Introduction

The moduli problem for algebraic curves of given genus g is a classical problem that goes back to Riemann. This problem, of a global nature, has already been solved. One knows that the moduli space is an open subset of an algebraic variety of dimension $3g - 3$ (if $g > 1$). The compactifications of this variety have also been studied. The most recent article on this subject is that of P. Deligne and D. Mumford (Publ. Math. de l'I.H.E.S., no. 36).

In this course, we study the analogous local problem of the moduli space of branches of the *same equisingularity class*, where *analytical equivalence* replaces birational equivalence which is used in the global problem. Very little has been written on this problem. We cite two articles:

(1) S. Ebey, *The classification of singular points of algebraic curves*, Trans. of the AMS (1965).

(2) K. Wolffhardt, *Variation of complex structure in a point*, Amer. J. of Math. (1968).

The problem of the complete description of the moduli space M of a given equisingularity class is entirely open, and the few examples of Chapter V show that M has a structure that is too complex to hope for a complete solution to the problem.

The somewhat more restrictive question of the determination of the dimension of the "generic component" of M is not solved. We give however some partial results for the characteristic sequence $(n; m)$ where n, m are relatively prime.[1]

Chapter VI is dedicated entirely to this particular problem. We determine the dimension q of the "generic component" when $m = n + 1$. We have also been able to find q in the case $m \equiv 1 \pmod{n}$ but we give the formula without proof.

In Chapter V we give a *complete* description of the moduli space for the characteristic sequence $(n; m) \in \{(2; m), (3; m), (4; 5), (5; 6), (6; 7)\}$. In the case $(6; 7)$, the most interesting, one observes several special features that the moduli space can exhibit.

In Chapter IV we show that M is *quasi-compact* (M is always non-separable unless it is a single point) only if the characteristic sequence is $(n; m)$ (m, n relatively prime) or $(4; 6, 2s + 1)$.

In Chapters I, II, III, we treat the structure of a singular branch and the principal numerical invariants of an equisingularity class (the exponent of the conductor c, the characteristic of the class, the semigroup Γ of positive integers associated to the class, etc.). The majority of these notions are classic and well known, but it was essential for us, and useful—we believe—for the younger reader to include them together here.

[1] On this subject, see the article by Ch. Delorme in the Groupe d'étude des singularités 1974-75. Publ. du Dept. de Math., Université de Paris-Sud Orsay.

We hope that this course will stimulate new work on the subject. One of the problems that we recommend to the reader is that of the identification of the moduli space M with a constructible subset of a variety V.

Recall that a constructible subset C of a variety V is a finite union of sets of the form $C_i - D_i$ where C_i is an irreducible subvariety of V and D_i is a strict subvariety of C_i. We have the following characterization of the structure of a constructible set (non-published):

There exists one and only one decomposition

$$C = (C_1 - D_1) \cup (C_2 - D_2) \cup \cdots \cup (C_k - D_k)$$

of a constructible subset C satisfying the following conditions:

(a) the decomposition is not redundant (i.e. none of the terms $C_i - D_i$ can be omitted);

(b) the subvarieties C_i are irreducible and distinct;

(c) the D_i are minimal: more precisely if $D'_1, D'_2, \ldots, D'_k$ are strict subvarieties of $C_1, C_2, \ldots, C_k$ such that for each i, $D'_i \subset D_i$ and

$$C = (C_1 - D'_1) \cup (C_2 - D'_2) \cup \cdots \cup (C_k - D'_k),$$

then $D'_i = D_i$ for each i;

(d) if $C_j \subset C_i$, then $C_j \subset D_i$.

The k varieties C_i are called the irreducible component of C, and C_i is called an immersed component if there exists C_j such that $C_i \subset C_j$.

Let C' be another constructible set, and let

$$C' = (C'_1 - D'_1) \cup (C'_2 - D'_2) \cup \cdots \cup (C'_h - D'_h)$$

be the decomposition satisfying (a)–(d). We say that C and C' are isomorphic if the following properties are satisfied:

1) $h = k$;

2) by an appropriate ordering of the irreducible components, we have $C_i \subset C_j$ if and only if $C'_i \subset C'_j$;

3) there exists an isomorphism $\varphi : \bigcup_{i=1}^{h} C_i \to \bigcup_{i=1}^{h} C'_i$ such that $\varphi(C_i) = C'_i$ and $\varphi(D_i) = D'_i$ for each $i \in \{1, \ldots, h\}$.

This result and the above definition give a precise meaning to our question concerning the isomorphism between M and a given constructible set. Since the topology of M is not the topology of a constructible set (be it the classical or Zariski topology), the answer to the preceding question is only possible with a suitable choice of topology for the constructible set.

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