sum_{i=1}^d a_i < \deg(b)$ or $(\deg(a) = \deg(b)$ and $a <_{lex.} b))$.
This ordering verifies all the conditions of Definition 3.4. It extends to a good ordering, denoted by $\leq_{d.lex.}$ on $\mathbb{Q}^d$.

One way to introduce an ordering $\preccurlyeq$ on a group G is to introduce a subset of the group as the subset of the positive elements, $G_{\succ 0} = \{g \in G : 0 \prec g\}$. For example we have

$$G_{>_{d.lex.}0} = \{\mathbf{u} \in \mathbb{Q}^2 : u_1 + u_2 > 0\} \bigcup \{\mathbf{u} \in \mathbb{Q}^2 : u_1 > 0, u_1 + u_2 = 0\}.$$

Lemma 3.10 *Consider an ordering $\preccurlyeq$ on the group $\mathbb{Q}^d$. We have:*
1) *It refines the partial ordering $\leq$ iff $\mathbb{Q}^d_{>0} \subset \mathbb{Q}^d_{\succ 0}$.*
2) *It is a total ordering iff for any $\mathbf{u} \in \mathbb{Q}^d$: $\{\mathbf{u}, -\mathbf{u}\} \bigcap \mathbb{Q}^d_{\succ 0} \neq \varnothing$.*
3) *Its restriction on $\mathbb{Z}^d_{\geq 0}$ is a well-ordering iff this restriction refines the partial ordering $\leq$ on $\mathbb{Z}^d_{\geq 0}$.*

proof. The items 1) and 2) are easy to prove. For a proof of 3) we refer to [**?**]. $\square$

As a corollary one can give another characterization of the good orderings.

Corollary 3.11 *An ordering $\preccurlyeq$ on the group $\mathbb{Q}^d$ is a good ordering if $\mathbb{Q}^d_{>0} \subset \mathbb{Q}^d_{\succ 0}$ and for any $\mathbf{u} \in \mathbb{Q}^d$ we have $\{\mathbf{u}, -\mathbf{u}\} \bigcap \mathbb{Q}^d_{\succ 0} \neq \varnothing$.*

As another corollary we can give another description of the construction given in Proposition 3.6.

Corollary 3.12 *Consider a good ordering $\preccurlyeq$ of the group G with respect to the inclusion $\mathbb{Z}^d \subseteq G$. It has a natural extension to a good ordering on $\mathbb{Q}^d$, with respect to the natural embedding $\mathbb{Z}^d \subseteq G \subseteq \mathbb{Q}^d$, which we denote with the same notation.*

We define this extension by the set of its positive elements: consider the positive cone in $\mathbb{R}^d$ based on the set of positive elements of $\preccurlyeq$ in $\mathbb{Z}^d$. The set of positive elements in $\mathbb{Q}^d$ will be the intersection of this cone with $\mathbb{Q}^d$. Moreover, this extension coincides with the extension defined in Proposition 3.6.

Definition 3.13 For any two orderings $\preccurlyeq$ and $\preccurlyeq'$ on a group G, we define the set

$$G_+(\preccurlyeq, \preccurlyeq') = (G_{\succ 0} - G_{\succ' 0}) \bigcup (G_{\succ' 0} - G_{\succ 0}).$$

We say the sequence $\{\preccurlyeq_k\}_{k=1}^{\infty}$ of orderings on the group G converges to the ordering $\preccurlyeq$ iff

$$G_+(\preccurlyeq_1, \preccurlyeq) \supset G_+(\preccurlyeq_2, \preccurlyeq) \supset \dots \text{ and } \bigcap_{k=1}^{\infty} G_+(\preccurlyeq_k, \preccurlyeq) = \varnothing.$$

In this case we write $\lim_{k\to\infty} \preccurlyeq_k = \preccurlyeq$.

Example 3.14 For any $\omega \in \mathbb{R}_{>0}$ define a good ordering $\preccurlyeq_\omega$ on $\mathbb{Q}^2$ by

$$\mathbb{Q}^2_{\succ_\omega 0} = \{\mathbf{u} \in \mathbb{Q}^2 : u_1 + \omega.u_2 > 0\} \bigcup \{\mathbf{u} \in \mathbb{Q}^2 : u_1 + \omega.u_2 = 0, u_1 > 0\}.$$

One can easily prove that this ordering verifies the conditions of the last corollary and it is a good ordering.

Example 3.15 Take a sequence $\{\omega_r\}_{r=1}^{\infty}$ of positive irrational numbers that are increasing and convergent to -1. With the notations of the previous example, construct the sequence of orderings $\{\preccurlyeq_{\omega_r}\}_{r=1}^{\infty}$. This is a sequence of good orderings. Then it is easily seen that

$$\lim_{r\to\infty} \preccurlyeq_{\omega_r} = \leq_{d.lex.} .$$

It is interesting to note that $\mathbb{Q}^2$ with the ordering $\preccurlyeq_{\omega_r}$ does not have non-trivial isolated subgroups. On the contrary if G is such an isolated subgroup then take $0 \prec_{\omega_r} g \in G$. The group G should contain all the rational points in the section between the line joining the origin to the point g, in the plane, and the line $u_1 + \omega.u_2 = 0$. The group generated by this last set is $\mathbb{Q}^2$. In Example 4.5 we see that $\mathbb{Q}^2$ with ordering $\leq_{d.lex.}$ has a nontrivial isolated subgroup. As a result we have constructed a sequence of orderings on $\mathbb{Q}^2$ with $rank(\mathbb{Q}^2_{\preccurlyeq_{\omega_r}}) = 1$ which converges to the ordering $\leq_{d.lex.}$ with $rank(\mathbb{Q}^2_{\leq_{d.lex.}}) = 2$.

Alternatively, in the above example one could take the ω_r to be rational numbers and define the same constructions and the same limit. Everything is the same as the argument given in Example 4.5 except that $rank(\mathbb{Q}^2_{\preccurlyeq_{\omega_r}}) = 2$.

4 The valuation and the examples

Given any good ordering $\preccurlyeq$ on $\mathbb{Q}^d$, we define the ring $k[[X^{Q^d_{\succcurlyeq 0}}]]$, which is the ring of power series $z(X) \in k[[X^{Q^d_{\succcurlyeq 0}}]]$, in which the set of exponents are well-ordered with respect to $\preccurlyeq$. This is in fact a valuation ring (see [**?**], Chap. 6, Section 3, n° 4, Exemple 6). We denote this valuation by ν. Notice that in view of Proposition 2.2 we have an injective morphism

$$\begin{array}{rl} \Theta_\zeta : k[X,Y] & \hookrightarrow k[[X^{Q^d_{\succcurlyeq 0}}]] \\ X & \mapsto X \\ Y & \mapsto \zeta(X). \end{array}$$

Now, we define the valuation induced by the transcendental element $\zeta(X)$ on the ring $k[X,Y]$, with respect to a fixed good ordering $\preccurlyeq$ on $\mathbb{Q}^d$:

Definition 4.1 We define a mapping $\nu : k[X,Y] \setminus \{0\} \longrightarrow \mathbb{Q}^d_{\geq 0}$ by:

$$\nu(f) = \nu(\Theta_\zeta(f)).$$

This mapping is a valuation on the ring $k[X,Y]$.

The next proposition shows that this valuation is approximated by the intersection indices of the quasi-ordinary hypersurfaces $f^{(i)}$.

Proposition 4.2 *For any unitary irreducible quasi-ordinary $g \in k[[X]][Y]$, which is comparable with the $f^{(i)}$, we have:*

$$\nu(g) = \lim_{i\to\infty} \frac{(f^{(i)},\ g)}{deg_Y(f^{(i)})}.$$

Proof. We notice that if i is chosen so large that $\kappa < \lambda_i$ (with the notations of the Proposition 2.6) then for any $j > i$ we have:

$$\frac{(f^{(i)}, g)}{\deg_Y(g).\deg_Y(f^{(i)})} = \frac{(f^{(j)}, g)}{\deg_Y(g).\deg_Y(f^{(j)})}.$$

As a result, the limit is well defined. For the equality, it suffices to note that:

$$\mathcal{N}(g(X, \zeta^{(i)}(X))) = \mathcal{N}(\prod_{r=1}^{m^{(i)}} g(\zeta_r^{(i)})) = \deg_Y(f^{(i)}).\mathcal{N}(g(\zeta^{(i)})) = \mathcal{N}(\mathrm{Res}_Y(f^{(i)}, g)),$$

where the $\zeta_r^{(i)}$ are all the $m^{(i)}$ roots of $f^{(i)} = 0$. $\square$

The following proposition gives an effective way to compute the value $\nu(g)$, for an arbitrary $g \in k[X,Y]$. It also gives essentially another definition of this valuation. We extend the definition of ν to the ring $k[[X]][Y]$ by the same formula.

Proposition 4.3 *We have:*
1) *For any $g \in k[X,Y]$, the values of the terms of its $(f_{[\infty]}) - adic$ expansion $g = \sum c_\ell (f_{[\infty]})^\ell$ are distinct elements of $\mathbb{Q}^d_{\geq 0}$. Therefore, we have:*

$$\nu(g) = min_\ell\{\nu(c_\ell (f_{[\infty]})^\ell)\}.$$

2) *We have:*

$$\nu(f^{(i)}) = \gamma_{i+1}.$$

3) *We have:*

$$\nu((f^{(j-1)})^{n_j}) = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \ldots + l_{j-1}^{(j)}\gamma_{j-1},$$

where the $l_k^{(j)}$ and $\alpha^{(j)}$ are defined in Lemma 3.2. Moreover, there is exactly one term in the $(f_{[\infty]}) -$ adic expansion of $(f^{(j-1)})^{n_j}$ with this value, if ℓ_* is the index of this term then $\ell_* = (l_1^{(j)}, \ldots, l_{j-1}^{(j)}, 0)$.

proof. The first claim is a direct consequence of Lemma 2.12 and the properties of the good orderings. The second one is a consequence of Proposition 4.2. The third one is a consequence of the last step and Lemma 3.3. Alternatively, we can prove the third result directly and as a consequence, yield another proof of Lemma 3.3; We note that by Lemma 2.8, we have $\mathcal{N}((f^{(j-1)})^{n_j}) = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \ldots + l_{j-1}^{(j)}\gamma_{j-1} + \mathbb{R}^d_{\geq 0}$, which gives the first claim of 3). By 1) there is a unique term, say with index ℓ_*, in the $(f_{[\infty]}) - adic$ expansion of $(f^{(j-1)})^{n_j}$ such that $\nu((f^{(j-1)})^{n_j}) = \nu(c_{\ell_*}(f_{[\infty]})^{\ell_*}) = \alpha^{(j)} + l_1^{(j)}\gamma_1 + \ldots + l_{j-1}^{(j)}\gamma_{j-1}$. Using the uniqueness of the representation of the elements of Γ_{j-1}, one can show that ℓ_* is of the claimed form. $\square$

We note that the monomial which appears in the first case of the above proposition is not necessarily a vertex of the Newton polyhedron of $g(\zeta)$.

Corollary 4.4 *The semigroup $\nu(k[X,Y]\setminus 0)$ of the valuation is equal to Γ_ζ. The value group is equal to the subgroup of $\mathbb{Q}^d$ generated by Γ_ζ. We denote this value group by Φ_ζ.*

The next example shows that for suitably chosen ζ the value group will be $\mathbb{Q}^d$. In order to simplify the notations, the example is stated in the case $d = 2$.

Example 4.5 In the set of natural numbers start from $s_1 = 2$ and pick up all the numbers that are power of a prime. Denote by $\{s_i\}_{i=1}^{\infty}$ the resulting sequence. The first elements are:

$$s_1 = 2, s_2 = 3, s_3 = 4, s_4 = 5, s_5 = 7, s_6 = 8, \ldots.$$

We define:

$$\gamma_1 = (\frac{1}{s_1}, 1), \gamma_2 = (s_2, s_2 + \frac{1}{s_1}),$$

$$\text{and for } i \geq 1: \begin{cases} \gamma_{2i+1} = (s_2 \ldots s_{2i+1} + \frac{1}{s_{i+1}}, s_2 \ldots s_{2i+1}) \\ \gamma_{2i+2} = (s_2 \ldots s_{2i+2}, s_2 \ldots s_{2i+2} + \frac{1}{s_{i+1}}). \end{cases}$$

One then defines the exponents λ_i using the inductive formula of Definition 2.4. These λ_i satisfy the conditions of Definition 2.1: By the construction and the computation of the integers n_i, which is given in the following, we have $\gamma_j > n_{j-1}\gamma_{j-1}$. This last inequality gives us $\lambda_j > \lambda_{j-1}$. The condition $\lambda_j \notin Q_{j-1}$ is a consequence of the fact that the components of the elements of Q_{j-1} have, as denominators, only $s_1, \ldots, s_{j-1}$. When s_i is a power of the prime p, we have $n_i = p$. As a result $m^{(i)} = \Pi_q q^{\alpha_q}$, where q runs through all the primes less than or equal to s_i and α_q is by definition the greatest power of q such that $q^{\alpha_q} \leq s_i$. By Definition 4.1, the series $\zeta(X) = \sum X^{\lambda_i}$ defines a valuation of $k[X,Y]$. We see, by induction, that $(\frac{1}{s_i}, 1), (1, \frac{1}{s_i})$ are in the value group Φ_ζ of this valuation. Therefore, by definition of the s_i we have $\Phi_\zeta = \mathbb{Q}^2$. If we give $\mathbb{Q}^2$ the order $\leq_{d.lex.}$, this valuation is of rank two: Define $G = \{(a, -a) : a \in \mathbb{Q}\}$, this is a subgroup of $\mathbb{Q}^2$. It is an isolated subgroup (see [**?**] for the definition of the isolated subgroups and its relation to the rank of a valuation), since if we take an arbitrary element $0 <_{d.lex.} (a, -a) \in G$ then for any $\mathbf{u} = (u_1, u_2) \in \mathbb{Q}^2$ from $0 <_{d.lex.} \mathbf{u} <_{d.lex.} (a, -a)$ we deduce $deg(\mathbf{u}) = 0$ and then $\mathbf{u} \in G$.

Remark 4.6 Consider the sequence of orderings introduced in Example 3.15. If we denote the semigroups that are attached to the valuations associated by the above example to each of these orderings by $\Gamma_{\zeta, \preccurlyeq_{\omega_r}}$ then as the choice of good ordering does not have any effect on the resulting semigroup, we have $\Gamma_{\zeta, \preccurlyeq_{\omega_r}} = \Gamma_{\zeta, \leq_{d.lex.}}$. Therefore, we have a sequence of orderings which converge to another one. All of these orderings impose the same semigroup but the dimension of the valuation ring for the elements of the sequence is one and the dimension of the valuation ring to which they converge is two.

Example 4.7 We generalize an example of Zariski in [**?**] and Example 4.22 of [**?**]. Take $c_1, \ldots, c_d \in \mathbb{N}\bigcup\{\infty\}$ such that at least one of them is ∞ and take d sequences of natural numbers $\{s_j^{(q)}\}_{j=1}^{c_q};\ 1 \leq q \leq d$, where $s_j^{(q)} > 1$ (for $q = 1, \ldots, d$ and $1 \leq j \leq c_q$), and complete these sequences by setting $s_{c_q+\ell}^{(q)} = 1$, for $\ell \geq 1$. Let $(e_k)_{1 \leq k \leq d}$ denote the standard basis of the vector space $\mathbb{Q}^d$, and let γ_0 be an arbitrary element of $\mathbb{Z}_{\geq 0}^d$. Define the following vectors:

$$\gamma_1 = \gamma_0 + \frac{1}{s_1^{(1)}} e_1,$$

Now, for $i \in \mathbb{N}$ set $i = dj + l$, where $j \in \mathbb{N}\bigcup\{0\}$ and $l = 1, \ldots, d$ then define:

$$\gamma_i = \begin{cases} s_{j+1}^{(l-1)} \gamma_{i-1} + \frac{1}{s_1^{(l)} \ldots s_{j+1}^{(l)}} e_l, & l \neq 1 \\ s_j^{(d)} \gamma_{i-1} + \frac{1}{s_1^{(l)} \ldots s_{j+1}^{(l)}} e_l, & l = 1 \end{cases}$$

By the definition of the γ_i it is clear that $n_i = s_{j+1}^{(l)}$. Drop the γ_i for which $n_i = 1$. As the above example construct the vectors λ_i. We have $\gamma_i - n_{i-1}\gamma_{i-1} = \frac{1}{s_1^{(l)} \ldots s_{j+1}^{(l)}} e_l > 0$, therefore $\lambda_i > \lambda_{i-1}$, and λ_i is not in the group Q_{i-1} of Definition 2.4. Consider the element $\zeta = \sum X^{\lambda_i}$, and the valuation attached to it by Definition 4.1. We see, by induction, that $\frac{1}{s_1^{(l)}\ldots s_{j_l}^{(l)}}e_l$ is in the value group of this valuation, Φ_ζ. Therefore, we have:

$$\Phi_\zeta=\{(\frac{p_1}{s_1^{(1)}\ldots s_{j_1}^{(1)}},\ldots,\frac{p_d}{s_1^{(d)}\ldots s_{j_d}^{(d)}}):p_1,\ldots,p_d\in\mathbb{Z},j_1\le c_1,\ldots,j_d\le c_d\}.$$

If we set $s_j^{(q)}=j$, for $q=1,\ldots,d$ and $j\in\mathbb{N}$, the resulting value group is $\Phi_\zeta=\mathbb{Q}^d$.

One may ask whether concerning the value groups the last example is the general situation? More precisely, let ζ be an element which verifies the conditions of Definition 2.1 and consider the valuation induced by it, as in Definition 4.1, with value group Φ_ζ. Does there exist another element $\zeta^{'}$ which comes from the construction of Example 4.7 such that $\Phi_\zeta=\Phi_{\zeta'}$? The answer is no if $d\ge 2$. Here is an example:

Example 4.8 Let $\overline{e}=e_1+\ldots+e_d$, where again the e_k form the standard basis of the vector space $\mathbb{Q}^d$. For $i\in\mathbb{N}$ we set:

$$\gamma_0=\overline{e},\gamma_i=2\gamma_{i-1}+\frac{1}{2^i}\overline{e},$$

As in the last two examples construct the vectors λ_i. One can show that these vectors verify the conditions of Definition 2.1 (here $n_i=2$). So, we can consider the element ζ attached to them. We show there is no element $\zeta^{'}$, which comes from a construction as in Example 4.7, such that $\Phi_\zeta=\Phi_{\zeta'}$. In contrary, let $\zeta^{'}$ be such an element and consider the first vector of the construction of $\zeta^{'}$, in Example 4.7, i.e., $\gamma_1'=\gamma_0'+\frac{1}{r}e_1$, where $\gamma_0'\in\mathbb{Z}^d_{\ge 0}$ and $r\in\mathbb{N}\backslash\{1\}$. Then we have $\gamma_1'\in\Phi_\zeta$ which implies that there exists a natural number n and integers $a_1,\ldots,a_n$ and a vector $\mathbf{b}\in\mathbb{Z}^d$ such that: $\sum_{j=1}^n a_j.\gamma_j+\mathbf{b}=\gamma_1'$. The γ_i can be written in the form: $\gamma_i=h_i\overline{e}+\frac{l_i}{2^i}\overline{e}$, $h_i,l_i\in\mathbb{N}$, where l_i is an odd number and $l_i<2^i$. So, the above equation implies: $\frac{1}{r}e_1-p\overline{e}\in\mathbb{Z}^d$, where $p=\sum_{j=1}^n\frac{a_jl_j}{2^j}\in\mathbb{Q}$. When $d>1$ this implies that $p,p-\frac{1}{r}\in\mathbb{Z}$, which is impossible. In fact, the semigroup Φ_ζ, can be given explicitly as follows:

$$\Phi_\zeta=\{\mathbf{b}+\frac{a_i}{2^i}\overline{e}:\mathbf{b}\in\mathbb{Z}^d,a_i\in\mathbb{Z},i\in\mathbb{N}\}.$$

For $d=1$, there will be no contradiction. Because in this case $\overline{e}=e_1$, therefore, $\frac{1}{r}e_1-p\overline{e}\in\mathbb{Z}^d$ only implies $p-\frac{1}{r}\in\mathbb{Z}$. We can construct the value group which it generates via the construction of Example 4.7. It suffices to set $s_j^{(1)}=2$, for $j\in\mathbb{N}$.

On the other hand, the following proposition shows that the transcendental elements are general enough to produce any totally ordered group G of rational rank d.

Proposition 4.9 *Suppose that $(G,\succcurlyeq)$ is a totally ordered group of rational rank d and $\succcurlyeq$ is a good ordering with respect to the inclusion $\mathbb{Z}^d\subseteq G$. Then there is a transcendental element ζ such that its associated valuation* (Definition 4.1) has the value group G, i.e., we have $G=\Phi_\zeta$.

Proof. Consider a set of generators of G, say $S=\{s_i\}_{i=1}^u$, such that $S\subset\mathbb{Q}^d_{\ge 0}$, where $u\in\mathbb{N}\bigcup\{\infty\}$. Let s_1' be the first element of S which is not in $G_0=\mathbb{Z}^d$ and set $\gamma_1=s_1',G_1=G_0+\mathbb{Z}\gamma_1,n_1=[G_1:G_0]$. Assume we have defined the elements $\{\gamma_j,s_j',n_j,G_j\}_{j=1}^i$. Let s_{i+1}' be the first vector of $S\setminus\{s_1',\ldots,s_i'\}$ which is not in G_i and set $\gamma_{i+1}=n_i\gamma_i+s_{i+1}',G_{i+1}=G_i+\mathbb{Z}\gamma_{i+1},n_{i+1}=[G_{i+1}:G_i]$. If this process goes for ever then we have infinitely many γ_i. Then as in the examples above construct the vectors λ_i. Using the inductive formula of Definition 2.4 and $\gamma_{i+1}-n_i\gamma_i=s_{i+1}'$ we see that: $\lambda_i=s_1'+\ldots+s_i'$. These vectors verify the conditions of Definition 2.1. Hence they define an element ζ, which is the desired element.

In the case where the process terminates after finitely many steps, we make use of the following lemma, which shows that any finite truncation of a quasi-ordinary series, corresponding to a finite initial set of generators $(\gamma_1,\ldots,\gamma_i)$ of the associated semigroup, can be viewed as a truncation of a transcendental quasi-ordinary series whose associated semigroup is generated by $(\gamma_1,\ldots,\gamma_i)$. $\square$

Lemma 4.10 [1] *Let $\gamma_1, \ldots, \gamma_i$ be an increasing sequence of elements of $\mathbb{Q}^d_{\succcurlyeq 0}$ satisfying the conditions of Lemma 3.2 and generating a subgroup $G_i \subset \mathbb{Q}^d$. Let $\lambda_1, \ldots \lambda_i$ be the increasing sequence of elements of $\mathbb{Q}^d_{\succcurlyeq 0}$ corresponding as in definition 2.4 to the γ_j. Given a finite sum*

$$\zeta_F(X) = \Sigma_{\lambda \in F} c_\lambda X^\lambda$$

having $\lambda_1, \ldots, \lambda_i$ as characteristic exponents in the sense of Definition 2.1, there exists a series of the form

$$\zeta(X) = \zeta_F(X) + \sum c_{\tilde{\lambda}} X^{\tilde{\lambda}}$$

which is transcendental over $k(X)$ and such that $\zeta_F(X)$ is a finite truncation of $\zeta(X)$ and the semigroup associated to $\zeta(X)$ is G_i.

Proof. Let us take an infinite increasing sequence of exponents $(\tilde{\lambda}_j)_{j\geq 1}$ tending to ∞ in $(\mathbb{Q}^d, \preccurlyeq)$, with $\tilde{\lambda}_1$ larger than all the exponents of ζ_F, and with $\tilde{\lambda}_1$ and all $\tilde{\lambda}_j - \tilde{\lambda}_{j-1}$ in the semigroup generated by the $(\gamma_k)_{k\leq i}$ for all $j > 1$. In addition we ask the following "large gaps" condition:
1) $\tilde{\lambda}_{j+1} > (j+1)\tilde{\lambda}_j$ for all j
and remark that because we assume that the $(\tilde{\lambda}_j)$ increase and tend to ∞ in $(\mathbb{Q}^d, \preccurlyeq)$ we have:
2) for any positive integer s we have $\lambda_\ell > s\mathbf{1}$ for sufficiently large ℓ, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{Q}^d_{\succcurlyeq 0}$.
Now the claim is that for arbitrary nonzero coefficients $c_{\tilde{\lambda}_j}$ the series

$$\zeta(X) = \zeta_F(X) + \Sigma_{j=i+1}^{\infty} c_{\tilde{\lambda}_j} X^{\tilde{\lambda}_j} \in k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$$

is transcendental over $k(X,Y)$ and provides a valuation of $k(X,Y)$ with group G_i.
Let $Q(X,Y)$ be a polynomial of degree $\leq s$ in Y and of total degree $\leq s$ in the X_k; by our assumptions we can choose an ℓ such that $\ell > s$ and $\lambda_\ell > s\mathbf{1}$. Set $u = \zeta_F(X) + \Sigma_{j=1}^{\ell} c_{\tilde{\lambda}_j} X^{\tilde{\lambda}_j}$ and $v = \Sigma_{j=\ell+1}^{\infty} c_{\tilde{\lambda}_j} X^{\tilde{\lambda}_j}$; we have $\zeta(X) = u + v$ in $k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$ and the equality:

$$Q(X, u+v) = Q(X,u) + vA_1(X) + v^2 A_2(X) + \cdots$$

with $A_k(X) \in k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$ and therefore by our choice of $\tilde{\lambda}_j$, denoting by ν the $X-adic$ order, we have:

$$\nu(vA_1(X) + v^2 A_2(X) + \cdots) \geq \nu(v) = \tilde{\lambda}_{\ell+1} > (\ell+1)\tilde{\lambda}_\ell.$$

On the other hand, since we consider finite sums, we can speak of the degree in X of a polynomial such as $Q(X,u) = Q_0(X) + Q_1(X)u + Q_2(X)u^2 + \cdots + Q_m(X)u^m$, with $m \leq s$ and $Q_m(X) \neq 0$; this degree is an element of $\mathbb{Q}^d_{\succcurlyeq 0}$.
Since if $\Sigma_1^d a_i \leq s$ we have the inequality $(a_1, \ldots, a_d) \leq s\mathbf{1} < \tilde{\lambda}_\ell$, for $i < m$, the degree of $Q_i(X)u^i$ is $\leq s\mathbf{1} + i\tilde{\lambda}_\ell < (i+1)\tilde{\lambda}_\ell \leq m\tilde{\lambda}_\ell$. On the other hand, the degree of the last term $Q_m(X)u^m$ is $\geq m\tilde{\lambda}_\ell$.

This shows that $Q(X,u)$ is not zero and since its valuation cannot exceed the highest power of X which appears, applying to $Q_m(X)u^m$ the same argument as for $i < m$, we see that it is $< (m+1)\tilde{\lambda}_\ell$.

So $Q(X, u+v) = Q(X, \zeta(X))$ is the sum of a polynomial with fractional exponents of valuation $\prec (m+1)\tilde{\lambda}_\ell \leq (s+1)\tilde{\lambda}_\ell$ and a series of valuation $\succ (\ell+1)\tilde{\lambda}_\ell > (s+1)\tilde{\lambda}_\ell$. It follows that $Q(X, \zeta(X))$ is of valuation $\prec (s+1)\tilde{\lambda}_\ell$ whenever ℓ satisfies the conditions stated above with respect to the degree of the polynomial $Q(X,Y)$. This proves that the series $\zeta(X)$ is not algebraic over $k(X)$. Moreover by construction, for polynomials $R(X,Y)$, the orders of the series $R(X, \zeta(X)$ belong to the semigroup generated by the $(\gamma_k)_{k\leq i}$, so that the value group is G_i. □

5 The sequence of key polynomials

The results of the last section can be interpreted as follows: given a totally ordered group $(G, \preccurlyeq)$ of rational rank d, and an inclusion $\mathbb{Z}^d \subset G$, we considered the valuation on the field $k(X)$ with values in $(\mathbb{Z}^d, \preccurlyeq)$ deduced from the valuation on $k[X]$ which associates to a polynomial the minimum for the order $\preccurlyeq$ of the exponents of its terms. Then we used a system of generators of G and the order to produce a transcendental series which determines, via an embedding $k[X,Y] \to k[[X^{\mathbb{Q}^d_{\succcurlyeq 0}}]]$ an extension of the valuation from $k(X)$ to $k(X,Y)$ having value group equal to G.

[1] This lemma and its proof were given by Bernard Teissier

In this section we explain the relation between the construction of the valuation ν and MacLane's method to extend a given valuation ν on the field k' to the field $k'(Y)$ via a sequence of key polynomials ([?]). In our case k' will be the field $k(X)$. In fact, we show that the quasi-ordinary hypersurfaces $f^{(i)}$ which attached to the valuation ν are a sequence of key polynomials in MacLane's terminology (Theorem 5.5). In order to prove this result, we give another way of defining the valuations ν_i (Proposition 5.6) which appear in MacLane's construction and prove several properties of these valuations. In particular they are given a geometric interpretation.

Throughout this section the value group Φ_ω is a sub-group of a totally ordered group G and the ordering on Φ_ω (as value group) is induced by the ordering on G. Consider an arbitrary valuation ω and a subring R of its valuation ring R_ω. We will sometimes say, in such a situation, that ω is a valuation of R. We set $\Gamma = \omega(R \setminus \{0\}) \subset \Phi_{\omega+} \bigcup \{0\}$; It is the semigroup of (R, ω). For $\phi \in \Phi_\omega$ set:

$$\mathcal{P}_\phi(R) = \{x \in R : \omega(x) \geq \phi\}$$

$$\mathcal{P}^+_\phi(R) = \{x \in R : \omega(x) > \phi\}.$$

The graded algebra associated to (R, ω) is defined by:

$$\mathrm{gr}_\omega R = \bigoplus_{\phi \in \Gamma} \frac{\mathcal{P}_\phi(R)}{\mathcal{P}^+_\phi(R)}.$$

It can be represented (see [?], Proposition 4.1) as a quotient of an infinite dimensional polynomial ring by a binomial ideal, so it is "essentially toric" (see [?], Subsection 4.2).

Definition 5.1 Given a valuation ω on a field K, and given a ring $R \subset R_\omega$, for any $a, b \in R$, we say they are equivalent if their image in $\mathrm{gr}_\omega R$ is the same. In this case we write $a \sim b$. We say b is equivalence-divisible in ω by a if there exists a $c \in R$ such that $b \sim ca$.

Definition 5.2 Given a valuation ω of $k'(Y)$, a non zero polynomial $\theta(Y) \in k'[Y]$ is a key polynomial for ω if it satisfies the following conditions:

- *Irreducibility.* If a product of polynomials is equivalence-divisible in $k'[Y]$ by $\theta(Y)$ then one of the factors is equivalence-divisible by $\theta(Y)$
- *Minimal degree.* Any non-zero polynomial equivalence-divisible in $k'[Y]$ by $\theta(Y)$ has a degree in Y not less than the degree of $\theta(Y)$.
- The leading coefficient of $\theta(Y)$ is 1.

Using such key polynomials MacLane introduces a new valuation based on ω : If ω is a valuation of $k'[Y]$ and $\theta(Y)$ is a key polynomial for ω then choose an arbitrary element $\mu \in G$ such that $\mu > \omega(\theta)$ and set $\omega_1(\theta) = \mu$. For any element $g \in k'[Y]$ with the $\theta - adic$ expansion $g = \sum_i g_i \theta^i$ define:

$$\omega_1(g) = \min_i[\omega(g_i) + i\mu].$$

Theorem 5.3 ([?]) With the notations above, the mapping ω_1 is a valuation on $k'[Y]$. The valuation ω_1 is called an augmented valuation and is denoted by

$$\omega_1 = [\omega, \omega_1(\theta) = \mu].$$

Definition 5.4 ([?]) An ith stage inductive valuation ω_i is any valuation of $k'[Y]$ obtained by a sequence of valuations $\omega_0 = \omega, \omega_1, \dots, \omega_i$, where for $j = 1, \dots, i$ we have $\omega_j = [\omega_{j-1}, \omega_j = \mu_j]$. Furthermore, for $j = 2, \dots, i$, the key polynomials θ_j must satisfy:

- $\theta_1(Y) = Y$
- $\deg\, \theta_j(Y) \geq \deg\, \theta_{j-1}(Y)$.
- $\theta_j(Y) \nsim \theta_{j-1}(Y)$ for ω_{j-1}.

We can symbolize this valuation thus:

$$\omega_i = [\omega_0, \omega_1(\theta_1) = \omega_1, \omega_2(\theta_2) = \mu_2, \ldots, \omega_i(\theta_i) = \mu_i].$$

In the special case that for any $g \in k'[Y]$ there exists some i such that for any $j \geq i$ we have $\omega_j(g) = \omega_i(g)$, one can define the limit augmented valuation:

$$\omega_\infty(g) = \lim_{i \to \infty} \omega_i(g).$$

The relation with the construction of the valuation ν of the last section is as follows:

Theorem 5.5 *Consider the valuation ν of the last section and suppose the transcendental element which is attached to this valuation (Definition 2.1) be ζ and the $f^{(i)}$ be the quasi-ordinary hypersurfaces attached to it. Then the sequence $\{\theta_i = f^{(i-1)}\}_{i=1}^{\infty}$ is a sequence of key polynomials for the sequence of inductive valuations*

$$\nu_i = [\nu_0 = \nu, \nu_1(\theta_1) = \gamma_1, \nu_2(\theta_2) = \gamma_2, \ldots, \nu_i(\theta_i) = \gamma_i].$$

Moreover, the limit valuation $\lim_{i \to \infty} \nu_i(g)$ exists and is equal to ν. Here ν_0 is a valuation which comes from fixing a good ordering $\preccurlyeq$ on the group $\mathbb{Q}^d$.

We define the valuations ν_i of Theorem 5.5 in another way which reflects the relation between different *adic* representations and also the relation between the valuations ν_i and ν. Using the properties of this new definition we are able to prove Theorem 5.5.

Proposition 5.6 *Define the mapping $\nu_i : k[X,Y] \setminus 0 \to \mathbb{Q}^d$ as follows; For any $g \in k[X,Y]$, with the $(f_{[i-1]}) - adic$ expansion $g = \sum c_\ell (f_{[i-1]})^\ell$, set:*

$$\nu_i(g) = min_\ell \{\nu(c_\ell (f_{[i-1]})^\ell)\}.$$

1) *The mapping ν_i defines a valuation.*
2) *For any $j < i$, we have: $\nu_i(f^{(j)}) = \nu(f^{(j)})$.*
3) *For any $g \in k[X,Y]$, we have: $\nu_1(g) \preccurlyeq \nu_2(g) \preccurlyeq \ldots \preccurlyeq \nu(g)$. Moreover, for this g there exists an i such that $\nu_i(g) = \nu(g)$. Therefore, for any $j \geq i$, we have: $\nu_j(g) = \nu(g)$.*
4) *The value semigroup of ν_i is: $\nu_i(k[X,Y] \setminus 0) = \Gamma_i$.*
5) *The valuations ν_i which are defined in this proposition are equal to the corresponding valuations defined in the Theorem 5.5.*

Proof. For 1), we show that for any $g, h \in k[X,Y] \setminus 0$ we have $\nu_i(g+h) \succcurlyeq \min\{\nu_i(g), \nu_i(h)\}$ and $\nu_i(gh) = \nu_i(g) + \nu_i(h)$. The first one is a direct consequence of the definition and the uniqueness of the $(f_{[i-1]}) - adic$ representation. For the second one, we show that the monomials in the $(f_{[i-1]}) - adic$ representations of g and h, with minimum value, can not cancel each other in the product $g.h$, through the process of getting the $(f_{[i-1]}) - adic$ representation of $g.h$ from this product. Let $g = \sum_t u_t (f^{(i-1)})^{n_i.t}$ and $h = \sum_t u'_t (f^{(i-1)})^{n_i.t}$ be the unique representations of g and h in $\mathrm{gr}_{\nu_i} k[[X]]][Y]$, which comes from Lemma 5.9. Now, consider the product $g.h = \sum_{t''} \sum_{\substack{t,t' \\ t+t'=t''}} u_t.u'_{t'} (f^{(i-1)})^{n_i.t''}$.
We do the replacements using Lemma 3.3, in each monomials of $g.h$, for those $f^{(j)}$ such that their power is greater than n_j, where $j < i-1$. By Lemma 5.10, such a replacement cannot change the power of $f^{(i-1)}$ in the uniquely generated monomial with minimal value in $\mathrm{gr}_{\nu_i} k[[X]][Y]$. Therefore, these replacements for the unique minimum t''_0, which in turn refers to the unique minima t_0 and t'_0, produces a monomial in the $(f_{[i-1]}) - adic$ representation of $g.h$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$ with value equal to $\nu_i(g) + \nu_i(h)$.
For 2), we note that it is a direct consequence of Proposition 4.2.
For 3), it is sufficient to note that we can write the $(f_{[i+1]}) - adic$ representation of an element from its $(f_{[i]}) - adic$ representation, using the equations given in Lemma 3.3. Moreover, in this process the value of the monomials in the representation can not decrease. As we noted earlier these equations do not change the minimum value.
The two last claims are clear. $\square$

Remark 5.7 The comparison of the propositions 4.3, 4.2, and 5.6 gives us two interpretations of the fact that the valuation ν is the limit of valuations ν_i. The first by associating each ν_i to a specific truncation of the series $\zeta(X)$, the second by associating it to the *adic* expansion in terms of the $f^{(i)}$. The next section unifies these interpretations.

Now, we can give a generalization of Proposition 4.9:

Corollary 5.8 *Given any totally ordered subgroup G of rational rank d, ordered by a good ordering, there is an element $\zeta(X)$ which verifies the conditions of Definition 2.1 such that for a unique $i \in \mathbb{N}\bigcup\{\infty\}$ we have $G = \Phi_{\nu_i}$, where the ν_i are those of Theorem 5.5.*

We now give the two lemmas used in the proof, with the notation of Theorem 5.5. Notice that $\theta_{[i]} = f_{[i-1]}$.

Lemma 5.9 *Let $g = \sum_\ell c_\ell(\theta_{[i]})^\ell$ be the $(\theta_{[i]})-$adic representation of $g \in k[X,Y]$. Set $\mathrm{in}_{\nu_i}(g) = \sum_{\ell'} c_{\ell'}(\theta_{[i]})^{\ell'}$ which are the monomials of the $(\theta_{[i]})-$adic representation of g that have minimum ν_i-value. Then the power of θ_i in these monomials is a power of n_i and for any $t \in \mathbb{N}$ there exists at most one monomial in $\mathrm{in}_{\nu_i}(g)$ such that the power of θ_i for it is $n_i.t$. In other words, we can write*

$$\mathrm{in}_{\nu_i}(g) = \sum_t u_t \theta_i^{n_i.t},$$

where $t \in \mathbb{N}\bigcup\{0\}$. Here for every t there is a unique ℓ such that $n_i.t = \ell_i$ and $u_t.\theta_i^{n_i.t} = c_\ell.(\theta_{[i]})^\ell$.

Proof. It is sufficient to note that if $\nu_i(c_{\ell_1}(\theta_{[i]})^{\ell_1}) = \nu_i(c_{\ell_2}(\theta_{[i]})^{\ell_2})$ then $n_i \mid \ell_{1,i} - \ell_{2,i}$ and if $\ell_{1,i} = \ell_{2,i}$ then $\ell_1 = \ell_2$. □

Lemma 5.10 *Let $M = c_\ell(\theta_{[i]})^\ell$ be an arbitrary monomial. For an arbitrary $j < i$ with $\ell_j > n_j$, we replace $\theta_j^{n_j}$ by its adic expansion from Lemma 3.3. Let g be the resulting element then we have $\mathrm{in}_{\nu_i}(g) = c_{\ell'}(\theta_{[i]})^{\ell'}$, with ℓ' such that $\ell'_i = \ell_i$.*

Proof. It is sufficient to note that after replacement the monomials which change the power of θ_i have a greater ν_i-value than M. Moreover, there is exactly one monomial with minimal ν_i-value, which is the same as the ν_i-value of M. □

Proposition 5.11 *With the notations of Theorem 5.5, the element θ_{i+1} is irreducible in $\mathrm{gr}_{\nu_i} k[[X]][Y]$.*

Proof. By Lemma 3.3, we have $c_{i+1}\theta_{i+1} = \theta_i^{n_i} - sX^{\alpha^{(i)}}(\theta_{[i-1]})^{l^{(i)}}$, for some $s \in k$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$. Suppose that $\theta_{i+1} = a.b$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$, for some $a, b \in k[X,Y]$. Then by Lemma 5.9, we have $a = \sum_{t=0}^P u_t.\theta_i^{n_i.t}$ and $b = \sum_{t=0}^Q u'_t\theta_i^{n_i.t}$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$. From $\nu_i(a) + \nu_i(b) = \nu_i(\theta_{i+1}) = n_i\gamma_i$ we deduce that $P + Q = 1$. Hence, without loss of generality, we can assume that $P = 1$ and $Q = 0$. But then $a.b = u_0u'_0 + u_1u'_0\theta_i^{n_i}$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$. By Lemma 5.10, the element $u_1u'_0$ is a unit in $\mathrm{gr}_{\nu_i} k[[X]][Y]$, therefore, b is a unit in $\mathrm{gr}_{\nu_i} k[[X]][Y]$. □

Proposition 5.12 *If $\theta_{i+1} \mid g$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$ for some $g \in k[X,Y]$ then $\deg_Y(g) \geq \deg_Y(\theta_{i+1})$.*

Proof. We have $g = h\theta_{i+1}$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$ for some $h \in k[X,Y]$. By Lemma 5.10, we can write $g = \sum_{t=0}^P u_t.\theta_i^{n_i.t}$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$. Note that $\deg_Y(g) \geq \deg_Y(u_P) + n_i.P.\deg_Y(\theta_i)$. If $\deg_Y(g) < \deg_Y(\theta_{i+1}) = n_i.\deg_Y(\theta_i)$, we have two possibilities: Either, we have $P = 1$ and $u_1 = 1$, which is impossible because by Lemma 5.10, this implies that $h = 1$ in $\mathrm{gr}_{\nu_i} k[[X]][Y]$, or, we have $P = 1$; this is also impossible, because by Lemma 5.10, the product $h\theta_{i+1}$ is of the form $\sum_{t=0}^Q u'_t\theta_i^{n_i.t}$, such that $Q \geq 1$. □

Proof of Theorem 5.5. By induction, suppose that we have proved ν_i is a valuation. We prove that θ_{i+1} is a key polynomial for ν_i and then by Theorem 5.3 the mapping ν_{i+1} is a valuation. The irreducibility is a result of Proposition 5.11, the minimal degree property is a result of Proposition 5.12. Moreover, the sequence $\{\theta_i\}$ satisfies the conditions of Definition 5.4, hence, it is a sequence of key polynomials. Notice that the condition $\theta_{i+1} \nsim \theta_i$ (for ν_i) is a consequence of the fact that $\nu_i(\theta_{i+1}) = n_i\nu_i(\theta_i) \neq \nu_i(\theta_i)$. □

6 Specialization to the graded ring associated to the valuation

Through this section we fix an element $\zeta(X)$ as defined in Definition 2.1 and a sequence of elements $\zeta^{(k)}(X)$ attached to it (Definition 2.3). Following [?], subsection 4.4 and [?], in this section we give a geometric interpretation of the construction of the valuation ν and the element $\zeta(X)$ attached to it. Take an infinite sequence of indeterminates $U = (U_1, U_2, \ldots)$. Consider the infinite dimensional space $\mathcal{A} = \mathrm{Spec}(k[[X]][U])$, which will play the role of a regular ambient space. Note that for every element $h \in k[[X]][U]$ there is an $i \in \mathbb{N}$ such that $h \in k[[X]][U_1, \ldots, U_i]$.

We embed the variety $\mathcal{S} = \mathrm{Spec}(R)$, $R = k[[X]][\zeta(X)]$, in $\mathcal{A}$ and give a natural (possibly infinite) ordered system of equations $(H_i(X,U) = 0)$ for the image of this embedding in terms of the relations given in Lemma 3.3.

Moreover, we give an embedding of the quasi-ordinary hypersurfaces $f^{(r)}(X,Y) = 0$, defined in Definition 2.3, in the ambient space $\mathcal{A}$ such that the equations of the image of this embedding come from truncating the system of equations $(H_i(X,U) = 0)$.

A specialization of the variety $\mathcal{S}$ to the toric variety $\mathrm{Spec}(\mathrm{gr}_\nu R)$ (see [?], subsection 4.2) will be given via a suitable filtration on the ring $k[[X]][U]$. This filtration is naturally induced from the valuation ν.

The embedding of $\mathcal{S}$ in $\mathcal{A}$ comes from the following morphism:

$$\begin{array}{rl} \Psi : k[[X]][U] & \rightarrow R \\ X & \mapsto X \\ U_i & \mapsto f^{(i-1)}(X, \zeta(X)). \end{array}$$

Note that Ψ is surjective, because $U_1 \mapsto f^{(0)}(X,\zeta(X)) = \zeta(X)$.

The valuation ν on $k[[X]][Y]$ (see Definition 4.1) induces a weight on any element of the ring $k[[X]][U]$: For any monomial $X^\beta U^\nu$ we define $\omega(X^\beta U^\sigma) = \nu(\Psi(X^\beta U^\sigma)) = \beta + \sum \sigma_i \gamma_i = \beta + \gamma.\sigma$. For any $\omega \in \Gamma_\zeta$ we define the ideal $\mathcal{I}_\omega$ (res. $\mathcal{I}_\omega^+$) of the ring $k[[X]][Y]$ which contains all the elements with weight greater than or equal to (res. strictly greater than) ω. The sequence of the ideals $\{\mathcal{I}_\omega\}_{\omega \in \Gamma_\zeta}$ is a filtration. Note that the ordering on the index set Γ_ζ of this sequence is the fixed good ordering defined the valuation ν.

Proposition 6.1 *The morphism Ψ induces a surjective morphism of $k[X]-$algebras:*

$$\begin{array}{rl} \mathrm{gr}\Psi : \mathrm{gr}_\omega k[[X]][U] = k[X,U] & \rightarrow \mathrm{gr}_\nu R = k[X^{\Gamma_\zeta}] \\ X & \mapsto X \\ U_i & \mapsto \overline{f^{(i-1)}(X, \zeta(X))}. \end{array}$$

Moreover, with the notations of Lemma 3.2, we have $ker(\mathrm{gr}\Psi) = < h_1, h_2, \ldots >$, where:

$$\left\{ \begin{array}{lllll} h_1 & := & U_1^{n_1} & - & d_1 X^{\alpha^{(1)}}, \\ h_2 & := & U_2^{n_2} & - & d_2 X^{\alpha^{(2)}} U_1^{l_1^{(2)}}, \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ h_i & := & U_i^{n_i} & - & d_i X^{\alpha^{(i)}} U_1^{l_1^{(i)}} \ldots U_{i-1}^{l_{i-1}^{(i)}}, \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array} \right.$$

Proof. In coordinate free terms the morphism $\mathrm{gr}\Psi$ is defined by $\mathrm{gr}\Psi(\overline{a}) = \overline{\Psi(a)}$, for $a \in k[[X]][U]$. The equality $\mathrm{gr}_\omega k[[X]][U] = k[X,U]$ is clear from the definition of the filtration on $k[[X]][U]$ and the equality $\mathrm{gr}_\nu R = k[X^{\Gamma_\zeta}]$ follows from the Proposition 4.3. The proof of the Proposition 38 of [?] could be adapted to give a proof of the second part. $\square$

The above proposition shows that $\mathcal{Z}^{\Gamma_\zeta} := \mathrm{Spec}(k[X^{\Gamma_\zeta}])$ is embedded in the infinite dimensional space $\mathcal{A}$. Moreover, the equations defining this embedding are binomial. This is also a general fact, see [?], section 4.

Proposition 6.2 *The kernel of the map $\Psi : k[[X]][U] \to R$ has the following generators:*

$$\begin{cases} H_1 & := & U_1^{n_1} & - & d_1 X^{\alpha^{(1)}} & + & c_1 U_2 & + & r_1(U_1), \\ H_2 & := & U_2^{n_2} & - & d_2 X^{\alpha^{(2)}} U_1^{l_1^{(2)}} & + & c_2 U_3 & + & r_2(U_1, U_2), \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ H_i & := & U_i^{n_i} & - & d_i X^{\alpha^{(i)}} U_1^{l_1^{(i)}} \dots U_{i-1}^{l_{i-1}^{(i)}} & + & c_i U_{i+1} & + & r_i(U_1, \dots, U_i), \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{cases}$$

for $i \in \mathbb{N}$. The elements c_i are defined in Lemma 3.3 and the d_i are defined in the previous proposition. For any $j \in \mathbb{N}$ the weight of a term $X^\beta U^\nu$ appearing in $r_j(U)$ is strictly greater than $n_j \gamma_j$. The terms appearing in the expansion of $r_j(U)$ are determined explicitly by Lemma 3.3.

Proof. The H_i are analogous to the equations given in Lemma 3.3. Therefore, $H_i \in \mathrm{Ker}\Psi \forall i$. On the other hand, we notice that $in_\omega(H_i) = h_i$, and by the last proposition the h_i generate $\mathrm{Ker}(\mathrm{gr}\Psi)$. This gives us $\mathrm{Ker}(\mathrm{gr}\Psi) \subset \mathrm{gr}(\mathrm{Ker}\Psi)$, therefore, $\mathrm{Ker}(\mathrm{gr}\Psi) = \mathrm{gr}(\mathrm{Ker}\Psi)$. Let $g = g(U_1, \dots, U_{i_0}) \in \mathrm{Ker}\Psi$; we have to show that g is in the ideal generated by the H_i. We write g in the form $g = \sum_{(\beta_1,\beta_2)} X^{\beta_1} U^{\beta_2}$. For any i such that $(\beta_2)_i > n_i$ replace $U_i^{n_i}$ with $H_i + X^{\alpha^{(i)}} U_1^{\ell_1^{(i)}} \dots U_{i-1}^{\ell_{i-1}^{(i)}} - c_i U_{i+1} - r_i(U_1, \dots, U_i)$, by Lemma 6.6, this terminates after finitely many steps and we get a representation $g = f(H_1, \dots, H_k) + \sum_{(\beta_1,\beta_2)} X^{\beta_1} U^{\beta_2}$, where f is a polynomial with coefficients in $k[[X]][U]$ and $f(0) = 0$, moreover, $(\beta_2)_i < n_i$. Then we have $\sum_{(\beta_1,\beta_2)} X^{\beta_1} U^{\beta_2} = g - f(H_1, \dots, H_k) \in \mathrm{Ker}\Psi$. If $\sum_{(\beta_1,\beta_2)} X^{\beta_1} U^{\beta_2} = 0$ we are done, otherwise, $\mathrm{in}_\omega(\sum_{(\beta_1,\beta_2)} X^{\beta_1} U^{\beta_2}) \in \mathrm{gr}(\mathrm{Ker}\Psi) = \mathrm{Ker}(\mathrm{gr}\Psi)$, which is impossible because $\mathrm{in}_\omega(\sum_{(\beta_1,\beta_2)} X^{\beta_1} U^{\beta_2}) = X^{\beta_1^*} U^{\beta_2^*}$, for a unique pair (β_1^*, β_2^*), and $\mathrm{gr}\Psi(X^{\beta_1^*} U^{\beta_2^*}) = X^{\gamma \cdot \beta_2^* + \beta_1^*} \neq 0$. $\square$

Remark 6.3 Notice that, unlike what is done in [**?**], it is not possible to arrange the situation so that $d_i = 1$, because we start from a fixed system of semi-roots. Moreover, the equality $\mathrm{in}_\omega(H_i) = h_i$ shows that the ideal defining the embedding $\mathcal{S} \hookrightarrow \mathcal{A}$ specializes through the filtration to the ideal of the embedding $\mathcal{Z}^{\Gamma_\zeta} \hookrightarrow \mathcal{A}$.

Consider a monomial $M = U_1^{q_1} \dots U_j^{q_j} \dots$, and define $V(M) = (q_1, \dots, q_j, \dots)$, $W_2(M) = \frac{q_1}{n_1} + q_2$, $W_{j+1}(M) = \frac{W_j(M)}{n_j} + q_{j+1}$. After one replacement for some term U_j $(q_j \geq n_j)$ in M the monomials generated are of the form $M' = U_1^{q_1+m_1u_1} \dots U_{j-1}^{q_{j-1}+m_{j-1}u_1} U_j^{m_j u_1} U_{j+1}^{q_{j+1}+u_2} U_{j+2}^{q_{j+2}} \dots$, such that $m_h < n_h$ $(h \leq j)$ and $u_1 + u_2 = [\frac{q_j}{n_j}]$. For $h < j$ we have $\frac{q_h + m_h u_1}{n_h} \leq u_1 - \frac{u_1}{n_h}$.

Lemma 6.4 *With the notations above, for the monomials M' obtained from M after a sequence of replacements for the U_j (for a fixed i and $j \leq i$) we have: $[W_{i+1}(M')] \leq [W_{i+1}(M)]$. The inequality is strict if in at least one of the replacements $u_1 \neq 0$. Moreover, the maximum exponent possible of U_{i+1} in these M' exists and is less than or equal to $[W_{i+1}(M)]$.*

Proof. It is sufficient to prove it just for one such replacement and use induction. So, if M' is one of the monomials obtained from M by one replacement on U_j $(j \leq i)$ and $V(M') = (q'_1, \dots, q'_j, \dots)$ then for some $u_1, u_2 \in \mathbb{Z}_{\geq 0}$ such that $u_1 + u_2 = [\frac{q_j}{n_j}]$, we have

$$W_{i+1}(M') \leq W_{i+1}(M) - \tfrac{q_j}{n_j \dots n_i} + \underbrace{\left(\frac{u_1 - (\frac{u_1}{n_1})}{n_2 \dots n_i}\right) + \dots + \left(\frac{u_1 - (\frac{u_1}{n_j})}{n_{j+1} \dots n_i}\right) + \left(\frac{u_2}{n_{j+1} \dots n_i}\right)}_{\text{A}}.$$

We note that $n_{j+1} \dots n_i . \text{A} \leq u_1 + u_2 - \frac{u_1}{n_1 \dots n_j}$, therefore, $\text{A} \leq \frac{q_j}{n_j \dots n_i} - \frac{u_1}{n_1 \dots n_i}$. This proves the first and the second claim of the lemma. For the last part: suppose it is proved for i, we prove it for $i+1$. Suppose that a strategy of the replacements on U_j, for $j \leq i$, generate a maximum power for U_{i+1}. Then every replacement on a U_j can only change the power of U_p, for $p \leq j+1$. Therefore, in the course of the above strategy there is some step where the power of U_i is maximized. By induction this maximum is equal to $[W_i(M)]$. But in this step any replacement on U_j, for $j < i$, can not change the power of U_i, and as a result the power of U_{i+1}. Therefore, in this step without loss of generality we can suppose that the generated monomial is $M'' = U_i^{[W_i(M)]} U_{i+1}^{q_{i+1}}$. Now, by the proof of the first part of the lemma, no matter how the strategy continues the maximum power of U_{i+1} that can be generated is $\frac{[W_i(M)]}{n_i} + q_{i+1}$. $\square$

Corollary 6.5 *The greatest term U_{i_0} that can be generated by the replacements in the monomial M exists and is equal to the largest index j such that $W_j(M) \neq 0$.*

Lemma 6.6 *For any monomial M after finitely many replacements all the monomials M' that are generated are such that $q'_j < n_j$.*

Proof. We use induction on the first index i, such that $q_i \geq n_i$. We prove this index can be shifted, after finitely many replacements, one step to the right and then by the corollary above we are done. So, let $M = M(U_1, \ldots, U_i, \ldots)$ be a monomial such that $q_j < n_j$ $(j < i)$, we use another induction on q_i. We have $W_{i+1}(M) = \frac{q_i}{n_i} + q_{i+1}$. Consider a monomial M' which is obtained by just one replacement from M. If $u_1 = 0$ then for any $j \leq i$ we have $q'_i < q_i$, so, we are done in this case. Otherwise, by Lemma 6.4 we have $[W_{i+1}(M')] = [\frac{W_i(M')}{n_i} + q'_{i+1}] < [W_{i+1}(M)] = [\frac{q_i}{n_i} + q_{i+1}]$, but $q'_{i+1} \geq q_{i+1}$, therefore, $[W_i(M')] < q_i$. So, by induction hypothesis after finitely many replacements in all the monomials M'' that are generated we have $q''_i < n_i$. □

Remark 6.7 Fix a good ordering $\preccurlyeq$. Using some ideas of [?] and [?], we can give for any $d' \geq d+1$ and for any rational group G of rank d a valuation ν' of the field $k(X_1, \ldots, X_d, U_1, \ldots, U_{t-1})$, where $d' = d + t - 1$ and $t \geq 2$, with value group G. Let (γ_i) be the generators of the group G which are constructed in the proof of Proposition 4.9 and the relations between the γ_i which are explained in Lemma 3.2. Using the form of the equations introduced in Proposition 6.2, we define a morphism $\Psi' : k[X_1, \ldots, X_d][U] \to k[X_1, \ldots, X_d, U_1, \ldots, U_{t-1}]$ given by $U_{t+i-1} \mapsto U_i^{n_i} - X^{\alpha^{(i)}} U_1^{l_1^{(i)}} \ldots U_{i-1}^{l_{i-1}^{(i)}} + r'_i(U_1, \ldots, U_i)$, for $i \geq 1$, where $r'_i(U_1, \ldots, U_i) \in k[X, U_1, \ldots, U_i]$ and they satisfy formally the same conditions of the r_i of Proposition 4.9 (when we give the weight γ_i to U_i). Then the kernel of this morphism is generated by:

$$\begin{cases}
H'_1 & := & U_1^{n_1} & - & X^{\alpha^{(1)}} & + & r'_1(U_1) & + & U_t \\
H'_2 & := & U_2^{n_2} & - & X^{\alpha^{(2)}} U_1^{l_1^{(2)}} & + & r'_2(U_1, U_2) & + & U_{t+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
H'_i & := & U_i^{n_i} & - & X^{\alpha^{(i)}} U_1^{l_1^{(i)}} \ldots U_{i-1}^{l_{i-1}^{(i)}} & + & r'_i(U_1, \ldots, U_i) & + & U_{t+i-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{cases}$$

The construction of the valuation ν' is as follows: We set $\nu'(X_i) = e_i$, where the e_i are the elements of the standard basis of the vector space $\mathbb{Q}^d$, and $\nu'(U_i) = \gamma_i$. We can consider Ψ' as a graded morphism by the grades which come from the $\nu'-$values. For any $g(X, U_1, \ldots, U_{t-1}) \in k[X, U_1, \ldots, U_{t-1}]$ we use the H'_i and the Lemma 6.6 to represent g in the form $g = \sum_{\alpha,\beta} c_{\alpha,\beta} X^\alpha U^\beta (\text{mod} H'_i)$, such that for any β and $j \in \mathbb{N} : \beta_j < n_j$. This representation is unique because the H'_i generate $\text{Ker}\Psi'$ (see [?]). We say this is the $U - adic$ representation of g subject to the conditions $H' = 0$. Then the valuation is defined as:

$$\nu'(g) = \min_{\alpha,\beta} \nu'(X^\alpha U^\beta).$$

The proof of Proposition 6.2 shows that this minimum exists and there is a unique monomial with this minimum exponent. Moreover, for any g, h we have: $\nu'(g+h) \succcurlyeq min\{\nu'(g), \nu'(h)\}$ and $\nu'(gh) = \nu'(g) + \nu'(h)$. The first one is a direct consequence of the definition and the uniqueness of the $U - adic$ representation. The second one is also a direct consequence of uniqueness and the fact that each replacement, using some relation $H'_i = 0$, in a monomial does not change the minimum value.

Remark 6.8 The equations $H'_i = 0$ of the last remark can be viewed as a sequence of key polynomials. Transferring the U_{t+j} to the other side of the equations we get a set of equations which introduces the U_{t+j} as polynomials in $k[X, U_1, \ldots, U_{j+1}]$. Using the results of the last section we see that they are a sequence of key polynomials (with respect to the weights γ_i) and there is a sequence of valuations ν_i attached to this sequence.

We can unify the content of the last remarks and Corollary 5.8 in the following theorem:

Theorem 6.9 *Given any abelian group G of rational rank less than or equal to d, totally ordered by a good ordering, there exists a valuation of the field $k(X_1, \ldots, X_d, Y)$ with value group G and with residue field k. This valuation can be described by assigning values to the elements of a suitably chosen sequence of key polynomials in the ring $k[X, Y]$.*

In the next proposition we give the explicit embedding of the sequence of the quasi-ordinary hypersurfaces defined by $f^{(r)}(X,Y)=0$ in the space $\mathcal{A}$, and the relation between these equations and the equations of the embedding in Proposition 6.2.

Proposition 6.10 *There exists an embedding of the quasi-ordinary hypersurface* $f^{(r)}(X,Y)=0$ *(Definition 2.3) in the the space* $\mathrm{Spec}(k[[X]][U_1,\ldots,U_r])$, *in such a way that a set of generators for the ideal of this embedding is given by a process of truncation of the system of equations of the embedding* $\mathcal{S}\hookrightarrow\mathcal{A}$ *which is given in the Proposition 6.2.*

Proof. Consider the embedding:

$$\begin{array}{rl} \Psi_r : k[[X]][U_1,\ldots,U_r] & \to \frac{k[[X]][Y]}{f^{(r)}(X,Y)} \\ X & \mapsto X \\ U_j & \mapsto f^{(j-1)}(X,\zeta^{(r)}(X)). \end{array}$$

Consider the H_i introduced in Proposition 6.2. Fix a natural number j and truncate the system of equations H_i at the jth step in the following sense: Keep $H_1,\ldots,H_{r-1}$, in the equation of H_r delete the term c_rU_{r+1}, and drop all the following H_i.

Now, the proof of Proposition 6.2 can be repeated to give: the kernel of the embedding Ψ_r is generated by the truncated elements, i.e.,

$$H_1^{(r)}=H_1,\ldots,H_{r-1}^{(r)}=H_{r-1},H_r^{(r)}=H_r-c_rU_{r+1}.$$

The point is that the equations of H_j, given in the proof of Proposition 6.2, come from the *adic* expansion of the $(f^{(j-1)})^{n_j}$ in Lemma 3.3. These expansions are independent of the parametrizations $\zeta^{(i)}(X)$. These equations give us exactly $H_j^{(r)}$. For $j=r$ notice that by definition we have $f^{(r)}(X,\zeta^{(r)}(X))=0$, hence, the *adic* expansion of $(f^{(r-1)})^{n_r}$ which gives the equation $H_r=0$ translates to $H_r^{(r)}=0$. $\square$