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Valuations, deformations, and toric geometry

Bernard Teissier
Institut mathématique de Jussieu, UMR 7586 du CNRS
16 Rue Clisson, 75013 Paris
teissier@math.jussieu.fr

This paper is dedicated to David Rees, on the occasion of his eightieth Birthday, and to Pierre Samuel, on the occasion of his seventy-seventh Birthday

Summary

Given a nœtherian local integral domain R and a valuation ν of its field of fractions which is non negative on R, i.e., an inclusion $R \subset R_\nu$ of R in a valuation ring, I study a geometric specialization of R to the graded ring $\mathrm{gr}_\nu R$ determined by the valuation. If R_ν dominates R and the residue field extension $k = R/m \to R_\nu/m_\nu$ is trivial, this graded ring corresponds to an essentially toric variety, of Krull dimension $\leq \dim R$ but possibly of infinite embedding dimension; it is of the form:

$$\mathrm{gr}_\nu R = k[(U_i)_{i\in I}]/(U^m - \lambda_{mn}U^n)_{(m,n)\in E} \quad , \quad \lambda_{mn} \in k^*,$$

where I and E are countable sets and $U^m = U_{i_1}^{m_{i_1}} \dots U_{i_r}^{m_{i_r}}$. In order to apply this fact to a characteristic-blind proof of local uniformization by deformation of a partial resolution of singularities of $\mathrm{Specgr}_\nu R$, in the case where R is equicharacteristic and excellent, with k algebraically closed, I explore the following strategy:
1) Extend the valuation ν to a valuation $\hat{\nu}$ of a suitable nœtherian *scalewise* ν-adic completion $\hat{R}^{(\nu)}$ of R such that the natural map

$$\mathrm{gr}_\nu R \to \mathrm{gr}_{\hat{\nu}} \hat{R}^{(\nu)} \qquad \text{is } \textit{scalewise} \text{ birational.}$$

2) Obtain a presentation, that is a surjective continuous morphism of k-algebras

$$k[\widehat{(w_j)_{j\in J}}] \to \hat{R}^{(\nu)},$$

where the left hand term is a suitable *scalewise* completion of the polynomial ring, and the kernel is generated up to closure by elements whose initial forms for the term order t deduced from $\hat{\nu}$ are the binomial equations defining $\mathrm{gr}_{\hat{\nu}} \hat{R}^{(\nu)}$, that is, which are of the form:

$$w^m - \lambda_{mn} w^n + \sum_{t(s)>t(m)=t(n)} c_s^{(mn)} w^s \qquad \text{with } (m,n) \in \hat{E}, \lambda_{mn} \in k^*, \ c_s^{(mn)} \in k.$$

3) Show that all but finitely many of these equations serve only to express the images of all w_j's in the nœtherian ring $\hat{R}^{(\nu)}$ in terms of finitely many of them. Then show that a toric map in the coordinates $(w_j)_{j\in J}$ which resolves the singularities of the finitely many binomial equations of $\mathrm{gr}_{\hat{\nu}} \hat{R}^{(\nu)}$ involving these finitely many variables will uniformize $\hat{\nu}$ on $\hat{R}^{(\nu)}$, and that such resolving toric maps exist.
4) Use the excellence of R and the birationality of the map $\mathrm{gr}_\nu R \to \mathrm{gr}_{\hat{\nu}} \hat{R}^{(\nu)}$ to lift this to a uniformization of ν on R.

CONTENTS

1 Introduction

Lord Calversham: Do you always really understand what you say, sir?
Lord Goring (after some hesitation): Yes, father, if I listen attentively.

Oscar Wilde, *An ideal husband*, third Act.

This text is an attempt to clarify the problems which one meets when trying to prove local uniformization of a valuation at a singular point of an algebraic or analytic variety along the lines of what is done for plane branches in [G-T], by re-embedding the singularity in such a way that the valuation can be uniformized (in an *embedded* manner) by a *single* toric modification of the ambient space. It will help the reader to be aware of the basics of valuation theory contained in [V1] and [Cu], of the theory of toric varieties found in [Cox] and of the results of [T1] and [G-T].

Originally intended for the proceedings of "Mountains and singularities, Working Week on Resolution of singularities," and in agreement with the spirit that Herwig Hauser wished to give to that Working Week, this text is somewhat programmatic, even speculative in nature since there are several statements which may not be in their final form, and some proofs are only sketched, or even absent. The corresponding statements are between asterisks. I am grateful to the Editors of this volume who have kindly agreed to publish it in the same spirit, with some asterisks left.

There is therefore no claim to give here a complete detailed proof of local uniformization for an excellent equicharacteristic local ring with an algebraically closed residue field, but only to sketch a new and possibly useful way of looking at the problem of local uniformization and, perhaps more importantly, at a simultaneous uniformization process for all valuations "close enough" to a given one.

Given an integral domain R with fraction field K, a valuation of R is a valuation $\nu\colon K^* \to \Phi$ of K, with values in a totally ordered group Φ, which is non-negative on R. It is also the datum of a valuation ring R_ν of K containing R. To such a datum (R, ν) are associated the semigroup $\Gamma = \nu(R \setminus \{0\}) \subset \Phi_+ \cup \{0\}$ and a filtration by ideals $\mathcal{P}_\phi(R) = \{x \in R/\nu(x) \geq \phi\}$, the "successor" of $\mathcal{P}_\phi(R)$ being $\mathcal{P}^+_\phi(R) = \{x \in R/\nu(x) > \phi\}$, and therefore a graded algebra

$$\mathrm{gr}_\nu R = \bigoplus_{\phi\in\Gamma} \frac{\mathcal{P}_\phi(R)}{\mathcal{P}^+_\phi(R)}.$$

The problem of local uniformization is, given the local ring R of an algebraic variety or more generally an excellent local ring, and assuming that it is an integral domain, to find for each valuation ν of R a *regular* local R-algebra R', essentially of finite type over R and contained in R_ν. We wish R' to be obtained by localizing an affine chart of a proper algebraic map $Z \to \mathrm{Spec}R$ which is described as precisely as possible, for example as a composition of blowing ups with non singular centers or, according to what is proposed here, as a proper and birational toric map with respect to some system of generators of the maximal ideal of R.

The basic idea here is to view our local ring R as a *deformation* of the graded ring $\mathrm{gr}_\nu R$ of R with respect to the filtration associated to the valuation, and to obtain the uniformization of the valuation ν as a deformation of an *initial partial* *toric uniformization* of the valuation $\nu_{\rm gr}$ induced by ν on ${\rm gr}_\nu R$. In the case where the residue field of R is algebraically closed, this initial partial uniformization of $\nu_{\rm gr}$ should contain all the combinatorics associated to the uniformization of ν.

The motivating example is the case of complex plane branches studied as deformations of monomial curves in [T1] and [G-T]. The analytic algebra of a plane branch has a unique valuation, since its normalization is the valuation ring $\mathbf{C}\{t\}$. It has a semigroup Γ and a graded ring as above, and we chose to abandon the simplicity of embedding dimension two in exchange for the simplicity of dealing only with the monomial curve C^Γ with algebra $\mathbf{C}[t^\Gamma]$ in affine $(g+1)$-space, where $g+1$ is the number of generators of the semigroup of the branch (in characteristic zero, g is the number of its Puiseux exponents), because the resolution of singularities of the monomial curve is a purely combinatorial problem. We proved that after re-embedding a plane branch C in $\mathbf{C}^{g+1}$ in the way given by elements of its algebra whose valuations generate the semigroup it specializes to the monomial curve C^Γ with the same semigroup, which corresponds to the graded algebra ${\rm gr}_\nu R$ of the ring R of C with respect to its unique valuation.

The equations of the corresponding family of curves in $\mathbf{C}^{g+1}$ are of a special form which makes it possible to prove, using the implicit function theorem, that a toric embedded resolution of $C^\Gamma \subset \mathbf{C}^{g+1}$ is also an embedded resolution for $C \subset \mathbf{C}^{g+1}$.
In higher dimensions, the first serious difficulty is that the graded ring ${\rm gr}_\nu R$ associated to the filtration of R determined by the valuation ν is not nœtherian in general; its structure depends on the semigroup of the values which the valuation takes on the ring, which is not always finitely generated.

The good news is that for rational valuations, that is for valuations of a local ring R such that the inclusion $R \subset R_\nu$ in the ring R_ν of the valuation induces a trivial residue field extension[1], the space ${\rm Specgr}_\nu R$ is, up to a very simple deformation, a toric variety, of finite Krull dimension but possibly of infinite embedding dimension (see Proposition 4.2); it is this toric variety which generalizes the monomial curve of [T1] and [G-T]. The uniformization of the valuation $\nu_{\rm gr}$ induced by ν on ${\rm gr}_\nu R$ amounts to resolving the singularities of this toric variety, which is a combinatorial problem (see §6) with a simple solution in the finitely generated case (see subsection 6.1) and no solution in the usual sense otherwise, but fortunately we need only to partially resolve in that case; see below. Moreover, it is blind to the characteristic. The combinatorial structure of the binomial equations defining the toric variety reflects the complexity of the valuation in R, including its rational rank and its rank (or height). The fact that however complicated this structure the variety defined by the binomial equations involving finitely many variables can be resolved by a toric map is crucial.

In order to relate the uniformizations of ν and $\nu_{\rm gr}$, the main tool is the *valuation algebra* (see 2.1) associated to a valuation ν of a domain R; its spectrum is the total space of a (faithfully flat) specialization of ${\rm Spec}R$ to ${\rm Specgr}_\nu R$ (see Proposition 2.3).

However, just like in the plane branch case, where we had to re-embed our branch in $\mathbf{C}^{g+1}$ before we could write the specialization, I need to write explicit equations for this deformation, in suitable variables.

[1]this meaning is different from that used by Abhyankar, e.g., in [A3], where it means "of rational rank one". Abhyankar writes "residually rational" for what I call "rational".

This leads to the second serious difficulty, that of finding suitable completions of R, inside which one can lift generators of the graded k-algebra $\mathrm{gr}_\nu R$ to coordinates, as Cohen's Theorem does for the usual graded algebra $\mathrm{gr}_m R$ and the usual m-adic completion. Here the key result is that Laurent monomials $\overline{\eta}_j = \overline{\xi}^{d(j)}$ in the generators $(\overline{\xi}_i)_{i\in I}$ of the graded k-algebra $\mathrm{gr}_\nu R$ generate the graded algebra of a suitable completion $\hat{R}^{(\nu)}$ of R with respect to an extension $\hat{\nu}$ of the valuation ν, and can be lifted to elements $(\eta_j)_{j\in J}$ of $\hat{R}^{(\nu)}$, which are in some sense "coordinates"; the surjection

$$k[(W_j)_{j\in J}] \longrightarrow \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}, \quad W_j \mapsto \overline{\eta}_j$$

expressing the fact that the $\overline{\eta}_j$ generate the k-algebra $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ lifts to a surjection

$$k[\widehat{(w_j)_{j\in J}}] \longrightarrow \hat{R}^{(\nu)}, \quad w_j \mapsto \eta_j.$$

Here the first candidate for $\hat{R}^{(\nu)}$ is the completion $\hat{R}^\nu$ of R with respect to the topology defined by the rational valuation ν. However this is not enough because the quotients of this ν-adic completion are not complete for the quotient topology in general and therefore ν-adic completeness does not suffice to ensure convergence of series with terms of increasing valuation (see subsection 5.2). This difficulty is circumvented by extending the valuation ν to a valuation $\hat{\nu}$ of a suitable quotient $\hat{R}^{(\nu)}$ of the m-adic completion $\hat{R}^m$ of R; it is complete for the $\hat{\nu}$-adic topology and its quotients are also complete, while the natural morphism $\mathrm{gr}_\nu R \to \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ is birational in a *scalewise* sense if the valuation ν is rational. The ring $k[\widehat{(w_j)_{j\in J}}]$ is a *scalewise* completion of the polynomial ring (see 5.4).

The next important fact is that one can describe by equations the specialization of $\hat{R}^{(\nu)}$ to $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$. One can choose generators (up to closure) of the kernel of the surjection $k[\widehat{(w_j)_{j\in J}}] \to \hat{R}^{(\nu)}$ such that their initial forms in a suitable sense are the binomials which generate the kernel in $k[(W_j)_{j\in J}]$ of the surjective homomorphism of graded k-algebras $k[(W_j)_{j\in J}] \to \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ sending W_j to the j-th generator.
These generators of the kernel whose initial forms generate the kernel of the associated graded map may be interpreted as a generalized Gröbner, or standard, basis for this kernel with respect to the monomial order on the -countably many-variables w_j defined by deciding that $w^s < w^t$ if the valuation of the image of w^s in $\hat{R}^{(\nu)}$ is less than the valuation of the image of w^t. From the viewpoint taken here, however, the important thing is that they can be resolved by toric maps.

This presentation of the scalewise complete nœtherian local ring $\hat{R}^{(\nu)}$ as a quotient of a generalized power series ring may be seen as a valuative analogue of Cohen's structure theorem, and manifests an important difference with Spivakovsky's and Zariski's approaches. I do not start from a set of coordinates and equations for the singularity and try to improve them, then follow them through a sequence of blowing-ups, but I produce directly from the ring of the singularity, given a valuation, a "natural" embedding in a (possibly infinite-dimensional) space equipped with a natural class of coordinates $(w_j)_{j\in J}$, liftings of (Laurent monomials in) the generators of the graded algebra $\mathrm{gr}_\nu R$, and a "natural" set of equations, which are as simple as one might wish (deformations of binomials) and in particular are "nondegenerate" in the natural coordinates (see Propositions 5.48 and 5.49). A substantial part of the work is then to show that they truly are coordinates and equations in a useful sense with respect to the given valuation.

This viewpoint is illustrated in subsection 4.4 by a geometric interpretation of MacLane's "Key polynomials" associated to a valuation of $k(x)[y]$ extending the x-adic valuation of $k(x)$ and having value group $\mathbf{Q}$ as equations of plane approximations of a transcendental plane curve which is a deformation of an essentially monomial curve of infinite embedding dimension.

The specialization of the space defined by all the equations to the space defined by their binomial initial forms, inside the space with coordinates $(w_j)_{j\in J}$, is the explicit form of the geometric specialization of the ring $\hat{R}^{(\nu)}$ to its associated graded ring with respect to the valuation $\hat{\nu}$.

Finally the nœtherian and henselian properties of the completion $\hat{R}^{(\nu)}$ are used again to ensure that the lifting to $\hat{R}^{(\nu)}$ of finitely many steps of the (possibly infinite) combinatorial process of resolution of $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ suffices to uniformize the extension $\hat{\nu}$ of ν to $\hat{R}^{(\nu)}$.

Here the main point is that all but finitely many of the equations defining $\mathrm{Spec}\hat{R}^{(\nu)}$ in the possibly infinite-dimensional affine space with coordinates $(w_j)_{j\in J}$ serve only to express in $\hat{R}^{(\nu)}$ the coordinates $(w_j)_{j\notin F}$ in terms of finitely many coordinates $(w_j)_{j\in F}$ which suffice to generate its maximal ideal and by their valuations, to rationally generate the value group, and have some other properties. If there were also infinitely many equations not serving this purpose, they would indefinitely add singularities, and the method could not work. This finiteness implies that it suffices that a toric modification *partially* resolves $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$, in a precise sense, to uniformize $\hat{\nu}$. This uniformization occurs in spite of the fact that $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ may be of a smaller Krull dimension than $\hat{R}^{(\nu)}$, because of the “abyssal” phenomenon generated by the form of the equations defining $\mathrm{Spec}\hat{R}^{(\nu)}$; see example 4.20 in subsection 4.4, subsections 5.5, 6.4 and 7.2, and Proposition 5.62.

In general, for a rational valuation ν the Krull dimension of $\mathrm{gr}_\nu R$ is equal, by a theorem of Piltant (see 3.1), to the rational rank $\mathrm{r}(\nu)$ of ν. By Abhyankar's inequality we have $\mathrm{r}(\nu)=\mathrm{dimgr}_\nu R\leq \mathrm{dim}R$. The cases of strict inequality may seem to contradict the fact that there is a specialization of R to $\mathrm{gr}_\nu R$, which is even faithfully flat. But the semicontinuity of fiber dimensions is proved only for morphisms which are essentially of finite type. The abyssal phenomenon provides explicit examples of failure of this semicontinuity in the general case.

It is also necessary to make precise what *initial partial resolution* of $\mathrm{Specgr}_\nu R$ means, that is, to approximate in a useful way the (possibly infinite-dimensional) toric map which resolves our toric variety by finite-dimensional ones. This is possible thanks to the fact that for any valuation ring R_ν with residue field k_ν and finite rational rank $\mathrm{r}(\nu)$, the graded algebra $\mathrm{gr}_\nu R_\nu$ is the direct limit of a system of essentially toric injective maps between polynomial k_ν-subalgebras in $\mathrm{r}(\nu)$ variables, and this system can be described rather explicitly; for archimedian valuations, it is essentially an avatar of Perron's algorithm (see section 4).

This result may also be seen as a graded version of (non-embedded) local uniformization, since local uniformization in the rational case over a field k means that a valuation ring over k with residue field k is a direct limit of regular local k-subalgebra with the same field of fractions which are essentially of finite type.
Perron's algorithm also appears in Zariski's proof, in an apparently different role.

Then there remains to show that this process can be extended to a toric modification of R provided that it is excellent. This last step uses the scalewise birationality of the map $\mathrm{gr}_\nu R\to \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ and an observation, also used by Spivakovsky, which implies in the language used here that if R is excellent, Spec$\hat{R}^{(\nu)}$ is not contained in the singular locus of Spec$\hat{R}^m$ (see Proposition 7.10).

Another difference with Zariski's approach is that I must go directly to the "worst" valuations from the viewpoint of the value group, the rational ones, to take advantage of the toric nature of the corresponding graded algebra, while Zariski and Spivakovsky both begin by the case of height one (archimedian) valuations (see [Z1], top of page 858). That the center of gravity of the proof, as Zariski puts it, is now the uniformization of rational valuations has the great advantage that it makes sense to assume that the residue field is algebraically closed, whereas in Zariski's approach, in order to deal with valuations of height > 1, one localizes, and so one cannot make such an assumption. We may therefore, in this first phase, confine our attention to the case of an algebraically closed residue field. The rational valuations are also the only ones which appear naturally in the complex-analytic case.

The cases of valuations which are not rational are essentially deformations of the rational case, as I show using the simple behavior of the valuation algebra under composition of valuations. The reason is that, if we assume that the residue field of R is algebraically closed, for any valuation ν_0 there are rational valuations ν composed with ν_0, of which ν_0 may be deemed to be a deformation.

The specialization to the graded ring changes the field of fractions, which neither Zariski's proof in [Z1] nor Spivakovsky in [S2] did, but it brings us to a situation where two of the principal complications have disappeared: they are the behaviour of transformed equations under translation after a birational map (see [Z1], pp. 884-885), and the proof that one reaches non singularity after finitely many steps, using the fact that some partial derivative is not zero (see [Z1], pp. 886-887). Both are replaced by the results of section 6, and the implicit function theorem then allows us, via the abyssal phenomenon, to pass from the graded ring to a suitable completion of the original ring. From there one finally gets back to the ring itself using excellence.

I believe that this strategy clarifies the nature of the problem by separating as much as possible combinatorial problems from choices of coordinates and equations. Once the toric setup is understood, finding centers of blowing-up in a given embedding of the singularity should be considerably easier.

This work was initially partly motivated by the desire to understand from the viewpoint of [T1] the inspiring attempt of Spivakovsky in [S2] to prove local uniformization and resolution of singularities; because of this change in approach, substantial differences soon appeared, which were outlined in [G-T] for the case of plane branches. The viewpoint developed here is therefore different from that of [S2], although I am indebted to its author for some specific statements, as the text evidences. Spivakovsky generalizes to valuations of height one the construction which for plane branches describes resolution as a composition of *plane* toric maps (see sections 7 and 8 of [G-T]), each of these being in fact described as a composition of blowing ups of points in the plane rather than as a toric map. The graded algebra $\mathrm{gr}_\nu R$ also plays an important, although quite different, role (see the introduction and §7 of [G-T], as well as [GP]). Many difficulties in this approach are due to the fact whose avatar for curves is that after each plane toric map one must choose again an origin and local coordinates in which the next toric map will be described. I take a systematically toric approach and allow from the start a change of embedding, even to infinite dimensions, such that in this new embedding a *single* toric map in natural coordinates will uniformize, and I do not seek, in the first phase, to construct a sequence of blowing-ups with non singular centers. The existence of such a sequence will follow from a theorem of De Concini-Procesi ([DC-P]) on the domination of toric birational maps by sequences of blowing ups.

In the redaction, some facts must be applied to valuation rings R_ν, which are not nœtherian in general, and so parts are written without that assumption. Basically, I try to do here what is needed for local uniformization of an arbitrary valuation on a local equicharacteristic excellent integral domain with an algebraically closed residue field. The case where the residue field is not algebraically closed requires other techniques in this approach. In the first sections I have to assume the existence of a base field, since it is only after completion that one is sure to have a field of representatives.

The point of view of this paper also avoids the use of the ramification and extension theory of valuations in the study of the local uniformization problem, replacing it by the relations between the ring and its associated graded ring. The ramification over non singular spaces is in general much worse for the graded ring, but that is of no importance since it corresponds to a toric variety and as such can be resolved by toric maps unaffected by the ramification. There have been remarkable advances in the ramification theory (see [C-P], [K1], [K-K]) and the extension theory (see [V2]), some of which indicate that the graded rings may play a useful role there too, especially when the graded ring extension associated to a separable extension of valued rings is inseparable.

It was during Heisuke Hironaka's wonderful Summer meeting at Ohnuma-Hokkaido in 1993 that this train of thought started. I wish to thank Antonio Campillo, Dale Cutkosky, David Eisenbud, Pedro González Pérez, Heisuke Hironaka, Franz-Viktor Kuhlmann, Monique Lejeune-Jalabert, Olivier Piltant, Mark Spivakovsky, Bernd Sturmfels and Michel Vaquié for stimulating comments, some important corrections, and suggestions. I am especially grateful to Mark Spivakovsky, whose work rekindled the ever living embers of my interest for the role of the monomial curve and the Riemann-Zariski manifold, and to Franz-Viktor Kuhlmann, Michel Vaquié and the referees who made very useful comments on a very preliminary version and later on a more elaborated one. I am also grateful to Nicolas Bourbaki for his incomparably careful and lucid exposition of completion problems.

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2 Valuative sorites

This section[2] introduces the specialization of a ring R to the graded ring associated to a valuation. It generalizes to filtrations indexed by a totally ordered group a construction found, for filtrations indexed by $\mathbf{Z}$, in the work of Gerstenhaber [G1], [G2], Rees [R], and in [T1]. One difference is that I do not use the valuation algebra as an algebraic construction describing the interaction of the filtration with the multiplication law, but rather as a way to encode a geometric object.

2.1 The valuation algebra. Recall that a valuation of a commutative field K is a map $\nu\colon K^* \to \Phi$ to a totally ordered commutative group Φ (the value group) such that $\nu(xy) = \nu(x) + \nu(y)$, $\nu(x+y) \geq \inf\big(\nu(x),\ \nu(y)\big)$, with equality if $\nu(x) \neq \nu(y)$. One may extend the map to K by giving to the element $0 \in K$ a valuation ∞ greater than any element of Φ. The set $\{x \in K \mid \nu(x) \geq 0\}$ is the ring R_ν of the valuation; it is local with maximal ideal $m_\nu = \{x \in K \mid \nu(x) > 0\}$. If $\nu\colon R \to \Phi_+ \cup \{0\} \cup \{\infty\}$ is defined on a commutative ring R and the first condition is relaxed to $\nu(xy) \geq \nu(x) + \nu(y)$, one has a *loose valuation* of R; the constructions explained below in this section are valid for $R/\nu^{-1}(\infty)$ if one replaces "valuation" by "loose valuation", the only difference being that the graded algebras are not integral domains in that case. Loose valuations are useful because under certain conditions a valuation of a ring R induces a loose valuation on a quotient ring. The trivial valuation has value group $\{0\}$.

Let R be an integral domain with field of fractions K and let ν be a valuation of K such that its valuation ring R_ν contains R. Except when there is explicit mention of the contrary, all valuations are assumed to be non trivial. Let us denote by Φ the totally ordered value group of the valuation ν and by k_ν its residue field R_ν/m_ν. Denote by Φ_+ (resp. Φ_-) the semigroup of positive (resp. negative) elements of Φ and set $\Gamma = \nu(R \setminus \{0\}) \subset \Phi_+ \cup \{0\}$; it is the *semigroup* of (R, ν); since Γ generates the group Φ, it is cofinal in the ordered set Φ_+.
For $\phi \in \Phi$, set

$$\begin{aligned}\mathcal{P}_\phi(R) &= \{x \in R \mid \nu(x) \geq \phi\}\\ \mathcal{P}^+_\phi(R) &= \{x \in R \mid \nu(x) > \phi,\}\end{aligned}$$

where we agree that $0 \in \mathcal{P}_\phi$ for all ϕ, since its value is larger than any ϕ, so that by the properties of valuations the $\mathcal{P}_\phi$ are ideals of R. Note that the intersection $\bigcap_{\phi\in\Phi_+} \mathcal{P}_\phi = (0)$ and that if ϕ is in the negative part Φ_- of Φ, then $\mathcal{P}_\phi(R) = \mathcal{P}^+_\phi(R) = R$. Consider the graded algebra *à la* Gerstenhaber-Rees (see the papers quoted above), which we may call the *valuation algebra* of (R, ν):

$$\mathcal{A}_\nu(R) = \bigoplus_{\phi\in\Phi} \mathcal{P}_\phi(R) v^{-\phi} \subset R[v^\Phi]$$

where $R[v^\Phi]$ is the group algebra of Φ with coefficients in R, the element $x_\phi[\phi]$ being written $x_\phi v^\phi$.

[2]Its title is that of an embryo version of this paper. The following was added at the request of a referee: the origin of the word *sorite* is an abbreviation (found e.g. in Galien) of the Greek σωρείτης συλλογισμός which refers to a syllogistic type of reasoning based on the accumulation of premises; σωρεύω means "I accumulate". It is used by some mathematicians to designate such an accumulation, adding a slightly depreciatory nuance. According to the Concise Oxford Dictionary, it also evokes "a form of sophism leading by gradual steps from truth to absurdity".

When there is no ambiguity we shall denote this algebra simply by $\mathcal{A}$; it is a domain, not nœtherian in general even if R is, as we shall see below. Note that when ν is the trivial valuation, with $\Phi=\{0\}$, we have $\mathcal{A}_\nu(R)=R$.

Remark Given a ring R and a group Φ, the datum of a graded R-subalgebra $\mathcal{G}$ of the group algebra $R[v^\Phi]$ is equivalent to the datum of a family of ideals $(I_\phi)_{\phi\in\Phi}$ of R such that $I_\phi.I_\psi\subseteq I_{\phi+\psi}$, the correspondence being described by the equality $\mathcal{G}=\bigoplus_{\phi\in\Phi} I_\phi v^{-\phi}$. If in addition Φ is a totally ordered group, and $I_\phi=R$ for $\phi\leq 0$, then $\mathcal{G}$ is a graded $R[v^{\Phi_+}]$-algebra if and only if we have $I_\phi\supseteq I_\psi$ whenever $\psi\geq\phi$.

2.2 The valuation $\mu_{\mathcal{A}}$**.** We now define a valuation $\mu_{\mathcal{A}}$ on $\mathcal{A}$ with value group Φ by

$$\mu_{\mathcal{A}}(\sum x_\phi v^{-\phi})=\min\bigl(\nu(x_\phi)-\phi\bigr).$$

By construction of $\mathcal{A}$ this valuation is non-negative on $\mathcal{A}$.
There are several rings connected with $\mathcal{A}_\nu(R)$: its quotient field is the quotient field $K(v^\Phi)$ of $K[v^\Phi]$, and it is not difficult to describe the valuation ring $\mathcal{R}_{\mu_{\mathcal{A}}}$ of $\mu_{\mathcal{A}}$; it is the set of quotients $(\sum x_\phi v^{-\phi})(\sum y_\psi v^{-\psi})^{-1}$ such that $\min\bigl(\nu(x_\phi)-\phi\bigr)\geq\min\bigl(\nu(y_\psi)-\psi\bigr)$.

We set $\mathcal{P}_\phi(R_\nu)=\{x\in R_\nu\mid\nu(x)\geq\phi\}$ as above, and similarly for $\mathcal{P}^+_\phi(R_\nu)$ and $\mathcal{P}_\phi(K)$, and remark that $\mathcal{P}_\phi(K)$, $\mathcal{P}^+_\phi(K)$ are R_ν-submodules of K, and that for $\phi\in\Phi_+$ we have $\mathcal{P}_\phi(K)=\mathcal{P}_\phi(R_\nu)$. With this notation, we have:

$$\mathcal{R}_{\mu_{\mathcal{A}}}\cap K[v^\Phi]=\mathcal{A}_\nu(K)=\bigoplus_{\phi\in\Phi_-}\mathcal{P}_\phi(K)v^{-\phi}\bigoplus R_\nu\bigoplus\bigoplus_{\phi\in\Phi_+}\mathcal{P}_\phi(R_\nu)v^{-\phi}$$

and

$$\mathcal{R}_{\mu_{\mathcal{A}}}\cap R_\nu[v^\Phi]=\mathcal{A}_\nu(R_\nu)=\bigoplus_{\phi\in\Phi}\mathcal{P}_\phi(R_\nu)v^{-\phi}\subset R_\nu[v^\Phi].$$

The maximal homogeneous ideal of $\mathcal{A}_\nu(R_\nu)$ containing m_ν is:

$$\mathcal{M}_\nu=\bigoplus_{\phi\in\Phi_-}R_\nu v^{-\phi}\bigoplus m_\nu\bigoplus\bigoplus_{\phi\in\Phi_+}\mathcal{P}_\phi(R_\nu)v^{-\phi},$$

so that the corresponding residue field is the residue field k_ν of the valuation, while the ideal of elements of positive value is

$$\tilde{\mathcal{Q}}^+_0=\bigoplus_{\phi\in\Phi_-}R_\nu v^{-\phi}\bigoplus m_\nu\bigoplus\bigoplus_{\phi\in\Phi_+}\mathcal{P}^+_\phi(R_\nu)v^{-\phi}.$$

The maximal homogeneous ideal of $\mathcal{A}_\nu(K)$ containing m_ν is

$$\bigoplus_{\phi\in\Phi}\mathcal{P}^+_\phi(K)v^{-\phi}$$

which induces $\tilde{\mathcal{Q}}^+_0$ on $\mathcal{A}_\nu(R_\nu)$. In any case, the ideal of elements of positive $\mu_{\mathcal{A}}$-value of $\mathcal{A}_\nu(R)$ is

$$\mathcal{Q}^+_0=\bigoplus_{\phi\in\Phi_-}Rv^{-\phi}\bigoplus(m_\nu\cap R)\bigoplus\bigoplus_{\phi\in\Phi_+}\mathcal{P}^+_\phi(R)v^{-\phi},$$

and we remark that it is exactly the ideal of $\mathcal{A}_\nu(R)$ generated by the family of elements (v^{Φ_+}). Indeed, it is generated by elements $x_\phi v^{-\phi}$ with $\nu(x_\phi)>\phi$; but then,

$$x_\phi v^{-\phi}=v^{\nu(x_\phi)-\phi}x_\phi v^{-\nu(x_\phi)}\in(v^{\Phi_+})\mathcal{A}_\nu(R).$$

The other inclusion is clear.

For any commutative ring A, let us denote by $A[v^{\Phi_+}]$ the algebra of the semigroup $\Phi_+ \cup \{0\}$ with coefficients in A.

2.3 The specialization to the graded algebra. The graded algebra associated with the valuation ν was introduced in ([L-T], [T1]) for the very special case of a plane branch (see [G-T]), and in [S1], [S2] in full generality. It is

$$\mathrm{gr}_\nu R = \bigoplus_{\phi\in\Gamma} \mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R).$$

For $\phi \notin \Gamma$, $\mathcal{P}_\phi(R) = \mathcal{P}^+_\phi(R)$; it is sometimes convenient to view it as graded by $\Phi_+ \cup \{0\}$, with a zero homogeneous component for $\phi \notin \Gamma$.
For each non zero element $x \in R$, there is a unique $\phi \in \Gamma$ such that $x \in \mathcal{P}_\phi \setminus \mathcal{P}^+_\phi$; the image of x in the quotient $(\mathrm{gr}_\nu R)_\phi = \mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R)$ is the *initial form* $\mathrm{in}_\nu x$ of x. When there is no ambiguity, I will often denote it by $\overline{x}$.
The graded ring $\mathrm{gr}_\nu R$ inherits naturally from (R,ν) a valuation ν_{gr}, defined by

$$\nu_{\mathrm{gr}}(\sum \overline{x_\phi}) = \min(\nu(x_\phi)),$$

where $x_\phi \in R$ is a representative of $\overline{x_\phi}$, and the valuation does not depend on its choice.

Remarks 2.1 1) Since R is an integral domain and ν is a valuation, the ring $\mathrm{gr}_\nu R$ is an integral domain.
2) If the value group Φ has finite rational rank, it is countable, since it is torsion free and hence the map $\Phi \to \mathbf{Q} \otimes_{\mathbf{Z}} \Phi = \mathbf{Q}^r$ is injective. Therefore the $R/(m_\nu \cap R)$-algebra $\mathrm{gr}_\nu R$ has a countable system of generators, and so does the k_ν-algebra $\mathrm{gr}_\nu R_\nu$.
3) Each $\mathcal{P}_\phi(R)/\mathcal{P}^+_\phi(R)$ is a torsion free $R/(m_\nu \cap R)$-module, which is of finite type if the ring R is nœtherian.
4) If R is a local ring with maximal ideal m, by definition of domination, the valuation ring R_ν *dominates* R if $m_\nu \cap R = m$, and we will sometimes be in that situation. Then $\mathrm{gr}_\nu R$ is a k_R-algebra, where k_R is the residue field of R. If we want to emphasize that R and R_ν have the same field of fractions, we say that R_ν *birationally dominates* R.

Let us denote by $(v^{\Phi_+})\mathcal{A}_\nu(R)$ the ideal generated in $\mathcal{A}_\nu(R)$ by the elements of (v^{Φ_+}).

Proposition 2.2 *a) The natural map*

$$\mathcal{A}_\nu(R) \longrightarrow \mathrm{gr}_\nu R$$

defined by

$$x_\phi v^{-\phi} \mapsto x_\phi \quad \mathrm{mod}.\mathcal{P}^+_\phi(R)$$

induces an isomorphism of graded rings

$$\mathcal{A}_\nu(R)/\mathcal{Q}^+_0 = \mathcal{A}_\nu(R)/(v^{\Phi_+})\mathcal{A}_\nu(R) \xrightarrow{\simeq} \mathrm{gr}_\nu R.$$

b) The natural inclusion $R[v^{\Phi_+}] \longrightarrow \mathcal{A}_\nu(R)$ obtained by considering only the part of negative degree (i.e., $\phi \in \Phi_-$) of $\mathcal{A}$ induces, after taking rings of fractions by the multiplicative subset (v^{Φ_+}), an isomorphism

$$(v^{\Phi_+})^{-1}R[v^{\Phi_+}] \xrightarrow{\simeq} (v^{\Phi_+})^{-1}\mathcal{A}_\nu(R);$$

the quotient $\mathcal{A}_\nu(R)/R[v^{\Phi_+}]$ is a torsion $\mathbf{Z}[v^{\Phi_+}]$-module.
c) Given a homomorphism $\phi \mapsto e(\phi)$ from Φ to the multiplicative group of units of R, it induces a surjection $\mathcal{A}_\nu(R) \to R$ defined by $\sum x_\phi v^{-\phi} \mapsto \sum x_\phi e(-\phi)$. The kernel of this surjection is the ideal $\big((v^\phi - e(\phi))_{\phi\in\Phi_+}\big)\mathcal{A}_\nu(R)$. If R contains a field k, this is true in particular for homomorphisms $\Phi \to k^*$.

Proof the map in a) is clearly surjective, and its kernel is homogeneous and generated by elements $x_\phi v^{-\phi}$ with $\nu(x_\phi) > \phi$; but we saw in 2.2 that it is $(v^{\Phi_+})\mathcal{A}_\nu(R) = \mathcal{Q}_0^+$.
To prove b), consider an element $v^{-\psi}\sum_{\phi\in\Phi_-} x_\phi v^{-\phi}$ of the ring of fractions $(v^{\Phi_+})^{-1}R[v^{\Phi_+}]$, and rewrite it as $v^{-\psi}\sum_{\phi\in\Phi_-} v^{-\nu(x_\phi)}x_\phi v^{-\phi+\nu(x_\phi)}$. Since $-\phi \geq 0$ and $\nu(x_\phi)\geq 0$, this is clearly an element of $(v^{\Phi_+})^{-1}\mathcal{A}_\nu(R)$, and every element can be written in this way; if $\sum x_\phi v^{-\phi} \in \mathcal{A}_\nu(R)$, let ϕ_+ be the largest degree appearing in this sum. If it is negative, the element is in $R[v^{\Phi_+}]$. Otherwise, rewrite the sum as $v^{-\phi_+}\sum x_\phi v^{-(\phi-\phi_+)} \in (v^{\Phi_+})^{-1}R[v^{\Phi_+}]$. This proves both assertions of b). Assertion c) follows from the remarks that any $x\in R$ of valuation ϕ is the image of $e(\phi)xv^{-\phi}$ and that if $\sum x_\phi e(-\phi) = 0$, then $\sum x_\phi v^{-\phi} = \sum x_\phi(v^{-\phi} - e(-\phi))$. □

To interpret this geometrically, let us assume in this section that R contains[3] a field k ; we have a composed map of algebras $k[v^{\Phi_+}] \to R[v^{\Phi_+}] \to \mathcal{A}_\nu(R)$. Then:

Proposition 2.3 *The $k[v^{\Phi_+}]$-algebra $\mathcal{A}_\nu(R)$ is faithfully flat.*

Proof The criterion for faithful flatness of [B3], Chap. I, §3, No. 7, Proposition 13 is that every solution $(y_k \in \mathcal{A}_\nu(R),\ 1\leq k\leq n)$ of a system

$$(*) \qquad \sum_{k=1}^{n} c_{ki}y_k = d_i \quad (1\leq i\leq m)$$

of linear equations with coefficients and right-hand sides in $k[v^{\Phi_+}]$ can be written in the form

$$y_k = x_k + \sum_{j=1}^{p} z_{jk}e_j \quad (1\leq k\leq n)$$

where (x_k) is a solution of $(*)$ in $k[v^{\Phi_+}]$, the e_j belong to $\mathcal{A}_\nu(R)$ and the z_{jk} for each fixed j are solutions in $k[v^{\Phi_+}]$ of the homogeneous system associated to $(*)$.
Writing $y_k = \sum y_{k\psi}v^{-\psi}$, we notice that the elements $y_{k\psi}$ with $\psi > 0$ are such that $\nu(y_{k\psi}) > 0$ and therefore if we write

$$y_k = \sum_{\psi\leq 0} y_{k\psi}v^{-\psi} + \sum_{\psi>0} y_{k\psi}v^{-\psi} = y_k^{\leq 0} + y_k^{>0}$$

we must have, since the d_i are in $k[v^{\Phi_+}]$ and so have all their coefficients of valuation zero,

$$\text{a)}\ \sum_k c_{ki}y_k^{\leq 0} = d_i \quad , \qquad \text{b)}\ \sum_k c_{ki}y_k^{>0} = 0 \quad (1\leq i\leq m).$$

We may view the system a) as a system of linear equations with coefficients and right-hand side in $k[v^{\Phi_+}]$, and for which we have a solution $y_k^{\leq 0} \in R[v^{\Phi_+}]$. By [B3], Chap. 1, §3, No.3, Prop. 5, the $k[v^{\Phi_+}]$-module $R[v^{\Phi_+}]$ is faithfully flat since it is

[3]In this paper, whenever I introduce a base field k, it will be tacitly assumed that the valuations studied are trivial on k.

obtained from the faithfully flat map $k \to R$ by the base extension $k \to k[v^{\Phi_+}]$. By the criterion quoted above, we may write

$$y_k^{\leq 0} = x_k^{\leq 0} + \sum_{j=1}^{p} z_{jk}^{\leq 0} e_j^{\leq 0} \quad (1 \leq k \leq n)$$

with the same conditions as above except that $\mathcal{A}_\nu(R)$ is replaced by $R[v^{\Phi_+}]$.

Let us now consider the second system; here we have to deal with restrictions on the valuations of the elements; a solution of this system defines a finite dimensional k-vector subspace V of R, generated by the coefficients $y_{k\psi}^{>0}$ of the solutions $y_k^{>0} = \sum_\psi y_{k\psi}^{>0} v^{-\psi}$ of system b). Let us choose a basis $(\tilde{e}_q^{>0})_{1 \leq q \leq r}$ of this vector space which is compatible with the finite filtration of V by the vector subspaces $\mathcal{P}_\phi \cap V$. We can now write

$$y_{k\psi}^{>0} = \sum_{q=1}^{r} w_{qk\psi}^{>0} \tilde{e}_q^{>0} \text{ with } \; w_{qk\psi}^{>0} = 0 \text{ if } \nu(\tilde{e}_q^{>0}) < \psi,$$

and the coefficients $w_{qk\psi}^{>0}$ satisfy the system of linear equations

$$\sum_k c_{ki} (\sum_\psi w_{qk\psi}^{>0} v^{-\psi}) = 0 \quad \forall q, \ \forall i.$$

This means that we may write

$$y_k^{>0} = \sum_q \big(\sum_\psi w_{qk\psi}^{>0} v^{\nu(\tilde{e}_q^{>0}) - \psi}\big) \tilde{e}_q^{>0} v^{-\nu(e_q^{>0})},$$

and this is indeed a combination of elements of $\mathcal{A}_\nu(R)$ with coefficients in $k[v^{\Phi_+}]$ satisfying b). Let us denote by $e_q^{>0}$ the product $\tilde{e}_q^{>0} v^{-\nu(\tilde{e}_q^{>0})}$ and by $z_{qk}^{>0}$ the sum $\sum_\psi w_{qk\psi}^{>0} v^{\nu(\tilde{e}_q^{>0}) - \psi} \in k[v^{\Phi_+}]$. Adding up the decompositions:

$$y_k = y_k^{\leq 0} + y_k^{>0} = x_k^{\leq 0} + \sum_{j=1}^{p} z_{jk}^{\leq 0} e_j^{\leq 0} + \sum_{q=1}^{r} z_{qk}^{>0} e_q^{>0},$$

we see that we have satisfied the criterion for faithful flatness. □

Remark 2.4 Let $(\xi_i)_{i \in I}$ be elements of R such that their images (initial forms) in $\mathrm{gr}_\nu R$ generate it as a $R/(m_\nu \cap R)$-algebra. Let us denote by S the k-subalgebra of R generated by the elements $(\xi_i)_{i \in I}$ and let $\mathcal{B}_\nu(S)$ the sub $k[v^{\Phi_+}]$-algebra of $\mathcal{A}_\nu(S)$ generated by the elements $\xi_i v^{-\nu(\xi_i)}$. The proof we have just seen can be adapted to show that the $k[v^{\Phi_+}]$-algebra $\mathcal{B}_\nu(S)$ is faithfully flat.

The inclusion $k[v^{\Phi_+}] \to \mathcal{A}$ defines a map of schemes

$$\mathrm{Spec}\mathcal{A} \longrightarrow \mathrm{Spec}k[v^{\Phi_+}].$$

Part a) of Proposition 2.2 is equivalent to the fact that the special fiber of this map is $\mathrm{Specgr}_\nu R$ and parts b) and c) are ways of stating that its generic and general fibers are respectively isomorphic to $\mathrm{Spec}(R \otimes_k (v^{\Phi_+})^{-1} k[v^{\Phi_+}])$ and to $\mathrm{Spec}R$. Since this map is faithfully flat by Proposition 2.3, we can say that we have made R appear as a deformation of its associated graded ring, in a family parametrized by $\mathrm{Spec}k[v^{\Phi_+}]$.

Remark 2.5 Without assuming that R contains a field of representatives, we have an isomorphism

$$\mathcal{A}_\nu(R)/(\bigoplus_{\phi\in\Phi_+}\mathcal{P}_\phi(R)v^{-\phi}+m\mathcal{A}_\nu(R))\simeq k[v^{\Phi_+}]$$

corresponding to an inclusion $\mathrm{Spec}k[v^{\Phi_+}]\subset \mathrm{Spec}\mathcal{A}_\nu(R)$.
The localization of $\mathrm{Spec}\mathcal{A}_\nu(R)$ at the general point corresponding to localization by (v^{Φ_+}) induces a general point of this subscheme and produces the "general fiber" seen above, which is essentially $\mathrm{Spec}R$, while the subscheme of $\mathrm{Spec}\mathcal{A}_\nu(R)$ defined by the inverse image in $\mathcal{A}_\nu(R)$ of the ideal $(v^{\Phi_+})k[v^{\Phi_+}]$ is $\mathrm{Specgr}_\nu R$.

The center of the valuation $\mu_{\mathcal{A}}$ on $\mathrm{Spec}\mathcal{A}_\nu(R)$ is the subscheme $\mathrm{Specgr}_\nu R$; as we saw, this subscheme is integral since ν is a valuation and not a loose valuation.

Let us now compute the associated graded ring of $\mathcal{A}=\mathcal{A}_\nu(R)$ with respect to the valuation $\mu_{\mathcal{A}}$. As usual I implicitely add $\{0\}$ to the semigroups.
Set $\mathcal{P}_\phi=\mathcal{P}_\phi(R)$ and

$$\mathcal{Q}_\delta=\{a\in\mathcal{A}\mid \mu_{\mathcal{A}}(a)\geq\delta\};$$

it is a graded ideal generated by the $\{x_\phi v^{-\phi}\mid \nu(x_\phi)-\phi\geq\delta\}$, and similarly for $\mathcal{Q}^+_\delta$. So we have

$$\mathcal{Q}_\delta=\mathcal{A}\cap\bigoplus_{\phi\in\Phi}\mathcal{P}_{\phi+\delta}v^{-\phi},$$

$$\mathcal{Q}^+_\delta=\mathcal{A}\cap\bigoplus_{\phi\in\Phi}\mathcal{P}^+_{\phi+\delta}v^{-\phi}.$$

So $\mathcal{Q}_\delta=\mathcal{A}$ for $\delta\leq 0$ and we can write:

$$\mathrm{gr}_{\mu_{\mathcal{A}}}\mathcal{A}=\bigoplus_{\delta\in\Phi_+\cup\{0\}}(\mathcal{Q}_\delta/\mathcal{Q}^+_\delta)=\bigoplus_{\delta\in\Phi_+}(\bigoplus_{\{\phi|\phi+\delta\in\Phi_+\cup\{0\}\}}\mathcal{P}_{\phi+\delta}/\mathcal{P}^+_{\phi+\delta}).$$

Thus $\mathrm{gr}_{\mu_{\mathcal{A}}}\mathcal{A}$ has a $\Phi_+\oplus\Phi_+$-graduation by $(\delta,\phi+\delta)$ if we give degree $\phi+\delta$ to the elements of $\mathcal{P}_{\phi+\delta}/\mathcal{P}^+_{\phi+\delta}$ as suggested by the graduation of $\mathcal{A}$.

Let us denote by $\overline{R}$ the quotient $R/(m_\nu\cap R)$, and remark that $\mathrm{gr}_\nu R$ is an $\overline{R}$-algebra.

We have a map of $\overline{R}$-algebras

$$\mathrm{gr}_{\mu_{\mathcal{A}}}\mathcal{A}\to(\mathrm{gr}_\nu R)\otimes_{\overline{R}}\overline{R}[v^{\Phi_+}]$$

induced by the map $\mathcal{Q}_\delta\to(\mathrm{gr}_\nu R)\otimes_{\overline{R}}\overline{R}[v^{\Phi_+}]$ defined by:

$$\sum_{\nu(x_\phi)-\phi\geq\delta}x_\phi v^{-\phi}\mapsto\sum_{\nu(x_\phi)-\phi=\delta}\overline{x_\phi}\otimes v^\delta,$$

where $\overline{x_\phi}$ is the initial form in $\mathcal{P}_{\phi+\delta}/\mathcal{P}^+_{\phi+\delta}$ of x_ϕ. This map is homogeneous with respect to the $\Phi_+\oplus\Phi_+$ graduation of $\mathrm{gr}_{\mu_{\mathcal{A}}}\mathcal{A}$ described above if we give $(\mathrm{gr}_\nu R)\otimes_{\overline{R}}\overline{R}[v^{\Phi_+}]$ the tensor product graduation where elements of degree $\delta\oplus\psi\in\Phi_+\oplus\Phi_+$ are tensor products with v^δ of elements of degree ψ of $\mathrm{gr}_\nu R$.

Proposition 2.6 *With the gradings just described, this map is a* $(\Phi_+\oplus\Phi_+)$*-graded isomorphism of* $\overline{R}$*-algebras*

$$\mathrm{gr}_{\mu_{\mathcal{A}}}\mathcal{A}\stackrel{\simeq}{\to}(\mathrm{gr}_\nu R)\otimes_{\overline{R}}\overline{R}[v^{\Phi_+}].$$

Proof The map is clearly surjective, and its injectivity is easily verified degree by degree. Note that upon taking $\phi + \delta = 0$, this map induces the identity of $\overline{R}[v^{\Phi_+}]$. Note also that the valuation $\mu_{\mathcal{A}}$ takes its positive values in Φ_+, and not in Γ. □

Remark 2.7 The meaning of this proposition is that the deformation of graded $\overline{R}$-algebras corresponding as in Proposition 2.2 to the valuation algebra associated to $\mathcal{A}$ and to $\mu_{\mathcal{A}}$ is "trivial" in the sense that the total space $\mathrm{Specgr}_{\mu_{\mathcal{A}}}\mathcal{A}$ is the product by the base space $\mathrm{Spec}\overline{R}[v^{\Phi_+}]$ of the special fiber $\mathrm{Specgr}_{\nu}R$.

2.4 The valuation $\nu_{\mathcal{A}}$. There is another natural valuation on the ring $\mathcal{A}_\nu(R)$: it is the valuation $\nu_{\mathcal{A}}$ defined by

$$\nu_{\mathcal{A}}(\sum x_\phi v^{-\phi}) = \min(\nu(x_\phi)).$$

Its center is the subscheme of $\mathrm{Spec}\mathcal{A}_\nu(R)$ defined by the ideal of $\mathcal{A}_\nu(R)$ generated by $(m_\nu \cap R, \bigoplus_{\phi\in\Phi_+} \mathcal{P}_\phi v^{-\phi})$. This subscheme is isomorphic to $\mathrm{Spec}\overline{R}[v^{\Phi_+}]$. In the special case where R is a local k-algebra dominated by R_ν and the extension $k \to R/m$ is trivial, this means that the center of $\nu_{\mathcal{A}}$ is the image of a section

$$\sigma\colon \mathrm{Spec}k[v^{\Phi_+}] \to \mathrm{Spec}\mathcal{A}_\nu(R).$$

Let us examine the valuation algebra of $\mathcal{A}_\nu(R)$ with respect to $\nu_{\mathcal{A}}$. Set

$$\mathcal{S}_\delta = \{\sum x_\phi v^{-\phi} \in \mathcal{A}_\nu(R) \mid \min(\nu(x_\phi)) \geq \delta\} = \bigoplus_{\phi\in\Phi} \mathcal{P}_{\max(\delta,\phi)}(R) v^{-\phi},$$

$$\mathcal{S}^+_\delta = \{\sum x_\phi v^{-\phi} \in \mathcal{A}_\nu(R) \mid \min(\nu(x_\phi)) > \delta\} = \bigoplus_{\phi\in\Phi} (\mathcal{P}^+_\delta(R) \cap \mathcal{P}_\phi(R)) v^{-\phi};$$

we have the equalities

$$\mathrm{gr}_{\nu_{\mathcal{A}}}\mathcal{A}_\nu(R) = \bigoplus_{\delta\in\Phi} \mathcal{S}_\delta/\mathcal{S}^+_\delta = \bigoplus_{\phi\in\Phi}\big(\bigoplus_{\delta\geq\phi}(\mathcal{P}_\delta(R)/\mathcal{P}^+_\delta(R))\big).$$

Let us define the "degree algebra" of a graded algebra $H = \bigoplus_{\delta\in\Phi_+\cup\{0\}} H_\delta$ in a way similar to the definition of the valuation algebra, replacing valuation by degree, and setting $H_\delta = 0$ for $\delta \in \Phi_-$:

$$\mathcal{A}_{\mathrm{deg}}(H) = \bigoplus_{\phi\in\Phi}\big(\bigoplus_{\delta\geq\phi} H_\delta\big) v^{-\phi} \subset H[v^\Phi].$$

This algebra is "trivial" in view of the:

Lemma 2.8 *The map*

$$\mathcal{A}_{\mathrm{deg}}(H) \to H \otimes_{H_0} H_0[v^{\Phi_+}]$$

determined by

$$\sum_\phi \big(\sum_{\delta\geq\phi} x_\delta\big) v^{-\phi} \mapsto \sum_{\phi,\delta\in\Phi,\ \delta\geq\phi} x_\delta \otimes_{H_0} v^{\delta-\phi}$$

is a graded isomorphism of H_0-algebras when one gives the right hand side the twisted $\Phi \oplus \Phi_+$ graduation for which a homogeneous element of degree $\delta \oplus \psi$ is of the form $x_\delta \otimes v^{\delta-\psi}$ where x_δ is of degree δ.

Proof Indeed, the inverse map is given by $x_\delta \otimes v^\psi \mapsto x_\delta v^{-(\delta-\psi)}$. □

In order to show that the valuation algebra $\mathrm{gr}_{\nu_{\mathcal{A}}}\mathcal{A}_\nu(R)$ is "trivial", as in the case of the valuation $\mu_{\mathcal{A}}$, it only remains to check the:

Lemma 2.9 *If we view* $\mathrm{gr}_\nu(R)$ *as a graded algebra indexed by* $\Phi_+\cup\{0\}$ *with a* 0 *component for* $\phi\notin\Gamma$*, we have the equality:*

$$\mathrm{gr}_{\nu_{\mathcal{A}}}\mathcal{A}_\nu(R)=\mathcal{A}_{\mathrm{deg}}(\mathrm{gr}_\nu(R)).$$

Proof It follows directly from the definitions. □

Finally, we have proved

Proposition 2.10 *With the twisted grading on* $(\mathrm{gr}_\nu R)\otimes_{\overline{R}}\overline{R}[v^{\Phi_+}]$ *described above, the map of Lemma 2.8 is a graded isomorphism of graded* $\overline{R}$*-algebras*

$$\mathrm{gr}_{\nu_{\mathcal{A}}}\mathcal{A}_\nu(R)\stackrel{\simeq}{\to}(\mathrm{gr}_\nu R)\otimes_{\overline{R}}\overline{R}[v^{\Phi_+}],$$

determined by the fact that the $\nu_{\mathcal{A}}$*-initial form of* $\xi v^{-\nu(\xi)}$ *is mapped to* $\overline{\xi}\otimes 1$*, where* $\overline{\xi}$ *is the* ν*-initial form of* ξ*.*
Note that

$$\mathcal{A}_\nu(R)/\mathcal{S}_0=\overline{R}[v^{\Phi_+}]=\overline{R}\otimes_{\overline{R}}\overline{R}[v^{\Phi_+}].$$

Remarks 2.11 1) The isomorphisms of the two graded algebras of $\mathcal{A}_\nu(R)$ with respect to $\mu_{\mathcal{A}}$ and $\nu_{\mathcal{A}}$ with $(\mathrm{gr}_\nu R)\otimes_{\overline{R}}\overline{R}[v^{\Phi_+}]$ have quite different geometric significations, as one can see by considering the analogous constructions for the Rees algebra $\mathcal{R}(I)=\bigoplus_{n\in\mathbf{Z}}I^nv^{-n}\subset R[v,v^{-1}]$ of an ideal I of a ring R, where $I^n=R$ for $n\leq 0$. The valuation μ corresponds to the v-adic filtration; it is the (loose) valuation associated to this filtration, while the valuation ν corresponds to the (loose) valuation induced by the I-adic filtration of $R[v,v^{-1}]$. The triviality of the $\mathrm{gr}_{\mu_{\mathcal{A}}}\mathcal{A}_\nu(R)$ generalizes the fact that since the special fiber Specgr_IR in this case is a divisor, its normal cone in $\mathrm{Spec}\mathcal{R}(I)$ is $\mathrm{Spec}(\mathrm{gr}_IR)[v]$. The triviality for $\nu_{\mathcal{A}}$ generalizes the fact, more specific to the Rees algebra, that the graded ring of $\mathcal{R}(I)$ with respect to the I-adic filtration is trivial in the sense that it is isomorphic to $\mathrm{gr}_IR\otimes_{R/I}R/I[v]$. This reflects the fact that in the specialization of $\mathrm{Spec}R$ to Specgr_IR provided by $\mathrm{Spec}\mathcal{R}(I)$, the graded ring is constant, or more precisely that the induced specialization on the graded ring is trivial.
2) We shall see below in subsection 3.2 that there is a valuation $\tilde{\nu}_{\mathcal{A}}$ on $\mathcal{A}_\nu(R)$ which is composed with the natural valuation ν_{gr} on the graded ring $\mathrm{gr}_\nu R$. Moreover, this valuation $\tilde{\nu}_{\mathcal{A}}$ induces in a natural way the valuation ν on R. This will gives a precise meaning to the intuition that ν_{gr} is the specialization of ν to $\mathrm{gr}_\nu R$.
3) The graded ring $\mathrm{gr}_{\nu_{\mathrm{gr}}}\mathrm{gr}_\nu R$ is isomorphic to $\mathrm{gr}_\nu R$.

An example: Let R be a discrete valuation ring, and u a generator of its maximal ideal. The value group of the valuation of R is $\mathbf{Z}$ and we have $\mathcal{P}_n(R)=u^nR$ for $n\geq 0$, $\mathcal{P}_n(R)=R$ for $n\leq 0$, so that

$$\mathcal{A}_\nu(R)=R[v,uv^{-1}]\subset R[v,v^{-1}].$$

In this case, $\mathrm{gr}_\nu R=(R/uR)[U]$, where U is the initial form of u. I leave it as an exercise to determine the maps $\mathcal{A}_\nu(R)\to R$ and $\mathcal{A}_\nu(R)\to\mathrm{gr}_\nu R$ in this case (hint: map uv^{-1} to U). This applies in particular when $R=\mathbf{C}\{u\}$, as we shall see in subsection 5.6, or to $k[[u]]$ where k is any field. It also applies to the non-equicharacteristic case, for example to $R=\hat{\mathbf{Z}}_p$, the ring of p-adic integers, and more generally when $R=W(k)$ is the ring of Witt vectors associated to a perfect field

k of characteristic p, with $u = p$ (see [B3], Chap. IX, §§1,2, [Ei], §7). We shall see below in subsection 5.3 that the Spectrum of the p-adic completion of $\mathcal{A}_\nu(W(k))$ contains both Spec$k[[t]]$ and Spec$W(k)$. This is the very first step in the extension of the method presented here to the non-equicharacteristic case.

3 Valuation algebras and composition of valuations

In this section I examine the behavior of valuation algebras when one changes a valuation for an other one with which it is composed, or for the corresponding residual valuation. This section also contains two results which are very useful in the sequel: Zariski's description of valuation ideals with respect to two consecutive valuations, and Piltant's theorem on the Krull dimension of $\mathrm{gr}_\nu R$.

3.1 The composition of valuations. In this paragraph, we study the relation of composition of valuations; if ν is composed with ν_1, both non-negative on our local ring R, it means that we have inclusions of local rings

$$R \subset R_\nu \subset R_{\nu_1},$$

and of ideals

$$m_{\nu_1} \subset m_\nu \subset R_\nu,$$

$$m_{\nu_1} \cap R \subseteq m_\nu \cap R \subseteq m.$$

I assume known the theory of composition of valuations, referring to ([V1], section 4, [Z-S], Vol 2, Ch. VI, [A1], [B2]), and fix notations.

Let us recall that the rank $\mathrm{r}(\Phi)$ of the $\mathbf{Q}$-vector space $\mathbf{Q} \otimes_{\mathbf{Z}} \Phi$ is called the *rational rank* of Φ; if Φ is the group of a valuation ν, it is also denoted by $\mathrm{r}(\nu)$. Recall also that in the case where R_ν dominates R, the transcendence degree $\mathrm{t}_{k_R}(\nu)$ of the extension $(k_\nu : k_R)$ is called the residual transcendence degree, and that the *height* $\mathrm{h}(\nu)$ (often also called the *rank*) of the valuation ν is the number (in general the ordinal type) of elements of a maximal chain

$$R_\nu \subset R_{\nu_1} \subset \cdots \subset R_{\nu_{h-1}}$$

of valuation rings of K containing R_ν; it is also the Krull dimension of the ring R_ν. Each inclusion $R_\nu \subset R_{\nu_i}$ corresponds to a monotone (non-decreasing) map of totally ordered groups $\lambda_i \colon \Phi \to \Phi_i$ such that $\nu_i = \lambda_i \circ \nu$. A monotone non-decreasing map means here that if $\phi \leq \psi$, then $\lambda(\phi) \leq \lambda(\psi)$. The kernel Ψ_i of such a map is an *isolated* (or *convex*) subgroup of Φ, so that $\mathrm{h}(\nu)$ is also the ordinal type of the chain of isolated subgoups of Φ; it depends only on Φ and we will speak of the height of Φ.

A totally ordered group of height one is archimedian and can be embedded as an ordered subgroup of $\mathbf{R}$. Following [A1], we will say that a group of height one is *discrete* if it is well ordered (i.e., isomorphic to $\mathbf{Z}$, see [A1]), and we will say that a group Φ of finite height is discrete if the quotient of two successive elements of the sequence

$$(0) \subset \Psi_{h-1} \subset \cdots \subset \Psi_1 \subset \Phi$$

of the isolated subgroups of Φ is discrete.

The additivity of the rational rank in exact sequences implies the inequality

$$\mathrm{h}(\nu) \leq \mathrm{r}(\nu)$$

and for a valuation ring R_ν birationally dominating a nœtherian local ring R, we have *Abhyankar's inequality* (see [V1], section 9, Th. 9.2)

$$\mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu) \leq \dim R.$$

It is important to note that given a valuation of height ν of a ring R, the prime ideals of R_ν form a chain

$$m_{\mathrm{h}(\nu)-1} \subset m_{\mathrm{h}(\nu)-2} \subset \cdots \subset m_{\nu_1} \subset m_\nu$$

of length $\mathrm{h}(\nu)$ but their intersections with R are not necessarily distinct, and the length of the chain of distinct prime ideals among the

$$m_{\mathrm{h}(\nu)-1} \cap R \subseteq \cdots \subseteq m_{\nu_1} \cap R \subseteq m_\nu \cap R,$$

which may be called the *height of* ν *in* R and denoted by $\mathrm{h}_R(\nu)$, is at most $\mathrm{h}(\nu)$. Abhyankar's inequality has a nice interpretation in terms of the graded algebra:

Proposition 3.1 (Piltant, [P]) *If R is a nœtherian local domain with uncountable residue field k_R, birationally dominated by the valuation ring R_ν, the Krull dimension of* $\mathrm{gr}_\nu R$ *is equal to* $\mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu)$*. If* $\mathrm{t}_{k_R}(\nu) = 0$*, the assumption on the cardinality is unnecessary.*

In the cases where k_R is uncountable, Abhyankar's inequality then becomes

$$\dim \mathrm{gr}_\nu R \leq \dim R.$$

Note that if we have $R \subset R_\nu$, the local ring R_ν dominates $R_{m_\nu \cap R}$.
Spivakovsky had remarked that the transcendence degree over R/m of $\mathrm{gr}_\nu R$ is equal to $\mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu)$ (see [S3], Remark 3.7).

I will use Piltant's result only in the case where $\mathrm{t}_{k_R}(\nu) = 0$. For the convenience of the reader, I sketch Piltant's proof of his result in this case. In the beginning, I suppose only that $m_\nu \cap R = m$, i.e. that the valuation ring R_ν dominates R. Let S be the multiplicative set of homogeneous elements of positive degree in $\mathrm{gr}_\nu R$, and let $[F]_0$ denote the part of degree zero of an homogenous ring of fractions F of a graded algebra by a multiplicative set of homogeneous elements.

Lemma 3.2 *The map*

$$[S^{-1}\mathrm{gr}_\nu R]_0 \to k_\nu, \quad \frac{\overline{x}_\phi}{\overline{y}_\phi} \mapsto \frac{x_\phi}{y_\phi} \mathrm{mod}.m_\nu$$

is well defined and is an isomorphism.

Proof This follows from a direct verification. □

Let now $(u_1, \ldots, u_r)$ be elements of R whose valuations $(\delta_1, \ldots, \delta_r)$ rationally generate the group Φ, where r is the rational rank $\mathrm{r}(\nu)$ of Φ; this means that $(\delta_1, \ldots, \delta_r)$ form a basis of the $\mathbf{Q}$-vector space $\Phi \otimes_{\mathbf{Z}} \mathbf{Q}$. Let $\Phi^{(u)} = \mathbf{Z}\delta_1 \oplus \cdots \oplus \mathbf{Z}\delta_r \subseteq \Phi$ be the subgroup generated by the δ_i; set $\Phi_+^{(u)} = \Phi^{(u)} \cap \Phi_+$ and define the subalgebra

$$\mathrm{gr}_\nu^{(u)} R = \bigoplus_{\phi \in \Phi_+^{(u)}} \mathcal{P}_\phi(R)/\mathcal{P}_\phi^+(R).$$

Let us denote by $S^{(u)}$ the intersection $S \cap \mathrm{gr}_\nu^{(u)} R$ and by $k^{(u)}$ the induced field $[(S^{(u)})^{-1}\mathrm{gr}_\nu^{(u)} R]_0$.

Lemma 3.3 *The extensions* $\mathrm{gr}_\nu^{(u)} R \to \mathrm{gr}_\nu R$ *and* $k^{(u)} \to k_\nu$ *are integral.*

Proof By [B3], Chap.V, No.1, Corollary to Prop.4, it suffices to check that any homogeneous element $\overline{x}_\phi \in \mathrm{gr}_\nu R$ has a power which is in $\mathrm{gr}_\nu^{(u)} R$. Since $\Phi^{(u)}$ rationally generates Φ, there is an integer n such that $n\phi \in \Phi^{(u)}$, so that $\overline{x}^n \in \mathrm{gr}_\nu^{(u)} R$. □

Given an element $\phi \in \Phi^{(u)}$, it can be written uniquely as

$$\phi = a_1\delta_1 + \cdots + a_r\delta_r \quad \text{with } a_i \in \mathbf{Z}.$$

Lemma 3.4 *With the notation just introduced, the map*

$$\iota_u \colon \mathrm{gr}_\nu^{(u)} R \to k^{(u)}[t^{\Phi_+^{(u)}}], \quad \overline{x}_\phi \mapsto \frac{\overline{x}_\phi}{\overline{u}_1^{a_1} \ldots \overline{u}_r^{a_r}} t^\phi$$

induces an isomorphism

$$(S^{(u)})^{-1} \mathrm{gr}_\nu^{(u)} R \xrightarrow{\simeq} k^{(u)}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}].$$

If k_ν is algebraic over k_R, the map ι_u is integral.

Proof For the first part, since $\Phi^{(u)}$ is a free abelian group of rank r, it suffices to show that $(S^{(u)})^{-1}\mathrm{gr}_\nu^{(u)} R \to k^{(u)}[t^{\Phi^{(u)}}]$ is an isomorphism. Remarking that $\iota_u(S^{(u)}) \subseteq \{\lambda t^\gamma,\ \gamma \in \Phi_+^{(u)},\ \lambda \in {k^{(u)}}^*\}$, we see that $(S^{(u)})^{-1}\mathrm{gr}_\nu^{(u)} R \subset k^{(u)}[t^{\Phi^{(u)}}]$. The surjectivity follows from the fact that $\Phi_+^{(u)}$ rationally generates $\Phi^{(u)} = \mathbf{Z}^r$.
If the extension $k_R \to k_\nu$ is algebraic, so is $k_R \to k^{(u)}$. To prove that ι_u is integral, it suffices to show that any element λt^γ with $\lambda \in k^{(u)},\ \gamma \in \Phi^{(u)}$, is integral. This follows from the fact that λ is algebraic over k_R. □

From these lemmas we deduce

Proposition 3.5 (Spivakovsky, Piltant) *The transcendence degree over k_R of $\mathrm{gr}_\nu R$ is equal to $\mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu)$.*

Proof Because the extension $\mathrm{gr}_\nu^{(u)} R \to \mathrm{gr}_\nu R$ is integral by Lemma 3.3, this transcendence degree is equal to that of $\mathrm{gr}_\nu^{(u)} R$, hence to that of $(S^{(u)})^{-1}\mathrm{gr}_\nu^{(u)} R$, which is that of $k^{(u)}[t^{\Phi^{(u)}}]$, and also of $k_\nu[t^{\Phi^{(u)}}]$ since the extension $k^{(u)} \to k_\nu$ is integral by Lemma 3.3. □

Now Piltant uses the following corollary to Cohen's dimension inequality:

Proposition 3.6 *Let B be an integral domain containing a nœtherian ring A. Then*

$$\dim B \leq \dim A + \mathrm{t}_A(B),$$

where $\mathrm{t}_A(B)$ is the transcendence degree of B over A.

Proof The proof is like the classical one (see [Ei], Chap. 13) except that we do not suppose that B is finitely generated over A. But if $(0) = \mathbf{p}_0 \subset \mathbf{p}_1 \subset \mathbf{p}_2 \subset \cdots \subset \mathbf{p}_d$ is a chain of distinct prime ideals of B, choosing for each $i,\ 1 \leq i \leq d$ an element $y_i \in \mathbf{p}_i \setminus \mathbf{p}_{i-1}$, we see that the $\mathbf{p}_i \cap A[y_1, \ldots, y_d]$ form a chain of distinct prime ideals, so that $\dim A[y_1, \ldots, y_d] \geq d$, and we are reduced to the case of a finitely generated algebra. □

Applying this result with $A = k_R$ and $B = \mathrm{gr}_\nu R$, remarking that we have used the nœtherianity of R only to ensure that $\mathrm{r}(\nu) + \mathrm{t}_{k_R}(\nu) < \infty$ and using Proposition 3.5 gives the first part of the:

Proposition 3.7 (Piltant, [P]) *Whenever the valuation ring R_ν birationally dominates the local ring R and $\mathrm{r}(\nu)+\mathrm{t}_{k_R}(\nu)<\infty$, the following inequality holds:*

$$\mathrm{dim}\mathrm{gr}_\nu R \leq \mathrm{r}(\nu)+\mathrm{t}_{k_R}(\nu).$$

If $\mathrm{t}_{k_R}(\nu)=0$,

$$\mathrm{dim}\mathrm{gr}_\nu R = \mathrm{r}(\nu).$$

Proof To prove the second part, we use the lemmas above: the map

$$(S^{(u)})^{-1}\mathrm{gr}_\nu^{(u)}R \to k^{(u)}[x_1^{\pm 1},\ldots,x_r^{\pm 1}]$$

is an isomorphism, so these two rings have the same dimension $\mathrm{r}(\nu)$. Since dimension cannot increase by localization, this gives $\mathrm{dim}\mathrm{gr}_\nu^{(u)}R \geq \mathrm{r}(\nu)$ and finally, because the extension $\mathrm{gr}_\nu^{(u)}R \to \mathrm{gr}_\nu R$ is integral, using ([B3], Chap. VIII, No.3. Th.1), $\mathrm{dim}\mathrm{gr}_\nu R \geq \mathrm{r}(\nu)$. But we have the opposite inequality by the first part of the proposition. □

Remark 3.8 Piltant's result implies that if $\mathrm{r}(\nu)<\infty$ we have:

$$\mathrm{dim}\mathrm{gr}_\nu R_\nu = \mathrm{r}(\nu)$$

and if $\mathrm{t}_{k_R}(\nu)=0$,

$$\mathrm{dim}\mathrm{gr}_\nu R = \mathrm{t}_{k_R}(\mathrm{gr}_\nu R),$$

the transcendence degree of $\mathrm{gr}_\nu R$ over k_R. I will use this last equality in several examples. The first one should be contrasted with the fact that $\mathrm{dim}R_\nu = \mathrm{h}(\nu) \leq \mathrm{r}(\nu)$.

Question. Does the proof in the case where $\mathrm{t}_{k_R}(\nu)=0$ also show that $\mathrm{gr}_\nu R$ is catenary?
The proof of the fact that equality in the inequality of Proposition 3.7 continues to hold without the assumption that $\mathrm{t}_{k_R}(\nu)=0$ provided that k_R is uncountable is more delicate. Piltant has given in [P] an example of a valuation ν on the ring $R=\overline{\mathbf{Q}}[s,t,x,y]_{(s,t,x,y)}$ with value group $\mathbf{Z}^2_{lex}$ and residue field $\overline{\mathbf{Q}}(\frac{s}{t})$ such that $\mathrm{gr}_\nu R$ has Krull dimension $3=\mathrm{r}(\nu)+\mathrm{t}_{k_R}(\nu)$ but is not catenary; it possesses a saturated chain of prime ideals of length two. It would be interesting to decide whether the completion $\widehat{\mathrm{gr}}_\nu^{(\nu)}R$ with respect to ν_{gr} which we shall meet in subsection 5.3 is catenary.
Strict inequality in Abhyankar's inequality may happen, but this does not contradict the semi-continuity theorem for the dimensions of the fibers of the map $\mathrm{Spec}A_\nu(R)\to \mathrm{Spec}k[v^{\Phi_+}]$ since semicontinuity is proved only for morphisms which are essentially of finite type. Under the assumptions of Piltant's theorem, for $\mathcal{A}_\nu(R)$ to be a $k[v^{\Phi_+}]$-algebra of finite type it is necessary that equality holds in Abhyankar's inequality.

Since R is assumed to be nœtherian we will in this text remain in the situation where $\mathrm{r}(\Phi),\mathrm{t}_{k_R}(\nu)$, etc.. are finite.

Recall (see [B3], Chap. VI, §10, No. 2) that if Φ is finitely generated and if $\mathrm{r}(\nu)=\mathrm{h}(\nu)$, then Φ is order isomorphic to $\mathbf{Z}^{\mathrm{h}(\nu)}$ with the lexicographic order.

I will often make use of the following fact:

Proposition 3.9 (Krull, [Kr], Zariski; see ([Z-S], Appendix 3, [S2], remark 2.7, [V1], Prop. 9.1.) *If R is nœtherian, the value semigroup $\Gamma = \nu(R \setminus \{0\}) \subset \Phi_+ \cup \{0\}$ is well ordered and its ordinal type* (see [B5]) *is $\omega^{\mathrm{h}_R(\nu)}$, where $\mathrm{h}_R(\nu)$ is the height in R of the valuation ν.*

Proof Zariski's proof is in terms of valuation ideals of R, which are the distinct $\mathcal{P}_\phi(R)$ and therefore in monotonous bijection with the elements of Γ by the map $\gamma \mapsto \mathcal{P}_\gamma$ for the reverse inclusion ordering on ideals; a decreasing sequence of elements of Γ gives an increasing sequence of ideals $\mathcal{P}_\phi(R)$, which has to be stationary since R is nœtherian. The second part of the statement follows from [Z-S], Appendix 3, Proposition 2. □

Corollary 3.10 *For a valuation ν of a nœtherian ring R, the value semigroup admits a unique minimal system of generators*

$$\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_i, \ldots \rangle$$

satisfying

$$\gamma_i < \gamma_{i+1}.$$

It is indexed by ordinals $< \omega^{\mathrm{h}_R(\nu)}$. If the height of ν in R is one, in particular if the height of ν is one, given $\gamma \in \Gamma$, there are finitely many generators $\gamma_i \leq \gamma$.

Proof Define γ_1 to be the smallest non zero element of Γ, and then inductively γ_{i+1} to be the smallest non zero element of Γ which is not in the semigroup

$$\Gamma_i = \langle \gamma_1, \gamma_2, \ldots, \gamma_i \rangle$$

generated by the previous ones. If after finitely many steps, $\Gamma_i = \Gamma$, our semigroup is finitely generated; otherwise, the elements obtained after iterating the successor construction countably many times generate a semigroup $\Gamma_{<\omega} \subseteq \Gamma$. If it is not equal to Γ, let γ_ω be the smallest non zero element of Γ not contained in $\Gamma_{<\omega}$, set $\Gamma_\omega = \langle \Gamma_{<\omega}, \gamma_\omega \rangle$ and continue in this manner. This is clearly a minimal system of generators of Γ. If for some i we had $\gamma_{i+1} < \gamma_i$ it would contradict the definition of γ_i since $\gamma_{i+1} \notin \Gamma_{<i}$. The second assertion also follows [Z-S], Appendix 3, Lemma 4. □

Remark 3.11 The element γ_ω has no predecessor among the γ_i. We shall see below that if R is complete there are only finitely many elements without predecessor.

Viewing a valuation as a map $\nu\colon K^* \to \Phi$ as in subsection 2.1, we see that a surjective monotone non-decreasing map of ordered groups $\lambda\colon \Phi \to \Phi_1$ corresponds to a composite valuation $\nu_1 = \lambda \circ \nu$ and another valuation ring $R_{\nu_1} \supset R_\nu$. However the classical terminology, referring to a geometric notion of composition, is that ν is *composed* with ν_1 and we will respect it. The map λ extends to a map

$$\tilde{\lambda}\colon k[v^{\Phi_+}] \to k[w^{\Phi_{1+}}].$$

We have a map $\mathcal{A}_\nu(R) \to \mathcal{A}_{\nu_1}(R)$ and in fact a commutative diagram

$$\begin{array}{lccc} \tilde{\lambda}_{\mathcal{A}}(R)\colon & \mathcal{A}_\nu(R) & \longrightarrow & \mathcal{A}_{\nu_1}(R) \\ & \uparrow & & \uparrow \\ \tilde{\lambda}\colon & k[v^{\Phi_+}] & \longrightarrow & k[w^{\Phi_{1+}}] \end{array}$$

where the top arrow is induced by the inclusions $\mathcal{P}_\phi(R) \subset \mathcal{P}_{\lambda(\phi)}(R)$, i.e., $x_\phi v^{-\phi} \mapsto x_\phi w^{-\lambda(\phi)}$, and the bottom arrow is $\tilde{\lambda}$. The map $\tilde{\lambda}$ is surjective since $\lambda\colon \Phi \to \Phi_1$ is order preserving, and its kernel is the ideal generated by the $(v^\psi - 1)_{\psi\in\Psi_+}$ where Ψ is the kernel of the map $\Phi \to \Phi_1$. Let us show that $\tilde\lambda_{\mathcal{A}}(R)$ is also surjective: given $xw^{-\mu}$, where $\mu = \lambda(\phi)$ and $\nu_1(x) \geq \mu$, either $\nu(x) \geq \phi$ and $xw^{-\mu}$ is the image of $xv^{-\phi}$, or we have not chosen ϕ well, but since λ is order preserving, if $\nu(x) < \phi$, we have $\nu_1(x) = \lambda(\nu(x)) \leq \lambda(\phi) = \mu$ so that $\lambda(\nu(x)) = \mu$ and $\phi - \nu(x) \in \Psi$. Finally, we see that $xw^{-\mu}$ is the image of $xv^{-\nu(x)}$ and we have shown the surjectivity. The kernel is the ideal of $\mathcal{A}_\nu$ generated by $(v^\psi - 1)_{\psi\in\Psi_+}$.

Finally, we have shown that the formation of $k[v^{\Phi_+}] \to \mathcal{A}_\nu(R)$ commutes with the composition of valuations, as follows;

Proposition 3.12 *The natural morphism*

$$\begin{array}{ll} \mathcal{A}_\nu(R) \otimes_{k[v^{\Phi_+}]} k[w^{\Phi_{1+}}] & \to \mathcal{A}_{\nu_1}(R) \\ x_\phi v^{-\phi} \otimes w^{\phi_1} & \mapsto x_\phi w^{\phi_1 - \lambda(\phi)} \end{array}$$

is an isomorphism of graded $k[v^{\Phi_+}]$*-algebras.*

Remark 3.13 The fact that

$$\mathcal{A}_{\nu_1}(R) = \mathcal{A}_\nu(R)/\big((v^\psi - 1)_{\psi\in\Psi_+}\big)$$

is true without assuming that R contains a field.

Corollary 3.14 *The rings* $\mathrm{gr}_\nu R$ *and* $\mathrm{gr}_{\nu_1} R$ *are both specializations of the same ring over the polynomial ring* $k[(t_\psi)_{\psi\in\Psi_+}]$.

Proof The ring $\mathrm{gr}_{\nu_1} R$ is isomorphic to the quotient ring of $\mathcal{A}_\nu(R)$ by $\big((v^\psi - 1)_{\psi\in\Psi_+}, (v^\phi)_{\phi\in\Phi_+\setminus\Psi_+}\big)$, and $\mathrm{gr}_\nu R = \mathcal{A}_\nu(R)/\big((v^\phi)_{\phi\in\Phi_+}\big)$. These rings are specializations of

$$\mathcal{A}_\nu(R) \otimes_k k[(t_\psi)_{\psi\in\Psi_+}]/\big((v^\psi - t_\psi)_{\psi\in\Psi_+}, (v^\phi)_{\phi\in\Phi_+\setminus\Psi_+}\big).$$

□

Remarks 3.15 1) If we have a sequence of valuation rings as above, with $R \subset R_\nu$,

$$R_\nu \subset R_{\nu_1} \subset \cdots \subset R_{\nu_{h-1}},$$

where h is the height of the valuation ν, all the graded algebras $\mathrm{gr}_{\nu_s} R$ are specializations of the same algebra of the form

$$\mathcal{A}_\nu(R) \otimes_k k[t_\omega]/\big((v^\omega - t_\omega)_{\omega\in\Omega_1}, (v^\omega)_{\omega\in\Omega_2}\big),$$

and we can in a number of cases use a system of generators of $\mathrm{gr}_\nu R$ to produce a system of "formal" generators for each $\mathrm{gr}_{\nu_s} R$ (see below).
2) Denoting by $\lambda\colon \Phi \to \Phi_1$ the morphism of value groups, notice that the graph $(\phi, \lambda(\phi)) \subset \Phi \times \Phi_1$ is isomorphic to Φ and that we have an isomorphism of graded algebras:

$$\mathcal{A}_\nu(R) \overset{\simeq}{\to} \bigoplus_{\phi_1\in\Phi_1} \big(\bigoplus_{\lambda(\phi)=\phi_1} \mathcal{P}_\phi(R) w^{-\phi}\big) w_1^{-\phi_1}.$$

3.2 The valuation $\tilde\nu_{\mathcal A}$. Let us denote by $\tilde\nu_{\mathcal A}$ the valuation on $\mathcal A_\nu(R)$ with values in the group $\Phi\oplus\Phi$ ordered lexicographically, and defined by

$$\tilde\nu_{\mathcal A}(\sum x_\phi v^{-\phi})=\big(\mu_{\mathcal A}(\sum x_\phi v^{-\phi}),\nu_{\mathcal A}(\sum x_\phi v^{-\phi})\big).$$

According to what we saw in §1, the center of this valuation is defined by the ideal $\big((v^{\Phi_+})\mathcal A_\nu(R),\oplus_{\phi\in\Phi_+}\mathcal P_\phi v^{-\phi}\big)$ of $\mathcal A_\nu(R)$. Geometrically, this center is $\mathrm{Spec}\overline{R}$.

This valuation is composed with $\mu_{\mathcal A}$ in the manner corresponding to the first projection $\Phi\oplus\Phi\to\Phi$, and the residual valuation of $\tilde\nu_{\mathcal A}$ on the center $\mathrm{Specgr}_\nu R$ of $\mu_{\mathcal A}$ is ν_{gr}.

This valuation is of interest because it is a "specialization" of the valuation ν on R to the valuation ν_{gr}. Indeed, the valuation $\tilde\nu_{\mathcal A}$ extends to a valuation of $(v^{\Phi_+})^{-1}A_\nu(R)=(v^{\Phi_+})^{-1}R[v^{\Phi_+}]$ (Proposition 2.2, b)) which is the natural extension of ν to the algebra of the "generic fiber" of the faithfully flat map defined in §2.3, and as we saw it induces the natural valuation on the special fiber. Heuristically, we may think of the inclusion $\mathcal A_\nu(R)\to\mathcal A_\nu(R_\nu)$ as a uniformization of the valuation $\tilde\nu_{\mathcal A}$ (see below, §4), forgetting for a moment the fact that it is not an extension of finite type. The center of $\tilde\nu_{\mathcal A}$ in $\mathcal A_\nu(R_\nu)$ is the origin, and, still heuristically, $\mathcal A_\nu(R_\nu)$ is "regular". Then the corresponding map of schemes is a simultaneous uniformization for ν and ν_{gr} since the induced map $\mathrm{gr}_\nu R\to\mathrm{gr}_\nu R_\nu$ is also (in the same heuristic sense) a uniformization of ν_{gr}.

3.3 The valuation algebra of the residual valuation. Let us now come back to the inclusion $R_\nu\subset R_{\nu_1}$ associated to an monotone non-decreasing surjection $\lambda\colon\Phi\to\Phi_1$ of ordered groups, with kernel Ψ; it is a classical fact of valuation theory (see [A1], pp. 56-57 and [V1]) that $m_{\nu_1}\subset R_\nu$ and the image $\overline{R}_\nu=R_\nu/m_{\nu_1}\subset R_{\nu_1}/m_{\nu_1}$ is a valuation ring of the field R_{ν_1}/m_{ν_1}, corresponding to a valuation $\overline{\nu}$ with value group Ψ and center $m_\nu\cap R/m_{\nu_1}\cap R$ in $R/m_{\nu_1}\cap R$.

If we endeavor to find the valuation algebra of $\overline{R}_1=R/(m_{\nu_1}\cap R)$ with respect to $\overline{\nu}$, we see that if we denote by $\mathcal A_\Psi(R)$ the subalgebra $\bigoplus_{\phi\in\Psi}\mathcal P_\phi(R)v^{-\phi}$ of $\mathcal A_\nu(R)$, then $\mathcal A_{\overline{\nu}}(\overline{R}_1)$ is nothing else than the image of $\mathcal A_\Psi(R)$ in $\overline{R}_1[v^\Phi]$ by the canonical map $R[v^\Phi]\to\overline{R}_1[v^\Phi]$. It is important to note that, setting $\Gamma=\nu(R\setminus\{0\})$, the semigroup $\Gamma\cap\Psi$ may be zero. This happens if and only if $m_{\nu_1}\cap R=m_\nu\cap R$; in that case the valuation $\overline{\nu}$ is trivial on $\overline{R}_1$.

A typical example is the valuation on $R=k[u_1,u_2]_{(u_1,u_2)}$ with values in $\mathbf Z^2$ ordered lexicographically determined by setting $\nu(u_1)=(1,0),\ \nu(u_2)=(1,1)$ and deciding that the valuation of a polynomial is the smallest valuation of the monomials which it contains; it is composed with the (u_1)-adic valuation, Γ is the semigroup generated by $(1,0),\ (1,1)$ and Ψ is the subgroup $0\bigoplus\mathbf Z$. Note that the two valuations have the same center in R and that for a given $\phi\in\Phi_+$ there are only finitely many ideals $\mathcal P_{\phi'}$ between $\mathcal P_{\lambda(\phi)}$ and $\mathcal P^+_{\lambda(\phi)}$.
Indeed, except for $\phi=0$, all these ideals are equal to the maximal ideal of R, which is also the ideal $m_{\nu_1}\cap R$. The image in $\overline{R}_1[v^\Phi]=k[v^\Phi]$ of $\mathcal A_\Psi(R)$ is equal to k. This is as it should be since the valuation $\overline{\nu}$ is trivial.

3.4 Comparison of graded algebras for composed valuations. The result of Proposition 3.12 suggests that we seek a closer relation between $\mathrm{gr}_\nu R$ and

$\mathrm{gr}_{\nu_1}R$ when ν is composed with ν_1. First, let us remark that with the usual notations, we have for $\phi\in\Phi_+\cup\{0\}$ the following inclusions[4]

$$\mathcal{P}^+_{\lambda(\phi)}\subset\mathcal{P}^+_\phi\subset\mathcal{P}_\phi\subset\mathcal{P}_{\lambda(\phi)},$$

so that we get a filtration of each $(R/m_{\nu_1}\cap R)$-module

$$(\mathrm{gr}_{\nu_1}R)_{\phi_1}=\frac{\mathcal{P}_{\phi_1}}{\mathcal{P}^+_{\phi_1}}$$

by the submodules $(\mathcal{P}_\phi/\mathcal{P}^+_{\lambda(\phi)})_{\lambda(\phi)=\phi_1}$. In the case where $\phi_1=0$, it is nothing else than the filtration of $\overline{R}_1=R/(m_{\nu_1}\cap R)$ by the images of the $(\mathcal{P}_\phi)_{\phi\in\Psi}$ where $\Psi=\mathrm{Ker}\lambda$, i.e., the filtration corresponding to the ideals $(\overline{\mathcal{P}}_\psi)_{\psi\in\Psi}$ associated to the residual valuation $\overline{\nu}$ on $\overline{R}_1$. Let us denote by $\overline{\mathcal{P}}(\phi_1)$ this filtration, for each $\phi_1\in\Phi_{1+}\cup\{0\}$. We have immediately

Lemma 3.16 *The natural map*

$$\mathcal{P}_\phi\to\frac{\mathcal{P}_\phi}{\mathcal{P}^+_{\lambda(\phi)}}$$

induces an isomorphism

$$\mathrm{gr}_\nu R\overset{\simeq}{\to}\bigoplus_{\phi_1\in\Phi_{1+}\cup\{0\}}\mathrm{gr}_{\overline{\mathcal{P}}(\phi_1)}(\mathrm{gr}_{\nu_1}R)_{\phi_1}.$$

It is graded in the sense that an element of degree ϕ is mapped to an element of bidegree $(\phi,\lambda(\phi))$. Moreover, for $x\in R$, we have

$$\mathrm{in}_\nu x=\mathrm{in}_{\overline{\mathcal{P}}(\nu_1(x))}\big(\mathrm{in}_{\nu_1}x\big).$$

In particular, taking $\lambda(\phi)=0$, we find that it induces a graded isomorphism

$$\bigoplus_{\psi\in\Psi_+\cup\{0\}}(\mathrm{gr}_\nu R)_\psi\overset{\simeq}{\to}\mathrm{gr}_{\overline{\nu}}\overline{R}_1,$$

which can also be deduced from subsection 3.3. Note that when R is nœtherian, and so Γ is well ordered by Proposition 3.9, the topology on $R/(m_{\nu_1}\cap R)$ corresponding to the $\overline{\nu}$-adic filtration is the same as the image by the natural surjection $R\to R/(m_{\nu_1}\cap R)$ of the ν-adic topology on R. This follows from the inclusions $\mathcal{P}_\psi(R)\supset m_{\nu_1}\cap R$ for $\psi\in\Psi$. Thus, the image of $\mathcal{P}_\psi(R)$ in $R/(m_{\nu_1}\cap R)$ is the valuation ideal $\overline{\mathcal{P}}_\psi(\overline{R})$ for $\overline{\nu}$. Note that it may be the trivial topology; more generally, if ν and ν_1 have the same center in R, the filtrations $\overline{\mathcal{P}}(\phi_1)$ are finite ([Z-S], Vol. II, Appendix 3, Corollary).
I will make considerable use of the following:

Proposition 3.17 (Zariski) *With the same notations, assume ν is a valuation on the nœtherian local domain R and that the valuation ν_1 has height one less than ν, so that $\overline{\nu}$ is of height one. Given $\phi_1\in\Phi_{1,+}$, we have, whenever $\lambda(\phi)=\phi_1$ (see Lemma 3.16)*

$$\mathcal{P}^+_{\phi_1}\subseteq\mathcal{P}_\phi\subseteq\mathcal{P}_{\phi_1}.$$

There are only finitely many ideals $\mathcal{P}_{\phi'}$ between $\mathcal{P}_{\phi_1}$ and $\mathcal{P}_\phi$. The ideals $\mathcal{P}_\phi$ between $\mathcal{P}_{\phi_1}$ and $\mathcal{P}^+_{\phi_1}$ are finite in number if the centers of ν and ν_1 are equal, and form a simple infinite sequence if the centers of ν and ν_1 are different; in this last

[4]The fact that $\lambda(\phi)\in\Phi_1$ is deemed sufficient, here and in the sequel, to indicate that the first and last ideals correspond to the valuation ν_1.

case, denoting by $\phi(\phi_1)$ the least element of $\Gamma\cap\lambda^{-1}(\phi_1)$, the $R/(m_{\nu_1}\cap R)$-modules $(\mathcal{P}_{\phi(\phi_1)+\psi}/\mathcal{P}^+_{\phi_1})_{\psi\in\Gamma\cap\Psi_+}$ are cofinal in the sequence of the $(\mathcal{P}_\phi/\mathcal{P}^+_{\phi_1})_{\lambda(\phi)=\phi_1}$, and for each $\phi\in\Gamma\cap\lambda^{-1}(\phi_1)$ there is an integer q such that $(m_\nu\cap R)^q.\mathcal{P}_{\phi_1}\subset\mathcal{P}_\phi$.

Proof Let $\Gamma\subset\Phi_+\cup\{0\}$ be the semigroup of R. Given $\phi_1\in\Gamma_1=\lambda(\Gamma)$, for any $\phi\in\Gamma$ such that $\lambda(\phi)=\phi_1$ we can write $\phi=\phi(\phi_1)+\psi$ with $\psi\in\Psi_+\cup\{0\}$, and the result follows from the fact that if $\overline{\nu}$ is not trivial, $\Gamma\cap\Psi_+$ is cofinal in Ψ_+ (see subsection 2.1), and the fact that $\overline{\nu}$ has center $(m_\nu\cap R)/(m_{\nu_1}\cap R)$ and is archimedian. See [Z-S], Appendix 3, lemma 4 and its corollary. □

Remark 3.18 In fact, whenever we have a specialization $R\subset R_\nu\subset R_{\nu_1}$ of valuations of R, the valuation ν induces, as described above, a filtration on the ring $\mathrm{gr}_{\nu_1}R$, to which we can associate as in subsection 2.1 a "filtration algebra" which determines a specialization of $\mathrm{gr}_{\nu_1}R$ to $\mathrm{gr}_\nu R$. This specialization is the same as that coming from the embedding of $\mathrm{Spec}\mathcal{A}_{\nu_1}(R)$ in $\mathrm{Spec}\mathcal{A}_\nu(R)$.

Example 3.19 Let R be a local integral domain, and $f\in R\setminus\{0\}$ be such that $R/(f)$ is again integral and $\bigcap_{n\geq0}f^nR=\{0\}$. Let us choose a valuation $\overline{\nu}$ of $R/(f)$ with group Ψ. Since each element x of $R\setminus\{0\}$ can be written in a unique way $x=f^ny$ with $y\notin(f)$, we may define a valuation ν on R with group $\mathbf{Z}\oplus\Psi$ ordered lexicographically by setting

$$\nu(f^ny)=(n,\overline{\nu}(\overline{y}))\quad\text{where } \overline{y}=y \text{ mod.}(f).$$

The map $f^ny\mapsto n\in\mathbf{N}$ defines a valuation ν_1 on R with value group $\mathbf{Z}$ and with which ν is composed. With the notations introduced above, we have $\overline{R}=R/(f)$, and

$$\mathrm{gr}_{\nu_1}R=\overline{R}[F]\quad\text{with } F=\mathrm{in}_{\nu_1}f,$$

and from the considerations above we see that because it is a polynomial ring in one variable, we have:

$$\mathrm{gr}_\nu R=(\mathrm{gr}_{\overline{\nu}}\overline{R})[F].$$

We shall see the general form of this result in subsection 4.2.

3.5 Comparison of residual valuations. In this subsection I recall some direct consequences of the basic facts on the composition of valuations found in ([A1], pp. 56-57, [V1], section 4, [Z-S]). Keeping the notations of the beginning of this section, let us consider a sequence

$$R_\nu\subset R_{\nu_1}\subset R_{\nu_2}$$

of valuation rings; it corresponds to a sequence

$$\Phi\to\Phi_1\to\Phi_2$$

of value groups; let $\Psi_2\subset\Psi_1\subset\Phi$ be the kernels of the maps $\Phi\to\Phi_1$ and $\Phi\to\Phi_2$ respectively. To this situation is associated an inclusion

$$R_\nu/(m_{\nu_2}\cap R_\nu)\subset R_{\nu_1}/(m_{\nu_2}\cap R_{\nu_1})$$

of valuation rings of the residue field k_{ν_2} of R_{ν_2}. Note that in fact $m_{\nu_2}\cap R_\nu=m_{\nu_2}$ since $m_{\nu_2}\subset R_\nu$ by the general properties of composed valuations (see [V1]), and similarly for R_{ν_1}; recall also that R_{ν_i} is the localization of R_ν at the prime m_{ν_i}. The value group of the first valuation ring is Ψ_1, and the value group of the second is Ψ_1/Ψ_2. If R_{ν_1} and R_{ν_2} are consecutive valuation rings, in the sense that there is no other valuation ring between them, the group Ψ_1/Ψ_2 is of height one.

Assuming that this is the case, given a subring $R \subset R_\nu$, we see that ν_1 induces a valuation of height at most one, with valuation ring $R_{\nu_1}/(m_{\nu_2} \cap R_{\nu_1})$, on the quotient $R/(m_{\nu_2} \cap R)$; the center of this valuation is the ideal $(m_{\nu_1} \cap R)/(m_{\nu_2} \cap R)$, which is also the kernel of the natural morphism

$$R/(m_{\nu_2} \cap R) \to R/(m_{\nu_1} \cap R).$$

More precisely, it may happen that $m_{\nu_1} \cap R = m_{\nu_2} \cap R$, and then the residual valuation is trivial on $R/(m_{\nu_2} \cap R)$.

3.6 Specialization of valuations and rational valuations. Let k be a field, and (R, m) a local k-algebra which is an integral domain. Let R_{ν_0} be a valuation ring in the fraction field of R, containing R, with residue field k_{ν_0}, and assume that the transcendence degree over k of k_{ν_0} is greater than zero. Then, if $R/(m_{\nu_0} \cap R) \neq k_{\nu_0}$, there exists a k-valuation ring V of k_{ν_0} containing $R/(m_{\nu_0} \cap R)$ and different from k_{ν_0}. If we denote by $\pi \colon R_{\nu_0} \to k_{\nu_0}$ the canonical projection, we see that $\pi^{-1}(V)$ is a valuation ring R_{ν_1} containing R and contained in R_{ν_0}. The residue field of R_{ν_1} is equal to the residue field of V, and therefore its transcendence degree over k is less than that of k_{ν_0}. We may iterate this construction as long as the residue field of R_{ν_1} is transcendental over k and $R/(m_{\nu_1} \cap R) \neq k_{\nu_1}$. When we stop, we have built a sequence

$$R \subset R_{\nu_t} \subset R_{\nu_{t-1}} \subset \cdots \subset R_{\nu_0}$$

such that $t \leq \mathrm{t}_k(\nu_0)$ and either R_{ν_t} dominates R and the residual extension is trivial, or the residue field of R_{ν_t} is algebraic over k. In this case, the ring $R/(m_{\nu_t} \cap R)$ contains k and is contained in k_{ν_t}, so it is a field. This means that $m_{\nu_t} \cap R = m$, so R_{ν_t} dominates R. So at worse we end up the specialization process with a valuation ring dominating R and an algebraic residual extension.

This is a special case of the construction found in [Z-S], Vol. 2, Chap VI, §16.

In the case where R_ν dominates R and the residual extension is trivial, one says that the valuation ν on R is *rational*. So we have:

Proposition 3.20 *If the residue field of R is algebraically closed, we can specialize any valuation ν_0 of R to a rational valuation.*

Remarks 3.21 1) Assuming that R is excellent, and so in particular a Nagata ring, we may (see [V1], Proposition 10.1), at each step in the preceding construction, choose a discrete divisorial valuation; then the transcendence degree of the residual valuation drops exactly by one at each step, while the height of the value group increases exactly by one since the value group of V is the kernel of the map $\Phi_{\nu_1} \to \Phi_{\nu_0}$. Finally we have:

$$\mathrm{h}(\nu_t) = \mathrm{h}(\nu_0) + \mathrm{t}_{k_R}(\nu_0),$$

and since $\mathrm{h}(\nu_t) \leq \mathrm{r}(\nu_t)$, in order to prove Abhyankar's inequality in this case it suffices to check that $\mathrm{r}(\nu) \leq \dim R$ when R_ν dominates R, but this is readily done, at least in the algebraic or formal case, if we present the fraction field K of R as an algebraic extension of a field of rational functions of transcendence degree $\dim R$ over k_R, since extending valuations to algebraic extensions does not affect the rational rank. This suggests a proof of Abhyankar's inequality, not very different from that described in [V1].

3.7 Monomial valuations. This section begins to explain in what sense the liftings $(\xi_i)_{i\in I}$ of generators $(\overline{\xi}_i)_{i\in I}$ of the graded $R/(m_\nu\cap R)$-algebra $\mathrm{gr}_\nu R$ form a system of coordinates "adapted" to the valuation. The idea is that, at least in the case of trivial residual extension, valuations which are monomial in a certain system of coordinates can be uniformized by a single toric modification of the ambiant space in which this system of coordinates embeds our singularity.

Given a valuation ν of a local or graded ring R, we generalize and modify the definition given in [S2] for a regular local ring as follows

Definition 3.22 1) The valuation ν of the local (or graded) integral domain R is *algebraically monomial* with respect to a system of generators $(\xi_i)_{i\in I}$ of the center $m_\nu\cap R$ of ν in R if $(\xi_i)_{i\in I}$ is a generating sequence for ν in the sense (see [S2]) that any element $x\in R$ can be written as a finite sum

$$x=\sum_{\nu(\xi^\gamma)\geq\nu(x)} c_\gamma\xi^\gamma$$

with $\gamma\in I^b,$ for some $b\in\mathbf{N},\quad c_\gamma\in R.$
2) If the valuation ν is rational and R is complete with respect to the ν-adic topology and has a field of representatives, we say that ν is *analytically monomial* with respect to $(\xi_i)_{i\in I}$ if the same holds with "finite sum" replaced by "ν-adically convergent series" and "$c_\gamma\in R$" by "$c_\gamma\in k$". In particular, in both cases, $\nu(x)=\min_{\{\gamma/c_\gamma\neq 0\}}(\nu(\xi^\gamma))$.

Remarks 3.23 1) To say that ν is algebraically monomial with respect to the ξ_i is equivalent, when R is nœtherian, to asking that for each $\phi\in\Phi_+$ the ideal $\mathcal{P}_\phi(R)$ is generated by monomials in the $(\xi_i)_{i\in I}$, and implies that $\mathcal{A}_\nu(R)$ is generated as a $R[v^{\Phi_+}]$-algebra by the $\xi_i v^{-\nu(\xi_i)}$ and the initial forms $(\overline{\xi}_i)_{i\in I}$ in $\mathrm{gr}_\nu R$ of the elements $(\xi_i)_{i\in I}$ generate it as $R/(m_\nu\cap R)$-algebra. Analytic monomiality implies that they generate the k-algebra $\mathrm{gr}_\nu R$. Again if R is nœtherian, since the ideal generated by the exponents of a series is finitely generated, analytic monomiality implies algebraic monomiality. We shall later meet monomial valuations in non nœtherian rings. In particular, if the valuation ν is algebraically (resp. analytically) monomial with respect to a *finite* set of generators of $m_\nu\cap R$ (resp. the maximal ideal m), the $R/(m_\nu\cap R)$-algebra (resp. k-algebra) $\mathrm{gr}_\nu R$ is finitely generated. Since the expansion $x=\sum_{\nu(\xi^\gamma)\geq\nu(x)}c_\gamma\xi^\gamma$ is not unique in general, one must stress the fact that the definition requires the existence of one expansion with the stated properties.
2) If Φ is a totally ordered group of finite rational rank, if R a Φ_+-graded k-algebra generated by the elements ξ_i and we assume that the valuation coincides with the grading and that each homogeneous component of R is a finite-dimensional vector space, the valuation ν is monomial with respect to the ξ_i. Indeed, it suffices to check the definition for homogeneous elements. Every homogeneous element $\overline{x}_\phi$ can be written as a finite sum of monomials of the same degree $\overline{x}_\phi=\sum_\gamma c_\gamma\xi^\gamma$ with $c_\gamma\in R/\mathbf{p}$ and $\nu_{\mathrm{gr}}(c_\gamma\xi^\gamma)=\phi$, as required.
3) If R is a one-dimensional excellent henselian equicharacteristic local domain, and ν its unique non-trivial valuation, there are, as we shall see (subsection 6.2 and Proposition 5.51), finite systems of generators of the maximal ideal of R with respect to which ν is analytically monomial; *in this case, the valuation is also algebraically monomial with respect to any such system of generators in the strong*

sense given by the theory of semiroots of [PP]*, where the coefficients in R of the monomials ξ^α depend only on one variable.*

The argument in the case where R is complete is as follows: Let k be the residue field of R and let $(\xi_1,\ldots,\xi_{g+1})$ be elements of R whose images $(\overline{\xi}_i)$ constitute a system of generators of the graded k-algebra $\mathrm{gr}_\nu R$ (see subsection 6.2 below). Let us fix a field of representatives $k\subset R$. Then, by the usual method of successive approximations, using the fact that the ν-adic and m-adic topologies coincide in this case, we have a surjection of k-algebras

$$k[[u_1,\ldots,u_{g+1}]]\to R$$

mapping each u_i to ξ_i and inducing, by passing to the graded rings with respect to ν and the corresponding monomial order on $k[[u_1,\ldots,u_{g+1}]]$, the surjective map

$$k[U_1,\ldots,U_{g+1}]\to \mathrm{gr}_\nu R$$

mapping U_i to $\overline{\xi}_i$. Then each element x of R can be written

$$(**)\qquad x=\textstyle\sum_{\alpha/\nu(\xi^\alpha\geq\nu(x)} c_\alpha\xi^\alpha\quad \text{with } c_\alpha\in k,$$

which shows that $(\xi_1,\ldots,\xi_{g+1})$ constitute a system of generators of the maximal ideal of R with respect to which the valuation ν is analyticaly monomial. For each index $2\leq i\leq g+1$, in the one-dimensional subring $k[[\xi_1,\xi_i]]\subset R$ with the valuation induced by ν, the minimality of the valuation of ξ_1 implies that ξ_i is integral over the ideal $\xi_1 k[[\xi_1,\xi_i]]$, so that we have a relation $\xi_i^{s_i}+b_1^{(i)}(\xi_1,\xi_i)\xi_i^{s_i-1}+\cdots+b_{s_i}^{(i)}(\xi_1,\xi_i)=0$ with $b_\ell^{(i)}(\xi_1,\xi_i)\in(\xi_1^\ell)k[[\xi_1,\xi_i]]$. Using the Weierstrass preparation theorem, we see that we have relations

$$\xi_i^{s_i}+a_1^{(i)}\xi_i^{s_i-1}+\cdots+a_{s_i}^{(i)}=0\quad\text{with}\quad a_\ell^{(i)}\in(\xi_1)^\ell k[[\xi_1]].$$

Using this relation to eliminate from the representation $(**)$ all terms of order $\geq s_i$ in ξ_i for $i\geq 2$, we see that we can now write

$$x=\sum_{\text{finite}} a_\alpha(\xi_1)\xi_2^{\alpha_2}\ldots\xi_{g+1}^{\alpha_{g+1}},$$

where $a_\alpha(\xi_1)\in k[[\xi_1]]$ and the sum involves only terms such that

$$\nu(a_\alpha)+\sum_{i=2}^{g+1}\alpha_i\nu(\xi_i)\geq\nu(x).$$

Writing $a_\alpha(\xi_1)=\xi_1^{\alpha_1}v_\alpha(\xi_1)$ with $v_\alpha(\xi_1)$ a unit in $k[[\xi_1]]$, this shows that the $(\xi_i)_{1\leq i\leq g+1}$ form a system of generators for the maximal ideal of R with respect to which ν is algebraically monomial in the strong sense. The proof in the henselian case is similar. For a different proof in the case of plane branches, see [PP].

We shall see in Proposition 5.48 that if ν is a rational valuation, after extending it to a suitable completion $\hat R^{(\nu)}$ one always has a (possibly infinite) system of generators of the maximal ideal of $\hat R^{(\nu)}$ with respect to which this extension of ν is analytically monomial. One may then, by analogy with the curve case, choose a system of parameters $(\xi_1,\ldots,\xi_s)$ with $s=\dim\hat R^{(\nu)}$ such that the integral closure of the ideal $(\xi_1,\ldots,\xi_s)\hat R^{(\nu)}$ is the maximal ideal, and examine under which conditions the valuation is algebraically monomial in this stronger sense with respect to this system of generators and this system of parameters, or becomes so after a birational extension.

Example 3.24 1) Let (S, m) be a regular local ring and ν be a valuation of S having the property of the above definition with respect to a system of regular parameters $(\xi_0, \ldots, \xi_n)$ of S. If we denote by $\mathcal{R}_\phi$ the ideal of R generated by the monomials in $(\xi_0, \ldots, \xi_n)$ of valuation $\geq \phi$, we have $x \in \mathcal{R}_{\nu(x)}$ and therefore ν is monomial in the sense of [S2]. In this case $\mathrm{gr}_\nu R$ is a polynomial ring in $n+1$ variables, and this is clearly the *only case* where $\mathrm{gr}_\nu R$ is regular, that is, a polynomial ring.
2) If R is the algebra of a plane branch C, and ν its unique valuation, then ν is monomial with respect to two well chosen generators of the maximal ideal of R if and only if the semigroup $\nu(R \setminus \{0\})$ of C has two generators. In characteristic zero, this means that C has only one characteristic pair (see [G-T], [T1]).

Proposition 3.25 *Let R be an integral domain and ν a valuation on R, i.e., $R \subset R_\nu$. Set $\mathbf{p} = m_\nu \cap R$. The valuation ν_{gr} on $\mathrm{gr}_\nu R$ is monomial with respect to any system of homogeneous generators $(\xi_i)_{i \in I}$ of the graded $R/\mathbf{p}$-algebra $\mathrm{gr}_\nu R$.*

Proof by the definition of ν_{gr}, this follows from one of the remarks made above. □

4 The toric structure of the graded algebras of valuation rings and valued nœtherian rings

In this section I show that the graded ring $\mathrm{gr}_\nu R$ is, when ν is a rational valuation of finite rational rank for the local integral domain R, a quotient of a polynomial ring in countably many indeterminates over the residue field by a binomial ideal, and that $\mathrm{gr}_\nu R_\nu$ is the direct limit of a nested sequence of polynomial subalgebras in $\mathrm{r}(\nu)$ variables over the residue field k_ν of the valuation. This means in particular that it is "regular", and hence can be deemed to be a resolution of singularities of the algebra $\mathrm{gr}_\nu R$, if we forget for a moment that $\mathrm{gr}_\nu R_\nu$ is not an extension of finite type of $\mathrm{gr}_\nu R$, and remember only that they have the same field of fractions. It means also that the binomial equations defining $\mathrm{gr}_\nu R_\nu$ are of a very special form.

It turns out that the result for $\mathrm{gr}_\nu R_\nu$ is equivalent to its counterpart for $k[t^{\Phi_+}]$ where k is any field.

The result for $k[t^{\Phi_+}]$ in turn follows from the stronger statement that the semigroup Φ_+ is a direct limit of a system of linear maps between finitely generated free monoids, which will give us toric maps between the corresponding polynomial algebras.

We shall see that this is true, even with finitely generated free *submonoids*, whenever Φ is of finite height.

4.1 The graded algebra of a rational valuation is essentially toric. Here and in the sequel, unless otherwise specified, the notation V^m stands for a monomial $V_{j_1}^{m_{j_1}} \ldots V_{j_s}^{m_{j_s}}$, with $m_{j_i} \in \mathbf{N} \cup \{0\}$.

Proposition 4.1 *Assume that a valuation ring R_ν birationally dominates a local ring R and that the residual extension $k_R = R/m \to R_\nu/m_\nu = k_\nu$ is trivial. Then each homogeneous component of the graded k_R-algebra $\mathrm{gr}_\nu R$ is a vector space of dimension ≤ 1 over k_R.*

Proof Indeed, by definition of the ideals $\mathcal{P}_\phi(R)$, we have the graded inclusion

$$\mathrm{gr}_\nu R \subset \mathrm{gr}_\nu R_\nu$$

of k_R-algebras and the result is true for the second algebra since the initial forms of two elements of R_ν with the same valuation differ by a factor which is the initial form of a unit, i.e., an element of k_ν^*; each homogeneous component of $\mathrm{gr}_\nu R_\nu$ is a k_ν-vector space of dimension one. □

From Proposition 4.1 above follows by a direct generalization of a result of Korkina ([Ko], see also [E-S]) the

Proposition 4.2 *In the situation of Proposition 4.1, assuming that* $\mathrm{r}(\nu)$ *is finite, the graded* k_R*-algebra* $\mathrm{gr}_\nu R$ *is the quotient of a polynomial ring in countably many indeterminates* $k_R[(U_j)_{j\in J}]$ *by a binomial prime ideal.*

Proof By assumption, Φ is of finite rational rank, and therefore countable. Let $(\overline{\xi}_j)_{j\in J}$ be countable system of nonzero homogeneous generators for the k_R-algebra $\mathrm{gr}_\nu R$ (note that the number of these generators may be finite); consider the kernel K of the surjective homomorphism

$$k_R[(U_j)_{j\in J}] \to \mathrm{gr}_\nu R$$

determined by $U_j \mapsto \overline{\xi}_j$. It is a homogeneous ideal for the Φ_+-grading giving to each U_j the degree of $\overline{\xi}_j$. Let $P = \sum c_m U^m$ be a homogeneous generator of K, say of degree ϕ. All the monomials occurring in P have their images in the one dimensional k-vector space $(\mathrm{gr}_\nu R)_\phi$ and the sum of these images is zero. By induction on the number of monomials of P, we see that P is a sum of binomial terms $c_{m,m'}(U^m - \lambda_{m,m'}U^{m'})$ with $c_{m,m'} \in k,\ \ \lambda_{m,m'} \in k^*$.
The ideal K is generated by binomials of the form $U^m - \lambda_{m,m'}U^{m'}$ and there is one such for each (m,m') such that U^m and $U^{m'}$ have the same degree. By construction, all the $\lambda_{m,m'}$ are $\neq 0$. Since $\mathrm{gr}_\nu R$ is integral, the ideal K is prime. □

Corollary 4.3 *For any valuation ring* R_ν *with countable value group, the graded algebra* $\mathrm{gr}_\nu R_\nu$ *is a quotient of a polynomial algebra* $k[(U_i)_{i\in I}]$ *in countably many variables over* $k_\nu = R_\nu/m_\nu$ *by a prime ideal generated by binomials* $U^m - \lambda_{mn}U^n$ *with* $\lambda_{mn} \in k^*$.

Remark 4.4 This corollary is used in the proof of Proposition 4.15, but does not tell us anything precise about the form of the binomial equations.

Assume now that R is nœtherian.

Let $\nu(R\setminus\{0\}) = \Gamma \subset \Phi_+ \cup \{0\}$ be the semigroup of R. Since R is nœtherian, the semigroup Γ is well ordered (Proposition 3.9), and we have a countable system of generators, the valuations of the generators $\overline{\xi}_i$ of $\mathrm{gr}_\nu R$, i.e., $\gamma_i = \nu\mathrm{gr}(\overline{\xi}_i)$:

$$\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_j, \ldots \rangle \quad \text{with } \gamma_j \in \Phi_+,\ j \text{ an ordinal } < \omega^{\mathrm{h}_R(\nu)},\ \ \gamma_j < \gamma_{j+1}.$$

(Corollary 3.10).

To this situation is associated the map of groups

$$\mathbf{Z}^{\mathbf{N}} \to \Phi \qquad \text{determined by} \quad \sum \lambda_j e_j \mapsto \sum \lambda_j \gamma_j,$$

where e_j is the j-th basis vector, such that the image of $(\mathbf{Z}_{\geq 0})^{\mathbf{N}}$ is Γ. It defines a map of k_ν-algebras

$$k_\nu[(U_j)_{j\in J}] \to k_\nu[t^{\Phi_+}] \quad \text{determined by} \quad U_j \mapsto t^{\gamma_j}.$$

The image of this map is the semigroup algebra $k_\nu[t^\Gamma] \subset k_\nu[t^{\Phi_+}]$, and its kernel is the binomial ideal obtained from that of of Proposition 4.2 by setting all $\lambda_{m,m'}$ equal to 1.

Remark 4.5 We note that since the γ_i are assumed to be a minimal system of generators, none of the binomial equations is of the form

$$U_i - \lambda_i U^{m(i)} = 0.$$

Let (R, ν) be a valued nœtherian ring; assume that the center of ν in R is the maximal ideal m; set $k = R/m$ and let

$$k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R \quad \text{determined by} \quad U_i \mapsto \overline{\xi}_i$$

be a presentation of the k-algebra $\mathrm{gr}_\nu R$ corresponding to a minimal system of generators of the semigroup Γ of ν. Let $(\xi_i) \in R$ be a system of representatives of the $\overline{\xi}_i$. Let R_ν be the valuation ring of ν and consider all the images λ_{mn} in R_ν/m_ν of the ratios of valuation zero ξ^{m-n} of monomials in the ξ_i. For each λ_{mn} which is algebraic over the field k, consider the minimal polynomial $P_{mn} \in k[T]$ of λ_{mn}. Setting $T = \frac{U^m}{U^n}$, after multiplication by a suitable power of U^n, P_{mn} may be considered as the image in $k[(U_i)_{i\in I}]$ of an irreducible polynomial. The polynomials P_{mn} belong to the kernel I of the surjective map $k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R$. After extension of scalars to $k_\nu = R_\nu/\mathrm{m}_\nu$, the ideal $Ik_\nu[(U_i)_{i\in I}]$ is contained, in view of Proposition 4.2 in the ideal of $R_\nu/\mathrm{m}_\nu[(U_i)_{i\in I}]$ generated by all the binomials $U^m - \lambda_{mn}U^n$, with $\lambda_{mn} \in k_\nu$ and $\nu(\xi^{m-n}) = 0$. By the faithful flatness of the extension $k[(U_i)_{i\in I}] \to k_\nu[(U_i)_{i\in I}]$, the ideal I is contained in the intersection of that ideal with $k[(U_i)_{i\in I}]$, which is the ideal generated by all the polynomials $P_{mn}(U)$ for λ_{mn} algebraic over k; the ideal I is equal to the ideal generated by the $P_{mn}(U)$. For an arbitrary valuation, say with center $\mathbf{p}$, we may apply this to the extension of ν to the localization $R_\mathbf{p}$ with residue field $\kappa(\mathbf{p}) = R_\mathbf{p}/\mathbf{p}R_\mathbf{p}$:

Corollary 4.6 *Let (R, ν) be a valued nœtherian ring; let $\mathbf{p}$ be the center of ν in R and extend ν to the localization $R_\mathbf{p}$. The kernel I of the map of $\kappa(\mathbf{p})$-algebras $\kappa(\mathbf{p})[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R_\mathbf{p}$ determined by $U_i \mapsto \overline{\xi}_i$ is generated by the polynomials P_{mn} obtained by clearing denominators in the minimal polynomials over $\kappa(\mathbf{p})$ of the images in R_ν/m_ν of the ratios ξ^{m-n} of valuation 0, viewed as polynomials in $\frac{U^m}{U^n}$.*

Proposition 4.7 *Let $(U^m - \lambda_{mn}U^n)_{(m,n)\in E}$ be the defining binomial ideal in $k[(U_i)_{i\in I}]$ for the graded ring $\mathrm{gr}_\nu R$ of a rational valuation of a nœtherian ring. Let $(\Lambda_{mn})_{(m,n)\in E}]$ be a set of indeterminates and let Λ denote the multiplicative subset of $k_\nu[(\Lambda_{mn})_{(m,n)\in E}]$ generated by the Λ_{mn}. The natural map of k_ν-algebras*

$$\Lambda^{-1}k_\nu[(\Lambda_{mn})_{(m,n)\in E}] \to \Lambda^{-1}k_\nu[(\Lambda_{mn})_{(m,n)\in E}][(U_j)_{j\in J}]/(U^m - \Lambda_{mn}U^n)_{(m,n)\in E}$$

is faithfully flat and corresponds to a flat family of schemes having one fiber equal to $\mathrm{Spec}k_\nu[t^\Gamma]$ *and another to* $\mathrm{Specgr}_\nu R$.

Indeed, if the second algebra is graded by giving to U_j the degree γ_j, then by definition of the binomial relations, each of its homogeneous components becomes a free module of rank one over $\Lambda^{-1}k_\nu[(\Lambda_{mn})_{(m,n)\in E}]$.

This family parametrized by the $\Lambda_{m,n}$ has simultaneous resolution along the subspace defined by the vanishing of the $(U_i)_{i\in I}$ in any reasonable sense; it is locally analytically trivial along this subspace. We shall see below that it has simultaneous embedded resolution by a toric map; it follows from the results of subsection 6.1.

4.2 More on the structure of $\mathrm{gr}_\nu R$. Let Φ be a group of height h associated to a valuation of a local integral domain R centered at the maximal ideal m of R; set $k = R/m$. Let

$$(0) = \Psi_h \subset \Psi_{h-1} \subset \cdots \subset \Psi_1 \subset \Psi_0 = \Phi$$

be the sequence of isolated subgroups of Φ (including the trivial ones). Let Γ be the semigroup of ν on R (note that it may be equal to Φ_+, if $R = R_\nu$) and set $\Gamma_i = \Gamma \cap \Psi_i$. Considering the valuation ν_1 of height $h-1$ with center $\mathbf{p}_1$ with which ν is composed, we have seen in subsection 3.4 that we could identify $\mathrm{gr}_{\overline{\nu}}\overline{R}_1$, where $\overline{R}_1 = R/\mathbf{p}_1$, with the subalgebra $\bigoplus_{\psi\in\Psi_{h-1+}\cup\{0\}}(\mathrm{gr}_\nu R)_\psi$ of $\mathrm{gr}_\nu R$. If we distinguish between the generators $(U_i)_{i\in I_1}$, where $I_1 \subset I$, of the k-algebra $\mathrm{gr}_\nu R$ whose degree lies in Ψ_{h-1} and those whose degree lies in $\Phi \setminus \Psi_{h-1}$, we have:

Proposition 4.8 *If $R \subset R_\nu \subset R_{\nu_1}$, where R_ν dominates the local ring R without residual extension and ν_1 is of height one less than ν, the natural map*

$$\mathrm{gr}_{\overline{\nu}}\overline{R}_1[(U_i)_{i\in I\setminus I_1}] \to \mathrm{gr}_\nu R$$

is surjective and its kernel is generated by the images in $\mathrm{gr}_{\overline{\nu}}\overline{R}_1[(U_i)_{i\in I\setminus I_1}]$ of those binomials $U^n - \lambda_{mn}U^n$ which appear in Proposition 4.2 and which involve at least one variable U_i with $i \in I \setminus I_1$.

Proof The subalgebra $\mathrm{gr}_{\overline{\nu}}\overline{R}_1$ contains all the generators of the k-algebra $\mathrm{gr}_\nu R$ whose degree is in I_1; we need to add only the generators U_i whose degree lies in $I \setminus I_1$, and the relations between them are those indicated. □

Corollary 4.9 *If R_ν dominates R without residual extension, setting $I_t = \{i \in I/\mathrm{deg}U_i \in \Psi_{h-t} \setminus \Psi_{h-t+1}\}$, we can write*

$$\mathrm{gr}_\nu R = \Big(\cdots\Big(k[(U_i)_{i\in I_1}]/(U^n-\lambda_{mn}U^n)\Big)[(U_i)_{i\in I_2}]/(U^n-\lambda_{mn}U^n)\Big)\cdots\Big)[(U_i)_{i\in I_h}]/(U^n-\lambda_{mn}U^n),$$

where at each step t the binomials which are written involve only variables U_i with degrees in Ψ_{h-t} and must involve one which has degree in I_t. Each monomial of such binomials must then involve variables U_i with $i \in I_t$.

Proof The first part is only the iteration of the preceding Proposition. The second part follows from the fact that there can be no linear expression of an element of I_t in terms of those of Ψ_{h-t+1}. □

4.3 $\mathrm{gr}_\nu R_\nu$ as a limit of polynomial algebras. In this subsection, I show that $\mathrm{gr}_\nu R_\nu$ is the union of a nested sequence of polynomial subalgebras in $\mathrm{r}(\Phi)$ variables over k_ν, generated by homogeneous elements, the inclusions sending variables to terms, i.e., monomials multiplied by non zero constants. Technically this is essentially Perron's algorithm. This result, however, may be considered as a graded version of (non embedded) local uniformization; each finitely generated k_ν-subalgebra of $\mathrm{gr}_\nu R_\nu$ is contained in a polynomial subalgebra in $\mathrm{r}(\Phi)$ variables. However this inclusion is birational only in special cases, for example if our subalgebra contains a finitely generated $\mathrm{gr}_\nu R$ with $R \subset R_\nu$ birational and residually rational.

I first prove the corresponding result for semigroup algebras $k[t^{\Phi_+}]$, and I begin with an example: the case when $\Phi = \mathbf{Z}^d$ with the lexicographic order.

Indeed, consider for each $(d-1)$-uple $(r_1, r_2, \dots, r_{d-1})$ of positive integers the map

$$e_{r_1,r_2,\dots,r_{d-1}}: \mathbf{N}^d \to \mathbf{Z}^d$$

defined by

$$e_{r_1,r_2,\dots,r_{d-1}}(i_1,\dots,i_d) = (i_1, i_2 - r_1 i_1, \dots, i_d - r_{d-1} i_{d-1});$$

its image is in the semigroup $\mathbf{Z}^d_+$ of positive elements, and it is injective. It defines an injective map of semigroup algebras

$$e_{r_1,r_2,\dots,r_{d-1}}: k[z_1,\dots,z_d] \to k[t^{\mathbf{Z}^d_+}],$$

and clearly this last algebra is the union of these subalgebras as $r_1, r_2, \dots, r_{d-1}$ vary. Note that whenever $(r_1, r_2, \dots, r_{d-1}) \le (s_1, s_2, \dots, s_{d-1})$ in the sense that $r_j \le s_j$, $1 \le j \le d-1$, the map $e_{r_1,r_2,\dots,r_{d-1}}$ factors through a map of algebras $e_{s_1-r_1,s_2-r_2,\dots,s_{d-1}-r_{d-1}}: k[z_1,\dots,z_d] \to k[w_1,\dots,w_d]$ which is toric, i.e., given by monomials, and that $k[t^{\mathbf{Z}^d_+}]$ is the direct limit of this system of toric maps.

When Φ is a subgroup of $\mathbf{R}$, consisting of numbers of the form $\sum_{i=1}^m a_i\tau_i$ with given $\tau_i > 0$ and $a_i \in \mathbf{Z}$, it is also true that $k[t^{\Phi_+}]$ is a direct limit of polynomial subalgebras over k. This is again because the half space in $\mathbf{R}^m$ defined by $\sum_{i=1}^m a_i\tau_i \ge 0$ can be approximated by regular rational simplicial cones (generated by integral vectors which form a basis of the integral lattice).

More precisely, let ν be a valuation of height one, i.e., with archimedian value group $\Phi \subset \mathbf{R}$ (see [A1], p.46, [Z-S], Vol. II). Assume first that Φ is generated by m rationally independent real numbers $\tau_1, \dots, \tau_m$, which we may assume to be positive. I use the Perron algorithm as expounded in ([Z1], B. I, p. 861; see also [H-P], Chap. XVIII, §5), but with a somewhat different interpretation. The algorithm consists in writing

$$\tau_1 = \tau_m^{(1)}, \tau_2 = \tau_1^{(1)} + a_2^{(0)}\tau_m^{(1)}, \dots, \tau_m = \tau_{m-1}^{(1)} + a_m^{(0)}\tau_m^{(1)},$$

where

$$a_j^{(0)} = [\tau_j/\tau_1], \quad j = 2, \dots, m,$$

and repeating this operation after replacing $(\tau_1, \dots, \tau_m)$ by $(\tau_1^{(1)}, \dots, \tau_m^{(1)})$, and so on. After h steps, one has written

$$\tau_i = A_i^{(h)}\tau_1^{(h)} + \cdots + A_i^{(h+m-1)}\tau_m^{(h)}$$

or, if we denote by w the (weight) vector $(\tau_1, \dots, \tau_m) \in \mathbf{R}^m$ and by $A^{(h)}$ the vector $(A_1^{(h)}, \dots, A_m^{(h)})$,

$$w = \tau_1^{(h)}A^{(h)} + \tau_2^{(h)}A^{(h+1)} + \cdots + \tau_m^{(h)}A^{(h+m-1)}$$

where the $\tau_j^{(h)}$ are positive, the coefficients $A_i^{(j)}$ are non negative integers, and the matrix of the vectors

$$A^{(h)}, A^{(h+1)}, \dots, A^{(h+m-1)}$$

has determinant $(-1)^{h(m-1)}$. Moreover, as h grows the directions in $\mathbf{P}^{m-1}(\mathbf{R})$ of the vectors $A^{(h)}$ tend to the direction of w. So we have a sequence of vectors $A^{(h)}$ with positive integral coordinates whose directions in $\mathbf{P}^{m-1}(\mathbf{R})$ spiral to the direction of w and such that any consecutive m of them as above form a basis of the integral lattice such that w is contained in the convex cone $\sigma^{(h)} = \langle A^{(h)}, A^{(h+1)}, \dots, A^{(h+m-1)} \rangle$ which they generate. The convex dual $\check{\sigma}^{(h)}$ of $\sigma^{(h)}$ (see [Cox], §2, [E], V, 2, p. 149) is contained in the half space $\sum_{i=1}^m a_i\tau_i \ge 0$, the integral points of which form the semigroup Φ_+. The algebra of the semigroup $\check{\sigma}^{(h)} \cap \mathbf{Z}^m$ is a polynomial algebra $k[x_1^{(h)}, \ldots, x_m^{(h)}]$ (*loc.cit.*, VI,2) contained in $k[t^{\Phi_+}]$, and since by assumption there are no integral points on the hyperplane $\sum_{i=1}^m a_i\tau_i = 0$ except the origin, the semigroup Φ_+ is the union of the $\check{\sigma}^{(h)} \cap \mathbf{Z}^m$ as $h \to \infty$. This proves that $k[t^{\Phi_+}]$ is the union, or direct limit, of these polynomial subalgebras.

Note that by construction we have for each $h \geq 1$ the equality

$$A^{(h)} - A^{(h+m)} + a_2^{(h)} A^{(h+1)} + \cdots + a_m^{(h)} A^{(h+m-1)} = 0,$$

which shows, since the $a_j^{(h)} = [\tau_j^{(h)}/\tau_1^{(h)}]$ are non-negative, that we have

$$A^{(h+m)} \in \sigma^{(h)} = \langle A^{(h)}, \ldots, A^{(h+m-1)} \rangle$$

and therefore $\sigma^{(h+1)} \subset \sigma^{(h)}$, that is $\check{\sigma}^{(h)} \subset \check{\sigma}^{(h+1)}$ and

$$k[x_1^{(h)}, \ldots, x_m^{(h)}] \subset k[x_1^{(h+1)}, \ldots, x_m^{(h+1)}],$$

so that our direct system is in fact a nested sequence of polynomial subalgebras. The morphisms between these polynomial algebras correspond by duality to the expressions of the $A^{(h+m)}$ as linear combinations with non negative integral coefficients of $(A^{(h)}, \ldots, A^{(h+m-1)})$ and are therefore monomial as announced.

If we now consider a group with one more generator $\tau_{m+1} > 0$ which is rationally dependent on $\tau_1, \ldots, \tau_m$, Zariski shows in ([Z1], B. I, p. 862) that the new weight vector $w = (\tau_1, \ldots, \tau_m, \tau_{m+1}) \in \mathbf{R}^{m+1}$ is contained in a rational simplicial cone $\sigma \subset \mathbf{R}^{m+1}$ generated by m integral vectors $v_1, \ldots, v_m$ of the first quadrant forming part of a basis of the integral lattice. Indeed w is contained in a unique rational hyperplane. The dual cone $\check{\sigma} \subset \check{\mathbf{R}}^{m+1}$ is the product of an m dimensional strictly convex cone generated by vectors $e_1, \ldots, e_m$ by a 1-dimensional vector space (see [E], V, 2), generated by a primitive integral vector e_{m+1}, which is the dual of the rational hyperplane containing w. The vectors $e_1, \ldots, e_{m+1}$ are a basis of the integral lattice, and correspond to the variables $x_1, \ldots, x_{m+1}$ generating a polynomial ring. Note that the map $f : \mathbf{Z}^{m+1} \to \mathbf{R}$ defined by $(a_1, \ldots, a_{m+1}) \mapsto \sum_{i=1}^{m+1} a_i\tau_i$ is no longer injective; the primitive vector e_{m+1} corresponding to the variable x_{m+1} is in the kernel. Let us set $\widetilde{\Phi_+} = f^{-1}(\Phi_+ \cup \{0\})$. By refining as above by the Perron algorithm for w inside the linear span $< \sigma >$ of σ, starting with the coordinates of w in σ, we find a sequence of regular simplicial cones $\sigma^{(h)} \subset \sigma$ whose duals $\check{\sigma}^{(h)} \subset \widetilde{\Phi_+}$ correspond ([E], VI, Th. 2.12) to algebras of the form $k[x_1^{(h)}, \ldots, x_m^{(h)}, x_{m+1}^{\pm 1}] \subset k[t^{\widetilde{\Phi_+}}]$. The free semigroups $\check{\sigma}^{(h)} \cap \mathbf{Z}^{m+1}$ fill up $\widetilde{\Phi_+}$ as $h \to \infty$ since the only rational points of the hyperplane $\sum_{i=1}^{m+1} a_i\tau_i = 0$ are on the dual of the hyperplane containing w, which is contained in all the $\check{\sigma}^{(h)}$. So the direct limit of the images of the maps $k[t^f] \colon k[x_1^{(h)}, \ldots, x_m^{(h)}, x_{m+1}^{\pm 1}] \to k[t^{\Phi_+}]$ is $k[t^{\Phi_+}]$. But these images are isomorphic to $k[x_1^{(h)}, \ldots, x_m^{(h)}, x_{m+1}^{\pm 1}]/(x_{m+1} - 1)$ so that they are again polynomial rings $k[x_1^{(h)}, \ldots, x_m^{(h)}]$. If we have more generators rationally dependent on $\tau_1, \ldots, \tau_m$, we can repeat the argument after taking as new generators the coordinates of the weight vector with respect to the m primitive vectors of σ. In both examples, the fact that k is a field plays no role, so we have proved:

Lemma 4.10 *Let Φ be a totally ordered finitely generated group of height one (i.e., archimedian), or $\mathbf{Z}^d$ with the lexicographic order. For any commutative ring A the semigroup algebra $A[t^{\Phi_+}]$ of Φ_+ with coefficients in A is the direct limit of* *a direct system of graded subalgebras which are polynomial algebras $A[x_1, \dots, x_m]$ over A with $m = \mathrm{r}(\Phi)$.*

In addition, the maps between these algebras are toric maps, i.e., each variable of one is sent to a monomial in the variables of the other, and there is a cofinal subsystem which is a chain of nested subalgebras.

Remarks 4.11 1) In both cases, the smooth subalgebras are produced by an algorithm.
2) The result, especially in view of its algorithmic nature, is much more useful than its consequence that $\mathrm{Spec}A[t^{\Phi_+}]$ is a pro-object in the category of smooth affine toric schemes over $\mathrm{Spec}A$ with toric maps.
3) The proof also shows that the semigroup algebra of the semigroup

$$\widetilde{\Phi_+} = \{(a_1, \dots, a_m, a_{m+1}, \dots, a_{m+r}) \in \mathbf{Z}^{m+r} \mid \sum_{i=1}^{m+r} a_i\tau_i \geq 0\},$$

where $\tau_1, \dots, \tau_m$ are rationally independent and the others are rationally dependent upon them, is a toric direct limit of toric subalgebras of the form

$$A[x_1, \dots, x_m, x_{m+1}^{\pm 1}, \dots, x_{m+r}^{\pm 1}].$$

4) The result for subgroups of $\mathbf{R}$ holds without the finiteness assumption provided the rational rank is finite, since any abelian group is a direct limit of its subgroups of finite type. If we allow m to vary, even that last assumption is unnecessary.

Let now Φ be a totally ordered group of finite height $h > 1$. We have a surjective monotone non-decreasing map $\lambda\colon \Phi \to \Phi_1$ where Φ_1 is of height $h-1$, and the kernel Ψ of λ is of height 1. By induction on the height we may assume that Φ_{1+} is the union of sub-semigroups isomorphic to $\mathbf{N}^m$, and we know from the lemma above that the same is true for Ψ_+.

Let us denote the free semigroups that fill Φ_{1+} by F_i, and let $\tilde{F_i} \subset \Phi_+$ be the subsemigroup generated by elements $e_1, \dots, e_{r_i}$ which lift to $\Phi_+ \setminus \Psi$ the generators of F_i. Similarly let us denote by $G_j \subset \Psi_+$ free semigroups which fill Ψ_+, generated say by $f_1, \dots, f_{s_j}$. Note that for $\phi \in \Phi_+ \setminus \Psi,\ \psi \in \Psi,\ \phi + \psi \in \Phi_+$, and consider for r_is_j-tuples $n = (n_{s,t}, 1 \leq s \leq r_i,\ 1 \leq t \leq s_j)$ of non negative integers, the free semigroups $\tilde{F_i}(n) \subset \Phi_+ \cup \{0\}$ generated by $e_1 - \sum_t n_{1t}f_t, \dots, e_{r_i} - \sum_t n_{r_it}f_t$. Let us check that the direct sums of free semigroups $\tilde{F_i}(n) \oplus G_j$ fill up Φ_+; the proof generalizes that given in the case of the lexicographic $\mathbf{Z}^d$. Given $\phi \in \Phi_+$, there exists an index i and a $\phi_1 \in \tilde{F_i}$ such that $\phi - \phi_1 \in \Psi$. If we have $\phi - \phi_1 \in \Psi_+$, there exists an index j such that $\phi - \phi_1 \in G_j$ so that indeed $\phi \in \tilde{F_i} \oplus G_j$. This happens in particular if $\phi \in \Psi$. If $\phi - \phi_1 \in \Psi_-$, there exists an index j such that we can write:

$$\phi = \sum_{s=1}^{r_i} k_se_s - \sum_{t=1}^{s_j} \ell_tf_t \text{ with non negative integers } k_s,\ \ell_t \text{ and } f_t \in G_j.$$

Since $\phi \notin \Psi$, at least one of the k_s is not zero, say k_1. Choosing positive integers $\tilde{\ell}_t$ such that $k_1\tilde{\ell}_t > \ell_t$, we may rewrite ϕ as follows

$$\phi = k_1(e_1 - \sum_{t=1}^{s_j} \tilde{\ell}_tf_t) + \sum_{s=2}^{r_i} k_se_s + \sum_{t=1}^{s_j}(k_1\tilde{\ell}_t - \ell_t)f_t.$$

This shows that ϕ is indeed in $\tilde{F_i}(n) \oplus G_j$ with $n = \tilde{\ell}$.

Now if we assume, as we may by induction, that the F_i and G_j are nested sequences, and choose $n(i)$ given by $(n_{s,t} = i, 1 \leq s \leq r_i,\ 1 \leq t \leq s_j)$, we see that the corresponding groups $\tilde{F}_i(n(i)) \oplus G_i$ form a nested sequence which is cofinal in the direct system. So we have:

Proposition 4.12 *For any totally ordered group Φ of finite height, and any commutative ring A, the semigroup algebra $A[t^{\Phi_+}]$ is a direct limit of a direct system of graded polynomial subalgebras over A with monomial maps, and there are cofinal nested subsystems of such polynomial subalgebras. If Φ is of finite rational rank* $\mathrm{r}(\Phi)$*, all the polynomial subalgebras may be chosen isomorphic to $A[x_1, \ldots, x_{\mathrm{r}(\Phi)}]$.*

Proof There remains only to prove the last sentence. This is done by induction on the height; we have shown above that the result is true for valuations of height one. Assume that the result is true for Ψ and Φ_1, and so the rank of the free monoid $\tilde{F}_i(n) \oplus G_j$ is the sum of the rational ranks of Ψ and Φ_1. But since the rational rank is additive in an exact sequence because $\mathbf{Q}$ is a flat $\mathbf{Z}$-module, this is the rational rank of Φ. □

Corollary 4.13 *If the rational rank of Φ is finite, the semigroup algebra $A[t^{\Phi_+}]$ endowed with its natural grading is a quotient of a polynomial ring over A in countably many indeterminates $A[(V_j)_{j\in J}]$ graded by Φ_+ by a homogeneous binomial ideal of the form*

$$(V_j - V^{m(j)})_{j\in J'},$$

where J' is a subset of J and $|m(j)| \geq 2$ for $j \in J'$.

Proof We may choose as a system of homogeneous generators of the A-algebra $A[t^{\Phi_+}]$, the union of the generators of the polynomial subalgebras of which $A[t^{\Phi_+}]$ is the direct limit. The only relations between these generators are those corresponding to the toric inclusions $A[x_1, \ldots, x_r] \to A[y_1, \ldots, y_r]$, and they are of the announced type after we discard the trivial relations $x_i = y_j$ by removing some generators. □

Remarks 4.14 1) There is in general no minimal system of generators for $A[t^{\Phi_+}]$ and we can of course discard an arbitrary number of those generators which appear as V_j in a relation $V_j - V^{m(j)} = 0$. For example consider $\Phi = \mathbf{Z}^2_{lex}$.
2) We can apply this to the subalgebra $k[v^{\Phi_+}]$ of $\mathcal{A}_\nu(R)$, and view the v^ϕ as coordinates for Spec$k[v^{\Phi_+}]$ subjected to the binomial relations described in this Corollary.

Proposition 4.15 *Given the ring R_ν of a valuation of finite rational rank $r(\nu)$:*
a) the graded k_ν-algebra $\mathrm{gr}_\nu R_\nu$ is a quotient of a polynomial ring $k_\nu[(V_j)_{j\in J}]$ in countably many indeterminates over the residue field k_ν, graded by Φ_+, by a homogeneous binomial ideal of the form

$$(V_j - \lambda_j V^{m(j)})_{j\in J'}\ ,\ \lambda_j \in k_\nu^*$$

where J' is a subset of J and $|m(j)| \geq 2$ for $j \in J'$.
b) The graded k_ν-algebra $\mathrm{gr}_\nu R_\nu$ is the union of a nested sequence of graded polynomial subalgebras $k_\nu[x_1^{(h)}, \ldots, x_{\mathrm{r}(\nu)}^{(h)}]$, where the inclusions are given by maps sending each variable $x_i^{(h)}$ to a constant times a monomial in the $x_j^{(h+1)}$, $1 \leq j \leq \mathrm{r}(\nu)$.

Proof See Proposition 4.12 and the previous Corollary. We have seen in Proposition 4.2 that $\mathrm{gr}_\nu R_\nu$ is isomorphic to a quotient of a polynomial algebra $k[(V_i)_{i\in I}]$ by a binomial ideal whose generators are of the form $V^m - \lambda_{mn}V^n$. Setting all the constants λ_{mn} equal to one gives the semigroup algebra $k[t^{\Phi_+}]$, hence assertion a). Assertion b) follows from this and the correspondence between the direct system of polynomial subalgebras and the binomial equations exhibited in the proof of the Corollary to Proposition 4.12. □

Corollary 4.16 *Given any finite set of homogeneous elements $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_s$ in $\mathrm{gr}_\nu R_\nu$ and a cofinal nested system of polynomial subalgebras as above, there is an algebra in our nested system such that not only do we have $\overline{x}_i \in A[x_1^{(h)}, \ldots, x_{\mathrm{r}(\Phi)}^{(h)}]$ for $1 \leq i \leq s$, but the element of least degree divides all the others in this subalgebra.*

Remark 4.17 It is important to note that the direct system built in this way is rather precisely determined by Φ; we shall see on the next examples that at least in dimension two the structure of the binomial equations, or equivalently that of the direct system of polynomial subalgebras converging to $\mathrm{gr}_\nu R_\nu$, reflects the structure of the system of points in birational modifications of $\mathrm{Spec}R$ corresponding to the valuation ν, as well as the height and rational rank of ν.

Problem. It is an interesting problem to make these correspondences precise, especially in dimensions ≥ 3. A very useful reference for dimension two is [A2].

4.4 A selection of examples.

Example 4.18 1) I give a slightly different proof of the fact, which we saw at the beginning of this section, that the algebra of the semigroup $\mathbf{Z}_+^d$ for the lexicographic order is a toric union of polynomial algebras; the relation between the two proofs is the idea for the induction on the height shown above. Let Φ be $\mathbf{Z}^d$ with the lexicographic order; the semigroup Φ_+ is generated by the following elements
$e_j^{(1)} = (1, -j, 0, \ldots, 0), e_j^{(2)} = (0, 1, -j, \ldots, 0), \ldots, e_j^{(d-1)} = (0, 0, \ldots, 1, -j)$, $e_0^{(d)} = (0, 0, \ldots, 0, 1)$ for $j = 0, 1, 2, \ldots$. To check this, we proceed by induction on d; the case $d = 1$ is obvious, so let $d \geq 2$ and $\phi = (a_1, \ldots, a_d) \in \mathbf{Z}_+^d$. If we have $a_1 = 0$ we are reduced to $\mathbf{Z}^{d-1}$, so we may assume $a_1 > 0$. If $a_2 \geq 0$, we may write $\phi = a_1 e_0^{(1)} + (0, a_2, \ldots, a_d)$ and we are again reduced to the case of $\mathbf{Z}^{d-1}$. If $a_2 < 0$, let ℓ be the smallest positive integer such that $\ell a_1 + a_2 \geq 0$; we can write $\phi = a_1 e_\ell^{(1)} + (0, \ell a_1 + a_2, a_3, \ldots, a_d)$ and again we may conclude by induction. One can check, again by induction, that the relations are generated by $e_j^{(i)} = e_{j+1}^{(i)} + e_0^{(i+1)}$ for $1 \leq i \leq d-1,\ j \geq 0$. The semigroup algebra $A[t^{\mathbf{Z}_+^d}]$ is isomorphic to

$$A[(V_{j_1}^{(1)})_{j_1\in\mathbf{N}_0}, \ldots, (V_{j_{d-1}}^{(d-1)})_{j_{d-1}\in\mathbf{N}_0}, V_0^{(d)}]/\big((V_j^{(i)} - V_{j+1}^{(i)}V_0^{(i+1)})_{j\in\mathbf{N}_0, 1\leq i\leq d-1}\big),$$

where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Note that from these equations we can read the fact that the valuation is of height d; they show that for each $(i, j),\ 1 \leq i \leq d-1,\ j \geq 1$ and each integer n we have $V_j^{(i)} = V_{j+n}^{(i)}(V_0^{(i+1)})^n$, so that the value in Φ_+ of $V_j^{(i)}$ is greater than n times the value of $V_0^{(i+1)}$ for all $n \geq 1$. This implies the existence of $d-1$ non trivial convex subgroups of Φ; since we know that $\mathrm{h}(\Phi) \leq \mathrm{r}(\Phi)$, we have $\mathrm{h}(\Phi) = d$.

In particular, for $d=2$, we see that

$$A[t^{\mathbf{Z}^2_+}] = A[W, V_0, V_1, \ldots, V_j, \ldots]/\big((V_j - WV_{j+1})_{j\in \mathbf{N}\cup\{0\}}\big).$$

Upon taking for A a field k, we see that it corresponds to the sequence of infinitely near points in the plane $\mathbf{A}^2(k)$ obtained by endlessly repeating the following process: blow up the origin of an ordered set of coordinates (u_1, u_2), choose the chart of the blow up where the ideal (u_1, u_2) is generated by u_1, and in this chart, with coordinates $(u_1, u_2/u_1)$, blow up the origin. We see that the direct limit of the monomial maps $k[u_1, u_2] \to k[u_1, u_2/u_1]$ obtained by repeating this operation is $k[t^{\mathbf{Z}^2_+}]$ presented as above, with $V_j = u_2 u_1^{-j},\ W = u_1$.

By the correspondence between sequences of points in blowing ups and valuations ([V1]), this sequence of points corresponds to a valuation ν of the local ring $R = k[u_1, u_2]_{(u_1,u_2)}$, the monomial valuation in the coordinates (u_1, u_2) with value group $(\mathbf{Z}^2, \text{lex.})$ determined by $\nu(u_1) = (1,0),\ \nu(u_2) = (0,1)$. The graded algebra $\text{gr}_\nu R_\nu$ is equal to $k[t^{\mathbf{Z}^2_+}]$. The valuation ν is monomial with respect to the coordinates (u_1, u_2), so that we have $\text{gr}_\nu R = k[U_1, U_2]$, where U_1, U_2 are the initial forms of u_1, u_2. Things are quite similar for $d > 2$.

Example 4.19 Given a positive real number τ, which I assume to be irrational and > 1, let us denote by

$$\tau = [s_1, s_2, \ldots, s_i, s_{i+1}, \ldots] = s_1 + \frac{1}{s_2 + \cdots}$$

its continued fraction expansion. Let $p_i/q_i = [s_1, \ldots, s_i]$ be the i-th approximant. Using the inductive relations

$$p_{i+1} = p_{i-1} + s_{i+1}p_i,\quad q_{i+1} = q_{i-1} + s_{i+1}q_i \quad \text{with } p_0 = 1,\ q_0 = 0,\ p_1 = s_1,\ q_1 = 1,$$

the reader will see that Perron's algorithm applied to $(1, \tau)$ produces the vectors $A^{(h)} = (p_h, q_h) \in \mathbf{R}^2_+$ for $h \geq 1$. If Φ denotes the subgroup of $\mathbf{R}$ generated by $(1, \tau)$ endowed with the induced order, the algorithm gives the following presentation, for any commutative ring A:

$$A[t^{\Phi_+}] = A[(V_i)_{i\in\mathbf{N}}]/\big((V_i - V_{i+1}^{s_i}V_{i+2})_{i\in\mathbf{N}}\big).$$

It is equivalent to say that it presents $A[t^{\Phi_+}]$ as the direct limit of the maps

$$A[V_i, V_{i+1}] \to A[V_{i+1}, V_{i+2}] \text{ given by } V_i \mapsto V_{i+1}^{s_i}V_{i+2},\quad V_{i+1} \mapsto V_{i+1}.$$

The case where $\tau \in \mathbf{Q}_+$ is left as an exercise.
We shall see below in examples 4.20 and 4.21 the interpretation of these maps in terms of blowing ups, the corresponding valuation of $k[u_1, u_2]_{(u_1,u_2)}$ and the graded algebra of its valuation ring.

Let ν be the monomial valuation of $R = k[u_1, u_2]_{(u_1,u_2)}$ determined by $\nu(u_1) = 1,\ \nu(u_2) = \tau$. The group of the valuation is $\Phi = \mathbf{Z} + \mathbf{Z}\tau$ with the order induced by that of $\mathbf{R}$, so that $\text{gr}_\nu R_\nu$ is presented as above, with $A = k$:

$$\text{gr}_\nu R_\nu = k[(V_i)_{i\in\mathbf{N}}]/\big((V_i - V_{i+1}^{s_i}V_{i+2})_{i\in\mathbf{N}}\big).$$

The semigroup of the values of ν on R is $\Gamma = \{a + b\tau/a, b \in \mathbf{Z}_+ \cup \{0\}\}$ and the graded algebra $\text{gr}_\nu R$ is the polynomial algebra $k[U_1, U_2]$, where $U_1 = V_2$ is of degree one and $U_2 = V_1$ of degree τ.

Example 4.20 Recall the examples given in [Z-S]. Vol.2, Chap. VI, §15, p.102: let k be a field, $\{s_1, s_2, \ldots\}$ a sequence of positive integers such that the products $s_1 s_2 \cdots s_i$ tend to infinity with i, and $\{c_1, c_2, \ldots\}$ a sequence of elements of k^*. We define an infinite sequence of elements v_i in $k(u_1, u_2)$ by

$$v_1 = u_1, v_2 = u_2, \ldots, v_{i+2} = \frac{v_i - c_i v_{i+1}^{s_i}}{v_{i+1}^{s_i}}, \ldots.$$

Setting $R_i = k[v_{i-1}, v_i]$, $\;m_i = (v_{i-1}, v_i)R_i$, Zariski shows that the ring

$$R_\nu = \bigcup_{i=2}^{\infty} (R_i)_{m_i}$$

is the valuation ring of a valuation ν of the field $k(u_1, u_2)$ with value group

$$\Phi = \{\frac{n}{s_1 \cdots s_k}\ ,\ n \in \mathbf{Z},\ k = 1, 2, \ldots\},$$

that is, all rational numbers whose denominator is a product of s_i's.

One checks that the elements $\big((v_i)_{i\geq 1}\big)$ generate the maximal ideal of R_ν, and their initial forms (V_i) generate the k-algebra $\mathrm{gr}_\nu R_\nu$. As $\nu(v_{i+2}) > 0$, we have $\nu(v_i - c_i v_{i+1}^{s_i}) > \nu(v_{i+1}^{s_i})$, which gives the relations $V_i - c_i V_{i+1}^{s_i} = 0$ for $i \geq 1$; they generate all relations between the (V_i), so that the graded algebra of R_ν is given by

$$\mathrm{gr}_\nu R_\nu = k[V_1, V_2, \ldots, V_i, \ldots]/\big((V_i - c_i V_{i+1}^{s_i})_{i\geq 1}\big).$$

Since the rational rank of Φ is one, we expect the k-algebra $\mathrm{gr}_\nu R_\nu$ to be the direct limit of polynomial algebras in one variable, and indeed this shows that $\mathrm{gr}_\nu R_\nu$ is the direct limit of the system of maps $k[V_i] \to k[V_{i+1}]$ given by $V_i \mapsto c_i V_{i+1}^{s_i}$.
This corresponds, replacing k by a commutative ring A and taking $c_i = 1$ for all is, to a presentation of the semigroup algebra of the semigroup Φ_+ with coefficients in A as

$$A[t^{\Phi_+}] = A[(V_i)_{i\in \mathbf{N}}]/\big((V_i - V_{i+1}^{s_i})_{i\geq 1}\big).$$

This corresponds also to the fact that Φ_+ is the direct limit of its subsemigroups generated by $\frac{1}{s_1 \ldots s_i}$, the inclusion between two consecutive semigroups being given by multiplication by s_{i+1}.

Note that this valuation is not monomial for $R = k[u_1, u_2]_{(u_1,u_2)}$ with respect to the coordinates (u_1, u_2).

In fact it is an interesting exercise to begin to compute $\mathrm{gr}_\nu R$ as a subalgebra of $\mathrm{gr}_\nu R_\nu$ in this case.
Recall that $R = k[u_1, u_2]_{(u_1,u_2)}$, and assume for simplicity that all s_i are > 1 for $i \geq 2$. From the equations defining the v_i, we see that if we set $\nu(u_1) = 1$, then $\nu(u_2) = \frac{1}{s_1}$ and $u_2^{s_1} - c_1^{-1} u_1 = -c_1^{-1} u_2^{s_1} v_3 \in R$, I set $u_3 = -c_1^{-1} u_2^{s_1} v_3$, so we have the following equation defining u_3

$$u_2^{s_1} - c_1^{-1} u_1 = u_3,$$

and we see similarly that $u_3^{s_2} - c_2^{-1}(-c_1)^{s_2} u_2^{s_1 s_2 + 1} = -c_2^{-1} u_3^{s_2} v_4 \in R$. I set $u_4 = -c_2^{-1} u_3^{s_2} v_4$ and so we have the equation

$$u_3^{s_2} - c_2^{-1}(-c_1)^{s_2} u_2^{s_1 s_2 + 1} = u_4$$

which we may use to define u_4. If we apply the same method to find u_5, we first find the equation

$$u_4^{s_3} - (-1)^{s_3(s_2-1)} c_1^{s_2 s_3} c_2^{-s_3} c_3^{-1} u_3^{s_2 s_3} v_3 = (-1)^{s_3(s_2-1)} c_1^{s_2 s_3} c_2^{-s_3} c_3^{-1} u_3^{s_2 s_3} v_4^{s_3} v_5.$$

But the monomial $u_3^{s_2s_3}v_3$ is not in R; its valuation, however, is in the semigroup Γ of R; it is the valuation of $u_2^{s_3(s_1s_2+1)-s_1}u_3$. Working out the computations gives an equation

$$u_4^{s_3} - d_4u_2^{s_3(s_1s_2+1)-s_1}u_3 = *u_4^{s_3}v_5 + *u_4v_3 \prod_{\omega\in\mu_{s_3}\setminus\{1\}} (u_3^{s_2} - *\omega u_2^{s_1s_2+1}),$$

where d_4 and the asterisks are Laurent monomials in c_1, c_2, c_3. We choose the right-hand term as u_5, note that its valuation is that of $u_4^{s_3}v_5$, and continue in this way. The minimal system of generators of the semigroup Γ corresponding to the values of ν on R is

$$\Gamma = \langle \frac{1}{s_1}, 1 + \frac{1}{s_1s_2}, s_2 + \frac{1}{s_1} + \frac{1}{s_1s_2s_3}, \ldots, \gamma_i, \ldots \rangle,$$

where $\gamma_i = \nu(u_{i+1})$ and

$$\gamma_{i+1} = s_i\gamma_i + \frac{1}{s_1s_2\cdots s_{i+1}}.$$

From this definition follows that for all $i \geq 2$

$$s_i\gamma_i \in \langle \gamma_1, \ldots, \gamma_{i-1} \rangle.$$

We have seen the result for $i = 2$. Proceed now by induction on i, assuming the result true until γ_{i-1} and writing $s_{i-1}\gamma_{i-1} = \sum_{k=1}^{i-2} \tilde{\ell}_k^{(i-1)}\gamma_k$, we have

$$\begin{array}{l} s_i\gamma_i = (s_is_{i-1}+1)\gamma_{i-1} - \gamma_{i-1} + \frac{1}{s_1\ldots s_{i-1}} = (s_is_{i-1}+1)\gamma_{i-1} - s_{i-2}\gamma_{i-2} \\ = ((s_i-1)s_{i-1}+1)\gamma_{i-1} + (\tilde{\ell}_{i-2}^{(i-1)} - s_{i-2})\gamma_{i-2} + \sum_{k=1}^{i-3} \tilde{\ell}_k^{(i-1)}\gamma_k \end{array},$$

the last equality being provided by the induction assumption with the extra twist that $\tilde{\ell}_{i-2}^{(i-1)} > s_{i-2}$.
Since we know by our induction hypothesis that $(s_i - 1)s_{i-1}\gamma_{i-1} \in \langle \gamma_1, \ldots, \gamma_{i-2} \rangle$, we see that in fact $s_i\gamma_i - \gamma_{i-1} \in \langle \gamma_1, \ldots, \gamma_{i-2} \rangle$ for $i \geq 3$. So we have relations, which we may write

$$s_i\gamma_i = \gamma_{i-1} + \sum_{k=1}^{i-2} \ell_k^{(i+1)}\gamma_k \quad \text{with } \ell_k^{(i+1)} < s_k \text{ for all } k \geq 2.$$

The ℓ are indexed by $i + 1$ which corresponds to the variable U_i instead of i in order to give naturality to the equations given below. These relations give us the equations for the monomial curve associated to Γ in the infinite-dimensional space with coordinates $(U_j)_{j\geq 2}$ (note that we have eliminated U_1), and therefore also for its avatar $\mathrm{Specgr}_\nu R$; the latter are

$$U_3^{s_2} - d_3U_2^{s_1s_2+1} = 0, \text{ and } U_{i+1}^{s_i} - d_{i+1}U_i \prod_{k=1}^{i-2} U_{k+1}^{\ell_k^{(i+1)}} = 0 \text{ for } i \geq 3,$$

where the d_i are, up to sign, Laurent monomials in the c_k's.
Going back to our ring R, note that in the first few relations defining the u_i inductively another expression appeared naturally for the inclusion $s_3\gamma_3 \in \langle \gamma_1, \gamma_2 \rangle$. In any case the important fact is the following : the equations defining inductively the u_j are (with a small change in the indexation) of the form

$$u_j^{s_{j-1}} - d_ju_1^{\ell_1^{(j)}} \ldots u_{j-1}^{\ell_{j-1}^{(j)}} = u_{j+1}$$

with $d_j \in k^*$, the d_j's being, up to sign, Laurent monomials in the c_k's, and the term on the right of the equation has greater valuation than the monomials appearing in the left hand side.

When we compute $\mathrm{gr}_\nu R$ from these equations, we need to take as coordinates $U_2, U_3, \ldots$ but *not* U_1, because of the equation $U_2^{s_1} - c_1^{-1}U_1 = 0$. All the right-hand sides disappear because they have greater valuation, and the equations give algebraic relations between the initial forms U_j of u_j and those of $u_2, \ldots, u_{j-1}$, which decreases the transcendence degree over k, and therefore the Krull dimension, of $\mathrm{gr}_\nu R$ to one in agreement with Proposition 3.7 (see also the remark which follows it). In the ring R however each equation is captured, via its right-hand side u_{j+1}, in an abyssal[5] sequence of relations, and never gets a chance to decrease the dimension of R by one. If we stopped after a finite number of steps, setting

$$u_j^{s_{j-1}} - d_j u_1^{\ell_1^{(j)}} \ldots u_{j-1}^{\ell_{j-1}^{(j)}} = P(u_1, \ldots, u_j),$$

where P is any polynomial (for example 0) which is not a zero divisor modulo the ideal generated by $u_j^{s_{j-1}} - d_j u_1^{\ell_1^{(j)}} \ldots u_{j-1}^{\ell_{j-1}^{(j)}}$ and the previous equations, the dimension would decrease to one.

This example displays for us a basic phenomenon: the decrease of dimension of the special fiber in the specialization of R to $\mathrm{gr}_\nu R$ in the case where Abhyankar's inequality is strict. In both rings, all variables of index ≥ 4 are algebraically dependent upon u_2 and u_3, but in $\mathrm{gr}_\nu R$ these two variables are dependent, while in R they are independent and all others are polynomials in them, which shows "from the equations" that R is regular of dimension two. In fact, we have given a presentation

$$k[u_2, u_3] = k[u_2, u_3, \ldots, u_i, \ldots]/(u_3^{s_2} - c_1^{-s_2} c_2^{-1} u_2^{s_1 s_2 + 1} - u_4, u_4^{s_3} - \ldots)$$

of our polynomial ring which may seem complicated, but is in fact an explicit description of the valuation, allowing us to compute the valuation of every element, as follows:

Given a polynomial $P(u_2, u_3)$, replace every occurrence of $u_3^{s_2}$ by $c_1^{-s_2} c_2^{-1} u_2^{s_1 s_2 + 1} + u_4$, then every occurrence of $u_4^{s_3}$ by $d_4 u_2^{\ell_2^{(4)}} u_3^{\ell_3^{(4)}} + u_5$, still replacing occurences of $u_3^{s_2}$ as above, and so on. This stops after finitely many steps because the products $s_2 s_3 \cdots s_i$ eventually exceed the degree of the polynomial P. We obtain finally a polynomial $\tilde{P}(u_2, u_3, \ldots, u_k)$ where no variable u_i appears with an exponent $\geq s_{i-1}$ for $i \geq 3$, and now the valuation of P is the smallest valuation of the monomials of $\tilde{P}$ simply because relations between their initial forms in the graded ring have become impossible. We shall see below in subsection 5.5 that since ν is rational of height one, a similar result is always true at least after a suitable completion of the ring R, that is

$$\hat{R}^{(\nu)} = k[u_2, \widehat{u_3, \ldots}, u_i, \ldots]/\overline{(\mathbf{F}, (u_j^{n_j} u^{n(j)} - \lambda_{nm} u^{m(j)} + c_{j+1} u_{j+1} + \cdots)_{j \geq j_0})},$$

where the second term is a quotient of a suitable completion of the polynomial ring, $\mathbf{F}$ is a finitely generated ideal, the c_{j+1} are in k^* and the bar means a topological closure. If $\mathrm{h}(\nu) > 1$, a similar result is true, where the variables u_i are indexed by ordinals.

[5] From the Greek ἄβυσσος meaning *without bottom*; in spite of a discrepancy in meaning I have preferred this to *abysmal*, which does not exist in French.

Another important phenomenon seen here is that because of the minimality the initial form U_1 of u_1 does not appear among the generators of $\mathrm{gr}_\nu R$, so that if we wish to find generators of the maximal ideal of R whose initial forms are among a minimal system of generators of $\mathrm{gr}_\nu R$, we have to write $R = k[u_2, u_3]_{(u_2,u_3)}$. The valuation of u_3 is maximal among those of elements which, together with u_2, generate the center of the valuation. Heuristically, the process leads us to coordinates which have, with respect to the given valuation, a property analogous to Hironaka's maximal contact (see [H1] and below).

We can describe the valuation in terms of blowing-ups as follows: starting with $\mathbf{A}^2(k)$ with ordered coordinates (v_1, v_2), we perform the following sequence of steps.
Step 1: reverse the order of coordinates.
Step 2: blow-up the origin and localize in the chart where v_2, now the first coordinate, generates the ideal (v_1, v_2), taking as coordinates $(v_2, v_1 v_2^{-1})$.
Step 3: repeat this second step $s_1 - 1$ times.
Step 4: substract c_1 from the second coordinate so that the point with coordinate c_1 on the axis of the second coordinate (the exceptional divisor just created) is the origin of coordinates, and localize at this point.
Repeat from step 1, replacing (s_1, c_1) by (s_2, c_2), and so on.
The valuation ring R_ν is the direct limit of the direct system of local rings corresponding to the sequence of infinitely near points obtained in this way. Note that the strict transform of u_3 is a coordinate at the point obtained after step 4; and so on; it is one of the attributes of maximal contact.
According to ([Z-S], Vol. 2, p. 104), any subgroup Φ of the additive group $\mathbf{Q}$ of rational numbers can be obtained in this manner.
This applies in particular to $\Phi = \mathbf{Q}$, obtained by taking $s_i = i$ for all $i \geq 1$.

Another construction of valuations whose value groups are given subgroups of $\mathbf{Q}$ is given by Zariski in [Z3], p.648, which leads to the distinction between those prime numbers which divide only a finite number of the s_i's and those which divide an infinite number of them. If we worked over a field whose characteristic belongs to the second group and tried to effectively build inductively elements like the v_i by solving equations of the type $c_i v_{i+1}^{s_i} - v_i + \cdots = 0$, we would run into substantial difficulties. However, this is not at all what I do here; I only use the direct system as a means to approximate $\mathrm{gr}_\nu R_\nu$ by polynomial subalgebras. Note anyway that in these examples, $\mathrm{gr}_\nu R$ is not regular.

Example 4.21 One may ask what happens if, in the sequence of steps just described, one makes no translation; it amounts to setting all the c_i equal to zero. Since the equations describing the formation of the v_i are

$$v_i = v_{i+1}^{s_i}(c_i + v_{i+2}),$$

we see that the graded algebra of our new valuation ν_0 is

$$\mathrm{gr}_{\nu_0} R_{\nu_0} = k[V_1, V_2, \ldots, V_i, \ldots] / \big((V_i - V_{i+1}^{s_i} V_{i+2})_{i \in \mathbf{N}} \big).$$

It follows from what we saw in example 4.19 that ν_0 is the monomial valuation of height one and rational rank two defined by $\nu_0(u) = 1, \nu_0(v) = \tau$, where τ is the number with continued fraction expansion

$$\tau = [s_1, s_2, \ldots, s_i, s_{i+1}, \ldots].$$

Its value group is

$$\Phi_\tau = \mathbf{Z} + \mathbf{Z}\tau \ \subset \mathbf{R}$$

with the order induced by that of $\mathbf{R}$. We have $\mathrm{gr}_{\nu_0} R = k[U_1, U_2]$, with U_1 of degree 1 and U_2 of degree τ; it is regular.
In particular, the valuation with value group $\mathbf{Q}$ constructed above becomes, upon setting all the c_i equal to zero, the monomial valuation corresponding to $\nu_0(u) = 1,\ \nu_0(v) = \rho$, where

$$\rho = [1, 2, 3, \ldots, i, i+1, \ldots].$$

Problem. It would be interesting if, given any rational valuation ν of a nœtherian local integral domain R, one could specialize it (in the sense used just above, which differs from the usual one) to a valuation of R of rational rank equal to the dimension of R. Heuristically this means we can always "suppress the translations" continuously. As we shall see in Section 5 it is reasonable to try first for a valuation ν_0 of rational rank equal to the dimension of the scalewise ν-adic completion of R. Can one use the results of [A2] to verify this statement in dimension two?

Example 4.22 In [Z3], Zariski gives, in the case where the characteristic of k is zero, yet another construction, also found in [McL-S], of valuations on $k[u_1, u_2]_{(u_1,u_2)}$ with a value group $\Phi \subseteq \mathbf{Q}$ given in advance. Let $k[[u^{\mathbf{Q}_+}]]$ be the ring of power series with exponents forming a well ordered subset of $\mathbf{Q}_+$, and with the same s_i as above, given a sequence of integers m_i and a sequence of elements $e_i \in k^*$, consider the series

$$w(u) = e_1 u^{\frac{m_1}{s_1}} + e_2 u^{\frac{m_2}{s_1 s_2}} + \cdots + e_j u^{\frac{m_j}{s_1 \cdots s_j}} + \cdots .$$

Since k is of characteristic zero, the series $w(u)$ cannot be algebraic over $k(u)$ and the map $u_1 \mapsto u,\ u_2 \mapsto w(u)$ induces an injection

$$k[u_1, u_2]_{(u_1,u_2)} \subset k[[u^{\mathbf{Q}_+}]]$$

and the u-adic valuation induces on $k(u_1, u_2)$ a valuation having as group of values the group of rational numbers whose denominator is a product of s_i's.
The correspondence between this construction and the previous one is as follows.
In our set of equations $u_j^{s_{j-1}} - d_j u_1^{\ell_1^{(j)}} \ldots u_{j-1}^{\ell_{j-1}^{(j)}} = u_{j+1}$ for $k[[u_1, u_2]]$, stop at $j = \ell$ and set $u_{\ell+1} = 0$, then using the previous equations, eliminate the variables $u_i,\ i > 2$ to finally get an equation $Q_\ell(u_1, u_2) = 0$ in u_2 and u_1 which has as solution, if the characteristic of k is zero, a Puiseux series $u_2 = w_\ell(u_1)$ whose exponents have denominators $s_1 \ldots s_{\ell-1}$. As we let ℓ go to infinity, the series $w_\ell(u_1)$ converge to a series of the form $w(u_1)$ in $k[[u_1^{\mathbf{Q}_+}]]$. This is a generalization of the procedure given in [T1] to compute the equation of a branch having a given semigroup.

What I have just done is to describe the transcendental branch defined by $u_1 = u,\ u_2 = w(u)$ as a plane deformation of the avatar corresponding to the constants d_j of the monomial curve associated to the semigroup Γ described above, which lives in an infinite-dimensional space. One can see, with a little work, that the $Q_\ell(u_1, u_2)$, viewed as polynomials in u_2, are "key polynomials" in the sense of [McL2] and [Ka] for the valuation ν viewed as extending to $k(u_1)(u_2)$ the (u_1)-adic valuation of $k(u_1)$; the irreducibility of the Q_ℓ in the sense of *loc. cit.* follows from their construction and the results of [T1] on deformations of monomial curves, and the minimality of their degrees follows from the easily verified fact that for each index i, the smallest non zero integer d such that $d\gamma_i \in \langle \gamma_1, \ldots, \gamma_{i-1} \rangle$ is

s_i. This provides a geometric interpretation for these polynomials and also of the construction of §5 of [McL-S] which builds, in arbitrary characteristic, a rational valuation of $k[u_1,u_2]_{(u_1,u_2)}$ with a given group of values contained in $\mathbf{Q}$, and for some of the constructions of [F-J] and [V2].
Note that, as ℓ goes to infinity, the parametric representation $u_2 = w_\ell(u_1)$ of our approximating plane branches tend to a transcendental series, while their equations Q_ℓ tend to zero in the (u_1,u_2)-adic topology; this last fact explains why the algebraic description, which is blind to transcendental series, gives us the whole (formal germ of the) affine plane as a limit.

Example 4.23 A power series in one variable with rational exponents having unbounded denominators is not necessarily very transcendental in positive characteristic:
Let k be a perfect field of finite characteristic p and u an indeterminate; set $K = \bigcup_{n\geq 1} k(u^{\frac{1}{p^n}})$, the perfect closure of $k(u)$. There is a unique extension to K of the u-adic valuation of $k(u)$, and its valuation ring is *not* nœtherian. Consider the series

$$v = \sum_{i=1}^{\infty} u^{1-\frac{1}{p^i}} \in k[[u^{\mathbf{Q}_+}]];$$

it is a solution of the polynomial equation

$$v^p - u^{p-1}(1+v) = 0.$$

This equation is an Artin-Schreier equation: it is obtained from the standard Artin-Schreier $y^p - y = \frac{1}{u}$ by replacing y by $\frac{v}{u}$. If we set $L = K(v)$, it is shown in [K3] that the extension L/K has degree p and defect p. More precisely, the unique extension ν to K of the u-adic valuation of $k(u)$ has a unique extension ν' to L, with the same group of values, so that the ramification index $e = [\Phi' : \Phi]$ is equal to one, and no residual extension so that the inertia degree $f = [\kappa(\nu') : \kappa(\nu)]$ is also equal to one. The extension is of degree p so that the Ostrowski ramification formula (see [K3], [Roq]), which is $[L : K] = def$, where d is the defect, gives $d = p$. This defect complicates the parametrization but does not make it more difficult to create a non-singular model. We remark that our curve is a deformation of the monomial curve $v^p - u^{p-1} = 0$, and apply to this monomial curve the toric resolution process of [G-T] and subsection 6 below: it gives us a chart $u = y_1^p y_2, v = y_1^{p-1} y_2$. Our equation then becomes $y_1^{p(p-1)} y_2^{p-1}(y_2 - 1 - y_1^{p-1} y_2)$, so that the strict transform $y_2 - 1 - y_1^{p-1} y_2 = 0$ is non singular. It can be parametrized in a neighborhood of the exceptional divisor $y_1 y_2 = 0$ by $y_2 = \frac{1}{1-y_1^{p-1}}$, so that we have the following parametrization of our curve in a neighborhood of the origin:

$$u = \frac{y_1^p}{1 - y_1^{p-1}}; \quad v = \frac{y_1^{p-1}}{1 - y_1^{p-1}}.$$

The fact that the extension $K \subset K(v)$ has defect seems to be related to the fact that while the extension of fields $k(u) \to k(u)[v]/(v^p - u^{p-1}(1+v))$ is separable, the extension of graded rings associated to the u-adic valuation of $k[u]$ and its extension to $k[u,v]/(v^p - u^{p-1}(1+v))$, which is

$$k[U] \to k[U,V]/(V^p - U^{p-1})$$

is purely inseparable of degree p.

5 Completion problems

In this section I study the ν-adic completion of a nœtherian local ring R and show that it is nœtherian in certain cases. Although completions of valued fields are a classical topic, the algebraic structure of the ν-adic completions of nœtherian rings does not seem to have been studied. Perhaps the reason is that the nœtherian hypothesis is not natural from the viewpoint of the theory of valued fields, while completions other than those with respect to an ideal are not commonplace in algebraic geometry. This is a meeting place for the number-theoretic/henselian and the algebro-geometric traditions of valuation theory.

The general fact is that the completion of R for the ν-adic topology is a quotient of the completion of R for the topology of the symbolic powers of the center of the height one valuation with which ν is composed. If R is analytically irreducible this topology is finer than the m-adic topology.

Being complete for the ν-adic topology, however, does not suffice to imply the convergence of the sequences which occur when one tries to lift elements from the graded ring to R. In order to overcome this difficulty I consider, in the case of a rational valuation, a quotient $\hat{R}^{(\nu)}$ of the m-adic completion $\hat{R}^m$, which is complete for a valuation $\hat{\nu}$ extending ν and such that the inclusion $R \to \hat{R}^{(\nu)}$ induces a scalewise birational morphism $\mathrm{gr}_\nu R \to \mathrm{gr}_{\hat{\nu}} \hat{R}^{(\nu)}$. This ring $\hat{R}^{(\nu)}$ is not only complete for the ν-adic topology but also *scalewise* complete, which ensures the convergence of the sequences mentioned above. It is also henselian, and in the equicharacteristic case admits a field of representatives.

I also show how to specialize the ring $\hat{R}^{(\nu)}$ to a suitable (scalewise) $\hat{\nu}_{\mathrm{gr}}$-adic completion of $\mathrm{gr}_{\hat{\nu}} \hat{R}^{(\nu)}$ by a suitable completion of the valuation algebra, and the existence of a coordinate system and equations for the ring of this specialization, which is a fundamental fact in this approach.

5.1 The ν-adic completion of R. In general, even if the local ring R has a field of representatives, it is not possible to lift generators of the graded algebra $\mathrm{gr}_\nu R$ to generators of R; we need a hypothesis of completeness (see [B3], Chap. III). The arguments using the fact that a deformation preserves smoothness and transversality also use some form of the implicit function theorem, or Hensel's Lemma, the validity of which is another attribute of complete rings. This leads us to the study of the completion of a ring R with respect to the $(\mathcal{P}_\phi)_{\phi\in\Phi_+}$-filtration. We note that the quotients $R/\mathcal{P}_\phi$ form a countable projective system since the semigroup Φ_+ is countable and totally ordered. One can then define the ν-adic completion of R as

$$\hat{R}^\nu = \varprojlim_{\phi\in\Phi_+} R/\mathcal{P}_\phi.$$

There is a natural map $R \to \hat{R}^\nu$, which is injective since the filtration is separated. The ring $\hat{R}^\nu$ is an integral domain and the valuation ν has a canonical extension $\hat{\nu}$ to a valuation of $\hat{R}^\nu$ with values in Φ. To see this it is enough to apply directly the explicit form of the definition of the ν-adic completion as a subset of the product of the quotients $R/\mathcal{P}_\phi$. We note that $\hat{R}^\nu$ is the closure of R in the completion $\hat{K}^\nu$ of its field of fractions with respect to the topology $\mathcal{T}_\nu$ defined by ν (see [B3], Chap VI, §5, No. 3). In particular, we have:

$$R_{\hat{\nu}} = \hat{R}^\nu_\nu.$$

One sees also immediately that if we set

$$\hat{\mathcal{P}}_\phi = \{x \in \hat{R}^\nu \mid \hat{\nu}(x) \geq \phi\}, \quad \hat{\mathcal{P}}_\phi^+ = \{x \in \hat{R}^\nu \mid \hat{\nu}(x) > \phi\},$$

they are respectively the closure of $\mathcal{P}_\phi$, $\mathcal{P}_\phi^+$ in $\hat{R}^\nu$ ([B3], Chap. VI, §5 No. 2) and the natural inclusion $\mathcal{P}_\phi/\mathcal{P}_\phi^+ \subset \hat{\mathcal{P}}_\phi/\hat{\mathcal{P}}_\phi^+$ is an equality, so that the inclusion $R \subset \hat{R}^\nu$ induces an equality of graded rings

$$\mathrm{gr}_\nu R = \mathrm{gr}_{\hat{\nu}} \hat{R}^\nu.$$

We note that the semigroup of values $\hat{\nu}(\hat{R}^\nu \setminus \{0\}) \subset \Phi_+ \cup \{0\}$ is the same as that of ν on R, and so is well ordered when R is nœtherian (Proposition 3.9).

Let us assume that R is a nœtherian local ring with maximal ideal m and ν is a valuation of K of height one such that its valuation ring R_ν contains R. Set $\mathbf{p} = m_\nu \cap R$; since R is nœtherian, we have $\nu(\mathbf{p}) > 0$, say $\nu(\mathbf{p}) = \phi_0$, and since ν is of height one, its group Φ is archimedian, so that for any $\phi \in \Phi$ there exists an integer $N(\phi)$ such that $N(\phi)\phi_0 \geq \phi$, that is $\mathbf{p}^{N(\phi)} \subset \mathcal{P}_\phi$. Then we have a projective system of surjective maps

$$R/\mathbf{p}^{N(\phi)} \to R/\mathcal{P}_\phi,$$

which induces a continuous morphism of completions $\hat{R}^{\mathbf{p}} \to \hat{R}^\nu$, where $\hat{R}^{\mathbf{p}}$ is the completion of R for the $\mathbf{p}$-adic topology. Since $\hat{R}^\nu$ is an integral domain, the kernel of this map is a prime ideal of $\hat{R}^{\mathbf{p}}$. By the definition of completions, this kernel is $\bigcap_{\phi \in \Phi_+} \mathcal{P}_\phi \hat{R}^{\mathbf{p}}$, so that in the case where ν is of height one and R_ν dominates R, i.e., $\mathbf{p} = m$, it coincides with the ideal called the implicit ideal of $\hat{R}$ by Spivakovsky ([S2]).
Recall that the symbolic power $\mathbf{p}^{(n)} = \mathbf{p}^n R_{\mathbf{p}} \cap R$ is the set of elements $x \in R$ such that there exists $s \notin \mathbf{p}$ such that $sx \in \mathbf{p}^n$. Because our ring R is a nœtherian domain, the symbolic powers define a separated topology on R (see [Z-S], Chap. IV, Th. 23), for which there is a completion $\hat{R}^{(\mathbf{p})}$. Note that, since when $s \notin \mathbf{p}$ we have $\nu(sx) = \nu(x)$, with the notations just introduced we have $\mathbf{p}^{(N(\phi))} \subset \mathcal{P}_\phi$, so that we have natural continuous maps

$$\hat{R}^{\mathbf{p}} \to \hat{R}^{(\mathbf{p})} \to \hat{R}^\nu.$$

Let us denote by H the kernel of the map $\hat{R}^{(\mathbf{p})} \to \hat{R}^\nu$ and remark that $H \cap R = (0)$. We have the following

Proposition 5.1 *For a valuation ν of height one on a nœtherian local ring R, with the notations introduced above, the natural continuous injection* [6]

$$\hat{R}^{(\mathbf{p})}/H \to \hat{R}^\nu$$

is an isomorphism of topological rings inducing the identity on R. If the center of ν in R is the maximal ideal m, the natural map $\hat{R}^m \to \hat{R}^{(m)}$ is an isomorphism, so that $\hat{R}^\nu$ is a quotient of $\hat{R}^m$ and is nœtherian.
If R is a regular local ring, the map $\hat{R}^{\mathbf{p}} \to \hat{R}^{(\mathbf{p})}$ is an isomorphism for all prime ideals, $\hat{R}^\nu$ is a quotient of $\hat{R}^{\mathbf{p}}$ and therefore is nœtherian.

[6] This corrects an error in the statement of Proposition 1.3 of [T2].

Proof The injection

$$\hat{R}^{(\mathbf{p})}/H \subset \hat{R}^{\nu}$$

implies that the valuation ν extends to $\hat{R}^{(\mathbf{p})}/H$ as a valuation $\hat{\nu}$ of height one. The valuation ν is still ≥ 0 on the localization $R_{\mathbf{p}}$, and so we have an injection $\hat{R}_{\mathbf{p}}^{\mathbf{p}R_{\mathbf{p}}}/H_{\mathbf{p}} \subset \hat{R}_{\mathbf{p}}^{\nu}$ corresponding to $R_{\mathbf{p}}$. Let us replace R by $R_{\mathbf{p}}$ for a moment, so that $\mathbf{p}$ is the maximal ideal m. By [Z-S], Vol. 2, Appendix 3, Lemma 3 p. 343, the distinct valuation ideals (i.e., our $\mathcal{P}_{\phi}(\hat{R}^m/H)$ written without repetition) form a simple infinite descending chain of ideals, which are primary for the maximal ideal $\hat{m}$ of $\hat{R}^m/H$ by the archimedian property, and have intersection zero. I denote them by $\hat{\mathcal{P}}_j$. Now we can apply Chevalley's Theorem ([Z-S], Vol. 2, Chap. VIII, §5, Th. 13, p. 270), which asserts that there exists an integer-valued function $s(n)$ tending to infinity with n and such that for each valuation ideal $\hat{\mathcal{P}}_j$, we have $\hat{\mathcal{P}}_j \subset \hat{m}^{s(j)}$. This, added to the fact that the $\hat{\mathcal{P}}_j$ are primary for $\hat{m}$, proves that in the ring $\hat{R}^m/H$ the $\hat{\nu}$-adic topology coincides with the $\hat{m}$-adic topology, so that it is complete for both, and therefore has to be equal to $\hat{R}^{\nu}$.

Now coming back to the case where the center of ν is not necessarily m, we can remark that by definition of the symbolic powers the inclusion $R \to R_{\mathbf{p}}$ extends to an inclusion

$$\hat{R}^{(\mathbf{p})} \to \hat{R}_{\mathbf{p}}^{\mathbf{p}R_{\mathbf{p}}}.$$

In fact $\hat{R}^{(\mathbf{p})}$ is the closure of R in $\hat{R}_{\mathbf{p}}^{\mathbf{p}R_{\mathbf{p}}}$. The $\mathbf{p}^{(n)}$-adic topology of the first ring is induced by the $\mathbf{p}^n$-adic topology of the second since $\mathbf{p}^{(n)}\hat{R}^{(\mathbf{p})} = \mathbf{p}^n\hat{R}_{\mathbf{p}}^{\mathbf{p}R_{\mathbf{p}}} \cap \hat{R}^{(\mathbf{p})}$ as one can check with Cauchy sequences. The valuation ν extends to $R_{\mathbf{p}}$, and so by the special case where the center is the maximal ideal, the ν-adic completion of $R_{\mathbf{p}}$ is $\hat{R}_{\mathbf{p}}^{\mathbf{p}R_{\mathbf{p}}}/H_{\mathbf{p}}$ where $H_{\mathbf{p}} = \bigcap_{\phi\in\Phi_+}\mathcal{P}_{\phi}(R_{\mathbf{p}})\hat{R}_{\mathbf{p}}^{\mathbf{p}R_{\mathbf{p}}}$. Setting $H = H_{\mathbf{p}} \cap \hat{R}^{(\mathbf{p})}$, we have an injection

$$\hat{R}^{(\mathbf{p})}/H \to \hat{R}_{\mathbf{p}}^{\mathbf{p}R_{\mathbf{p}}}/H_{\mathbf{p}}.$$

The quotient topology on $\hat{R}^{(\mathbf{p})}/H$ is the $\mathbf{p}^{(n)}$-adic topology and is induced by the quotient topology of $\hat{R}_{\mathbf{p}}^{\mathbf{p}R_{\mathbf{p}}}$; this last one coincides with the $\hat{\nu}$-adic topology by the special case, so that in the end the $\hat{\nu}$-adic topology and the $\mathbf{p}^{(n)}$-adic topology coincide on $\hat{R}^{(\mathbf{p})}/H$. From this follows that this quotient is equal to $\hat{R}^{\nu}$. Since $\mathcal{P}_{\phi}(R)$ is finitely generated, its closure in $\hat{R}^{(\mathbf{p})}/H$ is $\mathcal{P}_{\phi}(R)\hat{R}^{(\mathbf{p})}/H$ and we have $H = \bigcap_{\phi\in\Phi_+}\mathcal{P}_{\phi}(R)\hat{R}^{(\mathbf{p})}$. This gives the first result in the general case.
We have already seen that $m^{(n)} = m^n$ and finally, in a regular ring the adic and symbolic topologies coincide for all ideals ([Ve], Th. 3.5). □

Remarks 5.2 1) This shows that the ring $\hat{R}^m/H$ to which Spivakovsky ([S2]) extends the valuation ν in the special case of a valuation of height one such that R_{ν} dominates R is nothing but $\hat{R}^{\nu}$.

2) In particular if R is a regular local ring, since its completion $\hat{R}^{\mathbf{p}}$ for the $\mathbf{p}$-adic topology is a nœtherian ring ([B3], Chap. III, §3, No.4, Prop.8), for every valuation of height one ν of the field of fractions of R whose valuation ring contains R, the completion $\hat{R}^{\nu}$ of R for the ν-adic valuation is nœtherian; as a topological ring it is in fact a Zariski ring with respect to the center of the valuation $\hat{\nu}$.

Proposition 5.3 *Let R be a nœtherian local ring contained in a valuation ring R_{ν} of its field of fractions. The completion $\hat{R}^{\nu}$ is isomorphic to the completion*

$\hat{R}^{\nu_1}$ of R with respect to the height one valuation ν_1 with which the valuation ν is composed, and is a quotient of the symbolic **p***-adic completion of R, where* **p** *is the center of ν_1 in R.*

Proof We saw above that is true for valuations of height one. Because of Abhyankar's inequality, the valuation ν is of finite height. Let us assume that $h(\nu) > 1$. By assumption we have a surjective monotone map $\lambda\colon \Phi \to \Phi_1$ with Φ_1 of height one and Kerλ of height $h-1$, a maximal isolated subgroup of Φ.

Denote by ν_1 the valuation $\lambda \circ \nu$. Then we have the inclusion $\mathcal{P}_\phi \subset \mathcal{P}_{\lambda(\phi)}$. Whenever $\phi_1 > \lambda(\phi)$, we have $\mathcal{P}_{\phi_1} \subset \mathcal{P}_\phi \subset \mathcal{P}_{\lambda(\phi)}$, which shows that the ν-adic topology on R coincides with the ν_1-adic topology, hence the equality of the completions. We are thus reduced to the case of height one treated above, and we see that the completion $\hat{R}^\nu$ is a quotient of the completion of R with respect to the symbolic $(m_{\nu_1} \cap R)$-adic filtration. □

Remarks 5.4 1) In a nœtherian domain R with a non negative valuation ν, the ideal of elements of R which are topologically nilpotent with respect to the ν-adic topology is the center in R of the height one valuation with which ν is composed.

2) The equality of the topologies determined on a ring by two valuations, one of which is composed with the other, is well known; in the nœtherian case, a more precise result can be found in [Z-S], Vol. 2, Lemma 4 of Appendix 3, p. 344.

Corollary 5.5 *Given a nœtherian local integral domain R and a valuation ν of its field of fractions, non negative on R, if ν is composed with a valuation ν_1, the natural map of completions*

$$\hat{R}^\nu \longrightarrow \hat{R}^{\nu_1}$$

induced by the inclusions $\mathcal{P}_\phi \subset \mathcal{P}_{\lambda(\phi)}$ is an isomorphism of topological rings. In particular, the valuation ν_1 extends to a valuation $\hat{\nu}_1$ on $\hat{R}^\nu$.

Corollary 5.6 *If R is a regular excellent local ring, for any valuation ν of its field of fractions which is non negative on R, the ring $\hat{R}^\nu$ is excellent.*

Proof By Proposition 5.3, and the second part of Proposition 5.1, the ν-adic completion $\hat{R}^\nu$ is a quotient of the completion $\hat{R}^{\mathbf{p}}$ of the local ring R with respect to an ideal, so it is excellent as a quotient of an excellent ring (see [Ro1], [Ro2]). □

When trying to understand how to complete some non-nœtherian rings, essentially semigroup algebras, one encounters questions such as this:

Question. (Exercise; Compare with [Ha], [Ka]) Let A be a commutative ring and Φ_+ (resp. Γ) the positive semigroup of a totally ordered archimedian abelian group of finite rational rank (resp. the value semigroup of a valuation of height one on a nœtherian ring R). When is it true that the completion of the semigroup algebra $A[t^{\Phi_+}]$ (resp. $A[t^\Gamma]$) with respect to the topology deduced from the canonical valuation $\nu(\sum a_\phi t^\phi) = \min\{\phi | a_\phi \neq 0\}$ is the ring of power series $A[[t^{\Phi_+}]]$ (resp. $A[[t^\Gamma]]$) in the sense of [McL-S]?
Remark that $A[[t^\Gamma]]$, which a priori is only a group, is a ring, since in Γ there are only finitely many ways to write an element as a sum of other elements (because Γ is well ordered by Proposition 3.9; see [Ka], [B2], [C-G], Theorem 1), so that we can multiply two series.

Example 5.7 1) Take the usual example of a rank one valuation not satisfying Abhyankar's equality (see above and [Z-S], Chap VI, §15, Example 2, [S1], [S2]): let k be a field, and take $R = k[u_1, u_2]_{(u_1,u_2)}$ so that $\hat{R} = k[[u_1, u_2]]$. Let $\sum_{i\geq 1} a_i u_1^i \in k[[u_1]]$ be transcendental over $k(u_1)$ and set $f(u_1, u_2) = u_2 - \sum_{i\geq 1} a_i u_1^i \in \hat{R}$. Define a valuation ν on R by means of the injective map $R \to k[[t]]$ determined by $u_1 \mapsto t$, $u_2 \mapsto \sum a_i t^i$. The residual transcendence degree of ν is 0 and its rational rank is one. The completion $\hat{R}^\nu$ is isomorphic to the quotient $k[[u_1, u_2]]/(f)$; it is one-dimensional.
This shows that the injective map $R \to \hat{R}^\nu$ is not flat in general. The group of the valuation is $\mathbf{Z}$ and for each $n \in \mathbf{N} \cup \{0\}$ we have $\mathcal{P}_n(R) = (t^n)k[[t]] \cap R$. We see that for each $n \geq 1$ we have $u_2 - \sum_1^{n-1} a_i u_1^i \in \mathcal{P}_n(R)$, which shows that there is no inclusion $\mathcal{P}_n(R) \subset m^{s(n)}$ with $s(n) > 1$. Note that we could have taken any irreducible element $f \in (u_1, u_2)k[[u_1, u_2]]$ not having a root algebraic over $k(u_1)$ and the composed map $k[u_1, u_2]_{(u_1,u_2)} \to k[[u_1, u_2]]/(f) \to k[[t]]$ where the last map is the normalization.

2) On the same ring R, consider the valuation ν with value group $\mathbf{Z}^2$ ordered lexicographically, which attributes value $(1, 0)$ to u_1, value $(0, 1)$ to u_2, and such that the valuation of a polynomial is the infimum of the valuations of its monomials. There is a natural monotone surjective map $\lambda = \mathrm{pr}_1 \colon \mathbf{Z}^2 \to \mathbf{Z}$, hence a valuation ν_1, by the u_1-adic order. The ideal $\mathcal{P}_{(i,j)}$ is the product ideal $u_1^i(u_1, u_2^j)$ of R, while the ideal $\mathcal{P}_i$ for ν_1 is (u_1^i). Since $(u_1^{i+1}) \subset u_1^i(u_1, u_2^j) \subset (u_1^i)$, these two filtrations define the same topology on R, so that in this case,

$$\hat{R}^\nu = \hat{R}^{\nu_1}.$$

Remark that the residual ideal $(\mathcal{P}_{(i,j)} : \mathcal{P}_i) = (u_1, u_2^j)$ is a simple ν-ideal in R in the sense of Zariski's theory (see [Z-S], Vol.2, Appendix 3). Notice however that, if we look at the primary decomposition as in *loc.cit.*, we have $\mathcal{P}_{(i,j)} = (u_1^{i+1}, u_2^j) \cap (u_1^i)$. In this case, the completions of $k[u_1, u_2]_{(u_1,u_2)}$ with respect to ν and ν_1 are isomorphic to

$$\widehat{k[u_1, u_2]_{(u_1,u_2)}}^{\nu} = \varprojlim_{i\geq 0} k[u_1, u_2]_{(u_1,u_2)}/(u_1)^i = k[u_2]_{(u_2)}[[u_1]];$$

the ideal $\mathbf{p}$ is $(u_1)k[u_1, u_2]_{(u_1,u_2)}$ and the ideal H is zero.
Remark that the the quotient $\hat{R}^\nu/\mathbf{p}\hat{R}^\nu = k[u_2]_{(u_2)}$ is not complete for the $\overline{\nu}$-adic (in this case the (u_2)-adic) topology.
3) (a non-discrete case) Given a subgroup $\Phi \subset \mathbf{R}$ ordered by the order of $\mathbf{R}$, consider, following H. Hahn ([Ha]), the ring $k[[t^{\Phi_+}]]$ of formal power series with coefficients in k and exponents in Φ_+, such that the set of their exponents is a well ordered subset of Φ_+ (see also [Ka], [B2]). This ring is naturally equipped with the t-adic valuation, with values in Φ_+. Choose algebraically independent series without constant term $u_i(t) \in k[[t^{\Phi_+}]]$, $1 \leq i \leq r$ (see [McL-S]) and let R be the localization of the sub k-algebra of $k[[t^{\Phi_+}]]$ generated by $u_1(t), \ldots, u_r(t)$ at the maximal ideal generated by these elements. Since R is the image of an injective morphism $\iota \colon k[u_1, \ldots, u_r]_{(u_1,\ldots,u_r)} \to k[[t^{\Phi_+}]]$ determined by $u_i \mapsto u_i(t)$, the t-adic valuation of $k[[t^{\Phi_+}]]$ induces a rational valuation of height one on $k[u_1, \ldots, u_r]_{(u_1,\ldots,u_r)}$. The morphism ι extends, by substitution in formal power series, to a map $k[[u_1, \ldots, u_r]] \to k[[t^{\Phi_+}]]$, with a certain kernel H, which describes the formal

relations between the $u_i(t)$. By Proposition 5.1, the completion $\hat{R}^\nu$ is the image of this map.

5.2 Scalewise completions of R**.** The gist of this subsection is the construction of ν-adic completions of R such that their quotients by the centers of the valuations with which $\hat{\nu}$ is composed are complete for the image of the $\hat{\nu}$-adic topology. The natural choice is a quotient of the m-adic completion of R.
The connexion with the previous subsection is provided by the following:

Proposition 5.8 (Zariski, [Z5], p.29) *If the nœtherian local ring R is analytically irreducible, for any prime ideal* $\mathbf{p}$ *high symbolic powers of* $\mathbf{p}$ *are contained in high powers of the maximal ideal m.* □

Corollary 5.9 *If the nœtherian local ring R is analytically irreducible, for any non-negative valuation ν, the ν-adic topology of R is finer than its m-adic topology.*

What I shall really use is the following

Proposition 5.10 *If the nœtherian local integral domain R is complete for the m-adic topology, for any non negative valuation ν it is complete for the ν-adic topology as well.*

Proof One can use Corollary 5.9 but also argue directly as follows: by Proposition 5.3 we may assume that ν is of height one, and then the distinct valuation ideals $\mathcal{P}_j$ form a simple sequence having intersection zero; since R is complete, by Chevalley's theorem we have $\hat{\mathcal{P}}_j \subset m^{s(j)}$ with $s(j)$ tending to infinity with j. Let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in R for the ν-adic topology. By the preceding observation, it is also a Cauchy sequence for the m-adic topology, and so converges, since R is complete, to some $x \in R$. Let us show that it converges to x for the ν-adic topology as well. Say that we have $x - x_n \in m^{s(n)}$, with $s(n)$ tending to infinity with n. Since (x_n) is a Cauchy sequence for the ν-adic topology, for $j \geq n$ we have $x_j - x_n \in \mathcal{P}_{t(n)}$, with $t(n)$ tending to infinity with n. Since $x - x_j \in m^{s(j)}$, for all $j \geq n$ we have $x - x_n \in \mathcal{P}_{t(n)} + m^{s(j)}$. But $\mathcal{P}_{t(n)}$ is closed for the m-adic topology since R is nœtherian, so letting j increase gives $x - x_n \in \mathcal{P}_{t(n)}$, which shows that (x_n) converges to x in the ν-adic topology. □

Since the property of being m-adically complete is preserved under taking quotients, it is reasonable to try to extend the valuation ν to an m-adically complete ring; it will then be complete for ν, and is quotients as well. The case of rational valuations of height one suggests to try to extend ν to a quotient of $\hat{R}^m$.

Definition 5.11 Let R be a nœtherian local domain. Given a family $(\xi_i)_{i\in I}$ of elements of the maximal ideal m of R, we say that a birational map $R \to R'$ to a local ring R' is a *birational toric extension* of R associated to the family if there exists a finite subset $F \subset I$ and monomials $(\xi^{b^j})_{j\in F}$ with $b^j \in \mathbf{Z}^F$ such that R' is a localization of $R[(\xi^{b^j})_{j\in F}]$ at a maximal ideal.

Remarks 5.12 1) In this text the elements ξ_i will be representatives in R of a system of generators $\overline{\xi}_i$ of the k-algebra $\mathrm{gr}_\nu R$ for a rational valuation ν of R. We will consider only toric extensions of the following form: there exists a regular rational cone $\sigma = \langle a^1, \ldots, a^N \rangle$ in $\mathbf{R}^N_+$ with $N = \#F$, such that if we denote by M the matrix with columns $(a^1, \ldots, a^N)$, the matrix with columns $(b^1, \ldots, b^N)$ is M^{-1}.

2) Each vector b^j can be written as a difference $b^j = b^j_+ - b^j_-$ of vectors with non negative coordinates. The equality $\nu(\xi^{b^j}) = 0$ means that there is a constant $c_j \in k^*$ such that $\overline{\xi}^{b^j_+} - c_j\overline{\xi}^{b^j_-} = 0$ in $\mathrm{gr}_\nu R$. If we have presented, according to Corollary 4.3, the graded algebra $\mathrm{gr}_\nu R$ as $k[(U_i)_{i\in I}]/\big((U^m - \lambda_{mn}U^n)_{(m,n)\in L}\big)$, this shows that the binomial $U^{b^j_+} - c_jU^{b^j_-}$ must belong to the binomial ideal, and c_j is a monomial in the λ_{mn}. Heuristically, this means that some of the binomial relations in $\mathrm{gr}_\nu R$ correspond to translations of the center of the valuation ν with respect to the origin of the new coordinates in toric extensions subordinate to ν, and the constants λ_{mn} correspond to the coordinates of these translations.

If R is endowed with a rational valuation ν, it will be assumed implicitely that $\nu(\xi^{b^j}) \geq 0$ for $j \in F$; the maximal ideal with respect to which we localize $R[\xi^{b^1}, \ldots, \xi^{b^N}]$ will then be the center of ν in this ring. In this manner we obtain a birational toric extension

$$R \subset R' \subset R_\nu.$$

In this case we say that the toric extension is *subordinate* to ν. Let us now consider a valuation ν_1 of R, with which ν is composed; denote by Φ_1 the group of ν_1, by $\mathbf{p}_1$ the center of ν_1 in R and by $\overline{\nu}$ the valuation on $R/\mathbf{p}_1$ which is the image of ν.

Lemma 5.13 *Let ν be a valuation of R and $R \to R'$ be a birational toric extension subordinate to ν associated to a finite family $(\xi_j)_{j\in F}$ of elements such that $\nu_1(\xi_j) = 0$ for $j \in F$. For each $\phi_1 \in \Phi_1 \cup \{0\}$, each element of the the kernel and cokernel of the natural map of $R/\mathbf{p}_1$-modules*

$$\mathcal{P}_{\phi_1}(R)/\mathcal{P}^+_{\phi_1}(R) \otimes_{R/\mathbf{p}_1} R'/\mathbf{p}_1R' \to \mathcal{P}_{\phi_1}(R')/\mathcal{P}^+_{\phi_1}(R')$$

is annihilated by a monomial in the $(\xi_j)_{j\in F}$. In particular they are torsion modules, and the kernel is the torsion submodule of $\mathcal{P}_{\phi_1}(R)/\mathcal{P}^+_{\phi_1}(R) \otimes_{R/\mathbf{p}_1} R'/\mathbf{p}_1R'$.

Proof Let

$$x = (\sum_i c_i\xi^{\alpha_{i+}-\alpha_{i-}})(\sum_k d_k\xi^{\beta_k})^{-1},$$

with $c_i, d_k \in R$, $\alpha_{i+} \geq 0$, $\alpha_{i-} \geq 0$, and $\nu(\sum_k d_k\xi^{\beta_k}) = 0$ be a representative of an element $\overline{x}$ of $\mathcal{P}_{\phi_1}(R')/\mathcal{P}^+_{\phi_1}(R')$. Set $\alpha_- = \max_i\alpha_{i-} = \alpha_{i_0-}$, $\alpha_+ = \alpha_{i_0+}$.
The element $\xi^{\alpha_-}(\sum_k d_k\xi^{\beta_k})x$ is in $\mathcal{P}_{\phi_1}(R)$, so that

$$\xi^{\alpha_+}x = \xi^{\alpha_+-\alpha_-}.\xi^{\alpha_-}(\sum_k d_k\xi^{\beta_k})(\sum_k d_k\xi^{\beta_k})^{-1}x$$

is in $\mathcal{P}_{\phi_1}(R)R'$. This shows that the cokernel of our map is a torsion module. The proof for the kernel is analogous if we notice that $\mathcal{P}_{\phi_1}(R')/\mathcal{P}^+_{\phi_1}(R')$ is a torsion-free $R/\mathbf{p}_1$-module.
This follows from the fact that the $R'/\mathcal{P}^+_0(R')$-module $\mathcal{P}_{\phi_1}(R')/\mathcal{P}^+_{\phi_1}(R')$ is torsion-free (see subsection 2.3) and the map $R/\mathbf{p}_1 \to R'/\mathcal{P}^+_0(R')$ is injective since by construction $\mathcal{P}^+_0(R') \cap R = \mathcal{P}^+_0(R) = \mathbf{p}_1$. The last sentence also follows from this remark. □

Remark 5.14 For $\phi_1 = 0$, our map becomes $R'/\mathbf{p}_1R' \to R'/\mathcal{P}^+_0(R')$. This map is clearly surjective; its kernel consists of the torsion elements of the $R/\mathbf{p}_1R$-module $R'/\mathbf{p}_1R'$. The ideals $\mathbf{p}_1R'$ and $\mathcal{P}^+_0(R')$ correspond respectively to the total transform and the strict transform under our toric extension of the center of ν_1 in Spec$R$.

Definition 5.15 Assertion TLU(d) (Toric local uniformization in dimension $\leq d$) is the following: For every local equicharacteristic excellent integral domain R of dimension $\leq d$, given a rational valuation ν on R, and elements $(\xi_i)_{i\in I}$ in R whose initial forms generate the graded k-algebra $\mathrm{gr}_\nu R$, there is a birational toric (in the ξ_i) extension $R\subset R'\subset R_\nu$ such that:
a) The local ring R' is regular, with a system of coordinates containing elements whose images in $\mathrm{gr}_\nu R'$ are part of a minimal system of generators.
b) The kernel of the scalewise completion map $\hat{R'}^{m'}\to \hat{R'}^{(\nu)}$ (see Proposition 5.19 below and the remarks which follow) is generated by part of a regular system of generators of the maximal ideal of the regular local ring $\hat{R'}^{m'}$; in particular the complete local ring $\hat{R'}^{(\nu)}$ is regular.

Definition 5.16 Assertion TP(d) (Toric principalization in dimension $\leq d$) is the following: For every local equicharacteristic excellent integral domain R of dimension $\leq d$, given a rational valuation ν and an ideal I of R there is a birational toric extension $R'\subset R_\nu$ such that IR' is of the form $(\xi')^\delta u'$, where u' is a unit of R', the ξ'_j are elements of the maximal ideal m' and their images in $\mathrm{gr}_\nu R'$ are part of a system of generators of this k-algebra.

Definition 5.17 Let k be a ring and Φ be a totally ordered group of height h and

$$(0)\subset\Psi_{h-1}\subset\cdots\subset\Psi_1\subset\Psi_0=\Phi$$

the ordered sequence of its convex subgroups. A graded map $G\to G'$ of Φ-graded k-algebras is said to be *scalewise graded* if for each Ψ_k the image of the subalgebra $G_k\subset G$ generated by homogeneous elements whose degree is in Ψ_k is contained in the subalgebra G'_k generated by homogeneous elements of G' whose degree is in Ψ_k.

Lemma 5.18 *If k is a field, if G and G' are Φ-graded integral k-algebras, if all homogeneous components of G and G' are k-vector spaces of dimension ≤ 1 and if the semigroups $\Gamma=\{\phi\in\Phi/G_\phi\neq(0)\}$ and $\Gamma'=\{\phi\in\Phi/G'_\phi\neq(0)\}$ generate Φ as a group, then a graded inclusion $G\to G'$ is birational.*

Proof It suffices to show that any non zero homogeneous element of G' is the quotient of the images of two homogeneous elements of G. Such an element is of the form cy_ϕ where y_ϕ is a generator of the k-vector space G'_ϕ and $c\in k^*$. Since ϕ is the difference of two elements of Γ, we may write $y_\phi=d_\phi x_{\phi_1}x_{\phi_2}^{-1}$ where x_{ϕ_i} is a generator of G_{ϕ_i} and $d_\phi\in k^*$. The result follows. □

A rather special case (valuations of height one and center m) of the first part of the following Proposition was proved *en passant* by Zariski (see [Z6], from the bottom of p. 63) and has been proved again by Spivakovsky ([S2]) with a different perspective, in both cases without the assumption that the residue field is algebraically closed; see the remark after Proposition 5.1. Spivakovsky has also communicated to me that he had stated a different extension of his result to arbitrary finite heights.

Proposition 5.19 **Let R be a local equicharacteristic excellent integral domain with maximal ideal m and an algebraically closed residue field, and let ν a rational valuation on R with value group Φ.
a) There exists a prime ideal H of the m-adic completion $\hat{R}^m$ of R such that ν extends to a valuation $\hat{\nu}$ of $\hat{R}^m/H$ in such a way that:
b) the associated graded map*

$$\mathrm{gr}_\nu R \to \mathrm{gr}_{\hat{\nu}} \hat{R}^m/H$$

*is a scalewise birational extension of Φ-graded R/m-algebras, uniquely determined by (R,ν).
If the valuation ν is of height one, the map $\mathrm{gr}_\nu R \to \mathrm{gr}_{\hat{\nu}} \hat{R}^m/H$ is an isomorphism.**

The proof is not yet complete and should be the subject of a subsequent paper.

Remarks 5.20 1) The fact that $\hat{\nu}$ extends ν implies that $H \cap R = (0)$.
2) The fact that the extension of the graded rings is scalewise birational implies that any toric modification we make in $\hat{R}^{(\nu)}$ with respect to representatives $(\eta_j)_{j\in J}$ of the generators of the graded algebra $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ can be viewed as coming from a toric modification in R with respect to the chosen system of representatives $(\xi_i)_{i\in I}$.
3) Although the ideal H is unique in the case considered by Zariski and Spivakovsky, it is not uniquely determined by (R,ν) in general; it is only the graded map associated to the map $R \to \hat{R}^m/H$ which is uniquely determined. Once H is fixed, the extension of ν is unique.
The ideal H depends on the choice of the representatives $\xi_i \in R$ of the generators of $\mathrm{gr}_\nu R$ and of the complementary generators of the maximal ideal of R.

Let $R = k[x_1,x_2,x_3]_{(x_1,x_2,x_3)}$, let $w(x_1) = \sum_{k=1}^{\infty} c_k x_1^k$ be a series which is transcendental over $k(x_1)$, consider the ring $\overline{R} = k[x_1,x_2]_{(x_1,x_2)}$, its injection $\iota\colon \overline{R} \subset k[[x_1]]$ defined by $x_1 \mapsto x_1,\ x_2 \mapsto w(x_1)$ and the valuation $\overline{\nu}$ of $\overline{R}$ defined by the restriction via the injection ι of the x_1-adic valuation. Let ν be the rational valuation of R composed of $\overline{\nu}$ and the x_3-adic valuation. As we have seen in example 5.7 of subsection 5.1, the ideal $\overline{H}$ is the ideal of $k[[x_1,x_2]]$ generated by $x_2 - w(x_1)$. For given series without constant terms $y(x_3), z(x_3)$ we may choose as representatives of x_1, x_2 the elements $x_1 + y(x_3),\ x_2 + z(x_3)$. The corresponding ideal $H_{y,z}$ of $k[[x_1,x_2,x_3]]$ is $(x_2+z(x_3)-w(x_1+y(x_3)))k[[x_1,x_2,x_3]]$. For each pair $y,z \in k[[x_3]]$ of series without constant term the valuation $\hat{\nu}$ on $k[[x_1,x_2,x_3]]/H_{y,z}$ which is composed of the valuation $\hat{\overline{\nu}}$ (which is the x_1-adic valuation on $k[[x_1]]$), and the $\tilde{x}_3$-adic valuation, where $\tilde{x}_3$ is the image of x_3 modulo $H_{y,z}$, induces the valuation ν on R.

In spite of this non uniqueness, I shall denote by $\hat{R}^{(\nu)}$ a quotient $\hat{R}^m/H$ as in the Proposition and sometimes even speak of "the" scalewise completion of (R,ν). In a sense they are all "equisingular" with respect to $\hat{\nu}$, just as general sections by non singular spaces transverse to an equisingularity stratum through a singular point are all equisingular.

4) A completion which does not induce an isomorphism on the associated graded rings may be somewhat surprising; let us take example 3.19 and assume that $\hat{R}^m$ is an integral domain but $f\hat{R}^m$ is not a prime ideal; say that we have $f\hat{R}^m = \hat{g}\hat{h}\hat{R}^m$ with $\hat{g}\hat{R}^m$ and $\hat{h}\hat{R}^m$ prime ideals. By Proposition 7.10 below, an ideal $\overline{H}$ of $\hat{R}^m/f\hat{R}^m$ as in the Proposition contains a unique minimal prime of $\hat{R}^m/f\hat{R}^m$.

Assume that we are in the simple situation where $\overline{H} = \hat{g}(\hat{R}^m/f\hat{R}^m)$. The valuation ν extends to a valuation on $\hat{R}^m$ which is composed of the $\hat{g}$-adic valuation, of height one, of $\hat{R}^m$, and the valuation $\hat{\overline{\nu}}$ on $\hat{R}^m/\hat{g}\hat{R}^m$ which extends the valuation $\overline{\nu}$ on $R/(f)$. Remark that this is a case where $\overline{H} \neq (0)$ but $H = (0)$. Now by example 3.19, the graded ring $\mathrm{gr}_{\hat{\nu}}\hat{R}^m$ is equal to $\mathrm{gr}_{\hat{\overline{\nu}}}(R^m/\hat{g}\hat{R}^m)[\hat{G}]$, where $\hat{G}$ is the initial form of $\hat{g}$ with respect to $\hat{\nu}$. The inclusion

$$\mathrm{gr}_{\nu}R = \mathrm{gr}_{\overline{\nu}}(R/fR)[F] \subset \mathrm{gr}_{\hat{\overline{\nu}}}(R^m/\hat{g}\hat{R}^m)[\hat{G}]$$

is described by saying that $\mathrm{gr}_{\overline{\nu}}(R/fR) \to \mathrm{gr}_{\hat{\overline{\nu}}}(R^m/\hat{g}\hat{R}^m)$ is a scalewise birational map by induction on the dimension, and $F \mapsto (\mathrm{in}_{\hat{\overline{\nu}}}\overline{\hat{h}})\hat{G}$, where $\overline{\hat{h}}$ is the image of $\hat{h}$ in $\hat{R}^m/\hat{g}\hat{R}^m$. This last step introduces a scalewise birational character in the graded rings extension if it was not already present for R/fR.

Corollary 5.21 *Abhyankar's inequality can be made more precise in the case of a rational valuation ν of an excellent local ring R; with the notations of section 3, we have:*

$$\mathrm{r}(\nu) \leq \dim \hat{R}^{(\nu)} \leq \dim R.$$

In particular, if $\mathrm{r}(\nu) = \dim R$ *the ideal H must be a minimal prime of $\hat{R}^m$ and if in addition R is analytically irreducible, we have $\hat{R}^{(\nu)} = \hat{R}^m$. In all cases, in view of Proposition 7.10, the ideal H contains a unique minimal prime of $\hat{R}^m$.*

Remarks 5.22 1) If ν is a valuation of height one whose ring birationally dominates R, we have $\hat{R}^{(\nu)} = \hat{R}^{\nu}$. One may ask, if we only assume that ν is of height one, what is the relationship of the ring $\hat{R}^{(\nu)}$ with the m-adic Hausdorff completion of $\hat{R}^{\nu}$.
2) The Proposition can be used to study extensions of a rational valuation ν of an excellent local ring R to the completion $\hat{R}^m$; by composition with $\hat{\nu}$, every valuation of the regular local ring $\hat{R}^m_H$ centered at is maximal ideal gives rise to a valuation of $\hat{R}^m$ extending ν. As we saw in Corollary 5.21, if $\mathrm{r}(\nu) = \dim R$, the ideal H must be a minimal prime of $\hat{R}^m$ and if R is analytically irreducible, ν extends to $\hat{R}^m$ with the same group. This extension is then unique since $\hat{\mathcal{P}}_\phi = \mathcal{P}_\phi\hat{R}^m$.

If $\mathrm{r}(\nu) = \dim R - 1$, and we assume that H is not a minimal prime of $\hat{R}^m$, then the localization $\hat{R}^m_H$ is one-dimensional because of the Proposition, and it is a regular local ring by Proposition 7.10 below. It is a valuation ring with group $\mathbf{Z}$. Denote by μ its valuation. The valuation $\tilde{\nu}$ on $\hat{R}^m$ with values in $\mathbf{Z} \oplus \Phi$ ordered lexicographically, defined by $\tilde{\nu}(x) = (\mu(x), \hat{\nu}(x \bmod. H))$ is an extension of ν to $\hat{R}^m$.

In order to check the uniqueness of the extension, assuming that R is analytically irreducible, remark that $\bigcap_{\phi\in\Phi_+}\mathcal{P}_\phi\hat{R}^m \subseteq H$ by construction of H. The first ideal is prime because the graded ring associated to the filtration of $\hat{R}^m$ by the $\mathcal{P}_\phi\hat{R}^m$ is equal to $\mathrm{gr}_\nu R$, which is a domain. Since R is analytically irreducible, and in view of the dimension condition, we have $\bigcap_{\phi\in\Phi_+}\mathcal{P}_\phi\hat{R}^m = H$. Then, given an extension $\tilde{\nu}\,'$ of ν to $\hat{R}^m$ with group $\tilde{\Phi}$, the ideal H appears as the set of elements $z \in \hat{R}^m$ such that $\tilde{\nu}\,'(z) > \phi \ \ \forall\, \phi \in \Phi \subset \tilde{\Phi}$, and $\Phi \subset \tilde{\Phi}$ is a convex subgroup of the group $\tilde{\Phi}$. Then $\tilde{\nu}\,'$ is composed of a valuation on the localization $\hat{R}^m_H$, which is regular of dimension one, and a valuation on $\hat{R}^m/H$, which is uniquely determined since its valuation ideals must be the $\mathcal{P}_\phi\hat{R}^m/H$, and coincides with the extension $\hat{\nu}$ of the Proposition. This shows that if we assume R analytically irreducible, in

the case where $\mathrm{r}(\nu) = \dim R - 1$ there is a unique extension of ν to $\hat{R}^m$, described as above. These two results on extensions of valuations to $\hat{R}^m$ in the case where it is a domain are due to Heinzer and Sally ([H-S]) under an assumption weaker than excellence.
3) After replacing R by $\hat{R}^{(\nu)}$ and ν by $\hat{\nu}$, we may assume that the local ring R is complete for the m-adic topology, with a valuation ν, so that it is henselian by Hensel's lemma, admits (in the equicharacteristic case) a field of representatives by Cohen's theorem and is complete for the ν-adic topology.

Definition 5.23 Given R and a rational valuation as above, we call a map $R \to \hat{R}^{(\nu)}$ such as we have built in Proposition 5.19 a *scalewise* ν-adic completion of (R, ν).

Definition 5.24 We say that the valuation ν is *normally flat in degree* ϕ if either $\mathcal{P}_\phi/\mathcal{P}^+_\phi$ is zero, or it is a flat $\overline{R}$-module, where $\overline{R} = R/\mathbf{p}$ with $\mathbf{p}$ the center of ν in R.

Question.: Given a rational valuation ν on R, and a valuation μ with semigroup Δ, with which ν is composed, is it true that for any $\delta \in \Delta$ there exits a birational toric extension $R \to R'$ compatible with ν, determined by elements of μ-valuation zero, and such that μ is normally flat in degree δ on R'?

I now show that with this definition of scalewise completion one has a Cohen-type theorem.

Lemma 5.25 *Let R be an excellent equicharacteristic local integral domain, and ν a rational valuation on R. Assume that the valuation ν is of height one. Lift a system of generators $(\overline{\xi}_i)_{i\in I}$ of the k-algebra $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ to elements $\xi_i \in R$ and choose a field of representatives $k \subset \hat{R}^{(\nu)}$. The k-algebra $k[(\xi_i)_{i\in I}]$ is dense in $\hat{R}^{(\nu)}$ for the $\hat{\nu}$-adic topology.*

Proof We retain the notations of Proposition 5.19, remarking that our assumption implies that $\hat{\nu}$ is trivial on k and remembering that in this case $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)} = \mathrm{gr}_\nu R$. If we pick $x \in \hat{R}^{(\nu)}$, its $\hat{\nu}$-adic initial form is in $\mathrm{gr}_\nu R$, and therefore $\mathrm{in}_{\hat{\nu}} x = P_1(\overline{\xi}_i)$, with $P_1 \in k[(U_i)_{i\in I}]$. The polynomial P_1 is in fact a term cU^a with $c \in k^*$ and U^a a monomial, but I will not use this here. Abbreviating $(\xi_i)_{i\in I}$ to (ξ), we set $x^{(0)} = x$, $x^{(1)} = x - P_1(\xi)$, and iterate this process. We have for all $j \geq 1$ the inequality $\hat{\nu}(x^{(j+1)}) > \hat{\nu}(x^{(j)})$ and thus $\hat{\nu}(x^{(j+1)} - x^{(j)}) = \hat{\nu}(x^{(j)})$. Now remark that since $\hat{R}^{(\nu)}$ is nœtherian by Proposition 5.19, the valuation $\hat{\nu}(\hat{m})$ is > 0, and since $\hat{\nu}$ is of height one, for any $\phi \in \Phi_+$ there is an integer $N(\phi)$ such that $\hat{m}^{N(\phi)} \subset \hat{\mathcal{P}}_\phi$. Then there are only finitely many distinct $\hat{\mathcal{P}}_{\phi'}$ containing $\hat{\mathcal{P}}_\phi$, so that in a finite number of steps we have $\hat{\nu}(x^{(j+1)}) > \phi$, which proves that the sequence $x - x^{(j)}$ of elements of $k[(\xi_i)_{i\in I}]$ converges to x, and the Lemma. □

Remark 5.26 Without the height one assumption, it could happen that the valuations of the differences $x - \sum_{j=1}^r P_j(\xi)$ increase at each step of the construction but remain bounded in the semigroup Φ_+.

I assume until the next Corollary that the local ring R is complete for its m-adic topology; see remark 2) at the end of subsection 5.1.

Example 5.27 Let k be a field, let R be a complete nœtherian local k-algebra with residue field k, and let $f \in R$ generate a prime ideal. We go back to example

3.19 of subsection 3.4 and its notations. Let $x \in R$ and let us write it in the form $x = af^\ell$ with $a \notin (f)$. We have $\nu_1(x) = \ell$ and its ν-initial form is $(\mathrm{in}_{\overline{\nu}}\ \overline{a})F^\ell$. Let us fix a field of representatives $k \subset R$; in general, the element a will not be a polynomial with coefficients in k in the liftings (ξ_i) to R of the generators of $\mathrm{gr}_{\overline{\nu}}\overline{R}$, so that we will have, denoting by $\tilde{a}$ a lifting in R of $\mathrm{in}_{\overline{\nu}}\overline{a}$ which is a polynomial in the (ξ_i), and choosing f as representative of ξ, a lifting $\widetilde{\mathrm{in}}_\nu x$ of $\mathrm{in}_\nu x$ which is polynomial in the ξ_i and f, and such that

$$x - \widetilde{\mathrm{in}}_\nu x = (a - \tilde{a})f^\ell \mod. f^{\ell+1}$$

and the ν_1-value will not have changed.

However, if we approximate $\overline{a}$ in $\overline{R}$ by a sequence of polynomials $P_s((\overline{\xi}_i)_{i\in I_1})$ in $k[(\overline{\xi}_i)_{i\in I_1}]$, we will have after replacing the $\overline{\xi}_i$ by their representatives ξ_i in R $\lim_{s\to\infty}\big(a - P_s((\xi_i)_{i\in I_1})\big) \in fR$ so that we find as limit of a sequence of polynomials $P_s((\overline{\xi}_i)_{i\in I_1})f^\ell$, with the notations of the first remark of this subsection, an element $z^{(1)} \in R$ congruent to af^ℓ modulo the ideal $f^{\ell+1}R$. We may write $z^{(1)} = a_\ell f^\ell$. Then we have to repeat the process with $x_1 = x - z^{(1)}$ and the next term. At each stage we build by the convergence of a series an element $z^{(t)} \in R$ and x is the limit of the series $\sum_{t\geq 1} z^{(t)}$. In fact, the element x then appears as the sum of a series $\sum_{k=\ell}^{\infty} a_k f^k$. I call this situation "scalewise density"[7] for the subalgebra $k[(\xi_i)_{i\in I};\ f]$ in R.

Now for the general case:

Let R be a *complete* integral nœtherian local ring with a field of representatives k, and ν a k-valuation of its field of fractions, positive on R and whose center is the maximal ideal m of R. Let $(\xi_i)_{i\in I}$ be a set of elements of R whose images in $\mathrm{gr}_\nu R$ generate it as a k-algebra. Consider the set of isolated subgroups of the value group Φ of ν

$$(0) = \Psi_h \subset \Psi_{h-1} \subset \cdots \subset \Psi_t \subset \cdots \subset \Psi_1 \subset \Psi_0 = \Phi,$$

and the corresponding valuations ν_t with value group Φ/Ψ_t indexed by their height. Let

$$\mathbf{p}_r \subset \mathbf{p}_{r-1} \subset \cdots \subset \mathbf{p}_1 \subset m$$

be the sequence of the distinct centers of the valuations ν_t in R. Let us partition the set I as follows: $I = \bigcup_{k=1}^{h} I_t$ where

$$I_t = \{i \in I \mid \nu(\xi_i) \in \Psi_{h-t} \setminus \Psi_{h-t+1}\},$$

remarking that some of the sets I_t may be empty. Let us also define the following sequence of subrings of R:
$\hat{R}_1$ is the closure of $k[(\xi_i)_{i\in I_1}]$ in R for the topology defined by the filtration by the ideals $(\mathcal{P}_\psi)_{\psi\in\Psi_{h-1}}$. We can continue and define inductively $\hat{R}_t$ to be the closure of $\hat{R}_{t-1}[(\xi_i)_{i\in I_t}]$ in R for the topology defined by the filtration by the ideals $(\mathcal{P}_\psi)_{\psi\in\Psi_{h-t}}$.

Proposition 5.28 *For each t, $1 \leq t \leq h$, the ring $\hat{R}_t$ is local and its maximal ideal is generated by the $(\xi_i)_{i\in\bigcup_{\ell=1}^{t} I_\ell}$.*

[7] F.-V. Kuhlmann has suggested to me that this idea could be related to Ostrowski's pseudo-Cauchy sequences (see [Roq], [Ka]) and Ribenboim's notion of "complet par étages" ([Ri]) in the theory of completion of valued fields. The problem is similar, but I did not compare the two definitions; there is a major difference in that I work with nœtherian rings, not valuation rings.

Proof The ring $\hat{R}_1$ is local because if $y=\sum a_\alpha \xi^\alpha$ is not in the closure of the ideal $(\xi_i)_{i\in I_1}$, we can write $y=a_0(1+\sum_{\alpha>0} a_0^{-1}a_\alpha\xi^\alpha)$ with $a_0\in k^*$, and to show that y^{-1} is in $\hat{R}_1$ we have to show that the series

$$1-\sum_{\alpha>0} a_0^{-1}a_\alpha\xi^\alpha+(\sum_{\alpha>0} a_0^{-1}a_\alpha\xi^\alpha)^2+\cdots+(-1)^n(\sum_{\alpha>0} a_0^{-1}a_\alpha\xi^\alpha)^n+\cdots$$

converges, which is clear because Ψ_{h-1} is a group of height one. The passage from $\hat{R}_{t-1}$ to $\hat{R}_t$ is similar, if we notice the following

Lemma 5.29 *If Ψ is an isolated subgroup of Φ such that Φ/Ψ is of height one, the semigroup $\Phi_+\setminus\Psi_+$ is archimedean.*

Proof It follows directly from the fact that Φ/Ψ is archimedean. □

So, if $\nu(\xi)\in I_t$, then as $n\in\mathbf{N}$ tends to infinity, $n\nu(\xi)$ eventually exceeds any element of Ψ_{h-t}. □

Definition 5.30 We say that $k[(\xi_i)_{i\in I}]$ is scalewise dense in R for the ν-adic topology if $\hat{R}_h$ is equal to R.

Proposition 5.31 *Let R_ν be the ring of a rational valuation of a complete equicharacteristic nœtherian local integral domain R. If we fix a field of representatives k of R and choose representatives $(\xi_i)_{i\in I}$ in R of a system of generators $(\overline{\xi}_i)_{i\in I}$ of the k-algebra $\mathrm{gr}_\nu R$, where $k=R/m=R/(m_\nu\cap R)$, the algebra $k[(\xi_i)_{i\in I}]$ is scalewise dense in R for the ν-adic topology.*

Proof It is useful to keep in mind Corollary 4.9, which gives us the structure of the initial forms which appear here. We have to prove that, given $x\in R$, there is a sequence $(r_j)_{j\in\mathbf{N}}$ of elements of $\hat{R}_{h-1}[(\xi_i)_{i\in I_h}]$ converging to x in the $\tilde{\nu_h}$-adic topology. Define the sequence of $x^{(j)}$ and $P_j(\xi_i)$ as in the proof of Lemma 5.25, and let us build from it a sequence $(r_j)_{j\in\mathbf{N}}$ of elements of $\hat{R}_{h-1}[(\xi_i)_{i\in I_h}]$ such that the valuations $\nu(x-r_j)$ exceed, for large j, any $\phi\in\Phi_+$ given in advance. Let s be the least integer such that for infinitely many values of the integer j we have $\nu_s(x-x^{(j+1)})>\nu_s(x-x^{(j)})$. If $s=1$, since the valuation ν_1 is of height one, for any $\phi_1\in\Phi_{1+}$, after finitely many steps we have $\nu_1(x-x^{(j)})>\phi_1$, hence for any $\phi\in\Phi_+$, after finitely many steps we have $\nu(x-x^{(j)})>\phi$, so we may take $r_j=x-x^{(j)}$ and get $\nu(x-\sum_{r=0}^j P_r(\xi_i))=\nu(x-r_j)>\phi$ and this shows that $x\in\hat{R}_h$. Assume now $s>1$, and let $x^{(1)}$ be the representative in $\hat{R}_{s-1}[(\xi_i)_{i\in I_s}]$ of the initial form $\mathrm{in}_{\nu_{s-1}}(x)\in\mathrm{gr}_{\nu_{s-1}}R$. This makes sense because we may assume by induction on the dimension of R that the subring $\hat{R}_{s-1}\subset R$ is a representative of $R/(m_{\nu_{s-1}}\cap R)$. We define inductively the $x^{(j)}$ as representatives in $\hat{R}_{s-1}[(\xi_i)_{i\in I_s}]$ of $\mathrm{in}_{\nu_{s-1}}(x-x^{(j-1)})$. By definition of s, there exists a j_0 such that $\nu_{s-1}(x-x^{(j+1)})=\nu_{s-1}(x-x^{(j)})$ for $j\geq j_0$. By substracting from x a polynomial in the ξ_i, we may assume that $j_0=1$. Note that the center of ν_{s-1} is necessarily distinct from the center of ν_s, since if the two centers are equal, there are only finitely many distinct ν_s-ideals between two consecutive ν_{s-1}-ideals of R, according to [Z-S], Vol. II, Appendix 3, Corollary p.345.

We consider the initial forms $\mathrm{in}_{\nu_{s-1}}(x-x^{(j)})\in\mathrm{gr}_{\nu_{s-1}}R$; for $j\geq j_0$; they all have the same degree, say ϕ_{s-1} and they are in $\mathcal{P}_{\phi_{s(j)}}/\mathcal{P}^+_{\phi_{s-1}}$ with $\phi_s(j)$ increasing strictly with j. The $R/\mathbf{p}_{s-1}$-submodules $\mathcal{P}_{\phi_s}/\mathcal{P}^+_{\phi_{s-1}}\subset\mathcal{P}_{\phi_{s-1}}/\mathcal{P}^+_{\phi_{s-1}}$ form a simple infinite sequence in view of Proposition 3.17. This sequence of submodules has intersection (0) and by the module-theoretic version of Chevalley's theorem ([B3], Chap. IV, §2, No.5, Cor.4) there is a sequence of integers $t(\phi_s(j))$ tending to infinity with the image of $\phi_s(j)$ in the height one group Ψ_{s-1}/Ψ_s and such that $\mathcal{P}_{\phi_s(j)}/\mathcal{P}^+_{\phi_{s-1}} \subset m^{t(\phi_s(j))}(\mathcal{P}_{\phi_{s-1}}/\mathcal{P}^+_{\phi_{s-1}})$. Since each homogeneous component $\mathcal{P}_{\phi_{s-1}}/\mathcal{P}^+_{\phi_{s-1}}$ of $\mathrm{gr}_{\nu_{s-1}}R$ is a $R/(m_{\nu_{s-1}}\cap R)$-module of finite type, it is complete for the $m/(m_{\nu_{s-1}}\cap R)$-topology. It is also complete for the $\overline{\nu}_s$ topology by (the proof of) Proposition 5.10. The sequence of the initial forms $\mathrm{in}_{\nu_{s-1}}(x-x^{(j)})$ then converges in $(\mathrm{gr}_{\nu_{s-1}}R)_{\phi_{s-1}}$ for the m-adic topology, and therefore also for the topology defined by the $\mathcal{P}_{\phi_s}/\mathcal{P}^+_{\phi_{s-1}}$, to a unique limit $\overline{x_1^{(1)}}$. By the definition of $\hat{R}_s$, we can lift $\overline{x_1^{(1)}}$ to an element $x_1^{(1)} \in \hat{R}_s$. This element has the property that for all $j \geq 1$, we have $\nu_{s-1}(x-x_1^{(1)}) > \nu_{s-1}(x-x^{(j)}) = \nu_{s-1}(x)$. Replacing now x by $x-x_1^{(1)}$, and repeating this construction, we build a sequence of elements $x_1^{(q)} \in \hat{R}_h$ such that $\nu_{s-1}(x-x_1^{(q)})$ increases at each step.

We may now repeat the whole process, replacing the sequence $x^{(j)}$ by $x_1^{(q)}$; the integer s is replaced by $s-1$ and ultimately we are reduced to the case where $s=1$, which is already settled. This proves the Proposition. □

I stop assuming that the local ring R is complete, but I assume until the end of this section that its residue field is algebraically closed.

Corollary 5.32 *Let ν be a rational valuation on the local equicharacteristic excellent integral domain R; let k be the residue field of R and let $(\xi_i)_{i\in I}$ be representatives in R of a system of generators $(\overline{\xi}_i)_{i\in I}$ of the graded k-algebra $\mathrm{gr}_\nu R$. Let $k \subset \hat{R}^{(\nu)}$ be a field of representatives in the scalewise completion of R built in Proposition 5.19 and let $(\eta_j)_{j\in J}$ be representatives in $\hat{R}^{(\nu)}$ of a system of generators $(\overline{\eta}_j)_{j\in J}$ of the graded k-algebra $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$, which are Laurent monomials in the $(\overline{\xi}_i)_{i\in I}$ (see 5.19). The subalgebra $k[(\eta_j)_{j\in J}]$ is scalewise dense in $\hat{R}^{(\nu)}$. The local subrings $\hat{R}_t \subset \hat{R}^{(\nu)}$ of Proposition 5.28 are complete and nœtherian, and the valuation ν restricted to $\hat{R}_t$ is analytically monomial with respect to the $(\eta_j)_{j\in\bigcup_{\ell=1}^t I_\ell}$.*

Proof The first part follows from the Proposition in view of the fact that $\hat{R}^{(\nu)}$ is complete for its $\hat{m}$-adic topology. To prove that the rings R_t are nœtherian the last statement, use descending induction on k: the ring $\hat{R}_h = \hat{R}^{(\nu)}$ is nœtherian, and each $\hat{R}_{t-1}$ is isomorphic to the quotient of $\hat{R}_t$ by the closure of the ideal generated by $(\xi_i)_{i\in I_t}$. It also reproves that the $\hat{R}_t$ are local.

To prove the last two assertions, we can check that $m_{\nu_{h-1}} \cap \hat{R}_1 = (0)$ so that an element of $R/(m_{\nu_{h-1}}\cap R)$, which is in the closure of $k[(\overline{\xi}_i)_{i\in I_1}]$, lifts uniquely to an element of this closure in R, and we have seen in Proposition 3.25 that the valuation $\overline{\nu}_1$ is monomial with respect to the $(\overline{\eta}_j)_{j\in J_1}$. One passes from R_{t-1} to R_t in the same way. □

5.3 The $\nu_\mathcal{A}$-adic completion of $\mathcal{A}_\nu(R)$. Given a system $(\overline{\xi}_i)_{i\in I}$ of generators of the k-algebra $\mathrm{gr}_\nu R$, giving rise to a surjective map $k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R$, a field of representatives $k \subset \hat{R}^{(\nu)}$ and representatives $(\xi_i)_{i\in I}$ in R of the $\overline{\xi}_i$, the main result of this subsection is the extension of the map $k[(u_i)_{i\in I}] \to \hat{R}^{(\nu)}$ mapping u_i to ξ_i

to a continuous surjective map

$$k[\widehat{(w_j)_{j\in J}}] \to \hat{R}^{(\nu)}$$

from the scalewise completion of the polynomial ring to the scalewise ν-adic completion of R. When $k[\widehat{(w_j)_{j\in J}}]$ is endowed with the term order obtained by giving to w_j the valuation of its image $\eta_j \in \hat{R}^{(\nu)}$, the associated graded map is the map

$$k[(W_j)_{j\in J}] \to \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$$

mapping W_j to $\overline{\eta}_j$, where each $\overline{\eta}_j$ is a Laurent monomial in the $\overline{\xi}_i$.
This is a ν-adic analogue, called here $TC(*)$, of Cohen's structure theorem ([B3], Chap. IX, §3, No.3, Th. 2, a)). Thus, $TC(*)$ is the generalization of the embedding of a plane complex branch in $\mathbf{C}^{g+1}$, while the similar result for the valuation algebra corresponds to the construction of the family degenerating the re-embedded branch to the monomial curve.
For brevity I construct the map directly in the case of the valuation algebra (see Proposition 5.48), which is only slightly more complicated than the ν-adic Cohen theorem.

Proposition 5.33 *In the situation of 5.1, the completion* $\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$ *is the closure of* $\mathcal{A}_{\hat{\nu}}(\hat{R}^\nu)$ *in the algebra of restricted power series* $\hat{R}^\nu\{v^\Phi\}$ *defined by*

$$\hat{R}^\nu\{v^\Phi\} = \varprojlim_{\phi\in\Phi_+}(R/\mathcal{P}_\phi)[v^\Phi].$$

Proof This follows from the definitions, the fact that $\mathcal{S}_\delta = \mathcal{P}_\delta R[v^\Phi]\cap\mathcal{A}_\nu(R)$ and [B4], Chap II, §3, No. 9, Cor.1. □

Note that the closure in $\hat{R}^\nu$ of $\mathcal{P}_\phi$ is $\hat{\mathcal{P}}_\phi$ and that, with the notations of subsection 2.4, we have:

$$\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}} = \varprojlim_{\delta\in\Phi_+}\mathcal{A}_\nu(R)/\mathcal{S}_\delta = \varprojlim_{\delta\in\Phi_+}\bigoplus_{\phi\in\Phi}\big(\mathcal{P}_\phi(R)/\mathcal{P}_\phi(R)\cap\mathcal{P}_\delta(R)\big)v^{-\phi}.$$

So we may view $\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$ as the subring of the $\hat{R}^\nu$-module $\hat{R}^\nu[[v^\Phi]]$ of formal power series $\sum x_\phi v^{-\phi}$, consisting of power series $\sum x_\phi v^{-\phi}$ such that $\hat{\nu}(x_\phi)\geq\phi$ and which are restricted, i.e., such that for any $\psi\in\Phi_+$, all but a finite number of the coefficients x_ϕ are in $\mathcal{P}_\psi$. This last condition implies that when we formally multiply two series

$$\big(\sum x_\phi v^{-\phi}\big)\big(\sum x_\psi v^{-\psi}\big) = \sum_{\eta\in\Phi}\big(\sum_{\phi+\psi=\eta}x_\phi y_\psi\big)v^{-\eta},$$

the coefficient of $v^{-\eta}$ is a convergent series in $\hat{R}^\nu$.

Lemma 5.34 *The closure*

$$\overline{(v^{\Phi_+})\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}}\quad\text{in}\quad\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}\quad\text{of the ideal }(v^{\Phi_+})\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$$

is the set of series

$$\sum x_\phi v^{-\phi}\in\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}\quad\text{such that}\quad\hat{\nu}(x_\phi)>\phi.$$

Proof The proof follows immediately from 5.33. □

Definition 5.35 With the notations of subsection 2.3, let us define on the model of ([B4], exerc. 27 for §2) the algebra

$$\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}\{(v^{\Phi_+})^{-1}\} = \lim_{\substack{\longleftarrow \\ \delta\in\Phi_+}} (v^{\Phi_+})^{-1}\mathcal{A}_\nu(R)/\mathcal{S}_\delta = \widehat{\mathcal{A}_{\hat\nu}(\hat R^\nu)}^{\hat\nu_{\hat{\mathcal{A}}}}\{(v^{\Phi_+})^{-1}\}.$$

This ring is also the Hausdorff completion of $(v^{\Phi_+})^{-1}\mathcal{A}_\nu(R)$ for the topology having as a fundamental system of neighborhoods the $(v^{\Phi_+})^{-1}\mathcal{P}_\delta$. I follow Bourbaki's convention according to which writing $A\{S^{-1}\}$ entails the completion of A, i.e., $A\{S^{-1}\} = \hat A\{S'^{-1}\}$ where S' is the image of S in $\hat A$. The inclusion $\mathcal{A}_\nu(R) \longrightarrow \mathcal{A}_\nu(R)\{(v^{\Phi_+})^{-1}\}$ is universal with respect to continuous maps $\mathcal{A}_\nu(R) \to B$ for linearly topologized Hausdorff and complete rings B such that the images in B of the elements of the multiplicative set (v^{Φ_+}) are invertible.
I also consider the Hausdorff completion $\widehat{\mathrm{gr}_\nu}R$ of the graded algebra $\mathrm{gr}_\nu R$ with respect to the valuation ν_{gr}.
According to the general properties of completions of linearly topologized modules ([Ma], Prop. in 23.I, p.167), the completion of $\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}/(v^{\Phi_+})\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}$ for the quotient topology is $\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}/\overline{(v^{\Phi_+})\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}}$; it is also the completion, for the quotient topology, of $\mathcal{A}_\nu(R)/(v^{\Phi_+})\mathcal{A}_\nu(R)$. I can now state

Proposition 5.36 *a) The natural map*

$$\mathcal{A}_\nu(R) \longrightarrow \mathrm{gr}_\nu R$$

defined by

$$x_\phi v^{-\phi} \mapsto x_\phi \mod.\mathcal{P}^+_\phi(R)$$

induces an isomorphism of filtered rings

$$\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}/\overline{(v^{\Phi_+})\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}} \xrightarrow{\simeq} \widehat{\mathrm{gr}_\nu}R.$$

b) The natural inclusion $R[v^{\Phi_+}] \longrightarrow \mathcal{A}$ obtained by considering only the part of negative degree (i.e., $\phi\in\Phi_-$) of $\mathcal{A}$ induces an isomorphism

$$R[v^{\Phi_+}]\{(v^{\Phi_+})^{-1}\} = \hat R^\nu\{v^{\Phi_+}\}\{(v^{\Phi_+})^{-1}\} \xrightarrow{\simeq} \widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}\{(v^{\Phi_+})^{-1}\}.$$

Here, as above, $\hat R^\nu\{v^{\Phi_+}\}$ denotes the ring of restricted power series in v^ϕ, $\phi\in\Phi_+$ and coefficients in $\hat R^\nu$.
c) Given a homomorphism $\phi\mapsto c(\phi)$ from Φ to the multiplicative group of units of R, it induces a surjection $\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}} \to \hat R^\nu$ defined by $\sum x_\phi v^{-\phi} \mapsto \sum x_\phi c(-\phi)$. The kernel of this surjection is the closure of the ideal

$$\big((v^\phi - c(\phi))_{\phi\in\Phi_+}\big)\widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}.$$

Proof Part a) follows from 5.33 and the general facts on completion, and Part b) follows from 2.2 and the remark following definition 5.35. Part c) follows from 2.2 which contains the fact that the corresponding result for $\mathcal{A}_\nu(R)$ is true. □

Corollary 5.37 *If ν is composed with a valuation ν_1, according to a surjective monotone non-decreasing group homomorphism $\lambda\colon \Phi\to\Phi_1$, setting $\Psi = \mathrm{Ker}\lambda$, we have:*

$$\widehat{\mathcal{A}_{\nu_1}(R)}^{\nu_{1,\mathcal{A}}} = \widehat{\mathcal{A}_\nu(R)}^{\nu_{\mathcal{A}}}/\overline{\big((v^\psi - 1)_{\psi\in\Psi_+}\big)},$$

where the bar denotes the closure.

Proof This follows from Proposition 5.33 and the fact that in view of 5.5 the continuous map $\hat{R}^{\nu}\{v^{\Phi}\} \to \hat{R}^{\nu_1}\{v^{\Phi_1}\}$ is surjective and its kernel is the ideal $\overline{((v^{\psi}-1)_{\psi\in\Psi_+})}$. □

In fact this corollary is the generalization to valuation algebras of the first corollary of Proposition 5.3.
In spite of the fact that none of this applies when ν_1 is the trivial valuation, we have:

$$\widehat{\mathcal{A}_{\nu}(R)}^{\nu_{\mathcal{A}}}/\overline{\left((v^{\phi}-1)_{\phi\in\Phi_+}\right)} = \hat{R}^{\nu},$$

because the natural map $\mathcal{A}_{\nu}(R) \longrightarrow R$ with kernel $(v^{\phi}-1)_{\phi\in\Phi_+}$ is continuous not only when R has the trivial topology given by the trivial valuation, but also when R has the ν-adic topology.
Assuming now, as we did in subsection 2.3, that R is a k-algebra for some field k, we have:

Proposition 5.38 *The $k[v^{\Phi_+}]$-algebra $\widehat{\mathcal{A}_{\nu}(R)}^{\nu_{\mathcal{A}}}$ is faithfully flat.*

Proof This is proved exactly like Proposition 2.3 except that we have to accept infinite sums $\sum y_{k\psi}v^{-\psi}$ and an infinite-dimensional vector space V in $\hat{R}^{\nu}$, but everything is convergent in $\widehat{\mathcal{A}_{\nu}(R)}^{\nu_{\mathcal{A}}}$ because we deal with restricted power series. □

Remark that since ν is trivial on k, we have $k[v^{\Phi_+}] = k\{v^{\Phi_+}\}$.

In order to control the specialization we need in fact to consider the scalewise completion of both $\mathcal{A}_{\nu}(R)$ and $\mathrm{gr}_{\nu}R$, which correspond to the scalewise completion $\hat{R}^{(\nu)}$ of R.
I only give the definitions and state the properties we need.
Let us denote by

$$\Phi \xrightarrow{\lambda_1} \Phi_1 \xrightarrow{\lambda_2} \dots \xrightarrow{\lambda_{h-1}} \Phi_{h-1}$$

the sequence of groups of the valuations with which ν is composed. Notice that the subset

$$\Lambda = \{(\phi,\phi_1,\dots,\phi_{h-1}) \in \Phi\times\Phi_1\times\cdots\times\Phi_{h-1}/\phi_i = \lambda_i(\phi_{i-1})\text{for } 1\leq i\leq h-1\}$$

is isomorphic to Φ. Iterating the last remark before subsection 3.2, we can write:

$$\mathcal{A}_{\nu}(R) = \bigoplus_{\phi_{h-1}\in\Phi_{h-1}} \Big(\bigoplus_{\lambda_{h-1}(\phi_{h-2})=\phi_{h-1}} \big(\dots\mathcal{P}_{\phi}(R)w^{-\phi}\dots\big)w_{h-2}^{-\phi_{h-2}}\Big)w_{h-1}^{-\phi_{h-1}}.$$

Definition 5.39 The scalewise completion of $\mathcal{A}_{\nu}(R)$ consists in completing first with respect to the $\nu_{h-1,\mathcal{A}}$-adic topology, then completing the space of coefficients of each $w^{-\phi_{h-1}}$ with respect to the $\nu_{h-2,\mathcal{A}}$-adic topology, and so on. Note that each behaves like the filtration associated to a valuation of height one. I will denote the result by

$$\widehat{\mathcal{A}_{\nu}(R)}^{(\nu_{\mathcal{A}})}.$$

I define similarly the scalewise completion of $\mathrm{gr}_{\nu}R$ after iterating the construction of Lemma 3.16 to obtain an isomorphism

$$\mathrm{gr}_{\nu}R \xrightarrow{\simeq} \bigoplus_{(\phi,\phi_1,\dots,\phi_{h-1})\in\Lambda} \left(\mathrm{gr}_{\overline{\mathcal{P}}}\mathrm{gr}_{\overline{\mathcal{P}}_1(\phi_1)}\cdots(\mathrm{gr}_{\nu_{h-1}}R)_{\phi_{h-1}}\right)_{\phi},$$

where $\mathrm{gr}_{\overline{\mathcal{P}}_i}$ is the graded ring corresponding to the filtration $\overline{\nu}_i$ on each homogeneous component of $\mathrm{gr}_{\nu_{i+1}}$ induced by ν_i. Again each behaves like in the case of height one.

Definition 5.40 The scalewise completion of $\mathrm{gr}_\nu R$ is obtained by successively completing with respect to the filtrations $\nu_{h-1,\mathrm{gr}}$, $\overline{\nu}_{h-2,\mathrm{gr}}$, ..., $\overline{\nu}_{\mathrm{gr}}$. It will be denoted by $\widehat{\mathrm{gr}}_\nu^{(\nu)} R$.

Proposition 5.41 **The $k[v^{\Phi_+}]$-algebra $\widehat{\mathcal{A}_{\hat{\nu}}(\hat{R}^{(\nu)})}^{(\nu_{\mathcal{A}})}$ is faithfully flat; the special fiber of the map*

$$\mathrm{Spec}\widehat{\mathcal{A}_{\hat{\nu}}(\hat{R}^{(\nu)})}^{(\nu_{\mathcal{A}})} \to \mathrm{Spec}k[v^{\Phi_+}]$$

*is $\mathrm{Spec}\widehat{\mathrm{gr}}_\nu^{(\nu)} R$ and its "general" fiber, over $\mathrm{Spec}(v^{\Phi_+})^{-1}k[v^{\Phi_+}]$, is isomorphic to $\mathrm{Spec}(v^{\Phi_+})^{-1}\hat{R}^{(\nu)}[v^{\Phi_+}]$.**

In any case, we shall see below why it is for the family

$$\mathrm{Spec}\widehat{\mathcal{A}_{\hat{\nu}}(\hat{R}^{(\nu)})}^{(\nu_{\mathcal{A}})} \to \mathrm{Spec}k[v^{\Phi_+}]$$

that we may indeed hope to have simultaneous *embedded* uniformization; the $k[v^{\Phi_+}]$-algebra $\widehat{\mathcal{A}_{\hat{\nu}}(\hat{R}^{(\nu)})}^{(\nu_{\mathcal{A}})}$ is a quotient of the scalewise completion of $k[v^{\Phi_+}][(w_j)_{j\in J}]$, which corresponds to a simultaneous embedding of its fibers. This will put us in a position to apply the implicit function theorem to extend a partial uniformization of ν_{gr} on $\mathrm{gr}_\nu R$ to a uniformization of $\hat{\nu}$ on $\hat{R}^{(\nu)}$.

5.4 The specialization of $\hat{R}^{(\nu)}$ to $\widehat{\mathrm{gr}}_\nu^{(\nu)} R$. By Proposition 2.10 we have a graded isomorphism

$$\mathrm{gr}_{\nu_{\mathcal{A}}}\mathcal{A}_\nu(R) \xrightarrow{\simeq} (\mathrm{gr}_\nu R) \otimes_{\overline{R}} \overline{R}[v^{\Phi_+}].$$

If we take a system of generators $\overline{\xi}_i$ of the $\overline{R}$-algebra $\mathrm{gr}_\nu R$, we obtain generators $\overline{v}_i = \overline{\xi}_i \otimes 1$ of the $\overline{R}[v^{\Phi_+}]$-algebra $\mathrm{gr}_{\nu_{\mathcal{A}}}\mathcal{A}_\nu(R) = \mathrm{gr}_{\hat{\nu}_{\mathcal{A}}}\widehat{\mathcal{A}_\nu(R)}^{(\nu_{\mathcal{A}})}$ and we can lift them to homogeneous elements $v_i \in \mathcal{A}_\nu(R) \subset \widehat{\mathcal{A}_\nu(R)}^{(\nu_{\mathcal{A}})}$.
Note that according to Proposition 2.10, each element v_i is of the form $\xi_i v^{-\nu(\xi_i)}$ where $\xi_i \in R$ has initial form $\overline{\xi}_i$.
Let us begin by assuming that R is local and has a field of representatives k. Then we may consider the morphism of $k[v^{\Phi_+}]$-algebras

$$k[v^{\Phi_+}][(u_i)_{i\in I}] \to \mathcal{A}_\nu(R)$$

determined by $u_i \mapsto v_i = \xi_i v^{-\nu(\xi_i)}$. We can consider the *scalewise completion* of $k[v^{\Phi_+}][(u_i)_{i\in I}]$ defined as follows: define, with the notations introduced in Subsection 4.2, the sequence of rings

$$\hat{S}_1 = k[v^{\Phi_+}][(u_i)_{i\in I_1}\widehat{]}^{\tilde{\nu}_1}, \ldots, \hat{S}_t = \hat{S}_{t-1}[(u_i)_{i\in I_t}\widehat{]}^{\tilde{\nu}_t}, \ldots, \hat{S}_h = \hat{S}_{h-1}[(u_i)_{i\in I_h}\widehat{]}^{\tilde{\nu}_h}$$

where $\tilde{\nu}_t$ is the monomial valuation on $\hat{S}_{t-1}[(u_i)_{i\in I_t}]$ whose values lie in Ψ_{h-t}/Ψ_{h-t+1} defined by $\tilde{\nu}_t(\sum a_\alpha u^\alpha) = \min(\nu_t(\xi^\alpha))$, and where the hat means the completion.

Definition 5.42 We can inductively define a valuation $\tilde{\nu}$ on $\hat{S}_h$ as follows: it is trivial on $k[v^{\Phi_+}]$ and, assuming that it has been defined on $\hat{S}_{t-1}$, for an element $\sum a_\alpha u^\alpha \in \hat{S}_t$ we set

$$\tilde{\nu}(\sum a_\alpha u^\alpha) = \min_\alpha\{\tilde{\nu}(a_\alpha) + \nu(\xi^\alpha)\} \in \Psi_{h-t},$$

which is well defined because the semigroup $\Gamma = \nu(R \setminus \{0\})$ is well ordered. It extends the monomial valuation on the polynomial ring with the $(u_i)_{i\in I}$ as system of generators and $\tilde{\nu}(u_i) = \nu(\xi_i) = \gamma_i$, the i-th generator of the semigroup Γ.

Note that $\tilde{\nu}$ is *not* a rational valuation in general.

We can apply the same procedure to $k[(u_i)_{i\in I}]$ instead of $k[v^{\Phi_+}][(u_i)_{i\in I}]$:

Definition 5.43 The ring $k[\widehat{(u_i)_{i\in I}}]$ obtained from the polynomial ring by the process just described is called the *scalewise completion* of the polynomial ring $k[(u_i)_{i\in I}]$.

Proposition 5.44 *The scalewise completion $k[\widehat{(u_i)_{i\in I}}]$ is a local ring and its maximal ideal is the closure of the ideal generated by the $(u_i)_{i\in I}$. The valuation $\tilde{\nu}$ is rational in this case and analytically monomial with respect to $(u_i)_{i\in I}$.*

Proof The proof of the first two statements is the same as that of Proposition 5.28, and the last one follows from the very definition of $\tilde{\nu}$. The graded algebra with respect to $\tilde{\nu}$ is given by

$$\mathrm{gr}_{\tilde{\nu}} k[\widehat{(u_i)_{i\in I}}] = k[(U_i)_{i\in I}].$$

□

Remarks 5.45 1) If the k-algebra $\mathrm{gr}_\nu R$ is finitely generated, say by $(\overline{\xi}_i)_{i\in F}$, the ring $k[\widehat{(u_i)_{i\in I}}]$ is the usual power series ring $k[[(u_i)_{i\in F}]]$, and $\tilde{\nu}$ is the usual monomial valuation determined by $\tilde{\nu}(u_i) = \nu(\xi_i) = \gamma_i$, the i-th generator of the semigroup Γ.
2) In the general case, but with Φ of height one say, $k[\widehat{(u_i)_{i\in I}}]$ contains elements such as $u_1 + u_2 + \cdots + u_i + \cdots$.

Proposition 5.46 *The ring $\hat{S}_h$ is complete for the topology defined by $\tilde{\nu}$.*

Proof Again this is proved by induction on t. For $t = 1$ is follows from the definitions. Assume the proposition true up to $t-1$ and let $\sum_\alpha a_\alpha^{(i)} u^\alpha \in \hat{S}_t$ be a Cauchy sequence. The minimum over α of the $\tilde{\nu}$ valuations of the differences $(a_\alpha^{(i)} - a_\alpha^{(j)})u^\alpha$ must become arbitrarily large in Ψ_{h-t} as $\min(i,j)$ grows. For each α the $a_\alpha^{(i)}$ must form a Cauchy sequence for the $\tilde{\nu}$ topology, which converges by the induction hypothesis. □

Definition 5.47 We will denote by $k[\widehat{v^{\Phi_+}][(u_i)}_{i\in I}]$ the ring $\hat{S}_h$ endowed with the topology determined by the $\tilde{\nu}$-adic filtration, and call it the *scalewise completion* of the ring $k[v^{\Phi_+}][(u_i)_{i\in I}]$.

Assume that the local ring R is complete and let $k \subset R$ be a field of representatives. The map $k[(u_i)_{i\in I}] \to R$ defined by $u_i \mapsto \xi_i v^{-\nu(\xi_i)}$ induces a map

$$k[v^{\Phi_+}][(u_i)_{i\in I}] \to \mathcal{A}_\nu(R); \quad v^{-\phi} \mapsto v^{-\phi}, \quad u_i \mapsto \xi_i v^{-\nu(\xi_i)},$$

which in turn induces a surjective map of $k[v^{\Phi_+}]$-algebras

$$k[v^{\Phi_+}]\widehat{[(u_i)_{i\in I}]} \longrightarrow \widehat{\mathcal{A}_\nu(R)}^{(\nu_{\mathcal{A}})},$$

where the first hat designates the scalewise completion defined above. The surjectivity follows from Proposition 5.31 or rather from repeating its proof in this situation where k is replaced by $k[v^{\Phi_+}]$. This map induces, modulo the *closures* of the ideals generated by (v^{Φ_+}), the surjective map

$$k[\widehat{(U_i)_{i\in I}}] \to \widehat{\mathrm{gr}}_\nu^{(\nu)}R$$

which extends to the completions the original map $U_i \mapsto \overline{\xi}_i$.
We can summarize this as follows

Proposition 5.48 * *Let R be a complete equicharacteristic nœtherian local integral domain with residue field k, and ν a rational valuation of R with value group Φ. Let $k \subset R$ be a field of representatives for R and $(\xi_i)_{i\in I}$ be representatives in R of a system of homogeneous generators $(\overline{\xi}_i)_{i\in I}$ of the k-algebra $\mathrm{gr}_\nu R$. Then the map of $k[v^{\Phi_+}]$-algebras $k[v^{\Phi_+}][(u_i)_{i\in I}] \to \mathcal{A}_\nu(R)$ determined by the injection $k[v^{\Phi_+}] \subset \mathcal{A}_\nu(R)$ deduced from $k \subset R$ and the applications $u_i \mapsto \xi_i v^{-\nu(\xi_i)}$ extends to a continuous surjective map of topological $k[v^{\Phi_+}]$-algebras*

$$k[v^{\Phi_+}]\widehat{[(u_i)_{i\in I}]} \to \widehat{\mathcal{A}_\nu(R)}^{(\nu_{\mathcal{A}})},$$

which induces, modulo the closures in each ring of the ideal (v^{Φ_+}), the map

$$k[\widehat{(U_i)_{i\in I}}] \to \widehat{\mathrm{gr}}_\nu^{(\nu)}R.*$$

□

(I write capital U's for the initial forms of the u's).

The kernel of this last map is the closure in $k[\widehat{(U_i)_{i\in I}}]$ of the binomial ideal $(U^m - \lambda_{mn}U^n)k[(U_i)_{i\in I}]$ defining $\mathrm{gr}_\nu R$; it is the ideal consisting of possibly infinite sums $\sum_{m,n} A_{m,n}(U)(U^m - \lambda_{mn}U^n) \in k[\widehat{(U_i)_{i\in I}}]$, with $A_{m,n}(U) \in k[\widehat{(U_i)_{i\in I}}]$.
Using the fact that $\widehat{\mathcal{A}_\nu(R)}^{(\nu_{\mathcal{A}})}$ is a faithfully flat $k[v^{\Phi_+}]$-algebra, one can show:

Proposition 5.49 * *a) In the situation of the preceding Proposition, there exist elements of the form*

$$F_{mn} = u^m - \lambda_{mn}u^n + \sum_s c_s^{(mn)}(v^\phi)u^s \in k[v^{\Phi_+}]\widehat{[(u_i)_{i\in I}]},$$

with, for all s appearing in F_{mn}, $\tilde{\nu}(u^s) > \tilde{\nu}(u^n) = \tilde{\nu}(u^m)$ and $c_s^{(mn)}$ a term (=constant times a monomial) in the v^ϕ, and such that the closure in $k[v^{\Phi_+}]\widehat{[(u_i)_{i\in I}]}$ of the ideal generated by the F_{mn} is the kernel of the surjection

$$k[v^{\Phi_+}]\widehat{[(u_i)_{i\in I}]} \to \widehat{\mathcal{A}_\nu(R)}^{(\nu_{\mathcal{A}})}.$$

*b) The F_{mn} may be chosen so that, with the notations of Proposition 5.28, if $U^m - \lambda_{mn}U^n \in k[(U_i)_{i\in\Psi_{h-t}}]$, then $F_{mn} \in k[v^{\Phi_+}]\widehat{[(u_i)_{i\in\Psi_{h-t}}]}$.**

Proof By the remark following Proposition 2.3, and with its notations, after replacing R by $\hat{R}^{(\nu)}$, the $k[v^{\Phi_+}]$-algebra $\mathcal{B}(S)$, which is the image of the map of $k[v^{\Phi_+}]$-algebras

$$k[v^{\Phi_+}][(u_i)_{i\in I}] \to \mathcal{A}_\nu(R), \quad u_i \mapsto \xi_i v^{-\nu(\xi_i)}$$

is faithfully flat. It follows that, denoting by J the kernel of this map, we can lift the generators $U^m - \lambda_{mn}U^n$ of the kernel of the map $k[(U_i)_{i\in I}] \to \mathrm{gr}_\nu R$ to elements $F_{mn} \in k[v^{\Phi_+}][(u_i)_{i\in I}]$ generating an ideal $J' \subseteq J$ such that $J \subseteq J' + (v^{\Phi_+}).J$. The equality $(v^{\Phi_+}) \cap J = (v^{\Phi_+}).J$ holds by flatness since it is equivalent to

$$\mathrm{Tor}_1^{k[v^{\Phi_+}][(u_i)_{i\in I}]}(k[v^{\Phi_+}][(u_i)_{i\in I}]/(v^{\Phi_+}), k[v^{\Phi_+}][(u_i)_{i\in I}]/J) = 0.$$

It means that we can write in $\mathcal{A}_\nu(R)$ generators of J which begin with

$$u^m - \lambda_{mn}u^n + \sum_s c_s(v^\phi)u^s + \sum_i v^{\phi_i}A_i(u^{m^i} - \lambda_{m^in^i}u^{n^i} + \cdots).$$

Now we observe that since this sum must give zero in $\widehat{\mathcal{A}_\nu(R)}^{(\nu_\mathcal{A})}$, the w-order of $\sum_i v^{\phi_i}A_i(u^{m^i} - \lambda_{m^in^i}u^{n^i} + \cdots)$ must be greater than that of u^m. Following through the process of scalewise completion, one sees that the closure in $k[v^{\Phi_+}]\widehat{[(u_i)_{i\in I}]}$ of the ideal J' is the kernel of the surjection

$$k[v^{\Phi_+}]\widehat{[(u_i)_{i\in I}]} \to \widehat{\mathcal{A}_\nu(R)}^{(\nu_\mathcal{A})}.$$

To prove part b), one needs only apply this argument to each $\hat{R}_t$. □

Before the next corollary, let us record the

Definition 5.50 Assertion TC(d) (Toric coordinatization in dimension $\leq d$) is the following: For every excellent equicharacteristic local domain R of dimension $\leq d$, given a rational valuation ν, there exists a quotient $\hat{R}^{(\nu)}$ of $\hat{R}^m$ by an ideal H such that $H \cap R = (0)$ and an extension $\hat{\nu}$ of ν to $\hat{R}^{(\nu)}$ inducing a scalewise birational map $\mathrm{gr}_\nu R \to \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ of graded algebras such that, given a field of representatives $k \subset \hat{R}^{(\nu)}$ of the residue field of R and representatives $(\eta_j)_{j\in J}$ in $\hat{R}^{(\nu)}$ of a system of generators of the k-algebra $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$, the map of k-algebras $k[(w_j)_{j\in J}] \to \hat{R}^{(\nu)}$ determined by $w_j \mapsto \eta_j$ extends to a continuous surjection for the $\tilde{\nu}$ and $\hat{\nu}$-adic topologies

$$k\widehat{[(w_j)_{j\in J}]} \to \hat{R}^{(\nu)}$$

with a surjective associated graded map

$$k[(W_j)_{j\in J}] \to \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)},$$

where the ring $k\widehat{[(w_j)_{j\in J}]}$ is the scalewise completion of the polynomial ring (see the definition before Proposition 5.44). We denote by $(W^m - \lambda_{mn}W^n)_{(m,n)\in\hat{E}}$ a system of generators for the kernel of the map $k[(W_j)_{j\in J}] \to \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$.

Corollary 5.51 Toric coordinatization: *(assuming that TLU(d-1) holds) *Given a rational valuation ν of the excellent local domain R of dimension $\leq d$, let c be a character of Φ i.e., an homomorphism $c\colon \Phi \to k^*$; replacing R by $\hat{R}^{(\nu)}$ and v^ϕ by $c(\phi)$ in 5.49 gives, in view of Propositions 5.36, c) and 5.10 a continuous surjection as in TC(d)*

$$\hat{c}\colon k\widehat{[(w_j)_{j\in J}]} \to \hat{R}^{(\nu)}, \quad w_j \mapsto c(-\hat{\nu}(\eta_j))\eta_j$$

where the hat means scalewise completion. The kernel of this map is the closure of the ideal generated by the elements

$$c(G_{mn}) = w^m - \lambda_{mn}w^n + \sum_s c_s^{(mn)}(c(\phi))w^s \quad (m,n) \in \hat{E}.$$

In particular, taking the trivial character c_1 *determined by* $c_1(\phi) = 1 \ \ \forall \phi \in \Phi$*, we have a surjective map*

$$\hat{c}_1 \colon k[\widehat{(w_j)_{j\in J}}] \to \hat{R}^{(\nu)} \qquad w_j \mapsto \eta_j$$

whose kernel is generated up to closure by elements

$$c_1(G_{mn}) = w^m - \lambda_{mn} w^n + \sum_s c_s^{(mn)}(1) w^s.*$$

As I have already noted, this may be thought of as a valuative analogue of Cohen's theorem stating that every complete equicharacteristic nœtherian local ring is a quotient of a power series ring in finitely many variables over its residue field, remembering that the associated graded map must then be surjective. The advantage here is that, when the residue field is algebraically closed, we know that the kernel of the map is generated, up to closure, by deformations of binomials. One may summarize a part of my approach by saying that the elements $\eta_j \in \hat{R}^{(\nu)}$ representing generators $\overline{\eta}_j$ of the k-algebra $\mathrm{gr}_{\hat{\nu}} \hat{R}^{(\nu)}$ form a system of generators of the maximal ideal of $\hat{R}^{(\nu)}$ in which the valuation $\hat{\nu}$ is analytically monomial in the sense of section 3.7. The elements $c(G_{mn})$ may be thought of as an infinite standard, or Gröbner, basis for the kernel of the map $k[\widehat{(w_j)_{j\in J}}] \to \hat{R}^{(\nu)}$ with respect to the the monomial order on the -countably many- variables w_j defined by deciding that $w^s < w^t$ if the valuation of the image of w^s in $\hat{R}^{(\nu)}$ is less than the valuation of the image of w^t.

Corollary 5.52 *a) With the notations of subsection 2.4, each ideal* $\mathcal{S}_\delta(\mathcal{A}_{\hat{\nu}}(\hat{R}^{(\nu)}))$ *is generated as a* $k[v^{\Phi_+}]$*-module, up to closure, by monomials in the* $\eta_j v^{-\hat{\nu}(\eta_j)}$.
b) Each ideal $\mathcal{P}_\phi(\hat{R}^{(\nu)})$ *is generated as a* k*-vector space, up to closure, by monomials in the* η_j.
c) If the k*-algebra* $\mathrm{gr}_{\hat{\nu}} \hat{R}^{(\nu)}$ *is finitely generated, say by* $(\overline{\eta}_j)_{j\in F}$*, the* $k[v^{\Phi_+}]$*-algebra* $\widehat{\mathcal{A}_{\hat{\nu}}(\hat{R}^{(\nu)})}^{(\hat{\nu}_{\mathcal{A}})}$ *is a quotient of* $k[v^{\Phi_+}][[(w_j)_{j\in F}]]$.

Proof : a) is a direct consequence of 5.48 and Proposition 5.44. Statement b) is a consequence of a) by evaluation via a character of Φ in k^*. For c), one checks by induction on the height that in the finitely generated case, the scalewise completion of $k[v^{\Phi_+}][(w_j)_{j\in F}]$ coincides with $k[v^{\Phi_+}][[(w_j)_{j\in F}]]$. ☐

Corollary 5.53 *Let* R *be a complete nœtherian local ring endowed with a rational valuation* ν *and let* $\mathbf{p}$ *be the center of the valuation* ν_1 *of height one with which* ν *is composed. For each* $\phi_1 \in \Phi_{1+}$ *there are finitely many elements* γ_i *of our minimal system of generators for the semigroup* Γ *of* ν *which are the valuations of elements of* $\mathcal{P}_{\phi_1} \setminus \mathcal{P}^+_{\phi_1}$.

Proof The $R/\mathbf{p}$-module $\mathcal{P}_{\phi_1}/\mathcal{P}^+_{\phi_1}$ is finitely generated. Each generator can be written as a series in the ξ^α with coefficients in k. Each of the monomials in the ξ_i which appear in this way can be written $\xi'^{\alpha'}\xi''^{\alpha''}$ with $\nu_1(\xi'^{\alpha'}) = 0$ and $\nu_1(\xi''^{\alpha''}) \geq \phi_1$. All the monomials $\xi''^{\alpha''}$ which are of valuation ϕ_1 generate a $R/\mathbf{p}$-submodule of $\mathcal{P}_{\phi_1}/\mathcal{P}^+_{\phi_1}$. Since $R/\mathbf{p}$ is nœtherian, this submodule is generated by finitely many of these monomials, and the ν-valuation of every element of $\mathcal{P}_{\phi_1} \setminus \mathcal{P}^+_{\phi_1}$

is a linear combination with coefficients in $\mathbf{N} \cup \{0\}$ of elements of the semigroup $\overline{\Gamma}$ of the residual valuation $\overline{\nu}$ and the valuations of these monomials. □

Corollary 5.54 *Let R be a complete nœtherian local ring endowed with a rational valuation ν of height $\mathrm{h}(\nu)$. In the minimal system of generators of the semigroup Γ of (R, ν) there are at most $\mathrm{h}_R(\nu) - 1$ elements without predecessor.*

Proof The proof is by induction on the height $\mathrm{h}_R(\nu)$ of ν in R. The result is true if ν is of height one because in this case every element has a predecessor by the second part of Corollary 3.10. Keeping the same notations as in Corollary 5.53 and assuming the result to be true for the valuation $\overline{\nu}$ of $R/\mathbf{p}$, we see that if γ_i has no predecessor and $\nu_1(\gamma_i) = \phi_1 > 0$, the only possibility is that there is an infinite sequence $\gamma_k < \gamma_{k+1} < \ldots$ of our generators of Γ whose ν_1-valuation is the predecessor of ϕ_1 in the image Γ_1 of Γ in Φ_1, which exists since ν_1 is of height one. By Corollary 5.53 this is impossible unless the predecessor of ϕ_1 is zero and therefore γ_i has to be the smallest element of Γ which is not in $\Gamma \cap \Psi$ where Ψ is the convex subgroup corresponding to ν_1. The number of elements of the well ordered set $(\gamma_1, \ldots, \gamma_i, \ldots)$ without predecessor in the set is therefore at most equal to $\mathrm{h}_R(\nu) - 1$. □

Remarks 5.55 1) Part b) of the Corollary means that the valuation $\hat{\nu}$ on $\hat{R}^{(\nu)}$ is analytically monomial in the variables $(w_j)_{j \in J}$.
2) Assume that the ring R contains a field of representatives. In spite of the flatness of the $k[v^{\Phi_+}]$-algebra $\mathcal{A}_\nu(R)$, it may happen that the k-algebra $\mathrm{gr}_\nu R$ is finitely generated while the $k[v^{\Phi_+}]$-algebra $\mathcal{A}_\nu(R)$ is not; if we go back to example 5.7 of subsection 5.1, we can see that for each integer n, the ideal $\mathcal{P}_n(R)$ is generated by $(u_2^n, u_1 - \sum_1^{n-1} a_i u_2^i)$, while $\mathcal{P}_n(\hat{R}^{(\nu)}) = u_2^n k[[u_2]]$ and $\mathrm{gr}_\nu R = \mathrm{gr}_{\hat{\nu}} \hat{R}^{(\nu)} = k[U_2]$.
3) It would be interesting to relate the fact that $\mathrm{gr}_{\hat{\nu}} \hat{R}^{(\nu)}$ is finitely generated with the same condition for $\mathrm{gr}_\nu R$.

5.5 The abyssal phenomenon. Let us keep the notations of Corollary 5.51. The key fact now is that $\hat{R}^{(\nu)}$ is nœtherian and complete for the m-adic topology. The images by $\hat{c}$ of the w_j generate the maximal ideal of $\hat{R}^{(\nu)}$, and therefore there exists a finite subset $L \subset J$, which in the case of a valuation of height one we may assume to consist of $\{1, 2, \ldots, q\}$, such that the images of $(w_\ell)_{\ell \in L}$ generate the maximal ideal of $\hat{R}^{(\nu)}$. This means that for $i \notin L$ and any character $c\colon \Phi_+ \to k^*$, we have a relation

$$w_j - \sum_{\ell \in L} R_\ell w_\ell \in \overline{\left(\{w^m - \lambda_{mn} w^n + \sum_s c_s^{(mn)}(c(\phi)) w^s\}_{(m,n)}\right)} \text{ in } k[\widehat{(w_j)_{j \in J}}],$$

and this implies that one of the equations $w^m - \lambda_{mn} w^n + \sum_s c_s^{(mn)}(c(\phi)) w^s$ must contain w_j linearly (compare with example 5.6). According to Remark 4.5, this w_j does not appear in the binomial term of G_{mn}.

The abyssal phenomenon is that, whenever we have among the equations defining $\hat{R}^{(\nu)}$ as a quotient of $k[\widehat{(w_j)_{j \in J}}]$ infinitely many consecutive indices j for which such an equation exists, these equations cannot create singularities, and at the same time they produce no decrease of the Krull dimension of $\hat{R}^{(\nu)}$ (compare with example 4.20). The creation of singularities, as well as the decrease in dimension, are pushed away to infinity, since there is "no equation", but an endless sequence

of substitutions. The proof of the abyssal phenomenon relies on an infinite dimensional implicit function theorem which is *not* proved here. It states that, using the fact that $\hat{R}^{(\nu)}$ is henselian, we may solve this equation for w_j in $\hat{R}^{(\nu)}$.
More precisely, what we need has the following form, where each variable w_j has weight $\nu(\eta_j) \in \Phi$:

Theorem 5.56 (Implicit functions Theorem) ** Let $(G_h)_{h\in H}$ be elements in $k[\widehat{((w_j)_{j\in J}}]$ of the form*

$$G_h = w^{n(h)} - \lambda_h w^{m(h)} + c_h w_h + \sum_s c_s w^s,$$

indexed by a subset $H \subset J$, with $c_h \in k^*$, $c_s \in k$ and the weight of w_h and all the w^s is greater than the weight of the terms of the initial isobaric binomial .
Then, these series generate, up to closure, the same ideal as the following:

$$c_h w_h - (w^{n(h)} - \lambda_h w^{m(h)} + \sum_t d_t w^t)$$

*where the weight of all the monomials w^t is greater than the weight of the initial binomial and no variable w_k for $k \in H$ appears in the monomials w^t.**

Proof This requires an infinite-dimensional implicit function theorem in $k[\widehat{((w_j)_{j\in J}}]$, somewhat similar to that of [VdH], see also ([B1], Chap. IV), to use the fact that all the $\partial_{w_h} c(G_h)$ are invertible in $k[\widehat{((w_j)_{j\in J}}]$. This implicit function theorem should also allow us to recognize from the partial derivatives of the G_h when a nœtherian quotient of $k[\widehat{((w_j)_{j\in J}}]$ by the *closure* of the ideal generated by elements $(G_h)_{h\in H}$ is regular.

Because of part b) of Proposition 5.49 and Lemma 5.29, the implicit function theorem can be proved by induction on the height, and at each step we have only a situation of height one type. □

Let us now consider a finite set of representatives $(\eta_j)_{j\in F} \in \hat{R}^{(\nu)}$ of generators $\overline{\eta}_j$ of the k-algebra $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ which has the following properties (by definition, as in Corollary 3.10 of subsection 3.1, the ordinal $i+1$ is the index of the generator $\hat{\gamma}_{i+1}$ of the semigroup $\hat{\Gamma}$ following $\hat{\gamma}_i$):
a) the representatives $(\eta_j)_{j\in F}$ form a system of generators of the maximal ideal of $\hat{R}^{(\nu)}$,
b) their valuations rationally generate the valuation group Φ of ν,
c)With the usual notation for convex subgroups of Φ, whenever the set $J_t = \{j \in J | \nu(\eta_j) \in \Psi_{h-t} \setminus \Psi_{h-t+1}\}$ is finite, it is contained in F.
d) For any $k \notin F$, the valuation $\hat{\gamma}_k$ of η_k has an immediate predecessor among the $\hat{\gamma}_i$ which is rationally dependent upon the $(\hat{\gamma}_j)_{j\in F}$.
e) If $i \in J$ satisfies $i \notin F$, then $i+1 \notin F$.

Definition 5.57 Such sets of generators of the maximal ideal of $\hat{R}^{(\nu)}$ will be called *sufficient sets of generators.*

Lemma 5.58 *Let ν be a valuation of a nœtherian local ring R, and $\Gamma = \langle\gamma_1, \ldots, \gamma_i, \ldots\rangle$ its semigroup of values. Let $\Delta = (\delta_1 < \ldots < \delta_j < \delta_{j+1} < \ldots)$ be an increasing sequence of elements of Γ. Given a finite subset F of Δ, there is a finite subset $\tilde{F}$ of Δ containing F and such that $\delta_j \notin \tilde{F}$ implies $\delta_{j+1} \notin \tilde{F}$.*

Proof If ν is of height one in R, it suffices to take for $\tilde{F}$ the set of elements of Δ smaller than or equal to the largest element of F; it is a finite set. Assume that the result is true for valuations of height $\leq h-1$, and let ν be a valuation of height h. Let $\lambda\colon \Phi \to \Phi_1$ be the map of groups corresponding to the valuation ν_1 of height $h-1$ with which ν is composed. Set $\Delta_1 = \lambda(\Delta)$ (as a set) and $F_1 = \lambda(F)$; by the induction hypothesis, we have a finite set $\tilde{F}_1$ containing F_1 and with the property of the Proposition with respect to Δ_1. Define $\tilde{F}$ as follows: it is the union of finite subsets $\tilde{F}_{\phi_1}$ of the $\lambda^{-1}(\phi_1)$ for $\phi_1 \in \tilde{F}_1$. If $\Delta \cap \lambda^{-1}(\phi_1)$ is finite (which is always the case if the center of ν_1 is equal to the center of ν, by [Z-S], Appendix 3, Corollary to Lemma 4, or Proposition 3.17), set $\tilde{F}_{\phi_1} = \Delta \cap \lambda^{-1}(\phi_1)$. If $\lambda^{-1}(\phi_1) \cap \Delta$ is infinite, define $\tilde{F}_{\phi_1}$ to be the set of elements of $\lambda^{-1}(\phi_1) \cap \Delta$ which are smaller than or equal to the largest element of F contained in $\lambda^{-1}(\phi_1) \cap \Delta$. By *loc. cit.*, Lemma 4, or Proposition 3.17, each of these sets if finite. If $\delta_j \notin \tilde{F}$, either $\lambda(\delta_j) \notin \tilde{F}_1$, and then $\lambda(\delta_{j+1}) \notin \tilde{F}_1$ so that $\delta_{j+1} \notin \tilde{F}$, or $\lambda(\delta_j) = \phi_1 \in \tilde{F}_1$ and $\lambda^{-1}(\phi_1) \cap \Delta$ is infinite, so that $\lambda(\delta_{j+1}) = \phi_1$ and $\delta_{j+1} \notin \tilde{F}$ by definition of $\tilde{F}_{\phi_1}$. ☐

Proposition 5.59 *Sufficient sets of generators for the maximal ideal of $\hat{R}^{(\nu)}$ exist.*

Proof That condition a) can be satisfied follows from the nœtherianity of $\hat{R}^{(\nu)}$. For condition b) it is the finiteness of the rational rank $\mathrm{r}(\nu)$, and for condition c) the finiteness of the height $\mathrm{h}(\nu)$, which shows that there are finitely many sets J_t. That condition d) can be satisfied follows from the finiteness of the number of generators of Γ without a predecessor; see Corollary 5.54. The only thing left to check is that e) can be satisfied; it is a consequence of the preceding Lemma. ☐

Let

$$\hat{\Gamma} = \langle \hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_i, \ldots \rangle \subset \Phi_+$$

be the semigroup of the values taken on $\hat{R}^{(\nu)} \setminus \{0\}$ by the valuation $\hat{\nu}$. For each $i \geq 1$, set $\hat{\Gamma}_i = \langle \hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_i \rangle$, and similarly, denote by $\hat{\Gamma}_{<i}$ the semigroup generated by elements $\hat{\gamma}_k$ with $\hat{\gamma}_k < \hat{\gamma}_i$; I will abbreviate this last inequality to $k < i$.
If the height of Φ is one, by Corollary 3.10, there is a smallest integer p_0 such that the group generated by $\hat{\Gamma}_{p_0}$ is equal to Φ. This means that for each $i > p_0$ we have an expression $t_i \hat{\gamma}_i = \sum_{1 \leq j \leq p_0} s_j^{(i)} \hat{\gamma}_j$ with $s_j^{(i)} \in \mathbf{Z}$. In the general case, we still have the following:

Proposition 5.60 *a) With the notations just introduced, there is a finite set of generators $(\hat{\gamma}_j)_{j \in F}$ which rationally generates Φ. For every $i \notin F$ there is a finite relation with non negative integral coefficients*

$$n_i \hat{\gamma}_i + \sum_{j \in F} n_j^{(i)} \hat{\gamma}_j = \sum_{j \in F} s_j^{(i)} \hat{\gamma}_j,$$

such that $n_i > 0$ is minimal among all such relations. In addition, each relation between the $\hat{\gamma}_j$ is equivalent, modulo these relations, to a relation between the $\hat{\gamma}_j$, $j \in F$. If the height of ν in R is one, in particular if the height of ν is one, there is an integer p_0 such that one may choose $F = \{1, \ldots, p_0\}$.
b) Given any relation between the generators $\hat{\gamma}_s$ of $\hat{\Gamma}$, and denoting by $\hat{\gamma}_i$ the generator of highest degree which appears in it, it is equivalent, modulo the relations

which exist between the $(\hat{\gamma}_k)_{k<i}$*, to a relation of the form:*

$$n_i\hat{\gamma}_i + \sum_{k<i} n_k^{(i)}\hat{\gamma}_k = \sum_{k<i} \ell_k^{(i)}\hat{\gamma}_k,$$

such that $n_i > 0$ *is minimal among all such relations.*
c) We may assume that no relation beween the $(\hat{\gamma}_k)_{k<i}$ *appears in the difference* $\sum_{k<i} n_k^{(i)}\hat{\gamma}_k - \sum_{k<i} \ell_k^{(i)}\hat{\gamma}_k$*, in the sense that no restriction of the sums gives a difference equal to zero.*

Proof The first two statements follow from the fact that the rational rank of Φ is finite; the cardinality of F may be taken to be $\mathrm{r}(\Phi)$. More precisely, we may take $\hat{\gamma}_1$ as the first element of F, then $\hat{\gamma}_{i_2}$ to be the smallest element of $\hat{\Gamma}$ rationally independent of $\hat{\gamma}_1$, $\hat{\gamma}_{i_3}$ to be the smallest element $> \hat{\gamma}_{i_2}$ which is rationally independent of $\hat{\gamma}_1, \hat{\gamma}_{i_2}$, and so on. If

$$(0) = \Psi_h \subset \Psi_{h-1} \subset \cdots \subset \Psi_1 \subset \Psi_0 = \Phi$$

are the convex subgroups of Φ, the first $\mathrm{r}(\Psi_{h-1})$ elements of this sequence will be in $\Psi_{h-1} \cap \hat{\Gamma}$, the next $\mathrm{r}(\Psi_{h-2}) - \mathrm{r}(\Psi_{h-1})$ will be in $(\Psi_{h-2} \setminus \Psi_{h-1}) \cap \hat{\Gamma}$, and so on. In this way we build a *finite* sequence $\hat{\gamma}_1 < \hat{\gamma}_{i_2} < \cdots < \hat{\gamma}_{i_r}$ which rationally generates $\hat{\Gamma}$. The notation is chosen to stress the fact that it is a subset of a generating sequence of the semigroup.
For $i \notin F$, if $\hat{\gamma}_{i_j}$ is the largest element in our sequence such that $\hat{\gamma}_{i_j} < \hat{\gamma}_i$, then $\hat{\gamma}_i$ is rationally dependant on $(\hat{\gamma}_1, \hat{\gamma}_{i_2}, \ldots, \hat{\gamma}_{i_j})$ and we may write

$$n_i\hat{\gamma}_i + \sum_{k<i} n_k^{(i)}\hat{\gamma}_k = \sum_{k<i} \ell_k^{(i)}\hat{\gamma}_k$$

a relation expressing the rational dependance of $\hat{\gamma}_i$ on the $(\hat{\gamma}_k)_{k<i}$; we may choose such a relation with minimal n_i. Let

$$m_i\hat{\gamma}_i + \sum_{k<i} m_k^{(i)}\hat{\gamma}_k = \sum_{k<i} p_k^{(i)}\hat{\gamma}_k$$

be another relation with non negative coefficients. By the minimality of n_i, we have $m_i \geq n_i$, and we may write $m_i = qn_i + r$ with $q \geq 1$ and $0 \leq r < n_i$. Adding $q\sum_{k<i} n_k^{(i)}\hat{\gamma}_k$ to both sides of the second relation, and substracting to each side of the equality q times the corresponding side of the first gives

$$r\hat{\gamma}_i + \sum_{k<i} m_k^{(i)}\hat{\gamma}_k + q\sum_{k<i} \ell_k^{(i)}\hat{\gamma}_k = \sum_{k<i} p_k^{(i)}\hat{\gamma}_k + q\sum_{k<i} n_k^{(i)}\hat{\gamma}_k.$$

By definition of n_i, we must have $r = 0$, so that we have proved that modulo the first relation, the second one becomes a relation between $(\hat{\gamma}_k)_{k<i}$. The last statement of a) follows from the fact that in the height one case, there are only finitely many $\hat{\gamma}_k$ smaller than any given element of $\hat{\Gamma}$.
Statement c) follows from the fact that among all relations with minimal n_i, we may choose one such that the number of $\ell_k^{(i)}$ which it contains is minimal. $\square$

It is important in the sequel to remember only the fact that for $i \notin F$ some multiple of $\hat{\gamma}_i$ to which we add a sum of the form $\sum_{k<i} t_k^{(i)}\hat{\gamma}_k$ is in $\hat{\Gamma}_{<i}$; choosing the smallest such multiple, we will then write:

$$(R_i) \qquad n_i\hat{\gamma}_i + \sum_{k<i} n_k^{(i)}\hat{\gamma}_k = \sum_{k<i} \ell_k^{(i)}\hat{\gamma}_k.$$

The coefficients of each such relation have no common divisor. The reason why it is important is that all relations between the $\hat{\gamma}_i$'s appear in this way, and the coefficient n_i is really the smallest which can occur in a relation involving $\hat{\gamma}_i$ with smaller terms. If we require that these smaller terms are actually in F, the exponent n_i is in general larger.

With the notations of subsection 7.1, given a finite sufficient set of generators $(w_j)_{j\in F}$ for the maximal ideal of $\hat{R}^{(\nu)}$, let us choose h large enough for $k[x_1^{(h)},\ldots,x_r^{(h)}]$ to contain the images in $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ of the $(W_j)_{j\in F}$.

Remark that the relations R_i just written provide us with the following binomial relations among those defining $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$; we have, for $i\notin F$:

$$(B_i)\qquad W_i^{n_i}\prod_{k<i}W_k^{n_k^{(i)}}-\lambda_i\prod_{k<i}W_k^{\ell_k^{(i)}}=0\quad\text{in } \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}\quad\text{with } \lambda_i\in k^*.$$

Recall that the products appearing here and below are finite.
Let us denote by $\mathbf{B}(F)$ the binomial equations between $(W_j)_{j\in F}$ which define the graded algebra $\mathrm{gr}_{\hat{\nu}}^{(F)}\hat{R}^{(\nu)}\subset\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$. The ideal $\mathcal{B}$ in $k[(W_j)_{j\in J}]$ which defines $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ contains the ideal $\mathcal{B}'$ generated by $\big(\mathbf{B}(F),(B_i)_{i\notin F}\big)$.

From now on, we take a finite subset $(\delta_j)_{j\in F}$ of the generators of the semigroup $\hat{\Gamma}$ of the values which $\hat{\nu}$ takes on $\hat{R}^{(\nu)}$, such that $(\eta_j)_{j\in F}$ is a sufficient set of generators for the maximal ideal of $\hat{R}^{(\nu)}$. Such a set exists by Proposition 5.59.

Proposition 5.61 *Let $j\notin F$ be the index of one of the variables w_j. By the definition of F the element j has a predecessor in J and we write $j=i+1$. The generator $\hat{\gamma}_i$ is rationally dependent upon the $(\hat{\gamma}_k)_{k\in F}$; there exists an equation among the G_{mn} of Corollary 5.51, and in which w_{i+1} appears linearly. The equation can be chosen to be such that its initial binomial is of the form $W_i^{n_i}W^{n(i)}-\lambda_iW^{m(i)}$.*

Proof The first observation is that in $\hat{R}^{(\nu)}$, by Cohen's theorem, each of the elements η_ℓ; $\ell\notin F$, is a power series in the $(\eta_j)_{j\in F}$, say $\eta_\ell=\sum_p d_p^{(\ell)}\eta^p$. Since it involves only finitely many variables, this series makes sense in $k[\widehat{(w_j)_{j\in J}}]$ and so we get elements $w_\ell-\sum_p d_p^{(\ell)}w^p$ which belong to the kernel T of the map $k[\widehat{(w_j)_{j\in J}}]\to\hat{R}^{(\nu)}$. We may, in the series G_{mn} which generate this kernel up to closure, substitute freely the series $\sum_p d_p^{(\ell)}w^p$ for every occurrence of w_ℓ outside of the initial binomial, without changing this kernel. This shows that we may choose the G_{mn} corresponding to binomials containing only variables with indices in F in such a way that all the variables w_j which they contain have indices $j\in F$ as well. Of course the initial forms of these generators of the kernel no longer necessarily generate the initial ideal $(W^m-\lambda_{mn}W^n)_{(m,n)\in\hat{E}}$. Another remark is that any system of generators, up to closure, of the ideal T must contain for each $\ell\notin F$ a generator in which the variable w_ℓ appears linearly; otherwise there could not be a relation such as $\eta_\ell=\sum_p d_p^{(\ell)}\eta^p$ in $\hat{R}^{(\nu)}$.
Now we use again the fact that each element η_j for $j\notin F$ belongs in $\hat{R}^{(\nu)}$ to the maximal ideal, generated by $(\eta_j)_{j\in F}$. Let now i denote the smallest index of a generator of the semigroup $\hat{\Gamma}$ such that $i\notin F$. By the Proposition and our choice of generators for the ideal defining $R^{(\nu)}$, we may choose a relation among those

which are such that w_i appears for the first time in their initial (binomial) part, with minimal n_i; it is of the form:

$$G_i = w_i^{n_i} \prod_{k<i} w_k^{n_k^{(i)}} - \lambda_i \prod_{k<i} w_k^{\ell_k^{(i)}} + \sum_s c_s^{(i+1)}(v^\phi) w^s = 0,$$

with the condition that the weight of each w^s must be greater than the weight of $w_i^{n_i} \prod_{k<i} w_k^{n_k^{(i)}}$, which is equal to that of $\prod_{k<i} w_k^{\ell_k^{(i)}}$, so that the variable w_{i+1} cannot appear in an equation G_{mn} whose initial binomial contains a variable of index $> i$. By what we have just seen, we may assume that w_{i+1} does not appear in the equations G_{mn} whose initial form contains only variables of index in F. We may assume that w_{i+1} appears linearly in the equation G_i, since it must appear in one of the equations whose initial binomial contains w_i and other variables of index $< i$. Now for any index $i \notin F$ we may assume that w_{i+1} has not appeared in any of the equations G_{mn} whose initial binomial contains variables of index $< i$ and consider similarly equations whose initial binomial comes from the expression of $\hat{\gamma}_i$ in terms of $\hat{\gamma}_j$ with $j < i$. Then w_{i+1} must appear linearly in at least one of these equations; we choose one where this occurs, which we call G_i. □

Turning now to $\hat{R}^{(\nu)}$, we note that the result just proved provides us with a totally ordered subset $(G_i)_{i \notin F}$ of the set of equations G_{mn}.

Proposition 5.62 (The abyssal phenomenon) **With the same notations, there exists a finite set F' containing F such that the kernel T of the map*

$$k[\widehat{(w_j)_{j\in J}}] \to \hat{R}^{(\nu)}$$

can be generated up to closure by :
- a finite set of equations involving the variables $(w_j)_{j\in F'}$ and whose initial forms are binomials generating the kernel of the natural map $k[(W_j)_{j\in F'}] \to \mathrm{gr}_{\hat{\nu}} \hat{R}^{(\nu)}$, and
*- equations $G_i = w_i^{n_i} w^{n(i)} - \lambda_i w^{m(i)} + c_{i+1} w_{i+1} + \sum_p c_p w^p$, $i+1 \notin F'$, $c_{i+1} \in k^*$.**

Proof We keep the same notations, but please remark that we passed from $i \notin F'$ to $i+1 \notin F'$. To achieve this is easy: first add to our sufficient set F the smallest element which is not in F. We may also transform, without modifying the ideal, all the other equations G_{mn} whose initial binomial contains only variables of index $\leq i$ into equations in the $(w_j)_{j\in F}$. Then we add the ordinal j to our set F and continue with its successor. By transfinite induction on $j \notin F$, using Proposition 5.61, we build a set of equations

$$G_i = w_i^{n_i} w^{n(i)} - \lambda_i w^{m(i)} + c_{i+1} w_{i+1} + \sum_p c_p w^p$$

in bijection with the variables w_i such that $i+1 \notin F$. Let us now consider the ideal $\mathbf{F}$ of $k[\widehat{(w_j)_{j\in J}}]$ generated by the equations G_{mn} whose initial binomial contains only variables with index in F. By the implicit function theorem, the quotient of $k[\widehat{(w_j)_{j\in J}}]$ by the closure of the ideal $(\mathbf{F}, (G_i)_{i\notin F})$ is a quotient of $k[[(w_j)_{j\in F}]]$ which maps onto $\hat{R}^{(\nu)}$. The kernel of this map is generated by the images of the equations G_{mn} whose initial form involves some W_j with $j \notin F$ and which we have not used in the construction of the G_i. Since the ring $k[[(w_j)_{j\in F}]]$ is nœtherian, the images of finitely many of the equations G_{mn} suffice to generate the kernel. We choose such a finite set and add to our set F the variables involved in the initial forms of these equations, and consider the finite set F' containing this new set and having the property that $i \notin F'$ implies $i+1 \notin F'$. We consider the new ideal $\mathbf{F}'$ obtained in this manner. The closure of the ideal generated by $\mathbf{F}'$ and the equations $(G_i)_{i\notin F}$ is now the kernel T. By construction, this set of generators of T has the property that the initial forms of those among its elements which involve only variables of index $i \in F'$ generate the prime ideal which is the kernel of the map $k[(W_j)_{j\in F'}] \to \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$. □

This is a fundamental fact: all but finitely many of the equations defining $\hat{R}^{(\nu)}$ serve only to express w_{i+1}, for $i+1 \notin F$, in terms of $(w_j)_{j\in F}$ and the initial binomials of the finitely many equations generate a prime ideal in a polynomial ring in finitely many variables. More precisely, keeping the notations introduced at the beginning of this subsection, we have:

Corollary 5.63 *Let us denote by $\mathbf{F}'$ the finite set consisting of those equations among the G_{mn} whose initial forms depend only on the $(W_j)_{j\in F'}$. Given an homomorphism $c\colon \Phi \to k^*$, the kernel of the surjective map*

$$\hat{c}\colon k[\widehat{(w_j)_{j\in J}}] \to \hat{R}^{(\nu)}, \qquad w_j \mapsto \eta_j c(-\hat{\gamma}_j)$$

is generated, up to closure, by the ideal $\big(c(\mathbf{F}'), c(G_i)_{i\notin F}\big)$. The initial binomials of these generators which depend only on the $(W_j)_{j\in F'}$ generate the kernel of the map of k-algebras $k[(W_j)_{j\in F'}] \to \mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ determined by $W_j \mapsto \overline{\eta}_j c(-\hat{\gamma}_j)$.

Definition 5.64 Given an excellent local domain R with a rational valuation ν, a finite finite subset of the index set of the minimal system of generators of the semigroup $\hat{\Gamma} = \hat{\nu}(\hat{R}^{(\nu)} \setminus \{0\})$ having the property described in Proposition 5.62 will be called a *quite sufficient* set.

Corollary 5.65 ** With the same notations, let F be a quite sufficient set of generators for the maximal ideal of $\hat{R}^{(\nu)}$. For each j such that $j+1 \notin F$, the image η_{j+1} in $\hat{R}^{(\nu)}$ of the element w_{j+1} can be written as the image of a series in the images of $(w_j)_{j\in F'}$; such a series, viewed in $k[\widehat{(w_j)_{j\in J}}]$, may be called an j-th semiroot for $\hat{R}^{(\nu)}$; semiroots are now indexed by ordinals $< \omega^{\mathrm{h}_R(\nu)}$. Its expression in $\hat{R}^{(\nu)}$ as the image of a series in $(w_k)_{k\le j}$ is:*

$$\eta_{j+1} = d_{j+1}\big(\eta_j^{n_j}\prod_{k<j}\eta_k^{n_k^{(j)}} - \lambda_j\prod_{k<j}\eta_k^{\ell_k^{(j)}} + \sum_s d_s^{(j)}\eta^s\big),$$

with $d_{j+1} \in k^*$ and $\hat{\nu}(\eta^s) > \hat{\nu}(\prod_{k<j}\eta_k^{\ell_k^{(j)}}) = \hat{\nu}(\eta_j^{n_j}\prod_{k<j}\eta_k^{n_k^{(j)}})$, obtained by solving for the $(w_{j+1})_{i\notin F}$ the equations

$$\big(c(G_j) = 0\big)_{j\notin F}.^*$$

This suggests a way to generalizes the theory of approximate roots, or rather of semiroots in the sense of [PP].

From a valuation theoretic viewpoint the idea of approximate roots appears already in the "key polynomials" of MacLane (see [McL1], [McL2] and [V2]), but in a different guise it was developed by Abhyankar-Moh, (see [A-M1], [A-M 2]), Cossart-Moreno (in preparation), Lejeune-Jalabert, (see [L]) (for curves, the idea being that of branches having "critical" contact with a given plane curve singularity), and related notions are studied by Spivakovsky (in [S1] for surfaces, and in [S2] for valuations of height one). I refer to [PP], especially to Corollary 1.5.4, to be compared with Proposition 5.48, and Corollary 1.5.5. The connection between the various viewpoints is explained in a special case at the end of subsection 4.4.

Corollary 5.66 *For $i+1 \notin F$, we have the inequality*

$$\gamma_{i+1} > n_i \gamma_i.$$

Remark 5.67 The choice of the finite set F made above is simple but far from economical; one could begin by taking elements whose valuations rationally generate Γ, in minimal number $\mathrm{r}(\nu)$. Then Lemma 3.3 implies that the elements of $\mathrm{gr}_\nu R$ are connected to them by binomial equations, and we can add enough such elements for their representatives to generate the maximal ideal. Whether one of the corresponding equations F_{mn} will contain a linear term then depends on the structures of Γ and R. The simplest exemple is the case of a plane branch.

5.6 An example: complex plane branches.

Example 5.68 Let R be the analytic algebra of a complex analytic plane branch (i.e., germ of analytically irreducible curve) $(X, 0)$; then it is dominated by only one valuation, the t-adic valuation ν induced by the injection $R \subset \mathbf{C}\{t\}$ of R into its normalization, and the valuation ring is $\mathbf{C}\{t\}$. Let $\Gamma = \nu(R \setminus \{0\}) \subset \mathbf{N} \cup \{0\}$ be the semigroup of values of ν on R; it has $g+1$ generators[8] $(\overline{\beta_0}, \ldots, \overline{\beta_g})$ and if we choose for each $0 \leq i \leq g$ an element $\xi_i(t) \in R$ with valuation $\overline{\beta_i}$, then the completion $\hat{\mathcal{A}}(R)$ of $\mathcal{A}(R)$ with respect to the (t)-adic valuation (our $\nu_{\mathcal{A}}$) (resp. the analyticization $\mathcal{A}(R)^{an}$ of $\mathcal{A}(R)$) is isomorphic to the (t)-adic completion $\hat{\mathcal{A}}$ (resp. the analyticization) of the sub $\mathbf{C}[v]$-algebra of $R[v, v^{-1}]$ generated by the elements $\xi_i(t) v^{-\overline{\beta_i}}$. This follows from ([T1]) and what we have seen above. Anyway, this is the algebra of a family M, parametrized by v, of branches, each fiber $M(v)$ for $v \neq 0$ being isomorphic to the branch $(X, 0)$, and the special fiber being isomorphic to the monomial curve described parametrically by $u_i = t^{\overline{\beta_i}};\ \ 0 \leq i \leq g$. There is a natural map which resolves its singularities; it is the map $\mathcal{A}^{an} \to \mathbf{C}\{v, t'\}$ given by

$$\xi_i(t) v^{-\overline{\beta_i}} \mapsto \xi_i(vt') v^{-\overline{\beta_i}}.$$

Indeed, the natural resolution of singularities for $\mathrm{Spec}\mathcal{A}(R)$ is $\mathrm{Spec}\mathcal{A}(\mathbf{C}\{t\})$. This last ring is a genuine Rees algebra for the ideal (t) in $\mathbf{C}\{t\}$

$$\mathcal{A}(\mathbf{C}\{t\}) = \mathbf{C}\{t\}[v] \bigoplus \bigoplus_{n \in \mathbf{N}} (t)^n v^{-n},$$

and its analytization maps isomorphically to $\mathbf{C}\{v, t'\}$ by $v \mapsto v,\ t \mapsto vt'$.
This shows that the parametric specialization described in ([T1], 1.10, p. 167) and ([G-T], §3) coincides with the specialization given by the valuation algebra.
It was shown in [T1] that the binomial equations

$$u_i^{n_i} - u_0^{\ell_0^{(i)}} \ldots u_{i-1}^{\ell_{i-1}^{(i)}} = 0 \ \ , \ 1 \leq i \leq g,$$

[8] In the complex plane branch case, the generators of the semigroup are often denoted by $\overline{\beta}_i$ to underline their kinship with the Puiseux exponents β_i, according to a tradition initiated by Zariski; we followed that tradition in [G-T].

defining the monomial curve in $\mathbf{C}^{g+1}$ extend to equations defining the family M in $\mathbf{C}\times\mathbf{C}^{g+1}$, which are of the form

$$\begin{array}{ll}
u_1^{n_1}-u_0^{\ell_0^{(1)}}+b_2vu_2+\sum_{w(s)>n_1\overline{\beta}_1}c_s^{(1)}(v)u^s & =0\\
u_2^{n_2}-u_0^{\ell_0^{(2)}}u_1^{\ell_1^{(2)}}+b_3vu_3+\sum_{w(s)>n_2\overline{\beta}_2}c_s^{(2)}(v)u^s & =0\\
\vdots & \vdots\\
u_{g-1}^{n_{g-1}}-u_0^{\ell_0^{(g-1)}}\cdots u_{g-2}^{\ell_{g-2}^{(g-1)}}+b_gvu_g+\sum_{w(s)>n_{g-1}\overline{\beta}_{g-1}}c_s^{(g-1)}(v)u^s & =0\\
u_g^{n_g}-u_0^{\ell_0^{(g)}}\cdots u_{g-1}^{\ell_{g-1}^{(g)}}+\sum_{w(s)>n_g\overline{\beta}_g}c_s^{(g)}(v)u^s & =0
\end{array}$$

where the $c_s^{(j)}(v)$ are in $(v)\mathbf{C}\{v\}$, $w(s)=\sum_0^g\overline{\beta}_js_j$ is the weight of the monomial u^s with respect to the weight vector $w=(\overline{\beta}_0,\dots,\overline{\beta}_g)$, i.e., $w(s)=\langle w,s\rangle$, and with all $b_j\neq 0$ in $\mathbf{C}$, which corresponds to the fact that our branch is plane. The other u^s appearing in the j-th equation are different from u_{j+1}. Note that since $n_i\overline{\beta}_i<\overline{\beta}_{i+1}$ (see [T1]), we add to each binomial only terms of higher weight, one of which is linear except in the bottom equation, which "creates" the singularity.
We see that we have a surjective map

$$\pi\colon \mathbf{C}\{v\}[[u_o,\dots,u_g]]\to\hat{\mathcal{A}}(R),$$

and its kernel is generated by the equations just written, which illustrates Proposition 5.49.
It was shown in [G-T] that a toric map $\pi\colon Z\to\mathbf{C}^{g+1}$ resolving the monomial curve extends to a toric map $\mathrm{Id}_{\mathbf{C}}\times\pi\colon \mathbf{C}\times Z\to\mathbf{C}\times\mathbf{C}^{g+1}$ inducing a resolution of M which is a strong simultaneous embedded resolution along $\mathbf{C}\times\{0\}$. In particular the map π gives an embedded resolution for our germ of curve $C\subset\mathbf{C}^{g+1}$, the embedding being given by the generators $(\xi_0,\dots,\xi_g)$ of the maximal ideal of R, or alternatively by the equations written just above, with $v=1$.
In this example, the valuation $\tilde{\nu}$ of subsection 3.2 has values in the group $\mathbf{Z}\oplus\mathbf{Z}$ ordered lexicographically, with the values $\tilde{\nu}(v)=(1,0)$, $\tilde{\nu}(t)=(1,1)$, this last value corresponding to $\tilde{\nu}(t')=(0,1)$ on $\mathbf{C}\{v,t'\}$, and so to the natural height two valuation on this last ring adapted to the coordinates (v,t').

By successive elimination in the equations shown above, and thanks to the usual implicit function theorem, we can express the image of u_2 in the ring R of our branch in the form

$$u_2=b_2^{-1}(u_1^{n_1}-u_0^{\ell_0^{(1)}}+\sum d_s^{(2)}u_0^{s_0}u_1^{s_1}),$$

and then

$$u_3=b_3^{-1}((u_1^{n_1}-u_0^{\ell_0^{(1)}}+\sum d_s^{(2)}u_0^{s_0}u_1^{s_1})^{n_2}-u_0^{\ell_0^{(2)}}u_1^{\ell_1^{(2)}}+\sum_s d_s^{(3)}u_0^{s_0}u_1^{s_1}),$$

and so on. The plane branches with equations $u_2=0,\ u_3=0,\dots,u_k=0$ for $2\leq k\leq g$ correspond to semiroots in the sense of [PP] of the Weierstrass polynomial defining our plane branch. Compare with Corollary 5.65; the fact that no abyssal phenomenon is involved here does not prevent the fact that all variables but two appear linearly in the equations from having its consequences.

6 Toric modifications

This section begins the study of partial uniformization, by a combinatorial process, of the valuation $\nu_{\rm gr}$ on $\mathrm{gr}_\nu R$ when ν is a rational valuation of the nœtherian local ring R. I show how to resolve singularities of an irreducible variety defined by a binomial ideal in finite-dimensional space, which is the case for $\mathrm{Specgr}_\nu R$ in the situation just described if in addition $\mathrm{gr}_\nu R$ happens to be a finitely generated k-algebra, and is the case for a finitely generated approximation of $\mathrm{Specgr}_\nu R$ in general. As far as I know, existing literature deals only with toric varieties associated to fans, which are normal. I begin with an example. $k = k_R$ is algebraically closed and that ν is a valuation of rational rank and height equal to $d = \dim R$ dominating R. By Abhyankar's inequality, the extension $k \to k_\nu$ is trivial, so we are in the situation of Proposition 4.2, and the ordered group Φ is isomorphic to $\mathbf{Z}^d$ with the lexicographic order (see the text before Proposition 3.9).

Then we can write:

$$\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_j, \ldots \rangle \text{ with } \gamma_j \in \mathbf{Z}^d_{\geq 0}.$$

The semigroup Γ is the image of $\mathbf{N}^{\mathbf{N}}$ by a surjective map

$$b\colon \mathbf{Z}^{\mathbf{N}} \to \mathbf{Z}^d$$

defined by sending the i-th basis vector to γ_i. Passing to semigroup algebras over k gives a map

$$k[(U_j)_{j\in J}] \to k[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \text{ determined by } \ U_j \mapsto t^{\gamma_j}$$

having $k[t^\Gamma]$ for image. This describes $\mathrm{Spec}k[t^\Gamma]$ as the closure of an orbit of the torus $(k^*)^d$ in the $\mathbf{N}$-dimensional affine space $\mathbf{A}^{\mathbf{N}}(k)$.
Note that since R and R_ν have the same field of fractions, the semigroup Γ generates Φ as a group. It will not be the case in general for a valuation with value group Φ that there is a finite index i_0 such that the finitely generated semigroup $\langle \gamma_1, \ldots, \gamma_{i_0}\rangle$ generates $\mathbf{Z}^d$ as a group. Note also that even in this special case where $\Phi = \mathbf{Z}^d$ we have when d is > 1 a difficulty which does not appear when $\Phi = \mathbf{Z}$; the γ_j do not have all their coordinates positive, so that our orbit closure is not in general the image of a map $\mathbf{A}^d(k) \to \mathbf{A}^{\mathbf{N}}(k)$.

By dualizing the map b, we obtain a map

$$w\colon \check{\mathbf{Z}}^d \to \check{\mathbf{Z}}^{\mathbf{N}}.$$

If the semigroup Γ was finitely generated and therefore the right-hand group was finite dimensional, say $\check{\mathbf{Z}}^N$, we would know how to find an embedded resolution of the singularities of $\mathrm{Spec}k[t^\Gamma] \subset \mathrm{Spec}k[U_1, \ldots, U_N]$ by a single toric map, as is explained in the next section.

6.1 Embedded toric resolution of orbit closures and binomial ideals in finite dimensions. I describe here the embedded resolution of singularities of torus orbit closures (in finite dimensions) as a generalization of what is done in [G-T] for monomial curves. I assume that the reader is aware of the basics of toric geometry, and especially of the fact that any fan can be refined to a regular simplicial fan, by iterated stellar subdivisions. I refer to David Cox's notes [Cox] and to Günter Ewald's book ([E], VI, No.8, p. 253), and I use the description of orbit closures given by Sturmfels in [St1], [St2]; see [Cox], §4. I keep the notations introduced above, so that our orbit closure is described by a map of semigroups

$b\colon \mathbf{N}^N \to \mathbf{Z}^d$, where N is finite and the image of b generates the group $\mathbf{Z}^d$. Choosing generators $(m^\ell - n^\ell)_{\ell\in\{1,\dots,L\}}$ for Ker$b$ gives us an exact sequence

$$\mathbf{Z}^L \to \mathbf{Z}^N \to \mathbf{Z}^d \to 0\ .$$

Note that the generators $m^\ell - n^\ell$ are necessarily primitive vectors in $\mathbf{Z}^N$, and we may choose them in such a way that m^ℓ and n^ℓ both have non-negative coordinates (see [St]). I assume from now on that they are chosen in this way.
By dualizing, we obtain an exact sequence

$$0 \to \check{\mathbf{Z}}^d \to \check{\mathbf{Z}}^N \to \check{\mathbf{Z}}^L,$$

where the image of the $i-th$ basis vector of $\check{\mathbf{Z}}^d$ is the vector

$$w^i = (\gamma_{1i}, \gamma_{2i}, \dots, \gamma_{Ni}).$$

Choosing one of the $m^\ell - n^\ell$ gives us an injective map $\mathbf{Z} \to \mathbf{Z}^N$, and a surjective $\check{\mathbf{Z}}^N \to \check{\mathbf{Z}}$ having kernel $H_\ell = \{a \in \check{\mathbf{Z}}^N \mid \langle a, m^\ell - n^\ell\rangle = 0\}$.
The intersection $\bigcap_{\ell\in\{1,\dots,L\}}(H_\ell \otimes_{\mathbf{Z}} \mathbf{R})$ is the $\mathbf{R}$-linear span W of the w^i. From now on I will write H_ℓ indifferently for $H_\ell \otimes_{\mathbf{Z}} \mathbf{R}$ or its integral points. The interpretation will be clear from the context.
By construction, we have $\langle w^i, m^\ell - n^\ell\rangle = 0$ for $i = 1, 2, \dots, d,$ and $\ell \in \{1, \dots, L\}$, hence also $\langle w, m^\ell - n^\ell\rangle = 0$ for $\ell \in \{1, \dots, L\}$ and all vectors $w \in W$. Remark that we may assume that W contains no basis vector; if it contained the i-th base vector, the variable U_i would appear in none of the equations, so we might as well remove it. The complement in $\check{\mathbf{R}}^N_+$ of the union of the hyperplanes H_ℓ is a union of rational N-dimensional convex cones, and the union of hyperplanes itself is the reunion of rational convex cones of smaller dimensions. Note that H_ℓ meets the interior of $\check{\mathbf{R}}^N_+$ since the coordinates of the vectors $m^\ell - n^\ell$ are not all of the same sign.
By the fundamental result on resolution of toric varieties (see [O], [Cox]), we can find a regular fan Σ with support $\check{\mathbf{R}}^N_+$ which is not only compatible with the w^i but also with the union of the hyperplanes H_ℓ in the sense that any $\sigma \in \Sigma$ meets any one of the H_ℓ along one of its faces. Then, Σ is also compatible with the intersection W of the H_ℓ. In the sequel I tacitly assume Σ to have these properties.

I am going to show that for a suitable choice of such a regular fan Σ, the toric map

$$\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^N(k)$$

induces an embedded resolution of the orbit closure $X \subset \mathbf{A}^N(k)$ corresponding to b.

Let k be an algebraically closed field and let $(U^{m^\ell} - U^{n^\ell})_{\ell\in\{1,\dots,L\}}$ be generators of the kernel of the map

$$k[b]\colon k[U_1, \dots, U_N] \to k[t_1^{\pm1}, \dots, t_d^{\pm1}] \ \text{ given by } \quad U_i \mapsto t^{\gamma_i},$$

where $U^m = U_1^{m_1} \cdots U_N^{m_N}$.
Since the ideal ker$k[b]$ defines an orbit closure, in view of ([E-S], Corollary 2.5) and the identity

$$U^{m+m'} - \lambda_{mn}\lambda_{m'n'}U^{n+n'} = U^{m'}(U^m - \lambda_{mn}U^n) + \lambda_{mn}U^n(U^{m'} - \lambda_{m'n'}U^{n'}),$$

the ideal ker$k[b]$ is generated by binomials $U^m - U^n$ such that the exponents $m - n$ belong to the lattice $\mathcal{L} \subset \mathbf{Z}^N$ generated by the $m^\ell - n^\ell$ for $\ell \in \{1, \dots, L\}$, and this lattice is a direct factor in $\mathbf{Z}^N$, which corresponds to the fact that the ideal ker$k[b]$ is prime. If we take $N - d$ of the $m^\ell - n^\ell$, they generate a sublattice $\mathcal{L}_1 \subset \mathcal{L}$. We can choose these elements in such a way that they rationally generate the lattice $\mathcal{L}$, which means that $\mathcal{L}/\mathcal{L}_1$ is a torsion $\mathbf{Z}$-module. Since $\mathcal{L}$ is a direct factor, it is the saturation of $\mathcal{L}_1$ in the sense of [E-S]; it is the lattice of elements of $\mathbf{Z}^N$ which possess a multiple in $\mathcal{L}_1$.

If we take a regular simplicial cone of maximum dimension $\sigma = \langle a^1, \dots, a^N \rangle \in \Sigma$, in the affine chart $U_\sigma \subset Z(\Sigma)$ with coordinates $Y_1, \dots, Y_N$ associated to σ by definition of the toric map $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^N(k)$, we have the following expression for $\pi(\Sigma)|U_\sigma \colon U_\sigma \to \mathbf{A}^N(k)$

$$U_i \mapsto Y_1^{a_i^1} \cdots Y_N^{a_i^N} \quad \text{and so} \quad U^m \mapsto Y_1^{\langle a^1, m\rangle} \cdots Y_N^{\langle a^N, m\rangle},$$

where $\langle a^i, m\rangle = \sum_{j=1}^N a_j^i m_j$.
Let us compute the transform by this map of one of our binomial generators, denoted by $U^m - U^n$. We may assume that $a^1, \dots, a^t$ are those among the a^j which lie on the hyperplane H dual to $m-n$, i.e., such that $\langle a^j, m-n\rangle = 0$, $1 \le j \le t$. Because our fan is compatible with H, all the other $\langle a^j, m-n\rangle$ are of the same sign, say $\langle a^j, m-n\rangle > 0$. We have then

$$U^m - U^n \mapsto Y_1^{\langle a^1, n\rangle} \cdots Y_N^{\langle a^N, n\rangle} (Y_{t+1}^{\langle a^{t+1}, m-n\rangle} \cdots Y_N^{\langle a^N, m-n\rangle} - 1).$$

Compare with [Z1], Th. 2, p.863 and [S2]. This essentially means that in toric geometry, there is no need for Hironaka's game (see [H1]); it is replaced by the existence of a fan compatible with the H_ℓ as above. Hironaka's game comes into play when one wants to dominate our toric map by the composition of a sequence of blowing-ups with non singular centers.
The exceptional divisor of the map $\pi(\sigma)$ is the union of the $y_j = 0$ for those j such that a^j is not a basis vector of $\check{\mathbf{Z}}^N$ (see [G-T]). For simplicity I will assume that none of the a^j is a basis vector, so that the exceptional divisor is $Y_1 \dots Y_N = 0$. If $t = 0$, the strict transform $Y_{t+1}^{\langle a^{t+1}, m-n\rangle} \cdots Y_N^{\langle a^N, m-n\rangle} - 1 = 0$ does not meet the exceptional divisor; therefore the strict transform of $U^m - U^n = 0$ meets the exceptional divisor only in those charts U_σ for which at least one of the primitive vectors a^i of σ is in H. More generally, we have.

Proposition 6.1 *If the strict transform by $\pi(\Sigma)$ of the subspace $X \subset \mathbf{A}^N(k)$ defined by the ideal $(U^{m^\ell} - U^{n^\ell})_{\ell \in \{1, \dots, L\}}$ is to meet the exceptional divisor in the chart $\pi(\sigma)\colon U_\sigma \to \mathbf{A}^N(k)$, where $\sigma = \langle a^1, \dots, a^N\rangle$, there must be a vector a^j such that* $\langle a^j, m^\ell - n^\ell\rangle = 0$ for $\ell \in \{1, \dots, L\}$, *i.e., $a^j \in W$.*

Proof Look at the equations of the strict transform. □

This generalizes the result of [G-T], 5.2.

The $\mathbf{R}$-vector space $\check{W}$ generated by the $m^\ell - n^\ell$ is of dimension $N-d$; let us choose $N-d$ of the vectors $m^\ell - n^\ell$ which generate $\check{W}$, say $m^1 - n^1, \dots, m^{N-d} - n^{N-d}$. I now have to check that whenever the strict transform of the subspace of $\mathbf{A}^N(k)$ defined by

$$U^{m^1} - U^{n^1} = \cdots = U^{m^{N-d}} - U^{n^{N-d}} = 0$$

is not empty it is non singular and transverse to the exceptional divisor, and that the strict transforms of all the other $U^{m^\ell} - U^{n^\ell} = 0$ vanish on one of its irreducible components.

The strict transforms of the equations have the form

$$\begin{array}{rcl} Y_1^{\langle a^1,m^1-n^1\rangle}\cdots Y_N^{\langle a^N,m^1-n^1\rangle}-1 &=& 0\\ Y_1^{\langle a^1,m^2-n^2\rangle}\cdots Y_N^{\langle a^N,m^2-n^2\rangle}-1 &=& 0\\ \vdots & \vdots & \vdots\\ Y_1^{\langle a^1,m^{N-d}-n^{N-d}\rangle}\cdots Y_N^{\langle a^N,m^{N-d}-n^{N-d}\rangle}-1 &=& 0 \end{array}$$

Note that this strict transform meets the component $Y_j = 0$ of the exceptional divisor if and only if $a^j \in W$; let us renumber the a^j so that $a^1, \dots, a^t$ are those which lie in W, i.e., $\langle a^i, m^s - n^s\rangle = 0$ for $1 \le i \le t$ and all s; if none of the a^i is in W, set $t = 0$; in this case, the strict transform does not meet any of the coordinate hyperplanes in the chart U_σ. As in ([G-T], §4), we can compute the jacobian matrix J of these equations by logarithmic differentiation, and find the equality of $N \times (N-d)$ matrices

$$Y_{t+1}\dots.Y_N J = Y_{t+1}^{\sum_s\langle a^{t+1},m^s-n^s\rangle}\dots Y_N^{\sum_s\langle a^N,m^s-n^s\rangle}(\langle a^j, m^s-n^s\rangle).$$

Proposition 6.2 *Given an irreducible binomial variety $X \subset \mathbf{A}^N(k)$, with the notations just introduced, the rank of the image in $\mathrm{Mat}_{N\times L}(k)$ of the matrix $(\langle a^j, m^s-n^s\rangle) \in \mathrm{Mat}_{N\times L}(\mathbf{Z})$ is $N-d$. The strict transform by $\pi(\Sigma)$ of the subspace $X \subset \mathbf{A}^N(k)$ defined by the ideal $(U^{m^\ell} - U^{n^\ell})_{\ell\in\{1,\dots,L\}}$ is regular and transversal to the exceptional divisor; it is also irreducible in each chart.*

Proof Let σ be a cone of of maximal dimension in the fan Σ. Let us denote as above by t the dimension of $\sigma \cap W$. Since $\dim W = d$, we must have $t \le d$; since the vectors a^j form a basis of $\mathbf{Q}^N$, and the space $\check{W}$ generated by the $m^s - n^s$ is of dimension $N-d$, the rank of the matrix $(\langle a^j, m^s-n^s\rangle)$ is $N-d$, which proves the lemma if k is of characteristic zero.
To prove the general case, view our matrix $(\langle a^j, m^s-n^s\rangle)$ as describing the composed map

$$\mathbf{Z}^L \to \mathbf{Z}^N \overset{M(\sigma)}{\longrightarrow} \mathbf{Z}^N;$$

we have to check that the $(N-d)\times(N-d)$-minors of this matrix have no common divisor. This follows from the:

Lemma 6.3 *In a sequence of maps of free $\mathbf{Z}$-modules as above, assume that the image of the first map is a direct factor of rank $N-d$ and that the determinant of $M(\sigma)$ is ± 1. Then the $(N-d)\times(N-d)$-minors of the matrix describing the composed map have no common divisor.*

Proof Let us consider the sequence of maps obtained by taking $(N-d)^{\text{th}}$ exterior powers:

$$\overset{N-d}{\Lambda}\mathbf{Z}^L \to \overset{N-d}{\Lambda}\mathbf{Z}^N \overset{\overset{N-d}{\Lambda}M(\sigma)}{\longrightarrow} \overset{N-d}{\Lambda}\mathbf{Z}^N.$$

Since the image of the first map is a direct factor, so is the image of its exterior power, which is of rank one and generated by the $(N-d)\times(N-d)$-minors of the matrix defining the first map. This image is therefore a primitive vector, which implies that the minors are coprime. We have to prove that the image of this primitive vector by $\overset{N-d}{\Lambda} M(\sigma)$ is again a primitive vector. This will be true if the determinant of that matrix is ± 1. Remark that its entries are the $(N-d)\times(N-d)$-minors extracted from $M(\sigma)$ and suitably ordered. By a result of Sylvester (see

[Ai], p.92), nowadays an exercise in linear algebra, this implies that its determinant is the $\binom{N-1}{d}$-th power of the determinant of $M(\sigma)$, and the result. $\square$

So we see that the rank of J is $N-d$ everywhere on the strict transform, and by Zariski's jacobian criterion ([Z4]) this strict transform is smooth and transversal to the exceptional divisor. Note however that it is not necessarily irreducible; let us show that the strict transform of our orbit closure is one of its irreducible components. Since the differences of the exponents in the total transform and the strict transform of a binomial are the same, the lattice of exponents generated by the exponents of all the strict transforms of the binomials $(U^{m^\ell} - U^{n^\ell})_{\ell\in\{1,\dots,L\}}$ is the image $M(\sigma)\mathcal{L}$ of the lattice $\mathcal{L}$ by the linear map $\mathbf{Z}^N \to \mathbf{Z}^N$ corresponding to the matrix $M(\sigma)$ with rows $(a^1,\dots,a^N)$. Similarly the exponents of the strict transforms of $U^{m^1} - U^{n^1},\dots,U^{m^{N-d}} - U^{n^{N-d}}$ generate the lattice $M(\sigma)\mathcal{L}_1$. The lattice $M(\sigma)\mathcal{L}$ is the saturation of $M(\sigma)\mathcal{L}_1$, and so according to [E-S], since we assume that k is algebraically closed, the strict transform of our orbit closure is one of the irreducible components of the binomial variety defined by the $N-d$ equations displayed above.
The charts corresponding to regular cones $\sigma\in\Sigma$ of dimension $< N$ are open subsets of those which we have just studied, so they contribute nothing new. $\square$

I now show that our toric map can be chosen so that it induces an isomorphism outside of the singular locus of our irreducible binomial variety, so that it is an embedded resolution and not only a pseudo-resolution in the sense of [G-T]. A part of this is the fruit of common reflexions with Pedro González-Pérez (see [GP-T]). The same argument may also be used to replace the jacobian rank argument just given, since it reduces the general toric case to the case of normal toric varieties, where the jacobian rank argument is unnecessary. See [GP-T] for a proof of the resolution of singularities of binomial varieties along these lines.

With the notations introduced at the beginning of this subsection, remembering that the image of the map b generates $\mathbf{Z}^d$, let us denote by $\gamma_1,\dots,\gamma_N$ the images by the map b of the canonical basis vectors $e_1,\dots,e_N$ of $\mathbf{Z}^N$, and by $\check\sigma$ the cone which they generate in $\mathbf{R}^d$. It depends only on b and we may also denote it by $\check\sigma(b)$. Its convex dual σ (or $\sigma(b)$) in $\check{\mathbf{R}}^d$ is a strictly convex cone of dimension d whose image in $\check{\mathbf{R}}^N$ by the map $\check b$ is contained in the intersection of W with the first quadrant of $\check{\mathbf{R}}^N$. Indeed, we know by basic properties of duality that the image of $\check{\mathbf{R}}^d$ in $\check{\mathbf{R}}^N$ consists of those linear forms on $\mathbf{R}^N$ which vanish on the kernel of b, and is therefore equal to W. The image of σ consists of those linear forms $\tilde f$ on $\mathbf{R}^N$ which are in W and such that $\tilde f(e_i) = f(\gamma_i)\geq 0,\ i=1,\dots,N$, where $f\in\sigma$. Therefore $\tilde f$ is in the dual of the first quadrant of $\mathbf{R}^N$, which is the first quadrant of $\check{\mathbf{R}}^N$. Conversely if $\tilde f(e_i)\geq 0$ and $\tilde f\in W$ then $\tilde f$ is in the image of σ. This shows that σ is exactly the intersection of W with the first quadrant of $\mathbf{R}^N$.

Definition 6.4 Given a map $b\colon \mathbf{N}^N \to \mathbf{Z}^d$ of semigroups, the strictly convex cone $\sigma(b)\subset W\cap\check{\mathbf{R}}^N_+$ is called the *weight cone* associated to b.

Definition 6.5 (Condition RES(b)) Let X be an orbit closure corresponding to a map $b\colon \mathbf{N}^N\to\mathbf{Z}^d$ as above. Let $\sigma(b)$ be the weight cone associated to b just above. A fan Σ with support $\check{\mathbf{R}}^N_+$ satisfies condition RES(b) if for some system of generators $(m^\ell - n^\ell)_{\ell\in\{1,\dots,L\}}$ of the kernel of the extended map $b\colon \mathbf{Z}^N\to\mathbf{Z}^d$, it is compatible with the hyperplanes H_ℓ dual to the vectors $(m^\ell - n^\ell)_{\ell\in\{1,\dots,L\}}$, and in addition all the regular faces of the cone $\sigma(b)$ belong to Σ.

Remark that by the classical results on resolution of normal toric varieties (see [Cox]), fans satisfying (RES) do exist.

Proposition 6.6 *Assuming that the field k is algebraically closed, let $X \subset \mathbf{A}^N(k)$ be an orbit closure corresponding to a morphism $b\colon \mathbf{Z}^N \to \mathbf{Z}^d$. The singular locus of X is a union of intersections of X with linear coordinate spaces. The toric map $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^N(k)$ associated to a regular simplicial fan Σ satisfying the condition* RES(b) *just defined is an embedded resolution of the singularities of X.*

Proof Let d be the dimension of our binomial variety X and $(m^\ell - n^\ell)_{\ell\in\{1,\dots,L\}}$ and Σ be a system of generators and a fan as given by condition RES(b). A straightforward computation using logarithmic differentials shows that the jacobian determinant $J_{I,L'}$ of rank $c = N - d$ of the generators $(U^{m^\ell} - \lambda_{m^\ell n^\ell} U^{n^\ell})_{\ell\in\{1,\dots,L\}}$ of our prime binomial ideal $P \subset k[U_1, \dots, U_N]$ associated to $I = (i_1, \dots, i_c)$ and a subset $L' \subseteq \{1, \dots, L\}$ of cardinality c satisfies the congruence

$$U_{i_1} \dots U_{i_c}.J_{I,L'} \equiv \big(\prod_{\ell\in L'} U^{m^\ell}\big)\mathrm{Det}_{I,L'}\big((\langle m - n\rangle)\big) \mod .P,$$

where $\big(\langle m-n\rangle\big)$ is the matrix of the vectors $(m^\ell - n^\ell)_{\ell\in\{1,\dots,L\}}$, and $\mathrm{Det}_{I,L'}$ indicates the minor in question. By Proposition 6.2, the rank of the image in $k^{N\times L}$ of the jacobian matrix is equal to c. This proves the first part of the Proposition. For example, the only possibility for the intersection of our variety with $U_i = 0$ not to be in the singular locus is that there exists such a minor such that all the m_i^ℓ (resp. n_i^ℓ) for $\ell \in L'$ are zero except one, which is equal to one. In that case one sees that one of the equations of our binomial variety is of the form

$$U_i U'^r - U'^s = 0,$$

and all the others are independent of U_i.
By ([St3], Corollary 13.6), the normalisation of our binomial variety X is $X_{\sigma(b)} = \mathrm{Spec} k[\check\sigma(b) \cap \mathbf{Z}^d]$. The proof is as follows: let P be the prime binomial ideal which is the kernel of the map of semigroup algebras

$$k[b] : k[U_1, \dots, U_N] \to k[t_1^{\pm1}, \dots, t_d^{\pm1}]$$

corrresponding to b. By construction of $\check\sigma(b)$ the image of this map is contained in $k[\check\sigma(b) \cap \mathbf{Z}^d]$. Finally we have an injection

$$k[U_1, \dots, U_N]/P \to k[\check\sigma(b) \cap \mathbf{Z}^d]$$

which is birational since the image of b generates $\mathbf{Z}^d$, and the algebra on the right is normal since the semigroup $\check\sigma(b)\cap\mathbf{Z}^d$ is saturated. A toric resolution of singularities of X is obtained by a regular refinement of the fan in $W \cap \check{\mathbf{R}}_+^N$ consisting of $\sigma(b)$ and its faces which does not affect the regular faces of $\sigma(b)$, since that toric map resolves the singularities of $X_{\sigma(b)}$ and is an isomorphism outside of the singular locus. Now we can choose a regular fan supported in $\check{\mathbf{R}}_+^N$, compatible with the hyperplanes H_{mn} and which contains the cones of our regular subdivision of $\sigma(b)$. Any such fan corresponds to a toric map $Z(\Sigma) \to \mathbf{A}^N(k)$ which is a toric embedded resolution of X, since the strict transform of X is transversal to the exceptional divisor by the argument given above, and the restriction to this strict transform is an isomorphism outside of the singular locus of X. This proves the result. □

Remarks 6.7 1) At least in characteristic zero, the equivariant resolution theorem implies that an irreducible binomial variety X as above can be resolved by a sequence of equivariant permissible blowing ups. The theorem of De Concini-Procesi ([DC-P]) implies that our toric resolution can be dominated by a sequence of equivariant blowing-ups. It would be interesting to prove directly that these blowing ups can be chosen to be permissible, in the sense that at each step the space blown up is normally flat along the center.
2) It has been remarked by P. González Pérez (see [GP]) that any fan Σ which is compatible with W and contains the regular cones of $\sigma(b)$ will also provide a resolution of singularities of the orbit closure correponding to b; this description depends only on b and not on the choice of equations. This is illustrated by the fact that in [G-T] a resolution of the monomial curve is obtained precisely with such fans.

Proposition 6.8 *Let $(u^{m_k})_{k\in K}$ be a finite collection of monomials in a polynomial ring $R = k[(u_i)_{i\in I}]$ endowed with a rational valuation μ. There is a birational toric extension $R \to R' = R[(u^{\alpha_j})_{j\in J}]_{m'}$ such that in R' the ideal $(u^{m_k})_{k\in K}R'$ is principal and generated by the monomial with the least valuation.*

Proof Let $J \subset I$ be the finite set of the variables appearing in the monomials considered. Consider the toric modification associated to a fan of the first quadrant of $\mathbf{R}_+^{|J|}$ which is compatible with the hyperplanes dual to the vectors $(m_k - m_{k'})_{k\neq k'}$; by the valuative criterion for properness the valuation μ picks a point in a chart of that modification, and the corresponding toric extension satisfies the condition of the proposition. □

Let us now suppose that we have an irreducible subspace of $\mathbf{A}^N(k)$ defined by binomials $U^{m^\ell} - \lambda_{m^\ell n^\ell} U^{n^\ell}$ for $\ell \in \{1,\dots,L\}$, where as usual $U^m = U_1^{m_1}\cdots U_N^{m_N}$ and $\lambda_{mn} \in k^*$. By ([E-S], Cor. 2.3), at least if our field k is algebraically closed, the subspace defined by the binomials obtained by changing each λ_{mn} to 1 is a torus orbit closure, and by Proposition 6.6 has an embedded resolution by a toric map $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^N(k)$ as above. In one of the charts of $Z(\Sigma)$, the ideal $(U^{m^\ell} - U^{n^\ell})_{\ell\in\{1,\dots,L\}}$ becomes the ideal generated (up to permutation of the coordinates) by

$$Y^{e(n^k)}(Y_{t+1}^{\langle a^{t+1}, m^k - n^k\rangle}\cdots Y_N^{\langle a^N, m^k-n^k\rangle} - 1); \qquad 1 \le k \le N-d,$$

where $e(n) = (\langle a^1, n\rangle, \dots, \langle a^N, n\rangle)$, and we are assuming as above that

$$\langle a^i, m-n\rangle = 0,\ 1\le i\le t \ \text{ and } \ \langle a^i, m-n\rangle > 0,\ t+1\le i\le N.$$

The nature of the computation shows that we have:

$$U^m - \lambda_{mn}U^n \mapsto Y^{e(n)}(Y_{t+1}^{\langle a^{t+1}, m-n\rangle}\cdots Y_N^{\langle a^N, m-n\rangle} - \lambda_{mn}).$$

The relations between the λ_{mn} associated to linear relations between the $m-n$, of the form (with the notations used above)

$$\lambda_{m^\ell n^\ell} = \prod_{k=1}^{N-d} \lambda_{m^k n^k}^{b_k^\ell},$$

are exactly what is needed to ensure that once we have chosen a system of generators $m^k - n^k$ as above, the transforms of the remaining binomials $U^m - \lambda_{mn}U^n$ will define an irreducible component of the non singular variety defined by

$$Y_{t+1}^{\langle a^{t+1}, m^k - n^k\rangle} \cdots Y_N^{\langle a^N, m^k - n^k\rangle} - \lambda_{m^k n^k}^{\epsilon(k)} = 0\ ; \qquad 1 \le k \le N-d,$$

where $\epsilon(k) = \pm 1$ is the common sign which the vectors $a^i,\ t+1 \le i \le N$ take on $m^k - n^k$, so that the subspace defined by our binomial ideal is also resolved by $\pi(\Sigma)$. Keeping the notations just introduced, we see that we have proved:

Proposition 6.9 *Assuming that the field k is algebraically closed, let $X \subset \mathbf{A}^N(k)$ be an irreducible binomial variety, defined by the ideal of $k[U_1, \ldots, U_N]$ generated by $(U^{m^\ell} - \lambda_{m^\ell n^\ell} U^{n^\ell})_{\ell \in \{1,\ldots,L\}},\ \lambda_{m^\ell n^\ell} \in k^*$. For any regular simplicial fan Σ satisfying the condition* RES(b) *with respect to the orbit closure defined by the binomials $(U^{m^\ell} - U^{n^\ell})_{\ell \in \{1,\ldots,L\}}$, the corresponding toric map $\pi(\Sigma)\colon Z(\Sigma) \to \mathbf{A}^N(k)$ is an embedded resolution of the singularities of $X \subset \mathbf{A}^N(k)$.*

Remark 6.10 After the description of the singular locus of a binomial variety given in the proof of Proposition 6.6, Zariski's jacobian criterion implies that the $(N-d) \times (N-d)$-minors of the matrix $\left(m^\ell - n^\ell\right)$ are coprime integers if the binomial variety determined by the vectors $(m^\ell - n^\ell)$ is reduced and irreducible over any algebraically closed field. The geometric fact that this is true if it is reduced and irreducible over $\mathbf{C}$ also follows, in view of [E-S], from the combinatorial fact expressed in Lemma 6.3.
In the case where $d = 1$ and our (equations-wise) binomial variety is the (parametrization-wise) monomial curve corresponding to the semigroup $\Gamma = \langle \overline{\beta_0}, \ldots, \overline{\beta_g}\rangle$ and is a complete intersection, as in the example of subsection 5.6, this Lemma 6.3 solves the minor difficulty mentioned at the end of §5 of [G-T] and shows that the minors of the matrix coincide, up to sign, with the generators of the semigroup.

6.2 An example: branches revisited. Let R be a one-dimensional excellent equicharacteristic analytically irreducible local ring, i.e., the local ring of a branch. Let us assume that the residue field k of R is algebraically closed. The ring R has only one non trivial valuation, whose valuation ring is the integral closure R_ν of R in its field of fractions. Since R is excellent, R_ν is a finite R-module ([EGA], Scholie 7.8.3, vii)). Therefore there exists an element $d \in R,\ d \neq 0$, such that $dR_\nu \subset R$ and $\mathbf{N} \setminus \Gamma$ is finite, so that the semigroup Γ of values of ν on $R \setminus \{0\}$ is finitely generated. Let us write $\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_N\rangle$. Since k is algebraically closed, the valuation ν is rational, and the completion $\hat{R}^m$ of R is equal to $\hat{R}^\nu$ and is a subring of $k[[t]]$, the valuation $\hat{\nu}$ being induced by the t-adic valuation. Let us choose elements $(\xi_i)_{1 \le i \le N}$ in R such that $\nu(\xi_i) = \gamma_i$. Their initial forms for the ν-adic valuation generate the graded ring

$$\mathrm{gr}_\nu R = k[t^{\gamma_1}, \ldots, t^{\gamma_N}] \subset k[t] = \mathrm{gr}_\nu R_\nu,$$

which is the affine algebra corresponding to the monomial curve C^Γ associated to Γ. Since the $(\xi_i)_{1 \le i \le N}$ in R are finite in number, they form a minimal quite sufficient set of generators for the maximal ideal of $\hat{R}^{(\nu)} = \hat{R}^m$.
By Proposition 5.48, the map $u_i \mapsto \xi_i v^{-\nu(\xi_i)}$ determines a surjective map

$$k[v][[u_1, \ldots, u_N]] \to \widehat{\mathcal{A}_\nu(R)} \subset k[v][[t]].$$

The kernel of this map is generated by series

$$F_{mn} = u^m - u^n + \sum_s c_s^{(mn)}(v) u^s$$

which are deformations of binomial generators $u^m - u^n$ of the kernel of the map

$$k[u_1,\dots,u_N]\to k[t^{\gamma_1},\dots,t^{\gamma_N}],\quad u_i\mapsto t^{\gamma_i}.$$

Denoting by $w=(\gamma_1,\gamma_2,\dots,\gamma_N)\in\mathbf{N}^N$ the weight vector, for each of these equations, we add to the binomial u^m-u^n only terms $c_p^{(mn)}(v)u^s$ such that $w(s) > w(m)=w(n)$. If now we choose a regular fan Σ in $\mathbf{R}^N_+$ compatible with the dual hyperplanes of the vectors $m-n$, and so in particular with the vector w, we obtain a toric map

$$\pi(\Sigma)\colon Z(\Sigma)\to\mathbf{A}^N(k)$$

which is a toric embedded resolution of the monomial curve $C^\Gamma\subset\mathbf{A}^N(k)$ according to what we saw in subsection 6.1. The map

$$\mathrm{Id}_{\mathbf{A}^1(k)}\times\pi(\Sigma)\colon \mathbf{A}^1(k)\times Z(\Sigma)\to\mathbf{A}^1(k)\times\mathbf{A}^N(k)$$

then induces an embedded resolution of singularities for the formal subspace defined in Spec$k[v][[u_1,\dots,u_N]]$ by the ideal generated by the F_{mn}, which is a simultaneous resolution for all the fibers of the projection to $\mathbf{A}^1(k)$. This is checked exactly as in [G-T]. It implies that the map $\pi(\Sigma)$ gives an embedded resolution of the singularities of the image of the formal completion of our curve embedded in $\mathbf{A}^N(k)$ by using the elements $\xi_i\in R\subset\hat{R}^\nu$ as coordinates. Now since $\pi(\Sigma)$ is a monomial map, it corresponds to adding to $\hat{R}^\nu$ certain monomials in the ξ_i, with some negative exponents, and then localizing the $\hat{R}^\nu$-algebra which they generate. This operation has a meaning in R itself, and builds a local ring R' which is essentially of finite type over R and birationally equivalent to R. The local ring R' is excellent since R is, and its completion is regular since it is the completion of the transform of $\hat{R}^\nu$, so R' itself is regular.

To illustrate the difference between this approach and Abhyankar's, consider, over an algebraically closed field k of characteristic p, the curve given parametrically by:

$$u_0=t^p+t^{p+1},\qquad u_1=t^{p^2+1}+t^{p^2+p+1}.$$

It has no Puiseux presentation of the form $u'_0=t^p,\ u'_1=u'_1(t)$. However, we may consider it as a deformation of a plane monomial curve, with an equation of the form:

$$u_1^p-u_0^{p^2+1}-u_0^{p^2-p+1}u_1+\cdots$$

Here we can remark that the irreducible initial binomial describes a curve which is purely inseparable over $k((u_0))$, while this is not the case for our original curve. However, if we resolve this binomial, say by a toric modification of $\mathbf{A}^2(k)$ having as one of its charts

$$u_0=y_0^py_1,\qquad u_1=y_0^{p^2+1}y_1^p,$$

we find that the transform of the equation is

$$y_0^{p(p^2+1)}y_1^{p^2}(1-y_1-y_0y_1+\cdots)$$

and therefore we have an embedded resolution of our curve.

The usual relations between Puiseux exponents and semigroup may fail in characteristic p. Consider, over an algebraically closed field k of characteristic p, the curve (see [Ca], p.114):

$$u_0=t^{p^3},\qquad u_1=t^{p^3+p^2}+t^{p^3+p^2+p+1}.$$

One can check that although only three exponents are visible on the expansions, its semigroup has four generators:

$$\Gamma = \langle p^3,\ p^3+p^2,\ p^4+p^3+p^2+p,\ p^5+p^4+p^3+p^2+p+1\rangle,$$

and the corresponding monomial curve has equations

$$U_1^p - U_0^{p+1} = 0, \quad U_2^p - U_0^{p(p+1)}U_1 = 0, \quad U_3^p - U_1^{p^3}U_2 = 0,$$

while the curve itself has, in its natural coordinates, equations

$$u_1^p - u_0^{p+1} - u_2 = 0,\ u_2^p - u_0^{p(p+1)}u_1 + u_3 = 0,\ u_3^p - u_1^{p^3}u_2 - u_0^{p+1}u_3^p = 0.$$

The following question and example are motivated by two problems:
1) If there is a natural toric specialization of a singularity, where are the regular points of the strict transform of the singularity by a toric resolution of the toric variety?
2) Of what nature is the complexity of the binomial relations defining $\mathrm{gr}_\nu R$? this is of course essential for the description of its initial partial toric resolution.

6.3 A question on surfaces. Let k be an algebraically closed field of characteristic p. Consider in $\mathbf{A}^3(k)$ the surface

$$F = z^{p+1} + y^{p-1}z + x^{p+1} = 0$$

studied by Abhyankar in ([A4], p.589); it is quasi-ordinary with respect to the projection onto the (x,y)-plane; the z-discriminant is $x^{p+1}=0$. For $p \geq 5$, however, its Galois group with respect to this projection is "very large and quite complicated" (*loc. cit.*) and not a cyclic group as one would expect in characteristic zero. Indeed, Abhyankar later showed (see [A5]) that the Galois group over $\hat{K} = k((y, x^{p+1}))$ of the polynomial $F \in \hat{K}[z]$ is $\mathrm{PGL}_2(\mathbf{F}_p)$.
Study the rational valuations, if any, for which the associated graded ring of this surface singularity is

$$k[X,Y,Z,U]/(Z^p + Y^{p-1}, ZU + X^{p+1}),$$

so that $k[x,y,z]/(z^{p+1} + y^{p-1}z + x^{p+1})$ appears as a deformation of its associated graded ring; its equations in natural coordinates are

$$z^p + y^{p-1} - u = 0, \quad zu + x^{p+1} = 0.$$

Note that the associated graded ring is *not* quasi-ordinary for the projection onto the (X,Y)-plane and that the field extension associated to this projection displays inseparability.

6.4 Another example, in dimension three.

Example 6.11 (partly inspired by an example of Spivakovsky) Let us give $\mathbf{Z}^2$ the lexicographic order and denote by $k((t^{\mathbf{Z}^2_{\mathrm{lex}}}))$ the field of Puiseux series associated to the ordered group $\mathbf{Z}^2_{\mathrm{lex}}$ and the field k as in [Ka], [B2]; it is the field of formal series with exponents forming a well ordered subset of $\mathbf{Z}^2_{\mathrm{lex}}$. It is naturally endowed with the t-adic valuation with values in $\mathbf{Z}^2_{\mathrm{lex}}$. Let us denote by $k[[t^{\mathbf{Z}^2_+}]]$ the corresponding valuation ring. Choose a sequence of pairs of positive integers $(a_i, b_i)_{i\geq 3}$ and a sequence of elements $(\lambda_i \in k^*)_{i\geq 3}$ such that $b_{i+1} > b_i$, the series $\sum_{i\geq 3}\lambda_i u_2^{b_i}$ is not algebraic over $k[u_2]$, and the ratios $\frac{a_{i+1}-a_i}{b_{i+1}}$ are positive and increases strictly with i. Let R_0 be the k-subalgebra of $k[[t^{\mathbf{Z}^2_+}]]$ generated by

$$u_1 = t^{(0,1)}, u_2 = t^{(1,0)}, u_3 = \sum_{i\geq 3} \lambda_i u_1^{-a_i} u_2^{b_i}.$$

There cannot be an algebraic relation between u_1, u_2, and u_3, so the ring $R_0 = k[u_1, u_2, u_3]$ is the polynomial ring in three variables. It inherits the t-adic valuation of $k[[t^{\mathbf{Z}^2_+}]]$. One checks that this valuation extends to the localization $R = k[u_1, u_2, u_3]_{(u_1,u_2,u_3)}$; it is a rational valuation of height two and rational rank two. Let us try to compute the semigroup Γ of the values that it takes on R. We have $\gamma_1 = (0,1),\ \gamma_2 = (1,0), \gamma_3 = (b_3, -a_3) \in \Gamma$. Set $\Gamma_3 = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$. Then we have $u_1^{a_3} u_3 - \lambda_3 u_2^{b_3} = \sum_{i\geq 4} \lambda_i u_1^{a_3 - a_i} u_2^{b_i} \in R$, so that $\gamma_4 = (b_4, a_3 - a_4)$ is in Γ. It is easy to deduce from our assumptions that *no multiple of* γ_4 *is in* Γ_3, and that it is the smallest element of Γ which is not in Γ_3. We set $u_4 = u_1^{a_3} u_3 - \lambda_3 u_2^{b_3}$, and continue in the same manner: $u_1^{a_4 - a_3} u_4 - \lambda_4 u_2^{b_4} = u_5$, ..., $u_1^{a_i - a_{i-1}} u_i - \lambda_i u_2^{b_i} = u_{i+1}$,... with the generators $\gamma_i = \nu(u_i) = (b_i, a_{i-1} - a_i)$ for $i \geq 4$. Finally we have:

$$\Gamma = \langle \gamma_1, \gamma_2, \ldots, \gamma_i, \ldots \rangle,$$

the initial forms of the u_i constitute a minimal system of generators of the graded k-algebra $\mathrm{gr}_\nu R$, and the equations (setting $a_2 = 0$)

$$u_1^{a_i - a_{i-1}} u_i - \lambda_i u_2^{b_i} = u_{i+1}, \quad i \geq 3$$

above describe $\hat{R}^{(\nu)}$. In fact they even describe R; it is clear that from them we can reconstruct the value of u_3 as a function of u_1, u_2 by (infinite) elimination. The binomial equations for $\mathrm{gr}_\nu R$ are the

$$U_1^{a_i - a_{i-1}} U_i - \lambda_i U_2^{b_i} = 0, \quad i \geq 3,$$

showing that all the U_i for $i \geq 3$ are rationally dependent on U_1, U_2. Using the remark following proposition 3.7, we see that the Krull dimension of $\mathrm{gr}_\nu R$ is two. The fact that R is regular of dimension three is due to the abyssal phenomenon that we met in example 4.20. Remark that in this example $\mathrm{gr}_\nu R$ is not regular.
From our assumption on the growth of the ratios we see moreover that *no multiple of* γ_i is in $\Gamma_{i-1} = \langle \gamma_1, \ldots, \gamma_{i-1} \rangle$. In fact γ_i is outside of the cone with vertex 0 generated by Γ_{i-1} in $\mathbf{R}^2$.
It is interesting to study in this case the construction of a system of regular fans which will resolve the singularities of $\mathrm{gr}_\nu R$.

Remark 6.12 Olivier Piltant has communicated to me an example of a valuation of height one on the ring $R = k[u_1, u_2, u_3]_{(u_1,u_2,u_3)}$ where it is also the case that no multiple of γ_i is in Γ_{i-1}. The valuation is induced on R from a rational height one valuation, of rational rank three, on the power series ring $k[[u_1, u_2/u_1, u_3/u_1]]$ which is monomial in the variables $u_1, u_2/u_1$ and $u_3/u_1 - (u_2/u_1)^2 - (u_2/u_1)^3$. It is related to the example of [C-G-P].

Problem. If $\Gamma = \langle \gamma_1, \ldots, \gamma_i, \gamma_{i+1}, \ldots \rangle$ is the semigroup of a valuation of a nœtherian local integral domain R, and $\Gamma_i = \langle \gamma_1, \ldots, \gamma_i \rangle$, can the fact that even for large i no multiple of γ_i is in Γ_{i-1} occur with R of dimension two? Hint: use [A2]. Compare with subsection 6.3.

7 Local uniformization (speculative sketch)

...and he will be too dazzled by the light to look at the objects whose shadows he was seeing a moment ago...

Plato, *The Republic*, VII, 515

Assume in this section that the local equicharacteristic excellent integral domain R has an algebraically closed residue field k.

Given a valuation ν_0 on the ring R, we may specialize ν_0 to a valuation ν whose ring R_ν dominates R with trivial residue field extension (see subsection 3.6, Proposition 3.20).
We first uniformize the valuation $\hat{\nu}$ on $\hat{R}^{(\nu)}$. In the last subsection I will sketch the descent from $\hat{R}^{(\nu)}$ to R.
After this descent, by the first Corollary to Proposition 5.36, which ensures the nice behaviour of the specialization above with respect to composition of valuations, the same toric map will also simultaneously uniformize the valuations in the fibers of the total space of the specialization of R to $\mathrm{gr}_{\nu_0}R$, which is the subspace of the preceding specialization defined by the ideal $(v^\psi - 1)_{\psi\in\Psi_+}$, and hence also uniformize ν_0 on R. The proof proceeds by induction in the dimension, since in order to prove the existence of a scalewise completion for a ring of dimension d I have used local uniformization in dimension $< d$. The proof actually follows the scheme $TLU(d-1) \implies TC(d) \implies TLU(d)$. On the way we use that $TLU(d-1) \implies TP(d-1)$ by Corollary 7.6 below.

Since ν is rational, by Proposition 4.2 the graded algebra $\mathrm{gr}_\nu R$ is a quotient of a polynomial ring by a binomial ideal. Using Proposition 5.19, we may assume that the completion $\hat{R}^{(\nu)}$ has a field of representatives $k \subset \hat{R}^{(\nu)}$ which we take as our base field. According to Propositions 5.48 and 5.49, we have a surjective map

$$k[\widehat{v^{\Phi_+}][(w_j)}_{j\in J}] \to \widehat{\mathcal{A}_{\hat{\nu}}(\hat{R}^{(\nu)})}^{(\nu_{\mathcal{A}})},$$

the kernel of which is the closure in $k[\widehat{v^{\Phi_+}][(w_j)}_{j\in J}]$ of the ideal generated by the elements

$$\tilde{G}_{mn} = w^m - \lambda_{mn}w^n + \sum_s c_s^{(mn)}(v^\phi)w^s,$$

where the tilda means that we have taken the varying (with coefficients in $k[v^{\Phi_+}]$) version of the equation. The idea for the proof is that if the k-algebra $\mathrm{gr}_\nu R$ is finitely generated, say with N generators, it is also finitely presented and one can proceed essentially as in [G-T]: we have in this case by Lemma 7.2 below $\mathrm{dim}\hat{R}^{(\nu)} = \mathrm{dimgr}_\nu R = \mathrm{r}(\nu)$, and a toric map $Z \to \mathbf{A}^N(k)$ which resolves $\mathrm{Specgr}_\nu R$ will also uniformize $\hat{\nu}$ on $\hat{R}^{(\nu)}$. If $\mathrm{gr}_\nu R$ is not finitely generated, we have to show that it suffices to resolve a finite approximation to $\mathrm{gr}_\nu R$, the subalgebra $\mathrm{gr}_\nu^{(h)}R$ of $k[x_1^{(h)},\dots,x_r^{(h)}]$ generated by the images of $(U_j)_{j\in F}$ for sufficiently large h, where h is a level of approximation for $\mathrm{gr}_\nu R_\nu$ in the sense of Proposition 4.15 and F is chosen according to Proposition 5.58 and the text which follows it as a quite sufficient initial generating set in Γ. Here the key point is that the abyssal phenomenon of subsection 5.5) tells us that if we consider the family defined in $k[\widehat{v^{\Phi_+}][(w_j)}_{j\in J}]$ by the equations $(\tilde{\mathbf{F}}, (\tilde{G}_i)_{i\notin F})$ it has "generic fiber" $\mathrm{Spec}\hat{R}^{(\nu)}$ and a special fiber which may be larger that $\mathrm{Spec}\hat{\mathrm{gr}}_{\hat{\nu}}R^{(\nu)}$, but coincides with it if one considers only the variables with index in F'. Then, because the equations $\tilde{G}_i$ create no new singularity it is sufficient to resolve the space defined by $\tilde{\mathbf{F}}$, and for that, it is sufficient to resolve the ideal generated by its initial binomials, which reduces us to the case of the finitely generated approximating algebra.

7.1 The approximation process. *A priori*, we must examine whether it makes sense to say that we resolve ν_{gr} by a single infinite dimensional toric modification of $k[(U_i)_{i\in I}]$, hence uniformize the completion $\hat{\nu}_{\mathrm{gr}}$ on $\widehat{\mathrm{gr}}_\nu^{(\nu)}R$. This toric modification should then uniformize $\hat{\nu}$ on the completion $\hat{R}^{(\nu)}$. Fortunately, as we shall see below, we do not have to overcome this difficulty, because a finite initial partial toric resolution of $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ suffices to uniformize $\hat{\nu}$ on $\hat{R}^{(\nu)}$.
Let us keep the notations of the preceding section for a valued ring (R,ν); we will apply the results to $(\hat{R}^{(\nu)},\hat{\nu})$. The variables U_i are ordered by the valuations $\gamma_i\in\Gamma$ of their images in $\mathrm{gr}_\nu R$. Let us consider the nested sequence of polynomial algebras $k[x^{(h)}]=k[x_1^{(h)},\dots,x_r^{(h)}]$ approximating $\mathrm{gr}_\nu R_\nu$ according to Proposition 4.15, b). Remember that r is the rational rank of the value group Φ of ν. For each finite set F of generators of the k-algebra $\mathrm{gr}_\nu R$, there is an integer h such that F is contained in $\mathrm{gr}_\nu R\cap k[x^{(h)}]$. The degrees of the elements of this subalgebra are in a free subsemigroup $\mathbf{N}^r\subseteq\Phi_+\cup\{0\}$ and the subalgebra $\mathrm{gr}_\nu^{(F)}R$ of $\mathrm{gr}_\nu R$ generated by F is the image of the map

$$k[(U_j)_{j\in F}]\to k[x^{(h)}]$$

sending U_j to its image in $k[x^{(h)}]$. The kernel of the map $k[(U_j)_{j\in F}]\to\mathrm{gr}_\nu^{(F)}R$ is a prime binomial ideal, and corresponds to a torus orbit closure in $\mathbf{A}^F(k)$ characterized by the map $\omega(F)\colon\mathbf{Z}^F\to\mathbf{Z}^r$ sending the j-th basis element to $\nu(\xi_j)\in\mathbf{N}^r\subseteq\Phi_+\cup\{0\}$. I apply the results of subsection 6.1 to this binomial ideal in finitely many variables, obtaining a regular fan in $\check{\mathbf{R}}_+^F$. Adding an element to F to get $F'=F\cup\{g\}$, we have a commutative diagram

$$\begin{array}{ccc}\mathbf{Z}^F & \longrightarrow & \mathbf{Z}^r\\ \downarrow & & \downarrow\\ \mathbf{Z}^{F'} & \longrightarrow & \mathbf{Z}^r\end{array}$$

and after dualizing

$$\begin{array}{ccc}\check{\mathbf{Z}}^r & \longrightarrow & \check{\mathbf{Z}}^{F'}\\ \downarrow & & \downarrow p_{F',F}\\ \check{\mathbf{Z}}^r & \longrightarrow & \check{\mathbf{Z}}^F\end{array}$$

Note that in the case of curves, $r=1$ and, taking for F the finite set of generators of the algebra $\mathrm{gr}_\nu R$, the image of $\check{\mathbf{Z}}$ in $\check{\mathbf{Z}}^F=\check{\mathbf{Z}}^N$ is the weight vector. Although no approximation is needed, we can illustrate the procedure by setting $e_i=\gcd(\gamma_1,\dots,\gamma_i)$ and considering the nested sequence of subalgebras

$$k[t^{e_1}]\subset k[t^{e_2}]\subset\cdots\subset k[t^{e_{N-1}}]\subset k[t].$$

The construction shown above then builds a sequence of vector spaces $(\mathbf{R}^i\to\mathbf{R}^{i-1})_{2\leq i\leq N}$ and a weight vector in each. In the case of plane branches,

where the relations between the γ_i are particularly simple, this suggests a procedure to build inductively a fan compatible with the hyperplanes corresponding to these relations; see subsection 5.6 and [G-T].
For each finite set F of variables U_j as above, we take a fan Σ_F in $\check{\mathbf{R}}^F_+$ compatible with the hyperplanes dual to generators of the kernel of $\omega(F)$. For each inclusion $F \subset F'$, we consider similarly fans $\Sigma_{F'}$ compatible with the generators of the kernel of $\omega(F')$ and with the inverse image by $p_{F',F}$ of Σ_F. In this manner, letting F grow, we obtain a projective system of vector spaces indexed by the filtering system of finite subsets of I, and a subordinate system of regular fans, which is what we would like to call a regular fan for the projective system of vector spaces.
We are led to the following definition of a possibly infinite dimensional fan:

Definition 7.1 Let P be a projective system of $\mathbf{R}$-linear maps between finite-dimensional real vector spaces, indexed by a filtering ordered set I.

$$P\colon \big((p_{i,j}\colon \mathbf{R}^{n(i)} \to \mathbf{R}^{n(j)})_{i,j\in I, i>j}\big),$$

where $n(i)$ is an integer valued function. A fan Σ subordinate to P is the datum for each $i \in I$ of a rational fan Σ_i in $\mathbf{R}^{n(i)}$ in such a way that for each $i > j$, the fan Σ_i is a refinement of $p^{-1}_{i,j}(\Sigma_j)$.
We say that the fan is of finite type if there exists a finite subset $F \subset I$ such that for $i, j \notin F$, $i > j$, the image of Σ_i is equal to Σ_j, and if we denote by $K_i \subset \mathbf{R}^{n(i)}$ the kernel of $p_{i,j}$, the map $\sigma \mapsto (p_{i,j}(\sigma), \sigma \cap K_i)$ is a bijection from the set of cones in $\Sigma_{n(i)}$ to the set of pairs consisting of a cone in Σ_j and one of the cones in $\Sigma_{n(i)}$ whose image by $p_{i,j}$ is 0. We say that Σ is simplicial (resp. regular) if all Σ_i are. We say that Σ is *rational* if there exists a finite subset $F \subset I$ such that for $i, j \notin F$, $i > j$, the map $p_{i,j}$ sends a basis of the integral lattice of $\mathbf{R}^{n(i)}$ to part of a basis of the integral lattice of $\mathbf{R}^{n(j)}$.

To such a system of fans is associated a system of toric maps $\pi(\Sigma_i)\colon Z(\Sigma_i) \to \mathbf{A}^{n(i)}(k)$; if we have a simple system of surjective maps $p_i\colon \mathbf{R}^{n(i+1)} \to \mathbf{R}^{n(i)}$, we get inclusions $\mathbf{A}^{n(i)}(k) \subset \mathbf{A}^{n(i+1)}(k)$, so that we get a compatible system of toric maps of an increasing sequence of affine spaces, which we may call an infinite-dimensional toric map. I leave it to the reader to interpret geometrically the various conditions defined for the system of fans.

7.2 Toric modifications and transforms of $\hat{R}^{(\nu)}$. Finally, in order to uniformize the valuation $\hat{\nu}$ on $\hat{R}^{(\nu)}$ it should suffice to resolve the singularities of a finite number of the equations F_{mn}, so that we may consider only finitely many elements of the projective system of toric blowing-ups which "resolves" $\mathrm{gr}_\nu R$.
To check this, we must now compute the effect on $\hat{R}^{(\nu)}$ of the toric modifications associated to our regular fan.

Using the specialization of $\hat{R}^{(\nu)}$ to $\widehat{\mathrm{gr}}^{(\nu)}_\nu R$, which we can write in a very explicit form (see 5.36) as the space associated to

$$k[v^{\Phi_+}]\widehat{[(w_j)_{j\in IJ}]}/\overline{\big(w^m - \lambda_{mn}w^n + \sum_s c_s^{(mn)}(v^\phi)w^s\big)},$$

with weight$(w^s) >$ weight$(w^n) =$ weight(w^m), the weight of a monomial being the valuation of its image in $\hat{R}^{(\nu)}$. For any homomorphism $c\colon \Phi_+ \to k^*$, say $\phi \mapsto c(\phi)$, we have an isomorphism

$$\hat{c}:\ k[v^{\Phi_+}][\widehat{(w_j)_{j\in J}}]/\overline{\big(w^m-\lambda_{mn}w^n+\sum_s c_s^{(mn)}(v^\phi)w^s,\ (v^\phi-c(\phi))_{\phi\in\Phi_+}\big)}\xrightarrow{\simeq}\hat{R}^{(\nu)}.$$

We will denote by $c(G_{mn})$ the series

$$c(G_{mn})=w^m-\lambda_{mn}w^n+\sum_s c_s^{(mn)}(c(v^\phi))w^s\in k[\widehat{(w_j)_{j\in J}}].$$

Let us first assume that the k-algebra $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ is of finite type, generated say by $(\overline{\eta}_1,\dots,\overline{\eta}_N)$. We have

Lemma 7.2 *If the k-algebra $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ is finitely generated, the equality $\dim\hat{R}^{(\nu)}=\mathrm{r}(\nu)$ holds.*

Proof Using Propositions 5.48, 5.49, and their corollaries (see Corollary 5.52, c)), we are reduced to the case of a family of algebras which are "formally of finite type". This shows that we can apply the semi-continuity of the dimensions of fibers to the map

$$\mathrm{Spec}\widehat{\mathcal{A}_{\hat{\nu}}(\hat{R}^{(\nu)})}^{(\nu_{\mathcal{A}})}\to\mathrm{Spec}k[v^{\Phi_+}]$$

so that by Piltant's Theorem (see 3.1), $\mathrm{r}(\nu)=\dim\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}\geq\dim\hat{R}^{(\nu)}$. The reverse inequality is Abhyankar's inequality applied to $\hat{R}^{(\nu)}$. □

Remarks 7.3 1) By classical results on curves (see [T1]) and the results of Spivakovsky in [S1], when $\dim\hat{R}^{(\nu)}\leq 2$, for a rational valuation ν the equality $\mathrm{r}(\nu)=\dim\hat{R}^{(\nu)}$ implies that the k-algebra $\mathrm{gr}_\nu R$ is finitely generated and the equality $\mathrm{r}(\nu)=\dim R$ implies that the $k[v^{\Phi_+}]$-algebra $\mathcal{A}_\nu(R)$ is finitely generated.
2) There remains the problem of characterizing among rational valuations of an excellent equicharacteristic local domain the Abhyankar valuations (characterized by the equality $\dim R=\mathrm{r}(\nu)$) and the "weakly Abhyankar" valuations (characterized by $\dim\hat{R}^{(\nu)}=\mathrm{r}(\nu)$) in terms of the abyssal presentation of $\hat{R}^{(\nu)}$ and the construction of the ideal H of $\hat{R}^m$. One may also ask whether the k-algebra $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ is finitely generated if $\mathrm{gr}_\nu R$ is. One difference between example 5.7 and example 4.20 is that the first one is weakly Abhyankar while the second one is not since the completion of R in this second case is $k[[u_1,u_2]]$.

Let us keep the notations of the preceding subsection, with a quite sufficient initial set of generators $(\hat{\gamma}_j)_{j\in F}$ (see subsection 5.5) and the corresponding variables W_j of degree $\hat{\gamma}_j$, while h is large enough for $k[x_1^{(h)},\dots,x_r^{(h)}]$ to contain the images in $\mathrm{gr}_\nu R_\nu$ of the $(W_j)_{j\in F}$, and keep the notations used in the proof of Proposition 6.9 as well. I number the variables $(W_j)_{j\in F}$ as $(W_{j_1},\dots,W_{j_N})$, with $N=\#F$. We must compute the strict transforms of the equations G_{mn} under the toric map associated to a regular fan Σ as in 6.9. By assumption $(\gamma_{j_1},\dots,\gamma_{j_N})$ are in the free submonoid $\mathbf{N}^r\subseteq\Phi_+$ and we may consider the map

$$\mathbf{Z}^N\to\mathbf{Z}^r,\quad(n_1,\dots,n_N)\mapsto\sum_{t=1}^N n_t\gamma_{j_t}\in\mathbf{Z}^r.$$

The image $E\subset\check{\mathbf{Z}}^N$ of the dual map is the r-dimensional free submodule generated by the r elements $e^i=(\gamma_{j_t i})_{1\leq t\leq N}\in\check{\mathbf{Z}}^N$.

We want to show that if w^s has value greater than w^n, after a toric modification corresponding to a regular fan compatible with the hyperplanes defining W, the transform of w^n divides the transform of w^s.
More generally, we have the:

Proposition 7.4 *Let $(w^{p_1},\dots,w^{p_c})$ be a finite set of monomials in $k[\widehat{(w_j)_{j\in J}}]$, arranged in increasing term order. There is a toric modification in the variables w_j appearing in these monomials such that, at the point of the strict transform of* $\mathrm{Spec}\hat{R}^{(\nu)}$ *picked by $\hat{\nu}$, the image of w^{p_1} divides the images of all the other w^{p_j}.*

Proof By construction, and in view of Proposition 4.15, the assumption means that at least if we have taken N large enough and $1\le i\le c-1$, $\langle w^j, p_{i+1}-p_i\rangle\ge 0$ for $1\le j\le r$ (see the Corollary following Proposition 4.12). In view of the fact that $U^{p_i-p_1}\in \mathrm{gr}_\nu R_\nu$ is in $k[x^{(h)}]$ for large enough h, after refining our fan Σ to a regular fan Σ' without changing its intersection with W, we may assume that these inequalities imply for each cone $\sigma=\langle a^1,\dots,a^N\rangle\in\Sigma$ such that $\sigma\cap W\neq\emptyset$ the inequalities $\langle a^k, p_i-p_1\rangle\ge 0,\;\; 1\le k\le N$. These inequalities precisely mean that in the chart of $Z(\Sigma')$ corresponding to σ, the image in the transform ${\hat{R}'}^{m,\nu}$ of $\hat{R}^{(\nu)}$ by the toric map corresponding to Σ' of the transform of u^{p_1} divides the image of the transform of the u^{p_i} (compare with [Z1], Th.2, p. 863). □

Corollary 7.5 *Let $h_1,\dots,h_p$ be elements of $\hat{R}^{(\nu)}$. There is a toric modification in finitely many of the variables w_j such that at the point of the strict transform of* $\mathrm{Spec}\hat{R}^{(\nu)}$ *picked by $\hat{\nu}$, the image of the ideal $(h_1,\dots,h_p)$ is generated by the element of least valuation.*

Proof The ideal of $\hat{R}^{(\nu)}$ generated by all the monomials appearing in the h_k's is finitely generated since $\hat{R}^{(\nu)}$ is nœtherian. It suffices to apply the Proposition to a finite set of generators of this ideal. □

Corollary 7.6 * TLU(d) implies TP(d): *Given finitely many elements $g_k\in R$, there exists a birational toric extension $R\to R'$ subordinate to ν such that in R' the ideal generated by the elements g_k is equal to a monomial in ξ'_i times a unit. Here the $\xi'_i\in R'$ lift generators of $\mathrm{gr}_\nu R'$. In particular, given finitely many elements of R, we can achieve that in the ring R' obtained from R by a birational toric extension the element with the smallest valuation divides the others.**

Proof Let us first assume that R is complete: by the preceding Corollary, it suffices to remark that in R the ideal generated by the monomials in the ξ_i which appear in the series representing the g_k is generated by finitely many of them.
In the general case, assuming TLU(d) for $d=\dim R$, we have a birational toric extension R' of R such that $\hat{R}'$ is regular, and ν extends to a valuation $\hat{\nu}$ of a regular quotient $\hat{R}'/H$ of $\hat{R}'$ with the same graded ring. We may assume, since this is a matter of refining a fan to make it compatible with hyperplanes, that in this toric extension the images of the elements g_k in $\hat{R}'/H$ generate an ideal of the form ${\xi'}^\alpha u$ where u is a unit in $\hat{R}'/H$. We need to show that u is in fact in R'. Since $\hat{R}'/H$ is regular we may view it as a subring of $\hat{R}'$, and argue as follows: ${\xi'}^\alpha u$ is in R', and so is ${\xi'}^\alpha$, so $u\in K\cap\hat{R}'$, which is equal to R' by the flatness of $\hat{R}'$ as an R'-module ([B3], Chap. 3, §3, No. 5, Corollary 4). □

Corollary 7.7 *If the graded k-algebra $\mathrm{gr}_{\hat\nu}\hat R^{(\nu)}$ is finitely generated, say by the initial forms $(\overline{\eta}_j)_{1\le j\le N}$ of elements $\eta_j \in \hat R^{(\nu)}$, a toric map $Z(\Sigma)\to \mathbf{A}^N(k)$ corresponding to a regular fan compatible with the hyperplanes H_{m-n} associated to the binomial equations defining $\mathrm{gr}_{\hat\nu}\hat R^{(\nu)}$ will produce, upon taking the strict transform of* $\mathrm{Spec}\hat R^{(\nu)}$ *embedded in $\mathbf{A}^N(k)$ via the elements η_j, an embedded uniformization of the valuation $\hat\nu$ on $\hat R^{(\nu)}$.*

The k-algebra $\mathrm{gr}_{\hat\nu}\hat R^{(\nu)}$ is also finitely presented and we have to deal with finitely many equations G_{mn}. We keep the fan Σ' of Proposition 7.4 and apply this to compute the strict transform of the G_{mn}. By the same argument as in the proof of Corollary 7.6, we need concern ourselves with only a finite set of the monomials appearing in F_{mn}. We may assume without loss of generality that the intersection of σ with W is of dimension r, so that $(a^1,\dots,a^r)$, say, are in W. Then in our monomial transformation

$$w_i = y_1^{a_i^1}\dots y_N^{a_i^N},\quad 1\le i\le N,$$

we find that the strict transform of each G_{mn} in that chart has the following expression, where I write $y^{\langle a,m\rangle}$ for $y_1^{\langle a^1,m\rangle}\dots y_N^{\langle a^N,m\rangle}$ except when I want to single out some of the y_i's

$$G_{mn}\circ\pi(\sigma) =$$
$$y_1^{\langle a^1,m\rangle}\dots y_r^{\langle a^r,m\rangle}\big(y_{r+1}^{\langle a^{r+1},n-m\rangle}\dots y_N^{\langle a^N,n-m\rangle}-\lambda_{mn}+\textstyle\sum_p c_p^{(mn)}(v^\phi)y^{\langle a,p-m\rangle}\big).$$

It shows that the strict transform of $\hat R^{(\nu)}$ is non singular, as a deformation of the strict transform of $\mathrm{gr}_{\hat\nu}\hat R^{(\nu)}$. Now the refinement of Σ to Σ' corresponds to a birational map $Z(\Sigma')\to Z(\Sigma)$ which induces an isomorphism outside the divisor $\prod_{j=r+1}^N y_j=0$, which the strict transform of our space does not meet, so that we may descend the result of simultaneous resolution from $Z(\Sigma')$ to $Z(\Sigma)$. This is the same process as in [G-T].

Remark 7.8 I do not know whether the toric local uniformization map can be chosen in such a way that it induces an isomorphism outside of the singular locus of $\mathrm{Spec}\hat R^{(\nu)}$.

In the general case the first problem is to show that our valuation $\hat\nu$ on $\hat R^{(\nu)}$ can be uniformized after a toric modification which is sufficiently far in the projective system, but occurs after finitely many steps.

We keep the notations of subsection 5.5. Let us now consider the binomial ideal in $k[(W_j)_{j\in F'}]$ which is the kernel of the map to $\mathrm{gr}_{\hat\nu}^{(h)}\hat R^{(\nu)}$ sending W_j to $\overline{\eta}_j$. As we saw, it is a prime ideal, so that it corresponds to a torus orbit closure since our residue field is algebraically closed ([E-S]). We apply to it Proposition 6.9, and obtain a regular fan Σ of $\mathbf{R}_+^{F'}$ compatible with the hyperplanes dual to the $m-n$ where $(W^m-\lambda_{mn}W^n)_{mn}$ is a system of generators of the binomial ideal. We may choose a regular simplicial cone $\sigma\in\Sigma$ whose intersection with the space of weights $E\subset\mathbf{R}_+^{F'}$ is of the largest possible dimension, that is, the rational rank r of Φ. This corresponds to a monomial map

$$w_i\mapsto y_1^{a_i^1}\dots y_N^{a_i^N},\quad 1\le i\le N=\sharp F',$$

which we may extend by setting

$$w_k=y_k,\quad k\notin F'.$$

Now we compute the transforms of the equations of $\hat{R}^{(\nu)}$ under this transformation. By construction, using the same argument as we used in the case where the graded algebra is finitely generated, the equations G_{mn} whose initial forms involve only $(W_i)_{i\in F'}$, become

$$\begin{aligned} y^{\langle a,m^1-n^1\rangle}(y^{\langle a,n^1\rangle}-\lambda_{m^1n^1}+\cdots) &= 0\\ \vdots \qquad\qquad &\quad \vdots\\ y^{\langle a,m^t-n^t\rangle}(y^{\langle a,n^t\rangle}-\lambda_{m^tn^t}+\cdots) &= 0, \end{aligned}$$

while the G_i become

$$w_i^{n_i}\prod_{k\notin F',k<i} w_k^{n_k^{(i)}}y^{t(n^{(i)})}-\lambda_i\prod_{k\notin F,k<i} w_k^{\ell_k^{(i)}}y^{t(\ell^{(i)})}+\cdots+c^{(i)}(v^{\phi})w_{i+1}+\cdots=0$$

where $y^{t(n^{(i)})}$ (resp. $y^{t(\ell^{(i)})}$) is a monomial which is the transform of the monomial in the variables $(w_j)_{j\in F'}$ which affected $w_i^{n_i}$ (resp. appeared in the other term) in the original relations. Note that one can compute the valuations of the y_k from those of the w_j. Some of the y_k, say for $k\in T$, will have valuation zero, so that for $k\in T$ there is a constant $c_k\in k_R^*$ such that y_k-c_k has positive valuation; the equations $(y_k-c_k=0)_{k\in T},(y_k=0)_{k\in F\setminus T},(w_j=0)_{j\notin F}$ define the closed point which is the center of the valuation in the space obtained by the toric modification. By openness of transversality, if our toric map has resolved the $\mathrm{gr}_{\hat{\nu}}^{(h)}\hat{R}^{(\nu)}$, the first set of equations defines a regular ν-complete local ring at that point. By the nature of the equations, and the implicit function theorem mentioned in subsection 5.5, it will be the case for the whole set of equations and the strict transform of $\hat{R}^{(\nu)}$ will be regular.

Note that by the faitfull flatness of $\widehat{\mathcal{A}_{\hat{\nu}}(\hat{R}^{(\nu)})}^{(\nu_{\mathcal{A}})}$ over $k[v^{\Phi_+}]$, if the initial binomials of the equations G_{mn} do not constitute a regular sequence, relations between them will lift to relations between the G_{mn} so that the strict transforms of some of the G_{mn} will vanish on the nonsingular space defined by the strict transforms of the others, as in the proof of the second part of Proposition 6.2.

Remarks 7.9 1) If one compares with the case where $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)}$ is finitely generated, as in the example of plane branches, one see that the effect of the "abyssal" phenomenon is also to send singularities away to infinity: if we stopped at any finite step, replacing u_{i+1} by 0 or a polynomial in $(u_k)_{k\leq i}$ of higher weight than the binomial initial form, the new equation thus obtained from G_i would be singular.
b) The last equation written underlines the role of the $(u_i)_{i\in I}$ as semiroots in the sense of [PP]; after our toric modification the $(u_k)_{k\notin F}$ are still part of a natural coordinate system on the transformed ring for our valuation.
c) Note that there are at least $\mathrm{r}(\nu)$ independent variables, and possibly more if $\dim\hat{R}^{(\nu)}>\mathrm{r}(\nu)$, and that after substitution of the expressions for the $u_j,\ j\notin F$, some of the equations may become identities.

At this point we have proved local uniformization by a finite dimensional toric map for the rational valuation $\hat{\nu}$ on $\hat{R}^{(\nu)}$.

To deduce from this local uniformization for R, the fact that R is excellent is again crucial. In [S2] Spivakovsky used in a similar situation the following

Proposition 7.10 (see [Ma], §33, Spivakovsky [S2], [L-M], Chap. 15, Proof of Th. 15.7, p. 202) *If R is an excellent local integral domain, for any prime ideal* $\mathbf{q}$

of R and any prime ideal $\mathcal{H}$ of $\hat{R}^{\mathbf{q}}$ such that $\mathcal{H}\cap R=(0)$, the localization $\hat{R}^{\mathbf{q}}_{\mathcal{H}}$ is regular.

Proof Since we have $\mathcal{H}\cap R=(0)$, the ring $\hat{R}^{\mathbf{q}}_{\mathcal{H}}$ is a localization of the ring $\hat{R}^{\mathbf{q}}\otimes_R K$, where K is the field of fractions of R. It suffices to prove that this tensor product is regular. By ([EGA], Scholie 7.8.3, (v)), if R is excellent, the map $\mathrm{Spec}\hat{R}^{\mathbf{q}}\to\mathrm{Spec}R$ is regular. Our tensor product corresponds to the generic fiber of this map, so it is regular; it is even a geometrically regular K-algebra. □

Since the $\hat{\nu}$-initial forms of the elements η_j which we used to generate topologically the k-algebra $\hat{R}^{(\nu)}$ are Laurent monomials in the $\overline{\xi}_i$, with $\xi_i\in R$, we may perform in R the trace of the toric modification described above; it consists in adding to R the monomials (with some negative exponents) in the ξ_i which are the y_i and localizing at the point specified by the values of the y_i: this gives us a local ring R' with maximal ideal m'. If we denote by $\hat{R'}^{(\nu)}$ the scalewise completion of R', we have maps of R-algebras

$$R'\to R'\otimes_R\hat{R}^m\to\hat{R'}^{m'}\to\hat{R'}^{(\nu)}.$$

Following through constructions, we see that the image in $\hat{R'}^{m'}$ of the strict transform in $R'\otimes_R\hat{R}^m$ of the ideal H is contained in the ideal H' corresponding to ν in $\hat{R'}^{m'}$, so that we have

$$\mathrm{dim}\hat{R'}^{(\nu)}\leq\mathrm{dim}\hat{R}^{(\nu)}.$$

That strict inequality may occur has been shown by Spivakovsky, who discovered this “subanalytic” phenomenon in the case of valuations of height one (see [S2]). From this inequality follows the

Proposition 7.11 * *After replacing R by a toric transform, me may assume that the map*

$$R'\otimes_R\hat{R}^{(\nu)}\to\hat{R'}^{(\nu)}$$

*is the completion with respect to the maximal ideal $m'\otimes 1+1\otimes\hat{m}$, and $\hat{R'}^{(\nu)}$ is the completion of the transform of $\hat{R}^{(\nu)}$. In that case, the ideal H' of $\hat{R'}^{m'}$ is the strict transform of H.**

Proof Let $R\to R'$ be a toric modification, and consider the transform $\hat{R}^{(\nu)\prime}$ of $\hat{R}^{(\nu)}$, which is $R'\otimes_R\hat{R}^{(\nu)}$ localized at the prime ideal $n=m'\otimes 1+1\otimes m\hat{R}^{(\nu)}$. The extension $\hat{R}^{(\nu)}\to\hat{R}^{(\nu)\prime}$ is birational since R is a subring of $\hat{R}^{(\nu)}$, and the graded ring $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)\prime}$ is the strict transform of $\mathrm{gr}_{\hat{\nu}}R^{(\nu)}$ by the toric map defining our transform. We may choose as representatives in $\hat{R}^{(\nu)\prime}$ of generators of $\mathrm{gr}_{\hat{\nu}}\hat{R}^{(\nu)\prime}$ the transforms of positive valuation by our toric modification, as in subsection 5.2, of elements $\eta_j\in R^{(\nu)}$ whose initial forms generate $\mathrm{gr}_{\hat{\nu}}R^{(\nu)}$. Now the scalewise completion (Proposition 5.19) of $\hat{R}^{(\nu)\prime}$ may be a nontrivial quotient of its n-adic completion, and in that case $\mathrm{dim}{R'}^{(\nu)}<\mathrm{dim}\hat{R}^{(\nu)}$, but the remarks above show that given a toric transform R' of R, and another one R^* such that the dimension of $\hat{R^*}^{(\nu)}$ is minimal among all toric modifications of R, there is a toric modification of R' which dominates R^* by a toric map and therefore also reaches the minimum dimension of its scalewise completion. □

This implies that if the transform $\hat{R}^{(\nu)\prime}$ of $\hat{R}^{(\nu)}$ by a toric map is regular, so is $\hat{R'}^{(\nu)}$. Let $\hat{R}^{(\nu)}$ be the scalewise ν-adic completion of a local equicharacteristic excellent integral domain. We know by 5.1 that $\hat{R}^{(\nu)}$ is a quotient of the completion $\hat{R}^m$ by an ideal H. By Proposition 7.10, the assumption that R is excellent implies that the localization $\hat{R}^m_H$ is regular. If we have equality in Abhyankar's inequality, the kernel of the map $\hat{R}^m \to \hat{R}^{(\nu)}$ is a minimal prime and the same is true for the map $\hat{R'}^{m'} \to \hat{R'}^{(\nu)}$. We conclude by case 1) of the general case below.
For the general case, where Abhyankar's inequality is not assumed to be an equality, I use a variant of the method described by Spivakovsky ([S2], 1997, Lemma 6.4). We are reduced to the case where R is such that $\hat{R}^m_H$ and $\hat{R}^m/H$ are both regular, the first one by excellence and the second one because we have uniformized $\hat{\nu}$ on $\hat{R}^{(\nu)}$. We want to show that we can make $\hat{R}^m$ (i.e., its strict transform) non singular by a toric modification defined in R. Then R itself will be made non singular. As we have seen in Proposition 7.11, we may assume that the strict transform of H is equal to H'.
-case 1) If the ideal H contains only zero divisors of $\hat{R}^m$, it is a minimal prime ideal of $\hat{R}^m$, say $\hat{q}_0$. The blowing up in Spec$\hat{R}^m$ of the sum J of the minimal primes of $\hat{R}^m$ separates the strict transforms of the irreducible components. Let J_0 be the ideal of $\hat{R}^m/H$ which is the image of the intersection of the other minimal primes of $\hat{R}^m$. According to toric principalization, which we may apply in $\hat{R}^m/H$ because whe know that it is a quotient of $k[\widehat{(w_j)_{j\in J}}]$, there is a toric modification in the $(\xi_i)_{i\in I}$ which principalizes J_0. The corresponding toric map dominates the strict transform of Spec$\hat{R}^m/H$ by the blowing up of J. Therefore the completion of the transform R' of R by this toric map is an integral domain, the ideal H' associated to ν in $\hat{R'}^{m'}$ is zero and the kernel of the map $\hat{R}^m \to \hat{R'}^{m'}$ is the ideal H. Now R' has to be regular.
-case 2) If the ideal H contains non zero divisors, the regularity of $\hat{R}^m/H$ and of $\hat{R}^m_H$ implies the regularity of $\hat{R}^m$ if we know that H is generated by a regular sequence in $\hat{R}^m$. This is equivalent to saying that H/H^2 is a locally free $\hat{R}^{m'}/H$-module. It is enough to produce a toric modification of R such that the strict transform of H/H^2 is equal to $H'/{H'}^2$ and is locally free on Spec$\hat{R'}^{m'}/H'$. Note that since we may assume that H' is the strict transform of H by the map $\hat{R}^m \to \hat{R'}^{m'}$, the $\hat{R'}^{m'}/H'$-module $H'/{H'}^2$ is equal to $H/H^2 \otimes \hat{R'}^{m'}/H'$ divided by its torsion. Now the module $H'/{H'}^2$ will be locally free if the map $\hat{R}^m \to \hat{R'}^{m'}$ makes the Fitting ideal of the $\hat{R}^{(\nu)}$-module H/H^2 principal at the point picked by the valuation in the toric modification $R \to R'$. Using the Corollary to Proposition 7.4, we see that after a toric modification of R we can obtain that the Fitting ideal is principal. Thus, we reach the situation where R' is regular and we are done.

There will remain to translate the trace of the toric modification on the completion $\hat{R}$ (or $\hat{R}^{(\nu)}$) into a finite sequence of blowing ups with non singular centers coming from R. But one can prove a torus-equivariant version of Gruson-Raynaud's ([G-R]) and Hironaka's ([H2]) proper flattening theorem and use it to show that a toric modification is dominated by a sequence of toric blowing ups, that is, blowing ups with a torus-invariant non singular center. In fact a more general result is proved in [DC-P]; see also [O], Chap. 1, §1.7, p.39. We will also have to show that the centers of these blowing-ups can be chosen contained in the singular locus of Spec$R$ and its transforms, whereas all we know so far is that these centers are contained in the union of the coordinate hyperplanes (and their transforms), and although we know by Proposition 6.6 that our toric map can be chosen to be an isomorphism outside of the singular locus of $\mathrm{Specgr}_{\hat{\nu}}\hat{R}^{(\nu)}$, we do not know yet the position of these hyperplanes with respect to the singular locus of Spec$\hat{R}^{(\nu)}$.

Lemma 7.12 *Let ν be a rational valuation of a complete equicharacteristic nœtherian local ring R with algebraically closed residue field. Assume that for any $j \notin F$, in the equality*

$$\eta_{j+1} = d_{j+1}\big(\eta_j^{n_j}\prod_{k<j}\eta_k^{n_k^{(j)}} - \lambda_j\prod_{k<j}\eta_k^{\ell_k^{(j)}} + \sum_s d_s^{(j)}\eta^s\big)$$

of Corollary 5.65, there is no term $\prod_{k<j}\eta_k^{n_k^{(j)}}$. Then we may assume also that there appear no powers of η_j greater than n_j. If we assign value $\nu_0(\eta_j) = \nu(\eta_j)$ for $j \in F$ and the value defined inductively by $\nu_0(\eta_{j+1}) = \nu_0(\eta_j^{n_j})$ for $j \notin F$, there is in the series representing η_{j+1} a unique monomial with the minimal ν_0-value, namely $\eta_j^{n_j}$, and therefore ν_0 defines a monomial valuation on R.

Proof : for each monomial $\eta^s = \tilde{\eta}^p\eta_j^k$ in the series, we have

$$\nu(\tilde{\eta}^p) + k\nu(\eta_j) > n_j\nu(\eta_j)$$

and since $k \leq n_j$ this gives $(n_j - k)\nu(\eta_j) < \nu(\tilde{\eta}^p)$. The fact that $n_k\nu(\eta_k) < \nu(\eta_{k+1})$ for $k \notin F$ provides us with inequalities on the components $p_1, \dots, p_\ell$ of p, which give us the inequality $\nu_0(\eta^s) > \nu_0(\eta_j^{n_j})$. Since we must have $\nu_0(\eta_j^{n_j}) < \nu_0(\prod_{k<j}\eta_k^{\ell_k^{(j)}})$, this gives the result. □

Corollary 7.13 **Given a rational valuation ν of a complete equicharacteristic nœtherian local ring R with algebraically closed residue field, assuming that the conditions of Lemma 7.12 are satisfied, there exists a rational valuation ν_0 of R such that $\mathrm{gr}_{\nu_0}R$ is finitely generated, $\nu_0(x) \leq \nu(x)$ for $x \in R$, and toric resolutions of the binomial variety $\mathrm{Specgr}_{\nu_0}R$ provide (embedded) uniformizations of ν_0 which also uniformize ν on R.**

Proof Choose for ν_0 the monomial valuation of R, with respect to $(\xi_i)_{i\in F'}$, determined by $\nu_0(\xi_i) = \nu(\xi_i) = \gamma_i$ for $i \in F'$ as in Lemma 7.12 and apply Corollary 7.7. □

It remains to see whether the assumption of lemma 7.12 is necessary and whether a similar result holds for any excellent equicharacteristic local ring with an algebraically closed residue field.

A last example. An example where $\mathrm{gr}_\nu R$ is finitely generated is obtained by taking the first example after Proposition 5.3. The completion $\hat{R}^{(\nu)}$ is the ring of a branch, the associated graded ring is that of the corresponding monomial curve which may be desingularized as in subsections 5.6 and 6.2 by a toric map. The transform R' of R by this map has the property that it is a two-dimensional local ring with maximal ideal m' such that the maximal ideal of $\hat{R}'$ contains an element f' such that $\hat{R}'/(f')$ is a regular one-dimensional local ring and $(\hat{R}')_{(f')}$ is regular. So $\hat{R}'$ is regular and therefore so is R'. Essentially the same argument will work whenever $\dim\hat{R}^{(\nu)} = 1$, since once the singularity of this ring is resolved we deal with a discrete valuation ring, a case where the argument to bring the number of generators of H down to its height presents no difficulty.

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