TORIC GEOMETRY AND THE SEMPLE-NASH MODIFICATION

PEDRO D. GONZÁLEZ PÉREZ AND BERNARD TEISSIER

Abstract. This paper proposes some material towards a theory of general toric varieties without the assumption of normality. Their combinatorial description involves a fan to which is attached a set of semigroups subjected to gluing-up conditions. In particular it contains a combinatorial construction of the blowing up of a sheaf of monomial ideals on a toric variety. In the second part this is used to show that iterating the Semple-Nash modification or its characteristic-free avatar eventually resolves the singularities of any toric variety. This solves for toric varieties an old problem of singularity theory.

Introduction

In the first part of this paper we study abstract toric varieties without the assumption of normality. Since Sumihiro's Theorem on the existence of a covering of a toric variety by invariant affine varieties fails without the assumption of normality, we have to set the existence of such a covering as part of the definition of a toric variety. Then an abstract toric variety has a combinatorial description: it corresponds to certain semigroups in the convex duals of the cones of a fan, which satisfy a natural gluing-up condition. This generalizes the definition of [7] which concerns toric varieties equivariantly embedded in projective space. In spirit it is also a continuation of our previous work [8] on embedded normalization and embedded toric resolution of singularities of affine toric varieties. We can then define blowing-ups of sheaves of monomial ideals as toric varieties, and describe the corresponding operations on semigroups. We also provide the combinatorial description of torus-invariant Cartier divisors on a toric variety and the general versions of the classical criteria for ampleness and very-ampleness.

In the second part of the paper we use the description of blowing-ups given in the first part to show that one can, over an algebraically closed field, resolve the singularities of a toric variety by iterating the blowing-up of the logarithmic jacobian ideal introduced in [9]. If the field is of characteristic zero, this blowing-up is isomorphic to the Semple-Nash modification, so that a consequence of our result is that in characteristic zero one can resolve the singularities of a toric variety by iterating the Semple-Nash modification. Recall that this is a canonical modification of a reduced equidimensional space which replaces each point by the set of limit positions of tangent spaces at nearby non singular points. See the second part for details.

This answers in the special case of toric varieties an interesting question apparently first asked by Semple in [20]. One can present the relation of this proposed resolution process with "classical" resolution of singularities as follows: two types of proper birational correspondances naturally associate a singular variety to a non singular one: proper birational projections of an embedded non-singular algebraic variety to a smaller dimensional ambient space, and the taking of the *envelope* of a family of linear subspaces (of an affine or projective space) whose parameter space is a non singular algebraic variety. Hironaka's resolution shows that in characteristic zero all singularities may be created by the first process, and Semple-Nash resolution in general would show that at least all singularities of quasi-projective varieties in characteristic zero may be created by iterating the second process, if we allow singular spaces as parameter spaces. Moreover it would produce a canonical process for (non embedded) resolution in characteristic zero, as our results here do for toric varieties in a characteristic-free manner.

2000 *Mathematics Subject Classification.* 14M25, 14E15, 14B05.

Key words and phrases. Toric geometry, Semple-Nash modification, logarithmic jacobian ideal.

González Pérez is supported by *Programa Ramón y Cajal* and MTM2007-6798-C02-02 grants of MEC Spain. Teissier is partially supported by grant MTM2007-64704 of MEC Spain.

Part I: Toric varieties

The purpose of this part is to develop the combinatorial theory of toric varieties without the assumption of normality. We refer to [3], [6], [15], [18], and [19] for background on normal toric varieties, and to the books of Oda-Miyake ([18]), Gel′fand, Kapranov, and Zelevinsky ([7]) and Sturmfels ([22]) for certain classes of non necessarily normal toric varieties. We also point to previous work by H.M. Thompson towards the development of a general theory of toric varieties, see [26], [27] and, from the perspective of Log Schemes, [28]. We recommend [5] as a particularly accessible introduction to (normal) toric varieties, and mention the forthcoming book of D. Cox, J. Little and H. Schenk on the subject.

1. Semigroups and semigroup algebras

The theory of affine toric varieties over a field k is the geometric version of the theory of semigroup algebras over k. For part of the theory, one can omit the assumption that the semigroup is finitely generated, and replace the field k by a commutative ring.

Definition 1.1. A (commutative) semigroup Γ is a set equipped with an operation $+\colon \Gamma\times\Gamma \to \Gamma$ such that $\epsilon_1+\epsilon_2=\epsilon_2+\epsilon_1$, which satisfies the associativity property and is cancellative ($\epsilon_1+\epsilon_2=\epsilon_1+\epsilon_3$ implies $\epsilon_2=\epsilon_3$). We shall assume that Γ contains a zero element 0 such that $\epsilon+0=\epsilon$ and that no multiple of an element of Γ is zero. A system of generators of a semigroup is a subset (γ_i) of Γ such that each element of Γ is a (finite) linear combination of the γ_i with non negative integral coefficients. We denote by $\mathbf{Z}\Gamma$ the group generated by Γ (defined in a similar way as the field of fractions of an integral domain). The elements of $\mathbf{Z}\Gamma$ are finite linear combinations of the γ_i with integral coefficients. If the semigroup Γ is finitely generated, the group $\mathbf{Z}\Gamma$ is a lattice.

Examples of semigroups:
• Given finitely many coprime integers the set of all combinations of these integers with non negative integral coefficients is a subsemigroup Γ of the semigroup $\mathbf{N}$ of integers, and $\mathbf{N}\setminus\Gamma$ is finite. In fact any semigroup of integers is finitely generated.
• Let $(s_i)_{i\geq 1}$ be a sequence of integers such that $s_i\geq 2$ for $i\geq 2$. Define a sequence of rational numbers γ_i inductively by:

$$\gamma_1=\frac{1}{s_1},\quad \gamma_{i+1}=s_i\gamma_i+\frac{1}{s_1\ldots s_{i+1}}.$$

The set of integral linear combinations of the γ_i is a subsemigroup of $\mathbf{Q}_{\geq 0}$, which is not finitely generated. In fact the γ_i form a minimal set of generators.
• Let d be an integer and let $\check\sigma$ (the reason for the dual notation will appear below) be a convex cone of dimension d in $\check{\mathbf{R}}^d$. Denote by M the integral lattice of $\check{\mathbf{R}}^d$. Then the intersection $\check\sigma\cap M$ is a subsemigroup of the group M, which generates M as a group. By a Theorem of Gordan, if the convex cone $\check\sigma$ is rational in the sense that it is the intersection of finitely many half spaces determined by hyperplanes with integral coefficients, then the semigroup $\check\sigma\cap M$ is finitely generated.

Definition 1.2. If Δ is a subsemigroup of Λ the saturation of Δ in Λ is the semigroup Θ consisting of those elements of Δ which have a multiple in Λ. The semigroup Δ is saturated in Λ if $\Delta=\Theta$.

Lemma 1.3. *Let $\check\tau$ be a rational convex cone in $\check{\mathbf{R}}^d$ for the lattice M. The semigroup $\check\tau\cap M$ is saturated in M and the saturation of a subsemigroup Γ of M is $\check\sigma\cap M$ where $\check\sigma=\mathbf{R}_{\geq 0}\Gamma$ is the closed convex cone generated by Γ.*

Proof.: The first statement is clear. If $\mathbf{R}_{\geq 0}\Gamma=\check\sigma$, any element of $\check\sigma\cap M$ is a combination with rational coefficients of elements of Γ. Chasing denominators shows that an integral multiple of this element is in Γ. The converse is clear. □

Definition 1.4. Let Γ be a finitely generated commutative semigroup and A a commutative ring. The semigroup algebra $A[t^\Gamma]$ of Γ with coefficients in A is the ring consisting of finite sums $\sum_\gamma a_\gamma t^\gamma$ with $a_\gamma \in A$, endowed with the multiplication law

$$(\sum_\gamma a_\gamma t^\gamma)(\sum_\delta b_\delta t^\delta) \ = \ \sum_\zeta (\sum_{\gamma+\delta=\zeta} a_\gamma b_\delta) t^\zeta.$$

Proposition 1.5. *If Γ is a finitely generated subsemigroup of the lattice $M \subset \mathbf{R}^d$ such that $\mathbf{Z}\Gamma = M$ and $\check{\sigma} = \mathbf{R}_{\geq 0}\Gamma$ is the rational convex cone generated by Γ, the integral closure of $k[t^\Gamma]$ in its field of fractions is $k[t^{\check{\sigma}\cap M}]$.*

This follows directly from Lemma 1.3.

Remark 1.6. Quite generally, if k is a field the Krull dimension of $k[t^\Gamma]$ is equal to the *rational rank* of the semigroup Γ, which is the integer $\dim_{\mathbf{Q}} \Gamma \otimes_{\mathbf{Z}} \mathbf{Q}$ (see [25], Proposition 3.1).

Remark 1.7. If Γ is a semigroup the ideal of $A[t^\Gamma]$ generated by the $(t^\gamma)_{\gamma \in \Gamma\setminus\{0\}}$ is non trivial if and only if the cone $\mathbf{R}_{\geq 0}\Gamma$ is strictly convex. If k is a field, it is then a maximal ideal. We shall mostly be interested in the local study of the spectrum of semigroup algebras in the vicinity of the origin of coordinates, which corresponds precisely to that ideal.

The semigroup algebra has the following universal property: any semigroup map from Γ to the multiplicative semigroup of an A-algebra B extends uniquely to an homomorphism $A[t^\Gamma] \to B$ of A-algebras.

An additive map of semigroups $\phi\colon \Gamma \to \Gamma'$ induces a graded map of A-algebras $A[\phi]\colon A[t^\Gamma] \to A[t^{\Gamma'}]$ which is injective (resp. surjective) if ϕ is. If the semigroup Γ is torsion-free, the semigroup algebra $A[t^\Gamma]$ injects into $A[t^{\mathbf{Z}^d}] = A[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ and therefore is an integral domain if A is.

Proposition 1.8. *Let Γ, Γ' be two semigroups. The map of A-algebras*

$$A[t^{\Gamma\times\Gamma'}] \to A[u^\Gamma] \otimes_{\mathbf{A}} A[v^{\Gamma'}]; \quad t^{(\gamma,\gamma')} \mapsto u^\gamma \otimes_A v^{\gamma'}$$

is an isomorphism.

Proof. This follows immediately from the universal property. □

2. Algebraic tori

Let k be a field. The multiplicative group k^* of non zero elements of k is equipped with the structure of algebraic group over k, usually denoted by $\mathbf{G}_m := \mathrm{Spec} k[t^{\pm 1}]$. A d-dimensional *algebraic torus* over k is an algebraic group isomorphic to a $(k^*)^d$.

If M is a rank d lattice then $T^M := \mathrm{Spec} k[t^M]$ is an algebraic torus over k. If we fix a basis $m_1, \ldots, m_d$ of the lattice M we get a group isomorphism

$$\mathbf{Z}^d \to M, \quad a = (a_1, \ldots, a_d) \mapsto \sum_{i=1,\ldots d} a_i m_i$$

and isomorphism of k-algebras $k[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \to k[t^M]$ which induces an isomorphism $T^M(k) \to (k^*)^d$.

Remark 2.1. More generally one can consider the scheme $\mathrm{Spec} A[t^M]$, which is an algebraic torus over $\mathrm{Spec} A$ for any commutative ring A.

A *character* of the torus $T(k)$ is a group homomorphism $T(k) \to k^*$. The set of characters $\mathrm{Hom}_{\mathrm{alg.groups}}(T^M, k^*)$ of $T^M(k)$ is a multiplicative group isomorphic to the lattice M by the homomorphism given by $m \mapsto t^m$ for $m \in M$. We identify the monomials t^m of the semigroup algebra $k[t^M]$ with the characters of the torus.

By the universal property of the semigroup algebras applied to $k[t^M]$ we have a representation of k-rational points of T^M as group homomorphisms:

$$T^M(k) = \mathrm{Hom}_{\mathrm{groups}}(M, k^*) = N \otimes_{\mathbf{Z}} k^*,$$

where $N := \mathrm{Hom}\,(M, \mathbf{Z})$ is the dual lattice of M. We denote by $\langle\, ,\, \rangle \colon N \times M \to \mathbf{Z}$ the duality pairing between the lattices N and M.

A one parameter subgroup of $T^M(k)$ is group homomorphism $k^* \to T^M(k)$. Any vector $\nu \in N$ gives rise to a one parameter subgroup λ_ν which maps $z \in k^*$ to the closed point of $T^M(k)$ given by the homomorphism of semigroups $M \to k^*$, $m \mapsto z^{\langle \nu, m\rangle}$. The set of one parameter subgroups $\mathrm{Hom}_{\mathrm{alg.groups}}(k^*, T^M)$ forms a multiplicative group, which is isomorphic to N by the homomorphism given by $\nu \mapsto \lambda_\nu$.

3. Affine toric varieties

In this section we consider a finitely generated subsemigroup Γ of a free abelian group M of rank d. We assume in addition that the group $\mathbf{Z}\Gamma$ generated by Γ is equal to M. We denote by N the dual lattice of M. We introduce some useful notations.

Notation 3.1. We denote by $M_{\mathbf{R}}$ the d-dimensional real vector space $M \otimes_{\mathbf{Z}} \mathbf{R}$. The semigroup Γ, viewed in $M_{\mathbf{R}}$, spans the cone $\mathbf{R}_{\geq 0}\Gamma \subset M_{\mathbf{R}}$ which we denote also by $\check{\sigma}$. The dual cone of $\check{\sigma}$ is the cone $\sigma := \{\nu \in N_{\mathbf{R}} \mid \langle \nu, \gamma\rangle \geq 0,\ \forall \gamma \in \check{\sigma}\}$. We use the notation $\tau \leq \sigma$ to indicate that τ is a *face* of σ. Any face of $\check{\sigma}$ is of the form $\check{\sigma} \cap \tau^{\perp}$ for a unique face τ of σ, where $\tau^{\perp}$ is the linear subspace $\{\gamma \in M_{\mathbf{R}} \mid \langle \nu, \gamma\rangle = 0,\ \forall \nu \in \tau\}$.

Let $\gamma_1, \dots, \gamma_r$ be generators of Γ. Then the semigroup Γ is the image of $\mathbf{N}^r \subset \mathbf{Z}^r$ by the surjective linear map $b \colon \mathbf{Z}^r \to M$ determined by $b(e_i) = \gamma_i$ where the e_i, $1 \leq i \leq r$ form the canonical basis of $\mathbf{N}^r$. The kernel $\mathcal{L}$ of b is isomorphic to $\mathbf{Z}^{r-d}$.

Let us consider the map of semigroup algebras associated to the map $b|\mathbf{N}^r \colon \mathbf{N}^r \to \mathbf{Z}^d$, whose image is Γ. It is a map of A-algebras $A[U_1, \dots, U_r] \to A[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$. Its image is the subalgebra $A[t^\Gamma]$ of $A[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$.

An element $m \in \mathbf{Z}^r$ can be written uniquely $m = m_+ - m_-$ where m_+ and m_- have non negative entries and disjoint support.

By construction, the kernel of the surjection $A[U_1, \dots, U_r] \to A[t^\Gamma]$ is the ideal generated by the binomials $(U^{m_+} - U^{m_-})$ where $b(m_+) = b(m_-)$. It is the *toric ideal* associated to the map b. Note that it is not in general generated by the binomials associated to a basis of $\mathcal{L}$. Since the algebra $A[t^\Gamma]$ is an integral domain if A is, the toric ideal is a prime ideal in that case.

Conversely, assuming now that A is an algebraically closed field k, an ideal generated by binomials in $k[U_1, \dots, U_r]$ is called a *binomial ideal.* Those ideals are studied in [4], where it is shown that a prime binomial ideal $I \subset k[U_1, \dots, U_r]$ gives rise to a semigroup algebra $k[U_1, \dots, U_r]/I \simeq k[t^\Gamma]$, where $\Gamma = \mathbf{N}^r/_\sim$, and $\sim$ is an equivalence relation associated to the binomial relations. The *affine toric variety* $T^\Gamma := \mathrm{Spec}\ k[t^\Gamma]$ is the subvariety of the affine space $\mathbf{A}^r(k)$ defined by the binomial equations generating the toric ideal. By the universal property of the semigroup algebra, there is a bijection

$$\{\text{Closed points of Spec } k[t^\Gamma]\} \leftrightarrow \{\text{semigroup homomorphisms } \Gamma \to k\},$$

where k is considered as semigroup with respect to multiplication.

In particular, the torus $T^M(k) = \mathrm{Hom}_{\mathrm{groups}}(M, k^*)$ is embedded in T^Γ, as the principal open set where $t^{\gamma_1} \cdots t^{\gamma_r} \neq 0$.

From the description of closed points of T^Γ in terms of homomorphisms of semigroups we have an action of the torus $T^M(k)$ on $T^\Gamma(k)$. Another way to describe this action, which shows that it is algebraic, is to say that thanks to the universal property of semigroup algebras it corresponds to the composed map of k-algebras

$$k[t^\Gamma] \to k[t^\Gamma] \otimes_k k[t^\Gamma] \to k[t^M] \otimes_k k[t^\Gamma]$$

where the first map is determined by $t^\gamma \mapsto t^\gamma \otimes_k t^\gamma$ and the second by the inclusion $\Gamma \subset M$. The corresponding map $T^M \times T^\Gamma \to T^\Gamma$ is the action.

Let us now seek the invariant subsets of T^Γ under the torus action.

Definition 3.2. Given a semigroup Γ, a subsemigroup $F \subset \Gamma$ is a *face* of Γ if whenever $x, y \in \Gamma$ satisfy $x + y \in F$, then x and y are in F.

Let us remark that this condition is equivalent to the fact that the vector space of finite sums $\sum_{\delta \in \Gamma \setminus F} a_\delta t^\delta$ is in fact a prime ideal I_F of $k[t^\Gamma]$. It also implies that $\Gamma \setminus F$ is a subsemigroup of Γ (which in general is not finitely generated) and that the Minkowski sum $\Gamma + (\Gamma \setminus F)$ is contained in $\Gamma \setminus F$.

Lemma 3.3. *The faces of the semigroup Γ are of the form $\Gamma \cap \tau^\perp$, for $\tau \leq \sigma$.*

Proof. Let F be a face of the semigroup Γ. Then there is a face $\check{\sigma} \cap \tau^\perp$ of $\check{\sigma}$ which contains F and is of minimal dimension. Then F is also a face of the semigroup $\Gamma \cap \tau^\perp$ and there is an element $\gamma_0 \in F$ which belongs to the relative interior of the cone $\check{\sigma} \cap \tau^\perp$. Under these conditions is enough to prove that if $\tau = 0$ then $F = \Gamma$.

Notice that if $\gamma \in \Gamma$ and if $(\gamma + \Gamma) \cap \mathbf{Z}_{\geq 0}\gamma_0 \neq \emptyset$ then $\gamma \in F$ since F is a face and $\gamma_0 \in F$. By Theorem 1.9 [14] there is $\delta_0 \in \Gamma \cap \mathrm{int}(\check{\sigma})$ such that $\delta_0 + \check{\sigma} \cap M \subset \Gamma$. We deduce that the intersection $(\gamma + \delta_0 + \check{\sigma} \cap M) \cap \mathbf{Z}_{\geq 0}\gamma_0$ is non-empty, for any $\gamma \in \Gamma$, since $\gamma_0 \in \mathrm{int}(\check{\sigma}) \cap \Gamma$. □

Notation 3.4. If $\tau \leq \sigma$ the set $\Gamma \cap \tau^\perp$ is a subsemigroup of finite type of Γ. If $\tau \leq \sigma$ the lattice $M(\tau, \Gamma)$ spanned by $\Gamma \cap \tau^\perp$ is a sublattice of finite index of $M(\tau) := M \cap \tau^\perp$.

Remark 3.5. The torus of the affine toric variety $T^{\Gamma \cap \tau^\perp}$ is $T^{M(\tau,\Gamma)}$. If A is a commutative ring, the homomorphism of A-algebras $A[\Gamma] \to A[\Gamma \cap \tau^\perp] \cong A[\Gamma]/I_{\Gamma \cap \tau^\perp}$, is surjective and defines a closed embedding

$$i_\tau : T^{\Gamma \cap \tau^\perp} \hookrightarrow T^\Gamma$$

over Spec$A$. If $k = A$ the image by the embedding i_τ of a closed point $u \in T^{\Gamma \cap \tau^\perp}(k)$ (or $u \in T^{M(\tau,\Gamma)}(k)$) is the semigroup homomorphism $i_\tau(u) : \Gamma \to k$ given by

$$\gamma \mapsto \begin{cases} u(\gamma) & \text{if} \gamma \in \tau^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.6. *The map*

$$\tau \mapsto \mathrm{orb}(\tau, \Gamma) := i_\tau(T^{M(\tau,\Gamma)}) \quad (\textit{ resp. } \tau \mapsto i_\tau(T^{\Gamma \cap \tau^\perp}))$$

defines a bijection (resp. inclusion-reversing bijection) between the faces of σ and the orbits (resp. the closures of the orbits) of the torus action on T^Γ.

Proof. Let $u : \Gamma \to k$ be a semigroup homomorphism. Then $u^{-1}(k^*)$ is a face of Γ, hence of the form $\Gamma \cap \tau^\perp$ for some face τ of σ. Any such u extends in a unique manner to a group homomorphism $M(\tau, \Gamma) \to k^*$ defining an element of the torus $T^{M(\tau,\Gamma)}$ of the affine toric variety $T^{\Gamma \cap \tau^\perp}$. Conversely, given a group homomorphism $u \colon M(\tau, \Gamma) \to k^*$ we define a semigroup homomorphism $i_\tau(u) : \Gamma \to k$ as indicated above.

It follows that the orbit of the point defined by u by the action of T^M coincides with the image by i_τ of the orbit $T^{M(\tau,\Gamma)}$ of the point $u_{|\Gamma\cap\tau^\perp}: \Gamma \cap \tau^\perp \to k^*$ on the toric variety $T^{\Gamma\cap\tau^\perp}$. The rest of the assertion follows from Remark 3.5. □

The partition induced by the orbits of the torus action on T^Γ is of the form:

$$T^\Gamma = \bigsqcup_{\tau\leq\sigma} \mathrm{orb}(\tau,\Gamma). \tag{1}$$

Proposition 3.7. *If X is an affine toric variety with torus T^M then X is T^M-equivariantly isomorphic to T^Γ, where $\Gamma \subset M$ a semigroup of finite type such that $\mathbf{Z}\Gamma = M$.*

*P*roof. This is well-known (see Proposition 2.4, Chapter 5 of [7]). □

We characterize the affine T^M-invariant open subsets of T^Γ.

Definition 3.8. For any face τ of σ the set

$$\Gamma_\tau := \Gamma + M(\tau,\Gamma) \tag{2}$$

is a semigroup of finite type generating the lattice M.

Notice that the cone $\Gamma_\tau \mathbf{R}_{\geq 0}$ is equal to $\check\tau$ and if $\tau \leq \sigma$ the set $\mathrm{int}(\check\sigma \cap \tau^\perp) \cap \Gamma$ is non empty (int denotes relative interior).

Lemma 3.9.

i. *The minimal face of the semigroup Γ is a sublattice of M equal to $\Gamma \cap \sigma^\perp$.*

ii. *For any $m \in \Gamma$ in the relative interior of $(\check\sigma \cap \tau^\perp)$ we have that*

$$\Gamma_\tau = \Gamma + \mathbf{Z}_{\geq 0}(-m).$$

iii. *If $\tau \leq \theta \leq \sigma$ we have that $M(\tau,\Gamma_\theta) = M(\tau,\Gamma_\tau)$ and $\Gamma_\tau = \Gamma_\theta + M(\tau,\Gamma_\theta)$.*

Proof. i. By Lemma 3.3 the correspondence $\tau \mapsto \Gamma \cap \tau^\perp$ is a bijection between the faces of the cone σ and the faces of the semigroup Γ. By duality the minimal face of Γ is equal to $\Gamma \cap \sigma^\perp$. It is enough to prove that if Γ is a semigroup such that $\mathbf{Z}\Gamma = M$ and $\Gamma\mathbf{R}_{\geq 0} = M_\mathbf{R}$ then $\Gamma = M$. Since M is the saturation of Γ the assertion reduces to the case of rank one semigroups, for which it is elementary by Bezout identity.

ii. If $m \in \mathrm{int}(\check\sigma \cap \tau^\perp) \cap \Gamma$ then the semigroup $\Gamma + \mathbf{Z}_{\geq 0}(-m) \subset M$ spans the cone $\check\tau = \check\sigma + \tau^\perp \subset M_\mathbf{R}$. By i. the minimal face of this semigroup is the lattice $(\Gamma + \mathbf{Z}_{\geq 0}(-m)) \cap \tau^\perp$ which coincides by definition with the lattice $M(\tau,\Gamma)$.

iii. The lattices $M(\tau,\Gamma_\theta)$ and $M(\tau,\Gamma)$ are both generated by $\Gamma \cap \tau^\perp$ hence are equal. We have that $\Gamma_\tau = \Gamma_\theta + M(\tau,\Gamma_\theta)$ since $\theta^\perp \subset \tau^\perp$. □

Lemma 3.10. *If $\tau \leq \sigma$ the inclusion of semigroups $\Gamma \subset \Gamma_\tau$ determines a T^M-equivariant embedding $T^{\Gamma_\tau} \subset T^\Gamma$ as an affine open set. Conversely, if $X \subset T^\Gamma$ is a T^M-equivariant embedding of an affine open set then there is a unique $\tau \leq \sigma$ such that X is T^M-equivariantly isomorphic to T^{Γ_τ}.*

Proof. By Lemma 3.9 we have that $\Gamma_\tau = \Gamma + \mathbf{Z}_{\geq 0}(-m)$. More generally if $\gamma \in \Gamma$ and $f = t^\gamma$, the localization $T^\Gamma_f = \mathrm{Spec}\ \ k[\Gamma]_f$ is equal to $T^{\Gamma + (-\gamma)\mathbf{Z}_{\geq 0}}$ and it is embedded in T^Γ as a principal open set.

Conversely, an affine T^M-invariant open subset of T^Γ is an affine toric variety for the torus T^M hence it is of the form T^Λ, for $\Lambda \subset M$ a subsemigroup of finite type, such that $\mathbf{Z}\Lambda = M$ (see Proposition 3.7). We denote the cone $\mathbf{R}_{\geq 0}\Lambda$ by $\check\theta$. Since the embedding $T^\Lambda \subset T^\Gamma$ is T^M equivariant it is defined by the inclusion of algebras $k[t^\Gamma] \to k[t^\Lambda]$ corresponding to the inclusion of semigroups $\Gamma \subset \Lambda$. We deduce that $\check\sigma \subset \check\theta$ and hence that $\theta \subset \sigma$ by duality. We prove that if τ is the smallest face of σ which contains θ then $\Lambda = \Gamma_\tau$. It is enough to prove that if $\mathrm{int}(\theta) \cap \mathrm{int}(\sigma) \neq \emptyset$ then $\Lambda = \Gamma$.

Notice that the lattice $F = \sigma^\perp \cap M$ is the minimal face of Γ and the prime ideal I_F of $k[t^\Gamma]$ defines the orbit $\mathrm{orb}(\sigma, \Gamma)$, which is embedded as a closed subset of T^Γ. Let us consider a vector ν such that $\nu \in \mathrm{int}(\theta) \cap \mathrm{int}(\sigma)$. Then we get that $\sigma^\perp \cap M = \check{\sigma} \cap \nu^\perp \cap M$ is contained in $\check{\theta} \cap \nu^\perp \cap M = \theta^\perp \cap M$ hence $\Gamma \setminus (\sigma^\perp \cap M)$ is contained in $\Lambda \setminus (\theta^\perp \cap M)$ and therefore $1 \notin I_F k[t^\Lambda]$. Since $T^\Lambda \subset T^\Gamma$ is an open immersion $\mathrm{orb}(\sigma, \Gamma)$ is contained in T^Λ. By (1) and Proposition 3.6 the closure of any orbit contained T^Γ contains $\mathrm{orb}(\sigma, \Gamma)$ thus $T^\Gamma \subset T^\Lambda$. □

Remark 3.11. The immersion of T^M-invariant affine open subsets is compatible with normalization. By Lemma 3.10 any T^M-invariant affine open set of T^Γ is of the form X_f^Γ for $f = t^\gamma$, $\gamma \in \Gamma$. Then the following diagram commutes:

$$\begin{array}{ccc} T^{\check{\sigma} \cap M} & \hookrightarrow & T^\Gamma \\ \uparrow & & \uparrow \\ T_f^{\check{\sigma} \cap M} & \hookrightarrow & T_f^\Gamma, \end{array}$$

since $\Gamma + (-\gamma)\mathbf{Z}_{\geq 0}$ is saturated in $\check{\sigma} \cap M + (-\gamma)\mathbf{Z}_{\geq 0}$. The vertical arrows are embeddings as principal open sets while the horizontal arrows are normalization maps (see Proposition 1.5).

4. Toric varieties

Recall that a *fan* is a finite set Σ of strictly convex polyhedral cones rational for the a lattice N, such that if $\sigma \in \Sigma$ any face τ of σ belongs to Σ and if $\sigma, \sigma' \in \Sigma$ the cone $\tau = \sigma \cap \sigma'$ is in Σ. If $j \geq 0$ is an integer the subset of $\Sigma(j)$ of j-dimensional cones of Σ is called the *j-skeleton* of the fan. The *support* of the fan Σ is the set $|\Sigma| = \cup_{\sigma \in \Sigma} \sigma \subset N_{\mathbf{R}}$.

We give first a *combinatorial definition* of toric varieties.

Definition 4.1. A toric variety is given by the datum of a triple (N, Σ, Γ) consisting of lattice N, a fan Σ in $N_{\mathbf{R}}$ and a family of finitely generated subsemigroups $\Gamma = \{\Gamma_\sigma \subset | \sigma \in \Sigma\}$ of a lattice $M = \mathrm{Hom}(N, \mathbf{Z})$ such that:

i. $\mathbf{Z}\Gamma_\sigma = M$, for $\sigma \in \Sigma$.
ii. $\Gamma_\tau = \Gamma_\sigma + M(\tau, \Gamma_\sigma)$, for a each $\sigma \in \Sigma$ and any face τ of σ.

The corresponding toric variety T_Σ^Γ is the union of the affine varieties T^{Γ_σ} for $\sigma \in \Sigma$ where for any pair σ, σ' in Σ we glue up T^{Γ_σ} and $T^{\Gamma_{\sigma'}}$ along their common open affine variety $T^{\Gamma_{\sigma \cap \sigma'}}$.

Remark 4.2. The lattice N in the triple (N, Σ, Γ) is determined by Γ. We recall it by convenience. We omit the reference to the lattice N in the notation T_Σ^Γ.

Remark 4.3. This definition is consistent with the case of affine toric varieties. Let T^Γ be an affine toric variety in the sense of Section 3. If $\sigma' := \{\tau \mid \tau \leq \sigma\}$ and $\Gamma' := \{\Gamma_\tau \mid \tau \leq \sigma\}$, where Γ_τ is the semigroup defined by (2) for $\tau \leq \sigma$ then the conditions i. and ii. are satisfied by Lemma 3.9. Then T^Γ is T^M-invariantly isomorphic to $T_{\sigma'}^{\Gamma'}$.

Remark 4.4. A triple (N, Σ, Γ) determines similarly a toric scheme over Spec$A$, for any commutative ring A.

Lemma 4.5. *Let (Σ, Γ) as in Definition 4.1 define a toric variety T_Σ^Γ. Then we have:*

i. *If $\sigma, \theta \in \Sigma$ and if $\tau = \sigma \cap \theta$ then $\Gamma_\tau = \Gamma_\sigma + \Gamma_\theta$.*
ii. *The variety T_Σ^Γ is separated.*

Proof. The intersection $\tau = \sigma \cap \theta$ is a face of both σ and θ. By Lemma 3.9 we have that $M(\tau, \Gamma_\tau) = M(\tau, \Gamma_\sigma) = M(\tau, \Gamma_\theta)$. By axiom i. in the Definition 4.1 we get $\Gamma_\theta, \Gamma_\sigma \subset \Gamma_\tau$ and $\Gamma_\theta + \Gamma_\sigma \subset \Gamma_\tau$. Conversely, by the separation lemma for polyhedral cones, for any $u \in \mathrm{int}(\check{\sigma} \cap (-\check{\theta}))$ we have that $\tau = \sigma \cap u^\perp = \theta \cap u^\perp$. Notice that we can assume that $u \in \Gamma_\sigma \cap (-\Gamma_\theta) \cap \mathrm{int}(\check{\sigma} \cap (-\check{\theta})) \neq \emptyset$. Then by Lemma 3.9 we obtain $\Gamma_\tau = \Gamma_\sigma + \mathbf{Z}_{\geq 0}(-u)$. Hence Γ_τ is contained in $\Gamma_\sigma + \Gamma_\theta$ since $-u \in \Gamma_\theta$.

The homomorphism $k[t^{\Gamma_\theta}] \otimes_k k[t^{\Gamma_\sigma}] \to k[t^{\Gamma_\tau}]$ which sends $t^\gamma \otimes t^{\gamma'} \mapsto t^{\gamma+\gamma'}$ is surjective since $\Gamma_\sigma + \Gamma_\theta = \Gamma_\tau$. In geometric terms this implies that the diagonal map $T^{\Gamma_\tau} \to T^{\Gamma_\theta} \times T^{\Gamma_\sigma}$ is a closed embedding for any $\theta, \sigma \in \Sigma$ with $\tau = \theta \cap \tau$, hence the variety T_Σ^Γ is separated (see Chapter 2 of [12]). □

Remark 4.6. The morphisms corresponding to the inclusions $k[t^{\Gamma_\sigma}] \to k[t^{\Gamma_\sigma + \Gamma_{\sigma'}}]$ are open embeddings compatible with the normalization maps. The normalization of the toric variety T_Σ^Γ is the toric variety T_Σ corresponding to the fan Σ and the normalization map is obtained by gluing-up normalizations $T_\sigma := T^{\check{\sigma} \cap M}$ of the charts T^{Γ_σ}, for $\Gamma_\sigma \in \Gamma$ and $\sigma \in \Sigma$.

Lemma 4.7. *Let λ_v be a one-parameter subgroup of the torus T^M for some $v \in N$. Then $\lim_{z \to 0} \lambda_v(z)$ exists in the toric variety T_Σ^Γ if and only if v belongs to $|\Sigma| \cap N$.*

Proof. The statement is well-known in the normal case (see Proposition 1.6 [19]). The normalization map $n \colon T_\Sigma \to T_\Sigma^\Gamma$ is an isomorphism over the torus T^M. If $\lambda_v \colon k^* \to T^M \subset T_\Sigma^\Gamma$ is a one-parameter subgroup defined by $v \in N$ it lifts to the normalization, i.e., there is a morphism $\bar{\lambda}_v \colon k^* \to T^M \subset T_\Sigma$ in such a way that $n \circ \bar{\lambda}_v = \lambda_v$. Since the normalization is a proper morphism we get by the valuative criterion of properness that $\lim_{z \to 0} \lambda_v(z)$ exists in the toric variety T_Σ^Γ if and only if $\lim_{z \to 0} \bar{\lambda}_v(z)$ exists in T_Σ. □

Lemma 4.8. *Let X_Σ^Γ be a toric variety. Then the map*

$$\tau \mapsto \mathrm{orb}(\tau, \Gamma_\tau) := i_\tau(T^{M(\tau, \Gamma_\tau)})$$

defines a bijection between the faces of Σ and the orbits of the torus action on T_Σ^Γ.

Proof. This is consequence of the definitions and Lemma 3.6. □

In order to illustrate the combinatorial definition of a toric variety we describe the orbit closures as toric varieties.

Notation 4.9. If $\tau \in \Sigma$ we denote by N_τ the sublattice of N spanned by $\tau \cap N$ and by $N(\tau)$ the quotient N/N_τ. The lattice $N(\tau)$ is the dual lattice of $M(\tau) = M \cap \tau^\perp$. Since $M(\tau, \Gamma_\tau)$ is a sublattice of finite index $i(\tau, \Gamma_\tau)$ of $M(\tau)$ then the dual lattice $N(\tau, \Gamma_\tau)$ of $M(\tau, \Gamma_\tau)$ contains $N(\tau)$ as a sublattice of finite index equal to $i(\tau, \Gamma_\tau)$.

If $\sigma \in \Sigma$ and $\tau \leq \sigma$ the image $\sigma(\tau)$ of σ in $N(\tau)_{\mathbf{R}} = N_{\mathbf{R}}/(N_\tau)_{\mathbf{R}}$ is a polyhedral cone, rational for the lattice $N(\tau, \Gamma_\tau)$. The set $\Sigma(\tau) := \{\sigma(\tau) \mid \sigma \in \Sigma, \tau \leq \sigma\}$ is a fan in $N(\tau)_{\mathbf{R}}$. If $\sigma(\tau) \in \Sigma(\tau)$ we set $\Gamma_{\sigma(\tau)} := \Gamma_\sigma \cap \tau^\perp$. The set $\sigma(\tau) \subset N(\tau)_{\mathbf{R}}$ is the dual cone of the cone spanned by $\Gamma_\sigma \cap \tau^\perp$ in $M(\tau)_{\mathbf{R}}$. Let us denote by $\Gamma(\tau)$ the set $\{\Gamma_{\sigma(\tau)} \mid \sigma(\tau) \in \Sigma(\tau)\}$.

Lemma 4.10. *Let X_Σ^Γ be a toric variety. If $\tau \in \Sigma$ the triple $(N(\tau, \Gamma_\tau), \Sigma(\tau), \Gamma(\tau))$ defines a toric variety $T_{\Sigma(\tau)}^{\Gamma(\tau)}$. We have a closed embedding $i_\tau : T_{\Sigma(\tau)}^{\Gamma(\tau)} \to T_\Sigma^\Gamma$. The map*

$$\tau \mapsto i_\tau(T_{\Sigma(\tau)}^{\Gamma(\tau)})$$

defines a bijection between the faces of Σ and orbit closures of the action of T^M on T_Σ^Γ.

Proof. If τ is not a face of σ, for $\sigma \in \Sigma$ then $\mathrm{orb}(\tau, \Gamma_\tau)$ is does not intersect the affine invariant open set T^{Γ_σ}; if $\tau \leq \sigma$, for $\sigma \in \Sigma$ the closure of the orbit $\mathrm{orb}(\tau, \Gamma_\tau)$ in the affine open set T^{Γ_σ} is equal to $T^{\Gamma_\sigma \cap \tau^\perp}$ (see Lemma 3.6).

If $\tau \leq \theta \leq \sigma$ then $\theta(\tau) \leq \sigma(\tau)$ and $\theta^\perp \subset \tau^\perp$ hence $M(\theta, \Gamma_\sigma) = M(\theta(\tau), \Gamma_\sigma \cap \tau^\perp)$ is the sublattice spanned by $\Gamma_\sigma \cap \theta^\perp$.

If $\tau \leq \sigma, \sigma'$ and if $\theta = \sigma \cap \sigma'$ then we deduce from condition ii. in Definition 4.1 that:

$$\Gamma_\theta \cap \tau^\perp = \Gamma_\sigma \cap \tau^\perp + M(\theta(\tau), \Gamma_{\sigma(\tau)}) = \Gamma_{\sigma'} \cap \tau^\perp + M(\theta(\tau), \Gamma_{\sigma'(\tau)}).$$

We obtain that the triple $(N(\tau, \Gamma_\tau), \Sigma(\tau), \Gamma(\tau))$ satisfies the axioms in Definition 4.1 with respect to the torus $T^{M(\tau,\Gamma_\tau)}$.

We have also described an embedding $T^{\Gamma(\tau)}_{\Sigma(\tau)} \hookrightarrow T^\Gamma_\Sigma$ in such a way that the intersection of this variety with any affine chart containing $\mathrm{orb}(\tau, \Gamma)$ is the closure of the orbit $\mathrm{orb}(\tau, \Gamma)$ in the chart. The conclusion follows from Lemma 4.8. □

Remark 4.11. The non singular locus of the toric variety T^Γ_Σ is the union of the orbits $\mathrm{orb}(\tau, \Gamma)$ corresponding to *regular* cones $\tau \in \Sigma$ such their index $i(\tau, \Gamma_\tau)$ is equal to 1.

5. Blowing ups

The theory of normal toric varieties deals with normalized equivariant blowing ups, i.e., blowing ups of equivariant ideals followed by normalization. In this section we build blowing ups of equivariant ideals in toric varieties.

Let σ be a strictly convex rational cone in $N_{\mathbf{R}}$ and Γ a subsemigroup of finite type of the lattice M such that $\mathbf{Z}\Gamma = M$ and the saturation of Γ in M is equal to $\check{\sigma} \cap M$. For simplicity we assume that the cone σ is of dimension d hence $\check{\sigma}$ is strictly convex.

Let us consider a graded ideal $\mathcal{I}$ in $A[t^\Gamma]$, which is necessarily generated by monomials $t^{m_1}, \dots, t^{m_k}$. We build the corresponding *Newton polyhedron* $\mathcal{N}_\sigma(\mathcal{I})$, by definition the convex hull in $M_{\mathbf{R}}$ of the $m_i + \check{\sigma}$, which is also the convex hull of the set $|\mathcal{I}|$ of exponents of monomials belonging to the ideal $\mathcal{I}$ of $A[t^\Gamma]$. It is quite convenient to denote with the same letter $\mathcal{I}$ the set $\{m_1, \dots m_k\}$.

The set $\mathcal{I}$ determines the *order function*:

$$\mathrm{ord}_{\mathcal{I}} \colon \sigma \to \mathbf{R}, \quad \nu \mapsto \min_{m \in \mathcal{I}} \langle \nu, m \rangle. \tag{3}$$

The order function $\mathrm{ord}_{\mathcal{I}}$ coincides with the support function of the polyhedron $\mathcal{N}_\sigma(\mathcal{I})$. It is a gauge ($\mathrm{ord}_{\mathcal{I}}(\lambda u) = \lambda \mathrm{ord}_{\mathcal{I}}(u)$ for $\lambda > 0$) which is piecewise linear. The maximal cones of linearity of the function $\mathrm{ord}_{\mathcal{I}}$ form the d-skeleton of the fan $\Sigma(\mathcal{I})$ subdividing σ. Each such cone σ_i in the d-skeleton of $\Sigma(\mathcal{I})$ is the convex dual of the convex rational cone generated by the vectors $(m - m_i)_{m \in \mathcal{N}_\sigma(\mathcal{I})}$, where m_i is a vertex of $\mathcal{N}_\sigma(\mathcal{I})$. The correspondence $m_i \mapsto \sigma_i$ is a bijection between the set of vertices $\{m_1, \dots, m_s\} \subset \mathcal{I}$ of the polyhedron $\mathcal{N}_\sigma(\mathcal{I})$ and the d-skeleton of $\Sigma(\mathcal{I})$, such that

$$m_i \mapsto \sigma_i \text{ if and only if } \mathrm{ord}_{\mathcal{I}}(\nu) = \langle \nu, m_i \rangle \text{ for all } \nu \in \sigma_i.$$

Note that $\Gamma \subset \check{\sigma} \cap M \subset \check{\sigma}_i \cap M$. In each of the cones $\check{\sigma}_i$ we consider the semigroup

$$\Gamma_i = \Gamma + \langle m_1 - m_i, \dots, m_{i-1} - m_i, m_{i+1} - m_i, \dots, m_k - m_i \rangle \subset \check{\sigma}_i \cap M. \tag{4}$$

By Lemma 1.3, the saturation in M of this semigroup is equal to $\check{\sigma}_i \cap M$. We denote by $\Gamma(\mathcal{I})$ the set consisting of the semigroups Γ_i, together with $\Gamma_{i,\tau}$ (defined by equation (2)) for $\tau \leq \sigma_i$, $i = 1, \dots, s$.

Proposition 5.1. *The triple* $(N, \Sigma(\mathcal{I}), \Gamma(\mathcal{I}))$ *defines a toric scheme* B *over* Spec A. *The inclusions* $\Gamma \subset \Gamma_i$, $i = 1, \dots, s$, *determine a map of schemes*

$$\pi \colon B \to \mathrm{Spec}\ A[t^\Gamma]$$

over Spec A, *which is the blowing up of the ideal* $\mathcal{I}$.

Proof. We prove first that the triple $(N, \Sigma(\mathcal{I}), \Gamma(\mathcal{I}))$ verifies the compatibility conditions stated in Definition 4.1. By Lemma 3.9 it is enough to check them for the affine open sets corresponding to two vertices, say m_1 and m_2, of $\mathcal{N}_\sigma(\mathcal{I})$. Then, if $\tau = \sigma_1 \cap \sigma_2$ the condition we have to prove is that $\Gamma_{1,\tau} = \Gamma_{2,\tau}$.

Notice that the vector $m := m_2 - m_1 \in \Gamma_1$ belongs to the interior of $\check{\sigma}_1 \cap \tau^\perp$. By Lemma 3.9 and the definitions we get $\Gamma_{1,\tau} = \Gamma_1 + \mathbf{Z}_{\geq 0}(-m)$ and similarly $\Gamma_{2,\tau} = \Gamma_2 + \mathbf{Z}_{\geq 0} m$. Then the assertion follows since $\Gamma_{1,\tau}$, which is equal to

$$\Gamma + \mathbf{Z}(m_2 - m_1) + \sum_{j=2,\dots,k} \mathbf{Z}_{\geq 0}(m_j - m_1) = \Gamma + \mathbf{Z}(m_1 - m_2) + \sum_{j=2,\dots,k} \mathbf{Z}_{\geq 0}(m_j - m_2),$$

is the same semigroup as $\Gamma_{2,\tau}$.

It follows that the scheme B is covered by the affine sets Spec $A[t^{\Gamma_i}]$ for $i = 1, \dots, s$. Since each Γ_i contains Γ, there is a natural map π: Spec $A[t^{\Gamma_i}] \to$ Spec $A[t^\Gamma]$. The sheaf of ideals on B determined by the compositions with π of the generators of $\mathcal{I}$ is generated by $t^{m_i} \circ \pi$ in the chart Spec$A[t^{\Gamma_i}]$.

It is not difficult to prove that any semigroup Γ_i defined by (4), for $i > s$, that is when m_i is not a vertex of $\mathcal{N}_\sigma(\mathcal{I})$, is of the form $\Gamma_{j,\tau}$ for some $1 \leq j \leq s$ and $\tau \leq \sigma_j$. This means that the corresponding affine chart Spec $A[t^{\Gamma_i}]$ of the blowing up of $\mathcal{I}$ is in fact an affine open subset of Spec $A[t^{\Gamma_j}]$, where m_j is a vertex of $\mathcal{N}_\sigma(\mathcal{I})$. ☐

Corollary 5.2. *The blowing-up of an equivariant sheaf of ideals on a toric variety T_Σ^Γ is a toric variety. Its description above each equivariant open affine chart of T_Σ^Γ is given by Proposition 5.1.*

6. Toric morphisms

Recall that a morphism $\phi\colon T^{M'} \to T^M$ of algebraic tori gives rise to two group homomorphisms

$$\phi^* : M \to M' \text{ and } \phi_* : N' \to N$$

between the corresponding lattices of characters and between the corresponding lattices of one-parameter subgroups. The homomorphisms ϕ^* and ϕ_* are mutually dual and determine the morphism $\phi\colon T^{M'} \to T^M$ of algebraic tori. Note that ϕ is defined algebraically by

$$k[t^M] \to k[t^{M'}], \quad t^m \mapsto t^{\phi^*(m)}, \quad m \in M.$$

Now suppose that we have two toric varieties T_Σ^Γ and $T_{\Sigma'}^{\Gamma'}$ with respective tori T^M and $T^{M'}$ defined by the combinatorial data given by the triples (N, Σ, Γ) and (N', Σ', Γ') (see Definition 4.1).

Definition 6.1. The homomorphism ϕ_* is a *map of fans with attached semigroups* $(N, \Sigma, \Gamma) \to (N', \Sigma', \Gamma')$ if for any $\sigma' \in \Sigma'$ there exists $\sigma \in \Sigma$ such that $\phi^*(\Gamma_\sigma) \subset \Gamma'_{\sigma'}$.

Note then that ϕ_* is a *map of fans*, that is, for any $\sigma' \in \Sigma'$ there is a cone $\sigma \in \Sigma$ such that image by σ' by the $\mathbf{R}$-linear extension of ϕ_* is contained in σ. See Section 1.5 [19].

Proposition 6.2. *Let $\phi\colon T^{M'} \to T^M$ be a morphism of algebraic tori. If ϕ_* defines a map of fans with attached semigroups $(N, \Sigma, \Gamma) \to (N', \Sigma', \Gamma')$ then it gives rise to a morphism: $\bar{\phi}\colon T_{\Sigma'}^{\Gamma'} \to T_\Sigma^\Gamma$ which extends $\phi\colon T^{M'} \to T^M$ and is equivariant with respect to ϕ. Conversely, if $f\colon T_{\Sigma'}^{\Gamma'} \to T_\Sigma^\Gamma$ is an equivariant morphism with respect to ϕ then ϕ_* defines a map of fans with attached semigroups $(N', \Sigma', \Gamma') \to (N, \Sigma, \Gamma)$ and $f = \bar{\phi}$. In addition we have a commutative diagram*

$$\begin{array}{ccc} T_{\Sigma'} & \longrightarrow & T_\Sigma \\ \downarrow & & \downarrow \\ T_{\Sigma'}^{\Gamma'} & \longrightarrow & T_\Sigma^\Gamma \end{array}$$

where the vertical arrows are normalizations and the horizontal ones are the toric morphisms which extend $\phi : T^{M'} \to T^M$.

Proof. For any $\sigma' \in \Sigma'$ there exists a cone $\sigma \in \Sigma$ such that the restriction of ϕ^* determines a semigroup homomorphism $\Gamma_\sigma \to \Gamma'_{\sigma'}$. The corresponding homomorphism of k-algebras $k[\Gamma_\sigma] \to k[\Gamma'_{\sigma'}]$ defines a morphism:

$$\bar\phi_{\sigma',\sigma} \colon T^{\Gamma'_{\sigma'}} \to T^{\Gamma_\sigma} \text{ given on closed points by } \bar\phi_{\sigma',\sigma}(x) = x \circ \phi^*_{|\Gamma_\sigma},$$

where $x \in T^{\Gamma'_{\sigma'}}; x \colon \Gamma'_{\sigma'} \to k$ is a homomorphism of semigroups. The morphism $\bar\phi_{\sigma',\sigma}$ is equivariant through ϕ since for any $y \in T^{M'}$, $y \colon M' \to k^*$ group homomorphism and any $x \in T^{\Gamma'_{\sigma'}}$ we get:

$$\bar\phi_{\sigma',\sigma}(y \cdot x) = (y \cdot x) \circ \phi^*_{|\Gamma_\sigma} = (y \circ \phi^*) \cdot (x \circ \phi^*_{|\Gamma_\sigma}) = \phi(y) \cdot \bar\phi_{\sigma',\sigma}(x).$$

By gluing-up the affine pieces together we get a morphism $\bar\phi \colon T^{\Gamma'}_{\Sigma'} \to T^{\Gamma}_{\Sigma}$ which is equivariant with respect to ϕ.

For the converse, since f is assumed to be equivariant through ϕ the image by f of each orbit of the action of $T^{M'}$ on $T^{\Sigma'}_{\Gamma'}$ is contained in one orbit of the action of T^M on T^{Γ}_{Σ}. If $\tau' \leq \sigma'$ and $\sigma' \in \Sigma'$ then the orbit $\mathrm{orb}(\sigma', \Gamma'_{\sigma'})$ is contained in the closure of $\mathrm{orb}(\tau', \Gamma'_{\tau'})$ by Proposition 3.6. Then there exist $\sigma, \tau \in \Sigma$ such that

$$f(\mathrm{orb}(\sigma', \Gamma'_{\sigma'})) \subset \mathrm{orb}(\sigma, \Gamma_\sigma) \text{ and } f(\mathrm{orb}(\tau', \Gamma'_{\tau'})) \subset \mathrm{orb}(\tau, \Gamma_\tau).$$

Since f is continuous $\mathrm{orb}(\sigma, \Gamma_\sigma)$ must be contained in the closure of $\mathrm{orb}(\tau, \Gamma_\tau)$, hence τ is a face of σ by Proposition 3.6 and Lemma 3.5. By (1) it follows that $f(T^{\Gamma'_{\sigma'}}) \subset T^{\Gamma_\sigma}$. The restriction $f_{|T^{\Gamma'_{\sigma'}}} : T^{\Gamma'_{\sigma'}} \to T^{\Gamma_\sigma}$ is equivariant with respect to $\phi : T^{M'} \to T^M$. Hence $f_{|T^{\Gamma'_{\sigma'}}} : T^{\Gamma'_{\sigma'}} \to T^{\Gamma_\sigma}$ is defined algebraically by the homomorphism of k-algebras $k[t^{\Gamma_\sigma}] \to k[t^{\Gamma'_{\sigma'}}]$, which is obtained by restriction from the homomorphism of k-algebras $k[t^M] \to k[t^{M'}]$ which maps $t^m \mapsto t^{\phi^*(m)}$ for $m \in M$. This implies that $\phi^*(\Gamma_\sigma) \subset \Gamma'_{\sigma'}$ and also that $f = \bar\phi$.

Since ϕ_* is a map of fans it defines a toric morphism between the normalizations of $T^{\Gamma'}_{\Sigma'}$ and T^{Γ}_{Σ}. Finally, it is easy to check that the diagram above is commutative. □

It is sometimes useful to consider morphisms of toric varieties which send the torus of the source into a non dense orbit of the target: Let (N, Σ, Γ) and (N', Σ', Γ') be two triples defining toric varieties T^{Γ}_{Σ} and $T^{\Gamma'}_{\Sigma'}$. Let τ be a cone of Σ. Suppose that we have a morphism of algebraic tori $\phi : T^{M'} \to T^{M(\tau, \Gamma_\tau)}$ such that $\phi_* : N' \to N(\tau, \Gamma_\tau)$ defines a map of fans with attached semigroups $(N', \Sigma', \Gamma') \to (N(\tau, \Gamma_\tau), \Sigma(\tau), \Gamma(\tau))$. Then by Proposition 6.2 and Lemma 3.5 we have a toric morphism

$$\bar\phi : T^{\Gamma'}_{\Sigma'} \to T^{\Gamma(\tau)}_{\Sigma(\tau)}.$$

Let us denote by $n \colon T_\Sigma \to T^{\Gamma}_{\Sigma}$ the normalization map and by $\bar i_\tau \colon T_{\Sigma(\tau)} \to T_\Sigma$ the closed embedding of the closure of $\mathrm{orb}(\tau)$ in T_Σ. The following Proposition is consequence of Proposition 6.2 and Lemma 3.5.

Proposition 6.3. *The composite of $\bar\phi$ with the closed embedding $i_\tau : T^{\Gamma(\tau)}_{\Sigma(\tau)} \hookrightarrow T^{\Gamma}_{\Sigma}$ lifts to the normalization of T^{Γ}_{Σ}, i.e., there exists a toric morphism $\psi : T^{\Gamma'}_{\Sigma'} \to T_{\Sigma(\tau)}$ such that $i_\tau \circ \bar\phi = n \circ \bar i_\tau \circ \psi$ if and only if there is a lattice homomorphism $\varphi^* : M(\tau) \to M'$ such that $\varphi^*_{|M(\tau, \Gamma_\tau)} = \phi^*$ and then $\psi = \bar\varphi$.*

Example 6.4. By Proposition 6.3 the map $u \mapsto (u, 0, 0)$, which parametrizes the singular locus of the Whitney umbrella $\{x_1^2 x_2 - x_3^2 = 0\}$ does not lift to the normalization while $u \mapsto (u^2, 0, 0)$ does.

7. Abstract toric varieties

We recall the *usual* definition of toric variety.

Definition 7.1. A toric variety X is an irreducible (separated) algebraic variety equipped with an action of an algebraic torus T embedded in X as a Zariski open set such that the action of T on X is morphism which extends the action of T over itself by multiplication.

As stated in Proposition 3.7 any affine toric variety is the spectrum of certain semigroup algebra. Gel′fand, Kapranov, and Zelevinsky have defined and studied those projective toric varieties which are equivariantly embedded in the projective space, which is viewed as a toric variety, see [7], Chapter 5.

The following Theorem, which is consequence of a more general result of Sumihiro, provides the key to establish a combinatorial description of normal toric varieties.

Theorem 7.2. *(see* [23]*) Any normal toric variety X has a finite covering by T-invariant affine normal toric varieties.*

The statement of Theorem 7.2 does not hold if the normality assumption is dropped.

Example 7.3. Let $C \subset \mathbf{P}^2_{\mathbf{C}}$ be the projective nodal cubic with equation $y^2z - x^2(x+z)$. It is a rational curve with a node singularity at $P = (0 : 0 : 1)$ and only one point $Q = (0 : 1 : 0)$ at the line of infinity $z = 0$. The curve C is rational and has a parametrization $\pi : \mathbf{P}^1_{\mathbf{C}} \to C$ such that $\pi(0) = \pi(\infty) = P$ and $\pi(1) = Q$. Then we have that $\pi_{|\mathbf{C}^*} \colon \mathbf{C}^* \to C \setminus \{P\}$ is an isomorphism. The multiplicative action of $\mathbf{C}^*$ on $\mathbf{P}^1_{\mathbf{C}}$ corresponds by π to the group law action on the cubic hence it is algebraic. It follows that C is a toric variety with respect to Definition 7.1. Notice that C is the only open set containing P which is invariant by the action of $\mathbf{C}^*$. This example is also a projective toric curve which does not admit any equivariant embedding in the projective space (see [18] page 4 and [7] Chapter 5, Remark 1.6).

We modify the abstract definition of toric varieties as follows:

Definition 7.4. A toric variety X is an irreducible separated algebraic variety equipped with an action of an algebraic torus T embedded in X as a Zariski open set such that the action of T on X is morphism which extends the action of T over itself by multiplication and X has a finite covering by affine T-invariant Zariski open sets.

Theorem 7.5. *If X is a toric variety in the sense of Definition 7.4 with torus T, then there exists a triple (N, Σ, Γ) as in Definition 4.1 and an isomorphism $\varphi\colon T \to T^M$ such that the pair (T, X) is equivariantly isomorphic to $(T^M, T^{\Gamma}_{\Sigma})$ with respect to φ.*

Proof. We denote by M the lattice of characters of the torus T hence $T = T^M$ and N is the dual lattice of M.

By Proposition 3.7 an affine T^M-invariant open subset is of the form T^{Γ_σ} where Γ_σ is a subsemigroup of finite type of M such that $\mathbf{Z}\Gamma_\sigma = M$, and $\sigma \subset N_{\mathbf{R}}$ is the dual cone of $\check{\sigma} = \mathbf{R}_{\geq 0}\Gamma_\sigma \subset M_{\mathbf{R}}$. By Lemma 3.10 the open affine T^M-invariant subsets of T^{Γ_σ} are T^{Γ_τ}, for $\tau \leq \sigma$, where $\Gamma_\tau = \Gamma_\sigma + M(\tau, \Gamma_\sigma)$.

By definition X is covered by a finite number of T^M-invariant affine open subsets of the form $\{T^{\Gamma_\sigma}\}_{\sigma \in \Sigma}$. We can assume that if $\sigma \in \Sigma$ and if $\tau \leq \sigma$ then $\tau \in \Sigma$. We are going to show that Σ is a fan in $N_{\mathbf{R}}$, hence $T^{\Gamma_\sigma} \neq T^{\Gamma_{\sigma'}}$ if $\sigma \neq \sigma'$.

We have that for any $\sigma, \sigma' \in \Sigma$ the intersection $T^{\Gamma_\sigma} \cap T^{\Gamma_{\sigma'}}$ is an affine open subset of the separated variety X (see Chapter 2 of [12]). It is also a T^M-invariant affine subset of both T^{Γ_σ} and $T^{\Gamma_{\sigma'}}$, hence it is of the form T^{Γ_τ}. By Lemma 3.10 we obtain two inclusion of semigroups $\Gamma_\sigma \to \Gamma_\tau$ and $\Gamma_{\sigma'} \to \Gamma_\tau$. Since X is separated the diagonal map

$T^{\Gamma_\tau} \to T^{\Gamma_\sigma} \times T^{\Gamma_{\sigma'}}$ is a closed embedding (see Chapter 2 of[12]). Algebraically, this implies the surjectivity of the homomorphism

$$k[t^{\Gamma_\sigma}] \otimes_k k[t^{\Gamma_{\sigma'}}] \to k[t^{\Gamma_\tau}] \text{ , determined by } t^\gamma \otimes t^{\gamma'} \mapsto t^{\gamma+\gamma'}.$$

It follows that the homomorphism of semigroups $\Gamma_\sigma \times \Gamma_{\sigma'} \to \Gamma_\tau$, $(\gamma, \gamma') \mapsto \gamma + \gamma'$ is surjective. This proves that $\Gamma_\tau = \Gamma_\sigma + \Gamma_{\sigma'}$ thus

$$\mathbf{R}_{\geq 0}\Gamma_\tau = \check{\tau} = \mathbf{R}_{\geq 0}(\Gamma_\sigma + \Gamma_{\sigma'}) = \check{\sigma} + \check{\sigma'}$$

By duality we deduce that $\tau = \sigma \cap \sigma'$. By Proposition 3.10 we obtain

$$\Gamma_\tau = \Gamma_\sigma + M(\tau, \Gamma_\sigma) = \Gamma_{\sigma'} + M(\tau, \Gamma_{\sigma'}).$$

In conclusion, Σ is a fan in $N_{\mathbf{R}}$ and if $\Gamma := \{\Gamma_\sigma \mid \sigma \in \Sigma\}$ the triple (N, Σ, Γ) verifies the compatibility properties of Definition 4.1 and the variety X_Σ^Γ is T^M-equivariantly isomorphic to X. □

The following corollary is consequence of Proposition 6.2 and Theorem 7.5.

Corollary 7.6. *The category with objects the triples (N, Σ, Γ) of Definition 4.1 and morphisms those maps of fans with attached semigroups of Definition 6.1 is equivalent to the category with objets the toric varieties of Definition 7.4 and morphisms those equivariant morphism which extend morphisms of the corresponding algebraic tori; see Proposition 6.2.*

8. Invertible sheaves on toric varieties

In this section we describe how some of the classical results in the study of invariant invertible sheaves on a normal toric variety extend to the general case.

Let T_Σ^Γ denote a toric variety defined by the triple (N, Σ, Γ). Recall that if $\sigma \in \Sigma$ we denote by $T_\sigma = T^{\check{\sigma}\cap M}$ the normalization of the chart T^{Γ_σ} and by T_Σ the normalization of T_Σ^Γ.

A *support function* $h : |\Sigma| \to \mathbf{R}$ is a continuous function such that for each $\sigma \in \Sigma$ the restriction $h_{|\sigma} \colon \sigma \to \mathbf{R}$ is linear. We say that h is *integral* with respect to N if $h(|\Sigma| \cap N) \subset \mathbf{Z}$. We denote by $\mathrm{SF}(N, \Sigma)$ the set of support functions integral with respect to N. If h is a support function integral with respect to N then for any $\sigma \in \Sigma$ there exists $m_\sigma \in M$ such that

$$h(\nu) = \langle \nu, m_\sigma \rangle, \quad \text{for all } \nu \in \sigma.$$

Notice that by continuity we have that

$$m_\tau = m_\sigma \mod M(\tau) = M \cap \tau^\perp, \text{ for } \tau \leq \sigma, \sigma \in \Sigma. \tag{5}$$

The set $\{m_\sigma \mid \sigma \in \Sigma\}$ determines h but may not be uniquely determined.

Definition 8.1. A support function for the triple (N, Σ, Γ) is a support function $h : |\Sigma| \to \mathbf{R}$ integral with respect to N which in addition verifies the compatibility property

$$m_\tau = m_\sigma \mod M(\tau, \Gamma_\tau), \text{ for } \tau \leq \sigma, \sigma \in \Sigma. \tag{6}$$

We denote by $\mathrm{SF}(N, \Sigma, \Gamma)$ the additive group of support functions for the triple (N, Σ, Γ). It is a subgroup of $\mathrm{SF}(N, \Sigma)$. A vector $m \in M$ defines an element of $\mathrm{SF}(N, \Sigma, \Gamma)$ hence we have a homomorphism $M \to \mathrm{SF}(N, \Sigma, \Gamma)$, which is injective if the support of Σ spans $N_{\mathbf{R}}$ as a real vector space.

Any $h \in \mathrm{SF}(N, \Sigma)$ determines T^M-invariant *Cartier divisor* D_h on T_Σ by

$$D_{h|T_\sigma} = \mathrm{div}(t^{-m_\sigma}) \text{ for } \sigma \in \Sigma, \tag{7}$$

where $\mathrm{div}(g)$ denotes the *principal Cartier divisor* of the rational function g on an irreducible variety. Notice that D_h is independent of the possible choices of different

Cartier data $\{m_\sigma \mid \sigma \in \Sigma\}$ defining h. If $\sigma, \sigma' \in \Sigma$, $\tau = \sigma \cap \sigma'$ then $T_\tau = T_\sigma \cap T_{\sigma'}$ and (5) guarantees that $t^{-m_\sigma + m_{\sigma'}}$ and $t^{m_\sigma - m_{\sigma'}}$ are both regular functions on T_τ. Any T^M-invariant Cartier divisor on T_Σ is of the form D_h for $h \in \mathrm{SF}(N, \Sigma)$, i.e., it is defined by Cartier data.

Lemma 8.2. *If $h \in SF(N, \Sigma)$ is defined by the Cartier data $\{m_\sigma \mid \sigma \in \Sigma\}$ then it defines a T^M-invariant Cartier divisor on T^Γ_Σ if and only if (6) holds, that is, if and only if $h \in SF(N, \Sigma, \Gamma)$.*

Proof. The condition to determine a Cartier divisor is that for any $\sigma, \sigma' \in \Sigma$, $\tau = \sigma \cap \sigma'$ the transition function $t^{-m_\sigma + m_{\sigma'}}$ is an invertible regular function on $T^{\Gamma_\tau} = T^{\Gamma_\sigma} \cap T^{\Gamma_{\sigma'}}$. By Lemma 3.9 this is equivalent to (6). □

We have shown that the group $\mathrm{CDiv}_{T^M}(T^\Gamma_\Sigma)$ of T^M-invariant Cartier divisors on T^Γ_Σ can be seen as a subset of $\mathrm{CDiv}_{T^M}(T_\Sigma)$. The set $\{\mathrm{div}(t^m)\}_{m \in M}$ is a subgroup of $\mathrm{CDiv}_{T^M}(T^\Gamma_\Sigma)$ consisting of principal Cartier divisors.

The map

$$\mathrm{SF}(N, \Sigma, \Gamma) \longrightarrow \mathrm{CDiv}_{T^M}(T^\Gamma_\Sigma), \quad h \mapsto D_h.$$

is a group isomorphism. The inverse map sends a Cartier divisor D on T^Γ_Σ, given by the Cartier data $\{m_\sigma \mid \sigma \in \Sigma\}$, to the function

$$h_D := |\Sigma| \to \mathbf{R}, \quad h_D(\nu) = \langle \nu, m_\sigma \rangle \text{ if } \nu \in \sigma.$$

A Cartier divisor on T_Σ determines an *invertible sheaf* $\mathcal{O}_{T_\Sigma}(D)$. If U is an affine open set in which $D = \mathrm{div}(g_U)$ for some rational function g_U then the set of sections $H^0(U, \mathcal{O}_{T_\Sigma}(D))$ consists of those rational functions f which verify that $f g_U$ is a regular function on U.

We denote by $\mathcal{O}_{T^\Gamma_\Sigma}$ the structure sheaf on the toric variety T^Γ_Σ. The *invertible sheaf* of a T^M-invariant Cartier divisor D on T^Γ_Σ is the sheaf of $\mathcal{O}_{T^\Gamma_\Sigma}$-modules $\mathcal{O}_{T^\Gamma_\Sigma}(D)$. By (7) the set of sections of this sheaf on T^{Γ_σ} is

$$H^0(T^{\Gamma_\sigma}, \mathcal{O}_{T^\Gamma_\Sigma}(D)) = t^{m_\sigma} k[t^{\Gamma_\sigma}]. \tag{8}$$

We denote by P^Γ_Σ the following subset of M:

$$P^\Gamma_D := \bigcap_{\sigma \in \Sigma} m_\sigma + \Gamma_\sigma. \tag{9}$$

The set of global sections of the sheaf $\mathcal{O}_{T^\Gamma_\Sigma}$ is equal to

$$H^0(T^\Gamma_\Sigma, \mathcal{O}_{T^\Gamma_\Sigma}(D)) = \bigcap_{\sigma \in \Sigma} t^{m_\sigma} k[t^{\Gamma_\sigma}] = \bigoplus_{m \in P^\Gamma_D} k t^m. \tag{10}$$

Remark 8.3. As in the normal case, a T^M-invariant Cartier divisor D defines an equivariant line bundle $\mathcal{L}_D$ whose sections coincide with those of the invertible sheaf $\mathcal{O}_{T^\Gamma_\Sigma}(D)$. See [19], Chapter 2.

The *Picard group* $\mathrm{Pic}(X)$ of a variety X consists of the isomorphism classes of invertible sheaves in X.

Lemma 8.4. *Suppose that $|\Sigma| = N_\mathbf{R}$. For any Cartier divisor D on the toric variety T^Γ_Σ we have an $\mathcal{O}_{T^\Gamma_\Sigma}$-module isomorphism $\mathcal{O}_{T^\Gamma_\Sigma}(D) \cong \mathcal{O}_{T^\Gamma_\Sigma}(D_h)$ for some $h \in \mathrm{SF}(N, \Sigma, \Gamma)$. The following are equivalent for $h \in \mathrm{SF}(N, \Sigma, \Gamma)$.*

i. *$h \in M$*
ii. *D_h is a principal Cartier divisor.*
iii. *$\mathcal{L}_{D_h}$ is a trivial line bundle.*

iv. *The sheaf* $\mathcal{O}_{T^\Gamma_\Sigma}(D_h)$ *is isomorphic to* $\mathcal{O}_{T^\Gamma_\Sigma}$ *as* $\mathcal{O}_{T^\Gamma_\Sigma}$*-module.*

Proof. See Proposition 2.4 of [19]. □

Proposition 8.5. *Suppose that* $|\Sigma| = N_{\mathbf{R}}$*. Then we have canonical isomorphisms*

$$\mathrm{SF}(N,\Sigma,\Gamma)/M \to \mathrm{Pic}(T^\Gamma_\Sigma) \to \mathrm{CDiv}_{T^M}(T^\Gamma_\Sigma)/\{\mathrm{div}(t^m)\}_{m\in M},$$

from which we deduce a canonical injection $\mathrm{Pic}(T^\Gamma_\Sigma) \to \mathrm{Pic}(T_\Sigma)$*.*

Proof. This follows by using the same arguments as in Corollary 2.5 [19]. □

If ρ belongs to the 1-skeleton $\Sigma(1)$ of the fan Σ we denote by ν_ρ the primitive integral vector for the lattice N in the ray ρ, that is the generator of the semigroup $\rho \cap N$. We associate to $h \in \mathrm{SF}(N,\Sigma)$ the polyhedron

$$P_h := \{m \in M_{\mathbf{R}} \mid \langle \nu_\rho, m\rangle \geq h(\nu_\rho),\ \rho \in \Sigma(1)\}. \tag{11}$$

Recall that

$$P_{lh} = lP_h \text{ and } P_h + P_h = P_{h+h'} \tag{12}$$

for any integer $l \geq 1$ and $h, h' \in \mathrm{SF}(N,\Sigma)$.

Proposition 8.6. *Suppose that* $|\Sigma| = N_{\mathbf{R}}$*. The following are equivalent for* $h \in$ *SF*(N,Σ,Γ) *defining a Cartier divisor* $D = D_h$*.*

i. *The* $\mathcal{O}_{T^\Gamma_\Sigma}$*-module* $\mathcal{O}_{T^\Gamma_\Sigma}(D)$ *is generated by its global sections.*
ii. h *is upper convex, i.e.,* $h(\nu) + h(\nu') \leq h(\nu + \nu')$ *for all* $\nu, \nu' \in N_{\mathbf{R}}$*.*
iii. *The polytope* P_h *has vertices* $\{m_\sigma \mid \sigma \in \Sigma\}$*.*

If these conditions hold the convex hull of the set P^Γ_D *is the polytope* P_h *and* h *is the support function of the polytope* P_h*.*

Proof. The proof follows as in the normal case (see Theorem 2.7 [19]). □

If $|\Sigma| = N_{\mathbf{R}}$ the support function $h \in \mathrm{SF}(N,\Sigma,\Gamma)$, defined by the Cartier data $\{m_\sigma \mid \sigma \in \Sigma\}$, is *strictly upper convex* if it is upper convex and in addition

$$h(\nu) = \langle \nu, m_\sigma\rangle \text{ if and only if } \nu \in \sigma \text{ , for } \sigma \in \Sigma.$$

Suppose that $h \in \mathrm{SF}(N,\Sigma,\Gamma)$ verifies the equivalent conditions of Proposition 8.6. Set $D = D_h$. If $P^\Gamma_D = \{u_1,\dots,u_s\}$ we have a morphism

$$\Phi_D \colon T^\Gamma_\Sigma \longrightarrow \mathbf{P}^{s-1}_k, \quad \Phi_D = (t^{u_1} : \cdots : t^{u_s}). \tag{13}$$

(defined in homogeneous coordinates of $\mathbf{P}^{s-1}_k$). The morphism Φ_D is equivariant with respect the map of tori $\Phi_{|T^M} \colon T^M \to T^{M'}$, where $T^{M'}$ denotes the torus of $\mathbf{P}^{s-1}_k$ with respect to the fixed coordinates.

Proposition 8.7. *Suppose that* $|\Sigma| = N_{\mathbf{R}}$*. The following are equivalent for* $h \in$ *SF*(N,Σ,Γ) *defining a Cartier divisor* $D = D_h$*.*

i. D *is very ample.*
ii. h *is strictly upper convex and for all* $\sigma \in \Sigma(d)$ *the set* $\{m - m_\sigma \mid m \in P^\Gamma_D\}$ *generates the semigroup* Γ_σ*.*

Proof. Suppose that h is not strictly upper convex. Then there exists d-dimensional cones $\sigma, \sigma' \in \Sigma$ such that $\tau = \sigma \cap \sigma'$ is of dimension $d-1$ and $m_\sigma = m_{\sigma'}$. This implies that the section defined by t^{m_σ} in the open set $U = T^{\Gamma_\sigma} \cup T^{\Gamma_{\sigma'}}$ is no where vanishing.

By definition there exists $1 \leq i \leq s$ such that $m_\sigma = u_i$.

The restriction of Φ_D to U factors through the affine open set $\mathbf{C}^{s-1}$, where the i-th homogeneous coordinate does not vanish. It is of the form:

$$\Phi_{|U} : U \to \mathbf{C}^{s-1}, \text{ with } \Phi_{|U} = (t^{u_1 - m_\sigma}, \dots, t^{u_{i-1}-m_\sigma}, t^{u_{i+1}-m_\sigma}, \dots, t^{u_s - m_\sigma}).$$

By Lemma 3.5 the closure of the orbit $\mathrm{orb}(\tau,\Gamma)$ is a complete one-dimensional toric variety contained in U. The restriction $\Phi_{|\overline{\mathrm{orb}(\tau,\Gamma)}}$ must be constant hence Φ is not an embedding. This implies that if D_h is very ample h is strictly upper convex.

Suppose that h is strictly upper convex. If $\sigma \in \Sigma$ is a d-dimensional cone then m_σ belongs to $\{u_i\}_{i=1}^s$, say $m_\sigma = u_s$. The restriction of Φ to T^{Γ_σ} factors though the affine open set of $\mathbf{P}_k^{s-1}$ where the last homogeneous coordinate does not vanish. It is described algebraically by the homomorphism of k-algebras:

$$k[y_1,\dots,y_{s-1}] \to k[t^\Gamma], \quad y_i \mapsto t^{u_i - m_\sigma},\ i = 1,\dots,s-1.$$

This maps defines a closed immersion if and only if it is surjective. This happens if and only if the set of vectors $\{u_i - m_\sigma\}_{1\le i\le s-1}$ generate the semigroup Γ_σ. □

Proposition 8.8. *Suppose that $|\Sigma| = N_{\mathbf{R}}$. The following are equivalent for $h \in SF(N,\Sigma,\Gamma)$.*

i. *D_h is ample*
ii. *h is strictly upper convex.*

Proof. If D is ample then lD is very ample for $l \gg 0$. Since $lD = D_{lh}$ it follows that h is strictly upper convex if lh is and the assertion holds by Proposition 8.7.

Conversely suppose that h is strictly upper convex. We prove that lD_h is very ample for $l \gg 0$. By Proposition 8.7 it is sufficient to prove that there exists an integer $l \gg 0$ such that for each d-dimensional cone $\sigma \in \Sigma$ the semigroup Γ_σ is generated by $\{m - lm_\sigma \mid m \in P^\Gamma_{D_{lh}}\}$.

If $\sigma' \in \Sigma$, $\dim \sigma' = d$, $\tau = \sigma' \cap \sigma$ we have that $\Gamma_\tau = \Gamma_\sigma + \mathbf{Z}_{\ge 0}(-u)$ for any $u \in \Gamma_\sigma$ in the relative interior of the cone $\tau^\perp \cap \check\sigma$ (see Lemma 3.9). For instance we take $u = m_{\sigma'} - m_\sigma$. We obtain similarly that $\Gamma_\tau = \Gamma_{\sigma'} + \mathbf{Z}_{\ge 0}(u)$.

If $\gamma \in \Gamma_\sigma$ then γ belongs to Γ_τ and there exists $\gamma' \in \Gamma_{\sigma'}$ and an integer $p \ge 0$ such that $\gamma = \gamma' + pu$. If $l \ge p$ we obtain:

$$lm_{\sigma'} + \gamma' + (l-p)(m_\sigma - m_{\sigma'}) = lm_\sigma + \gamma. \tag{14}$$

If l is big enough, a formula of the form (14) holds for any γ in a finite set G_σ of generators of Γ_σ (where p and γ' vary with γ) and for any cone $\sigma' \in \Sigma(d)$. Since γ' and $m_\sigma - m_{\sigma'}$ belong to $\Gamma_{\sigma'}$ this implies $t^{lm_\sigma+\gamma}$ defines a section in $H^0(T^{\Gamma_{\sigma'}}, \mathcal{O}_{T^\Gamma_\Sigma}(D_{lh}))$ (see (8)) for any cone $\sigma' \in \Sigma(d)$. We deduce that for any $\gamma \in G_\sigma$ the vector $lm_\sigma + \gamma$ belongs to the set $P^\Gamma_{D_{lh}}$ and $t^{lm_\sigma+\gamma}$ defines a global section of $\mathcal{O}_{T^\Gamma_\Sigma}(D_{lh})$. □

Remark 8.9. Let $\mathcal{A} = \{u_1,\dots,u_s\}$ be a subset of a lattice M such that $\mathbf{Z}\mathcal{A} = M$, i.e., $\mathcal{A}$ spans M as a lattice. Gel′fand, Kapranov, and Zelevinsky [7] define a projective toric variety $X_{\mathcal{A}}$ as the closure of the image of the map

$$\varphi_{\mathcal{A}} = (t^{u_1} \colon \dots \colon t^{u_s}) \colon T^M \to \mathbf{P}_k^{s-1}.$$

Let us explain how their definition fits with our notion of projective toric variety. Let P be the convex hull of $\mathcal{A}$ in $M_{\mathbf{R}}$ and Σ de dual fan of P. Each $\sigma \in \Sigma$ of maximal dimension determines a vertex m_σ of P, which is necessarily an element of $\mathcal{A}$. We associate to σ the semigroup $\Gamma_\sigma := \sum \mathbf{Z}_{\ge 0}(u_i - m_\sigma)$. If $\tau \le \sigma$ we define Γ_τ by (2). The set $\Gamma := \{\Gamma_\theta \mid \theta \in \Sigma\}$ is well-defined and the triple (N,Σ,Γ) defines a toric variety T^Γ_Σ (the argument is the same as the one used in the proof of Proposition 5.1). The support function h of P belongs to $\mathrm{SF}(N,\Sigma,\Gamma)$ and is strictly upper convex. If $D = D_h$ we deduce from the definitions that $P^\Gamma_D = \mathcal{A}$. By Proposition 8.7 the Cartier divisor D is very ample, and the morphism (13) is an equivariant embedding of T^Γ_Σ in the projective space $\mathbf{P}_k^{s-1}$ such that $(\Phi_D)_{|T^M} = \varphi_{\mathcal{A}}$. It follows that $X_{\mathcal{A}} = \Phi_D(T^\Gamma_\Sigma)$.

Remark 8.10. If $F = \sum_{i=1}^{s} c_i t^{u_i} \in k[t^M]$ is a polynomial with $c_1 \ldots c_s \neq 0$, then F defines a global section of $\mathcal{O}_{T_\Sigma^\Gamma}(D)$ such that the closure of $\{F = 0\} \cap T^M$ in T_Σ^Γ does not meet any zero-dimensional orbit of T_Σ^Γ.

Part II: Resolution of toric varieties by Semple-Nash modifications

In this Part we prove that any toric variety has a canonical resolution of singularities by iterated blowing ups of logarithmic jacobian ideals. Recall that, as shown below, if k is an algebraically closed field of characteristic zero the blowing up of the logarithmic jacobian ideal of an affine toric variety T^Γ coincides with the Semple-Nash modification. This fact, originally due to Gonzalez Sprinberg in the normal case ([9]), is our starting point.

The sequence of logarithmic jacobian blowing-ups of a toric variety T_Σ^Γ is a sequence of toric varieties $T_{\Sigma^{(i)}}^{\Gamma^{(i)}}$ defined by a sequence $\Sigma^{(i)}$ of refinements (or subdivisions) of Σ with attached families $\Gamma^{(i)}$ of semigroups and we have to show that it eventually stabilizes. In most proofs of resolution the strategy is to attach to points an invariant which takes its minimal value only for regular points and then show that it can be made to decrease by successive blowing-ups. Our strategy is different: we show that the very nature of the blowing-up of the logarithmic jacobian ideals forces cones in the successive refinements $\Sigma^{(i)}$ of Σ to stabilize for i large enough, meaning that they are not subdivided in the $\Sigma^{(j)}$ for $j \geq i$. If one can stabilize the cones of maximal dimension the logarithmic jacobian blowing-ups are finite morphisms from then on, and it is easy to show that they resolve in finitely many steps (see Proposition 12.20). This stabilization is not measured by the constancy of some local invariant. The basic idea is to show stabilization by extending it from lower-dimensional cones to higher-dimensional ones, so that if one really insists on having an invariant, it should be the maximal codimension of stable cones of $\Sigma^{(j)}$; it is at most $d-1$ since edges are stable, and if it is zero, we are essentially done.

Here is a quick description of the structure of the proof: first we study a "local" problem with respect to the monomial valuation associated to a vector $\nu \in \sigma \cap N$, where $\sigma \in \Sigma(d)$. Such a vector determines a unique cone $\theta^{(j)} \in \Sigma^{(j)}$ for all j containing ν in its relative interior, and the first observation is that this sequence $(\theta^{(j)})_{j \geq 0}$ stabilizes for $j \geq j_1$ say; the limit $\theta^{(\infty)} = \theta^{(j_1)}$ is by definition a stable cone of $\Sigma^{(j_1)}$. This implies that the chart $T_{\theta^{(\infty)}}^{\Gamma^{(j)}}$ is non singular for $j \gg 0$ (see Propositions 12.20 and 12.21).
One of the difficulties is that the logarithmic jacobian blowing-up of the variety does not induce the logarithmic jacobian blowing-up of its lower-dimensional orbit closures. Nevertheless it is possible to show that the sequence of the fans of the corresponding orbit closures in the successive blowing-ups also stabilizes (Proposition 12.33). A key point in the proof of this result is that given a stable cone η, a nested sequence of cones $\zeta^{(j)} \in \Sigma^{(j)}$ containing η as a codimension one face necessarily stabilizes (Proposition 12.24). This uses the fact that for every one dimensional orbit closure associated to a stable cone of codimension one, the effect of the ambient blowing-up is very similar that of its logarithmic jacobian blowing-up (see Claim 12.25).
This controls the proliferation of the cones containing a given a stable cone η and corresponding to "new" orbits appearing in the successive blowing-ups.
By induction on the codimension of η, this enables us to prove that given a stable cone η, the set of cones of $\Sigma^{(j)}$ which contain it stabilizes. The final step is to globalize this argument to prove the stabilization of the sequence of fans $\Sigma^{(j)}$ thanks to the compactness of the projectivization of $|\Sigma|$ (Proposition 12.28).

9. The Semple-Nash modification: preliminaries

In [20], Semple introduced the Semple-Nash modification of an algebraic variety and asked whether a finite number of iterations would resolve the singularities of the variety. The same question was apparently rediscovered by Chevalley and Nash in the 1960's, and studied notably by Nobile (see [17]), Gonzalez Springberg (see [11] and [9]), Hironaka (see [13]), and Spivakovsky (see [21]). The best consequence so far of all this work is the Theorem, due to Spivakovsky, stating that by iterating the operation consisting of the Semple-Nash modification followed by normalization one eventually resolves singularities of surfaces over an algebraically closed field of characteristic zero.

Let X be a reduced algebraic variety or analytic space, which we may assume of pure dimension d for simplicity. Whenever we speak of the Semple-Nash modification, we assume that we are working over an algebraically closed field k of characteristic zero. Consider the Grassmanian $g\colon \mathrm{Grass}_d\Omega^1_X \to X$; it is a proper algebraic map, which has the property that its fiber over a point of x is the Grassmanian of d-dimensional subspaces of the Zariski tangent space $E_{X,x}$. The map g is characterized by the fact that $g^*\Omega^1_X$ has a locally free quotient of rank d and g factorizes in a unique manner every map to X with this property. Let X^o denote the non singular part of X, which is d-dimensional and dense in X by our assumptions. Since the restriction $\Omega^1_X|X^o$ is locally free the map g has an algebraic section over X^o and the Semple-Nash modification is defined as the closure NX of the image of this section, endowed with the natural projection $n_X\colon NX \to X$ induced by g. The map n_X is proper and is an isomorphism over X^o; it is a modification. Like the Grassmanian of Ω^1_X, it is defined up to a unique X-isomorphism.

A local description can be given for a chart $X|U$ of X embedded in affine space $\mathbf{A}^N(k)$ by taking the closure in $(X|U)\times \mathbf{G}(N,d)$ of the graph of the Gauss map $\gamma\colon (X|U)^o \to \mathbf{G}(N,d)$ sending each non singular point to the class of its tangent space in the Grassmanian of d-dimensional vector subspaces in $\mathbf{A}^N(k)$. For any point $x\in X$ the fiber $n_X^{-1}(x)$ is the subset of $\mathbf{G}(N,d)$ consisting of limit positions at $x\in X$ of tangent spaces to X along sequences of non singular points tending to x. In this guise, the Semple-Nash modification appears in a complex-analytic framework in the paper [29] of Hassler Whitney in connection with equisingularity problems.

Proposition 9.1. (Nobile), see [17] and [24]. *Let X be a reduced equidimensional space; if the map*

$$n_X\colon NX \to X$$

is an isomorphism, the space X is non singular.

For the convenience of the reader, we sketch the proof found in [24]:
If the map n_X is an isomorphism, the sheaf Ω^1_X has a locally free quotient of rank d. The problem is local, so it is enough to prove that the existence of a surjective map $\phi\colon \Omega^1_{X,x} \to \mathcal{O}^d_{X,x}$ implies, in characteristic zero, that $\mathcal{O}_{X,x}$ is regular. Passing to the completion and tensoring $\Omega^1_{X,x}$ by $\hat{\mathcal{O}}_{X,x}$ we may assume that $\mathcal{O}_{X,x}$ is complete. We consider the linear map $e\colon \mathcal{O}^d_{X,x} \to \mathcal{O}_{X,x}$ sending the first basis vector to 1 and the others to 0. The composition of e with the map ϕ gives a surjective map, so that there has to be an element $h\in\mathcal{O}_{X,x}$ such that the image of dh in $\mathcal{O}_{X,x}$ by $e\circ\phi$ is equal to 1, and then the k-derivation $D\colon \mathcal{O}_{X,x}\to\mathcal{O}_{X,x}$ corresponding to $e\circ\phi$ is such that $Dh=1$. In characteristic zero one can formally integrate this non vanishing vector field using the formal expansion of $\exp(-hD)$ to get an isomorphism $\mathcal{O}_{X,x}\simeq\mathcal{O}_1[[h]]$ where $\mathcal{O}_1\simeq\mathcal{O}_{X,x}/(h)$. By construction $\mathcal{O}_1$ satisfies the same assumptions as $\mathcal{O}_{X,x}$ in one less dimension. By induction we are reduced to dimension zero, but a reduced zero

dimensional complete equicharacteristic local ring is k in our case. We refer to [24] for details, and to [17] for the original proof.

Remark 9.2. We will see below in Section 12, Proposition 12.5 the characteristic-free version of this statement, which is that if the blowing-up of the logarithmic jacobian ideal is an isomorphism, the toric variety is smooth. Note that the Semple-Nash modification is defined in any characteristic but its being an isomorphism does not imply regularity in positive characteristic; it is the case for $y^p - x^q = 0$ with $(p,q) = 1$ in characteristic p. See [17].

10. The Semple-Nash modification in the toric case

The following is an extension to the case of not necessarily normal toric varieties of a result of Gonzalez Sprinberg ([9]; a summary of this work appeared in [10]) which was revisited by Lejeune-Jalabert and Reguera in the appendix to [16].
Let X be an affine toric variety over an algebraically closed field k. Using the notations of Section 3 we write its ring

$$R = k[U_1, \dots, U_r]/P,$$

where P is a prime binomial ideal $(U^{m^\ell} - U^{n^\ell})_{\ell \in \mathbf{L}}$ of the polynomial ring $k[U_1, \dots, U_r]$. Let d be the dimension of X and denote by $\mathcal{L} \subset \mathbf{Z}^r$ the lattice generated by the differences $(m^\ell - n^\ell)_{\ell \in \mathbf{L}}$; by [E-S], it is a direct factor of $\mathbf{Z}^r$ since X is irreducible and k is algebraically closed. Setting $c = r - d$, we may identify $\mathbf{L}$ with $\{1, \dots, L\}$ with $L = |\mathbf{L}|$ in such a way that the lattice generated by $(m^1 - n^1, \dots, m^c - n^c)$ has rank c. The quotient $\mathbf{Z}^r/\mathcal{L}$ is isomorphic to $\mathbf{Z}^d$ and we have an exact sequence

$$0 \to \mathcal{L} \xrightarrow{\psi} \mathbf{Z}^r \to \mathbf{Z}^d \to 0. \tag{15}$$

Our affine toric variety X is $\mathrm{Spec} k[t^\Gamma]$, where Γ is the semigroup generated in $\mathbf{Z}^d$ by the images $\gamma_1, \dots, \gamma_r$ of the basis vectors of $\mathbf{Z}^r$.

Proposition 10.1. (Generalizing [9] [10] and [16]) *Keeping the notations just introduced, let X be an affine toric variety over an algebraically closed field of characteristic zero. The Semple-Nash modification of X is isomorphic to the blowing-up of the ideal of R generated by the images of the products $U_{i_1} \dots U_{i_d}$ such that* $\mathrm{Det}(\gamma_{i_1}, \dots, \gamma_{i_d}) \neq 0$.

Proof. A straightforward computation using logarithmic differentials shows that the jacobian determinant $J_{K,\mathbf{L}'}$ of rank $c = r - d$ of the generators $(U^{m^\ell} - U^{n^\ell})_{\ell \in \{1,\dots,L\}}$ of our prime binomial ideal $P \subset k[U_1, \dots, U_r]$, associated to a sequence $K = (k_1, \dots, k_c)$ of distinct elements of $\{1, \dots, r\}$ and a subset $\mathbf{L}' \subseteq \{1, \dots, L\}$ of cardinality c, satisfies the congruence

$$U_{k_1} \dots U_{k_c} . J_{K,\mathbf{L}'} \equiv \big(\prod_{\ell \in \mathbf{L}'} U^{m^\ell}\big) \mathrm{Det}_{K,\mathbf{L}'}\big((\langle m - n \rangle)\big) \mod . P,$$

where $\big(\langle m-n \rangle\big)$ is the matrix of the vectors $(m^\ell - n^\ell)_{\ell \in \{1,\dots,L\}}$, and $\mathrm{Det}_{K,\mathbf{L}'}$ indicates the minor in question. By Lemma 6.3 of [25], the rank of the image in $k^{r \times L}$ of the matrix $\big(\langle m - n \rangle\big)$ is equal to c. Now we know that we may assume that the lattice generated by $(m^1 - n^1, \dots, m^c - n^c)$ has rank c, so that the binomial variety corresponding to the ideal $(U^{m^\ell} - U^{n^\ell})_{1 \leq \ell \leq c}$ is a complete intersection X_1 of dimension $r - c$ containing our binomial variety X. By [17], the Nash modification of X is isomorphic to the blowing up of the restriction to X of the jacobian ideal of X_1. Therefore the Semple-Nash modification of our binomial variety coincides with the blowing-up in X of the ideal generated as $K = (k_1, \dots, k_c)$ runs through the sets of c distinct elements of $(1, \dots, r)$ by the elements $J_{K,\mathbf{L}_0}$ satisfying the congruences

$$U_{k_1} \dots U_{k_c}.J_{K,\mathbf{L}_0} \equiv \big(\prod_{\ell \in \mathbf{L}_0} U^{m^\ell}\big)\mathrm{Det}_{K,\mathbf{L}_0}\big((\langle m-n \rangle)\big) \mod. P,$$

where $\mathbf{L}_0 = (1, \dots, c)$ and with the necessity that $J_{K,\mathbf{L}_0} = 0$ whenever the determinant on the right side is zero.

Now for each K let us multiply both sides by $U_{i_1} \dots U_{i_d}$, where $I = (i_1, \dots, i_d) = \{1, \dots, r\} \setminus K$. We obtain for each K the equality:

$$U_1 \dots U_r.J_{K,\mathbf{L}_0} \equiv U_{i_1} \dots U_{i_d}\big(\prod_{\ell \in \mathbf{L}_0} U^{m^\ell}\big)\mathrm{Det}_{K,\mathbf{L}_0}\big((\langle m-n \rangle)\big) \mod. P. \tag{16}$$

Taking exterior powers for the map ψ in the sequence (15) gives an injection

$$0 \to \bigwedge^{r-d} \mathcal{L} \xrightarrow{\bigwedge^{r-d} \psi} \bigwedge^{r-d} \mathbf{Z}^r$$

whose image is a primitive vector in $\bigwedge^{r-d} \mathbf{Z}^r$ since it is a direct factor.
Let $\mathcal{L}_0 \subset \mathcal{L}$ be the lattice generated by the differences $(m^1 - n^1, \dots, m^c - n^c)$, that is, corresponding to the first c binomial equations. The image of its $(r-d)$-th exterior power is a non zero multiple of the primitive vector $\bigwedge^{r-d} \mathcal{L}$; all the $c \times c$ minors of the matrix $\big(\langle m-n \rangle\big)$ involving vectors $m^\ell - n^\ell$ with $\ell > c$ are rationally dependent upon those which do not. Consider now the d-th exterior power of the map dual to the surjection $\mathbf{Z}^r \to \mathbf{Z}^d \to 0$ of (15):

$$0 \to \bigwedge^{d} \check{\mathbf{Z}}^d \to \bigwedge^{d} \check{\mathbf{Z}}^r.$$

The image of $\bigwedge^{d} \check{\mathbf{Z}}^d$ is a primitive vector in $\bigwedge^{d} \check{\mathbf{Z}}^r$.
By the natural duality isomorphism between $\bigwedge^{d} \check{\mathbf{Z}}^r$ and $\bigwedge^{r-d} \mathbf{Z}^r$ (see [2] §11, No. 11, Prop. 12) deduced from the pairings

$$\bigwedge^{d} \check{\mathbf{Z}}^r \otimes \bigwedge^{d} \mathbf{Z}^r \to \mathbf{Z}, \quad \bigwedge^{d} \mathbf{Z}^r \otimes \bigwedge^{r-d} \mathbf{Z}^r \to \mathbf{Z},$$

this vector correspond to the image of $\bigwedge^{r-d} \mathcal{L}$ in such a way that the coordinate which corresponds to the determinant of the vectors $\gamma_{i_1}, \dots, \gamma_{i_d}$ in $\mathbf{Z}^d$ is a rational multiple of the determinant $\mathrm{Det}_{K,\mathbf{L}_0}\big((\langle m-n \rangle)\big)$, which is non zero since our base field is of characteristic zero.

Equation 16 now shows that the ideal of R generated by the $J_{K,\mathbf{L}_0}$ differs from the ideal generated by the images of the products $U_{i_1} \dots U_{i_d}$ such that $\mathrm{Det}(\gamma_{i_1}, \dots, \gamma_{i_d}) \neq 0$ only by the product by invertible ideals, so that these two ideals determine isomorphic blowing ups, which proves the Proposition. □

Remark 10.2. The proof found in [LJ-R] is valid in the non-normal case; the proof given here makes explicit the connection of the logarithmic jacobian ideal with the usual one.

Remark 10.3. The isomorphism of Proposition 1.8 carries the logarithmic jacobian ideal of $k[t^{\Gamma \times \Gamma'}]$ onto the tensor product of the logarithmic jacobian ideals of the factors.

Remark 10.4. In the one-dimensional case the logarithmic jacobian ideal is the maximal ideal corresponding to the closed orbit. It is a classical fact that iterating the blowing-up of the singular point resolves the singularities of any branch.

11. The sheaf of logarithmic jacobian ideals on a toric variety

Let the pair (Σ, Γ) define a toric variety T_Σ^Γ as in Definition 4.1.

On the affine open set T^{Γ_σ}, $\sigma \in \Sigma$ we consider the ideal $\mathcal{J}_\sigma$ of $k[t^{\Gamma_\sigma}]$ generated by monomials of the form t^α, where α belongs to the set

$$|\mathcal{J}_\sigma| = \{\alpha_1 + \cdots + \alpha_d \mid \alpha_1, \ldots, \alpha_d \in \Gamma_\sigma \text{ and } \alpha_1 \wedge \cdots \wedge \alpha_d \neq 0\}.$$

The ideal $\mathcal{J}_\sigma$ is called the *logarithmic jacobian ideal of* T^{Γ_σ}.

Remark 11.1. If $\gamma_1, \ldots, \gamma_r$ are generators of Γ_σ the semigroup Γ_σ then the monomials t^α, for α in

$$\{\gamma_{i_1} + \cdots + \gamma_{i_d} \mid \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_d} \neq 0,\ 1 \leq i_1, \ldots, i_d \leq r\}, \tag{17}$$

generate the ideal $\mathcal{J}_\sigma$. Abusing notation we denote the set (17) with the same letter $\mathcal{J}_\sigma$, whenever the set of generators of Γ_σ is clear from the context.

Proposition 11.2. *The family $\{\mathcal{J}_\sigma \mid \sigma \in \Sigma\}$ defines a T^M-invariant sheaf of ideals $\mathcal{J}$ on T_Σ^Γ, which is called the sheaf of logarithmic jacobian ideals of T_Σ^Γ.*

Proof. It is sufficient to check that if $\tau \leq \sigma$, $\sigma \in \Sigma$, then the ideal $\mathcal{J}_\tau$ coincides with the extension $\mathcal{J}_\sigma k[t^{\Gamma_\tau}]$, induced by the inclusion $k[t^{\Gamma_\sigma}] \hookrightarrow k[t^{\Gamma_\tau}]$ defined by $\Gamma_\sigma \subset \Gamma_\tau$.

By Lemma 3.9 if $m \in \Gamma_\sigma$ belongs to the relative interior of the cone $\check{\sigma} \cap \tau^\perp$ then we have that $\Gamma_\tau = \Gamma_\sigma + \mathbf{Z}_{\geq 0}(-m)$ (such a vector m always exists).

If $\gamma_1, \ldots, \gamma_r$ are generators of Γ_σ then $\gamma_1, \ldots, \gamma_r, -m$ are generators of Γ_τ. This implies the inclusion $\mathcal{J}_\sigma \subset \mathcal{J}_\tau$. By Remark 11.1 an exponent α in $\mathcal{J}_\tau$ which does not belong to the set $\mathcal{J}_\sigma$ is of the form: $\alpha = \gamma_{i_1} + \cdots + \gamma_{i_{d-1}} - m$, with $\gamma_{i_1} \wedge \cdots \wedge \gamma_{i_{d-1}} \wedge (-m) \neq 0$. Then, the element $\beta := \gamma_{i_1} + \cdots + \gamma_{i_{d-1}} + m$ belongs to $\mathcal{J}_\sigma$ and we obtain that: $t^\alpha = t^{-2m} t^\beta \in \mathcal{J}_\sigma k[t^{\Gamma_\tau}]$, and $\mathcal{J}_\sigma k[t^{\Gamma_\tau}] = \mathcal{J}_\tau$. ☐

Lemma 11.3. *There is a continuous piecewise linear function* $\mathrm{ord}_{\mathcal{J}} \colon |\Sigma| \to \mathbf{R}$ *such that for each $\tau \in \Sigma$ the function* $\mathrm{ord}_{\mathcal{J}_\tau}$ *is the restriction of* $\mathrm{ord}_{\mathcal{J}}$ *to τ.*

Proof. This follows from the definition of $\mathrm{ord}_{\mathcal{J}_\sigma}$ (see (3)), by using that $\mathcal{J}$ is a sheaf of monomial ideals.

Remark 11.4. Note that Lemma 11.3 holds more generally if we replace $\mathcal{J}$ by any sheaf of monomial ideals $\mathcal{I}$ on T_Σ^Γ.

The will need the following lemma in Section 12.

Lemma 11.5. *Let $\theta^\perp \cap \Gamma$ be a face of the finitely generated semigroup $\Gamma \subset M$. The logarithmic jacobian ideal $\tilde{\mathcal{J}}$ of the image $\tilde{\Gamma}$ of Γ in the lattice $M/M(\theta)$ is equal to the image of the logarithmic jacobian ideal $\mathcal{J}$ of Γ.*

Proof Let us denote by $\tilde{\gamma}_i$ the images in $M/M(\theta)$ of the generators γ_i of Γ and by p the rank of the lattice $M/M(\theta)$. If $\tilde{\gamma}_{i_1}, \ldots, \tilde{\gamma}_{i_p}$ are linearly independent in $M/M(\theta)$, then $\gamma_{i_1}, \ldots, \gamma_{i_p}$ must be linearly independent from $\theta^\perp$. Remark that since $\theta^\perp \cap \Gamma$ is a face it must contain $d-p$ generators of Γ which are linearly independent, since $\theta^\perp \cap \Gamma$ spans the rank $d-p$ lattice $M(\theta, \Gamma)$. Choosing linearly independent generators $\gamma_{i_{p+1}}, \ldots, \gamma_{i_d} \in \theta^\perp \cap \Gamma$ of Γ gives us a generator $\gamma_{i_1} + \cdots + \gamma_{i_p} + \gamma_{i_{p+1}} + \cdots + \gamma_{i_d}$ of $\mathcal{J}$ whose image is $\tilde{\gamma}_{i_1} + \cdots + \tilde{\gamma}_{i_p}$, showing that $\tilde{\mathcal{J}}$ is contained in the image of $\mathcal{J}$. If we now take d independent generators $\gamma_{i_1}, \ldots, \gamma_{i_p}, \gamma_{i_{p+1}}, \ldots, \gamma_{i_d}$ of Γ, since they generate M, there must exist $d-p$ independent elements in their images, say $\tilde{\gamma}_{i_1}, \ldots, \tilde{\gamma}_{i_p}$. Then the image $\tilde{\gamma}_{i_1} + \cdots + \tilde{\gamma}_{i_p} + \tilde{\gamma}_{i_{p+1}} + \cdots + \tilde{\gamma}_{i_d}$ belongs to the logarithmic jacobian ideal $\tilde{\mathcal{J}}$, which shows that the image of $\mathcal{J}$ is equal to $\tilde{\mathcal{J}}$. ☐

12. Iterating the blowing-up of the logarithmic jacobian ideal

Let $\Gamma \subset M$ a finitely generated subsemigroup of a rank d lattice M such that $\mathbf{Z}\Gamma = M$. We assume in addition that the convex rational cone $\check{\sigma} := \mathbf{R}_{\geq 0}\Gamma$, which is d-dimensional since $\mathbf{Z}\Gamma = M$, is strictly convex, which is equivalent to saying that the dual cone $\sigma \subset N_{\mathbf{R}}$ is strictly convex of dimension d. The semigroup Γ determines the affine toric variety $T^\Gamma = \operatorname{Spec} k[t^\Gamma]$. We fix a finite set of generators $\gamma_1, \ldots, \gamma_r$ of Γ. We consider the set

$$\mathcal{J} := \{\gamma_{i_1} + \cdots + \gamma_{i_d} \mid \gamma_{i_1} \wedge \ldots \wedge \gamma_{i_d} \neq 0,\ 1 \leq i_1, \ldots, i_r \leq r\}$$

defining the logarithmic jacobian ideal of T^Γ.

The Newton polyhedron $\mathcal{N}_\sigma(\mathcal{J})$ of the monomial ideal $\mathcal{J}$ (see Section 5), is contained in the interior of $\check{\sigma}$, since the elements of $\mathcal{J}$ are sums of d-linearly independent elements in the d-dimensional cone $\check{\sigma}$. The set $\mathcal{J}$ determines the *order function* defined by (3). The maximal cones $\tau \subset \sigma$ of linearity of the function $\operatorname{ord}_{\mathcal{J}}$ form the d-skeleton of a fan Σ supported on σ. The map

$$\tau \mapsto m \text{ if } \operatorname{ord}_{\mathcal{J}}(\nu) = \langle \nu, m\rangle \text{ for all } \nu \in \tau. \tag{18}$$

is a bijection between the set $\Sigma(d)$ of d-dimensional cones of Σ and the set of vertices of the polyhedron $\mathcal{N}_\sigma(\mathcal{J})$.

We now consider the blowing up of the monomial ideal $\mathcal{J}$. A cone $\tau^{(1)} \in \Sigma(d)$ determines a vertex $m^{(1)}$ of $\mathcal{N}_\sigma(\mathcal{J})$ by (18) and also the finitely generated semigroup

$$\Gamma^{(2)}_{\tau^{(1)}} := \Gamma + \sum_{m \in \mathcal{J}} \mathbf{Z}_{\geq 0}(m - m^{(1)}) \subset \check{\tau}^{(1)} \cap M.$$

In view of the description recalled above of $\Sigma(d)$ in terms of $\mathcal{N}_\sigma(\mathcal{J})$, the cone $\mathbf{R}_{\geq 0}\Gamma^{(2)}_{\tau^{(1)}}$ is $\check{\tau}^{(1)}$. The affine toric variety $T^{\Gamma^{(2)}_{\tau^{(1)}}}$ is a chart of the blowing up of $\mathcal{J}$ and this toric variety is covered by charts of this form (see Section 5).

The semigroup $\Gamma^{(2)}_{\tau^{(1)}}$ is generated by $\{\gamma_1, \ldots, \gamma_r\} \cup \{m - m^{(1)}\}_{m \in \mathcal{J}}$. We denote also by $\mathcal{J}^{(2)}_{\tau^{(1)}}$ the finite subset of $\Gamma^{(2)}_{\tau^{(1)}}$ corresponding to the monomials generating the logarithmic jacobian ideal of $T^{\Gamma^{(2)}_{\tau^{(1)}}}$, by the same symbol this last ideal of $k[t^{\Gamma^{(2)}_{\tau^{(1)}}}]$, and by $\operatorname{ord}_{\mathcal{J}^{(2)}_{\tau^{(1)}}} : \tau^{(1)} \to \mathbf{R}$ the corresponding order function.

Remark 12.1. On the chart $T^{\Gamma^{(2)}_{\tau^{(1)}}}$ the pull back of the ideal $\mathcal{J}$ by the blowing up of $\mathcal{J}$ is the principal ideal $t^{m^{(1)}} k[t^{\Gamma^{(2)}_{\tau^{(1)}}}] = t^{\mathcal{J}} k[t^{\Gamma^{(2)}_{\tau^{(1)}}}]$. The Newton polyhedron

$$\mathcal{N}_{\tau^{(1)}}(\mathcal{J}) := \mathcal{J} + \check{\tau}^{(1)} = m^{(1)} + \check{\tau}^{(1)}$$

of $t^{\mathcal{J}} k[t^{\Gamma^{(2)}_{\tau^{(1)}}}]$ is *principal*, i.e., it has only one vertex $m^{(1)}$.

Lemma 12.2. *There is a continuous piecewise linear function* $\operatorname{ord}_{\mathcal{J}^{(2)}} \colon \sigma \to \mathbf{R}$ *such that for each* $\tau^{(1)} \in \Sigma(d)$ *the function* $\operatorname{ord}_{\mathcal{J}^{(2)}_{\tau^{(1)}}}$ *is the restriction of* $\operatorname{ord}_{\mathcal{J}^{(2)}}$ *to* $\tau^{(1)}$.

Proof. This follows from Lemma 11.3. □

As above, the maximal cones $\tau \subset \sigma$ of linearity of the function $\operatorname{ord}_{\mathcal{J}^{(2)}}$ form the d-skeleton of a fan $\Sigma^{(2)}$ supported on σ and subdividing the fan Σ. In particular, if $\tau^{(2)} \in \Sigma^{(2)}(d)$ is contained in $\tau^{(1)} \in \Sigma(d)$ then we denote by $m^{(2)}$ the vertex of the Newton polyhedron $\mathcal{N}_{\tau^{(1)}}(\mathcal{J}^{(2)}_{\tau^{(1)}})$ of $\mathcal{J}^{(2)}_{\tau^{(1)}}$ such that

$$\operatorname{ord}_{\mathcal{J}^{(2)}}(\nu) = \langle \nu, m^{(2)}\rangle \text{ for all } \nu \in \tau^{(2)}.$$

By iterating this construction we obtain a sequence of piecewise linear functions $\mathrm{ord}_{\mathcal{J}^{(j)}}$ on σ, together with the corresponding fans $\Sigma^{(j)}$, with $\mathcal{J} = \mathcal{J}^{(1)}$ and $\Sigma^{(1)} = \Sigma$, and such that $\Sigma^{(j)}$ is a subdivision of $\Sigma^{(j-1)}$ for all $j \geq 2$.

By definition a cone $\tau^{(j)} \in \Sigma^{(j)}(d)$ is contained in a unique cone $\tau^{(l)} \in \Sigma^{(l)}(d)$, for $0 \leq l \leq j-1$, where we set $\tau^{(0)} := \sigma$. Then we have unique vectors $m^{(l)} \in M$ such that

$$\mathrm{ord}_{\mathcal{J}^{(l)}}(\nu) = \langle \nu, m^{(l)} \rangle \text{ for all } \nu \in \tau^{(j)} \text{ and } 1 \leq l \leq j.$$

The cone $\tau^{(j)}$ corresponds to a chart of the blowing up of the logarithmic jacobian ideal $\mathcal{J}^{(j)}_{\tau^{(j-1)}}$ of $k[t^{\Gamma^{(j)}_{\tau^{(j-1)}}}]$. This chart is the affine toric variety defined by the semigroup

$$\Gamma^{(j+1)}_{\tau^{(j)}} = \Gamma^{(j)}_{\tau^{(j-1)}} + \sum_{m \in \mathcal{J}^{(j)}_{\tau^{(j-1)}}} \mathbf{Z}_{\geq 0}(m - m^{(j)}).$$

By induction this procedure also provides a system of generators of each semigroup $\Gamma^{j+1}_{\tau^{(j)}}$. We use also the notation $\mathcal{J}^{(j+1)}_{\tau^{(j)}}$ to refer to the finite set of generators of the logarithmic jacobian ideal of $k[t^{\Gamma^{(j+1)}_{\tau^{(j)}}}]$ (see Remark 11.1). The following inclusions, for $j \geq 2$, are consequence of the definitions:

$$\Gamma^{(j)}_{\tau^{(j-1)}} \subset \Gamma^{(j+1)}_{\tau^{(j)}}, \quad k[t^{\Gamma^{(j)}_{\tau^{(j-1)}}}] \subset k[t^{\Gamma^{(j+1)}_{\tau^{(j)}}}], \quad \mathcal{J}^{(j)}_{\tau^{(j-1)}} k[t^{\Gamma^{(j+1)}_{\tau^{(j)}}}] \subset \mathcal{J}^{(j+1)}_{\tau^{(j)}} k[t^{\Gamma^{(j+1)}_{\tau^{(j)}}}]. \tag{19}$$

By (19) we have that

$$\mathrm{ord}_{\mathcal{J}^{(j+1)}}(\nu) \leq \mathrm{ord}_{\mathcal{J}^{(j)}}(\nu) \text{ for all } \nu \in \sigma. \tag{20}$$

Remark 12.3. For $1 \leq l \leq j$ we deduce from Remark 12.1 that $\mathcal{J}^{(l)}_{\tau^{(l-1)}} k[t^{\Gamma^{(l+1)}_{\tau^{(l)}}}] = t^{m^{(l)}} k[t^{\Gamma^{(l+1)}_{\tau^{(l)}}}]$, hence the Newton polyhedron $\mathcal{N}_{\tau^{(l)}}(\mathcal{J}^{(l)}_{\tau^{(l-1)}}) = \mathcal{J}^{(l)}_{\tau^{(l-1)}} + \check{\tau}^{(l)} = m^{(l)} + \check{\tau}^{(l)}$ has only one vertex $m^{(l)}$.

Notation 12.4. We denote the Newton polyhedron $\mathcal{N}_{\tau^{(j-1)}}(\mathcal{J}^{(j)}_{\tau^{(j-1)}})$ simply by $\mathcal{N}(\mathcal{J}^{(j)}_{\tau^{(j-1)}})$ since there is no risk of confusion.

Proposition 12.5. *The affine toric variety T^Γ is non singular if and only if the blowing up of the logarithmic jacobian ideal is an isomorphism.*

Proof. We only have to prove that if the blowing up is an isomorphism the variety is smooth. Remembering that Γ generates the group M, let us choose a minimal system of generators $\gamma_1, \dots, \gamma_d, \gamma_{d+1}, \dots$ such that $m^{(1)} = \gamma_1 + \cdots + \gamma_d$ corresponds to the generator of the logarithmic jacobian ideal which, by our assumption, is principal. If there are more than d generators, then γ_{d+1} is linearly dependent on the previous ones which gives us another element $m = \gamma_1 + \cdots + \gamma_{i-1} + \gamma_{d+1} + \gamma_{i+1} + \cdots + \gamma_d$ of our ideal. Our assumption ensures that $m - m^{(1)} = \gamma_{d+1} - \gamma_i \in \Gamma$ which contradicts the assumption of minimality. Therefore Γ has d independent generators which generate M and $k[t^\Gamma]$ is a polynomial ring. □

Proposition 12.6. *The following assertions are equivalent:*

i. $\tau^{(j)} = \tau^{(j-1)}$

ii. *The blowing up of the ideal $\mathcal{J}^{(j)}_{\tau^{(j-1)}}$ of $T^{\Gamma^{(j)}_{\tau^{(j-1)}}}$ is a finite morphism.*

Proof. The hypothesis i. is equivalent to the following fact: the semigroups $\Gamma^{(j)}_{\tau^{(j-1)}}$ and $\Gamma^{(j+1)}_{\tau^{(j)}}$ have the same saturation in the lattice M; it is equal to $\check{\tau}^{(j-1)} \cap M = \check{\tau}^{(j)} \cap M$. This is equivalent to the following geometric statement: the composite of the normalization of $T^{\Gamma^{(j+1)}_{\tau^{(j)}}}$ with the blowing up of the logarithmic jacobian ideal of $T^{\Gamma^{(j)}_{\tau^{(j-1)}}}$ is the normalization map of $T^{\Gamma^{(j)}_{\tau^{(j-1)}}}$ and therefore this blowing up is finite. Conversely, if ii. holds, the blowing up morphism $T^{\Gamma^{(j+1)}_{\tau^{(j)}}} \to T^{\Gamma^{(j)}_{\tau^{(j-1)}}}$ induces an isomorphism of the normalizations, from which i. follows in view of Remark 4.6. □

Remark 12.7. The conditions of the Lemma are also equivalent to the fact that the Newton polyhedron of the ideal $\mathcal{J}^{(j)}_{\tau^{(j-1)}}$ has only one vertex $m^{(j)}$.

Definition 12.8. For any integer $j \geq 1$ we introduce a function

$$f^{(j)} : \{\tau \subset \sigma \mid 0 \neq \tau \text{ convex rational polyhedral cone } \} \to \mathbf{Z}_{\geq 1}.$$

If $\nu_1, \dots, \nu_s$ are the primitive integral vectors for the lattice N which span the edges of τ, then the value of $f^{(j)}(\tau)$ is defined by

$$f^{(j)}(\tau) := \sum_{i=1}^{s} \mathrm{ord}_{\mathcal{J}^{(j)}}(\nu_i).$$

Remark 12.9. Notice that if $0 \neq \tau$ is any rational polyhedral cone contained in $\tau^{(j)} \in \Sigma^{(j)}(d)$ then $f^{(j)}(\tau) = \sum_{i=1}^{s} \langle \nu_i, m^{(j)} \rangle$ and if τ is of dimension d then $f^{(j)}(\tau) \geq d$. Moreover, by (20) we obtain that

$$f^{(j)}(\tau) \leq f^{(j-1)}(\tau). \tag{21}$$

Lemma 12.10. *The following conditions are equivalent for $j \geq 1$:*

i. *The equality $f^{(j)}(\tau^{(j-1)}) = d$ holds.*
ii. *The cone $\tau^{(j-1)}$ is regular for the lattice N and $\Gamma^{(j)}_{\tau^{(j-1)}} = \check{\tau}^{(j-1)} \cap M$.*
iii. *The toric variety $T^{\Gamma^{(j)}_{\tau^{(j-1)}}}$ is smooth.*

Note that if the conditions of the Lemma are satisfied, the polyhedron $\mathcal{N}_{\tau^{(j-1)}}(\mathcal{J}^{(j)}_{\tau^{(j-1)}})$ has only one vertex $m^{(j)}$.

Proof. It is clear that ii. and iii. are equivalent. It is enough to prove the result for $j = 1$. Suppose first that i. holds. By hypothesis the fan $\Sigma^{(1)}$ is the fan consisting of the faces of σ. If $\nu_1, \dots, \nu_s$ are the primitive vectors for the lattice N which span the cone σ then $\langle \nu_i, m \rangle > 0$, $i = 1, \dots, s$ since $m = m^{(1)}$ belongs to the interior of $\check{\sigma}$. Since $f(\sigma) = d = \sum_{i=1}^{s} \langle \nu_i, m \rangle$ we get that $s = d$ and $\langle \nu_i, m \rangle = 1$.

By definition of $\mathcal{J}$ the vector m is sum of d generators of Γ which are linearly independent, say $m = \gamma_1 + \cdots + \gamma_d$. Since $\sum_{j=1}^{d} \langle \nu_i, \gamma_j \rangle = 1$ for $i = 1, \dots, d$ we obtain that, up to relabelling the ν_i, the vectors $\nu_1, \dots, \nu_d$ in $N_{\mathbf{R}}$ form the dual basis of $\gamma_1, \dots, \gamma_d$ in $M_{\mathbf{R}}$. Finally, notice that the parallelogram generated by the primitive vectors $\gamma_1, \dots, \gamma_d$ in $M_{\mathbf{R}}$ contains no integral points different from the vertices. It follows that $\gamma_1, \dots, \gamma_d$ form a basis of M.

Conversely, if ii. holds then we check from the definitions that i. holds. □

Proposition 12.11. *Suppose that $\tau^{(j)} \in \Sigma^{(j)}(d)$ is contained in $\tau^{(j-1)} \in \Sigma^{(j-1)}(d)$. The following equalities are equivalent:*

i. $f^{(j)}(\tau^{(j)}) = f^{(j-1)}(\tau^{(j)})$,
ii. $m^{(j)} = m^{(j-1)}$.

Proof. Notice that if $m^{(j)} = m^{(j-1)}$ then i. follows by Remark 12.9.

Suppose that the equality i. holds. By Remark 12.3 we have that

$$\mathcal{N}_{\tau^{(j-1)}}(\mathcal{J}^{(j-1)}_{\tau^{(j-2)}}) = m^{(j-1)} + \check{\tau}^{(j-1)} \text{ and } \mathcal{N}_{\tau^{(j)}}(\mathcal{J}^{(j)}_{\tau^{(j-1)}}) = m^{(j)} + \check{\tau}^{(j)}.$$

Since $\tau^{(j)}$ is contained in $\tau^{(j-1)}$ we get that $\check{\tau}^{(j-1)} \subset \check{\tau}^{(j)}$ and then $\mathcal{N}_{\tau^{(j)}}(\mathcal{J}^{(j-1)}_{\tau^{(j-2)}}) = m^{(j-1)} + \check{\tau}^{(j)}$. By (19) we get

$$m^{(j-1)} + \check{\tau}^{(j)} \subset m^{(j)} + \check{\tau}^{(j)}. \tag{22}$$

Let $\nu_1, \dots, \nu_s$ be the primitive integral vectors for the lattice N which span the cone $\tau^{(j)}$. The vector $\nu := \sum_{i=1}^{s} \nu_i$ belongs to the interior of the cones $\tau^{(j)}$ and $\tau^{(j-1)}$. By Remark 12.9 and the hypothesis we deduce

$$f^{(j-1)}(\tau^{(j)}) = \langle \nu, m^{(j-1)} \rangle = f^{(j)}(\tau^{(j)}) = \langle \nu, m^{(j)} \rangle.$$

This equality and the inclusion (22) imply that $m^{(j-1)} = m^{(j)}$. □

Proposition 12.12. *There exists an integer $l \geq 1$ such that for any cone $\tau \in \Sigma^{(l)}(d)$ if $f^{(1)}(\tau) > d$ then $f^{(1)}(\tau) > f^{(l)}(\tau)$.*

Proof. Let us assume that the assertion of the Proposition does not hold. This implies that there exists a infinite sequence of convex polyhedral cones

$$\sigma = \tau^{(0)} \supset \tau^{(1)} \supseteq \tau^{(2)} \supseteq \cdots \supseteq \tau^{(j)} \supseteq \cdots, \tag{23}$$

such that $\tau^{(j)} \in \Sigma^{(j)}(d)$ and

$$f^{(j)}(\tau^{(j)}) = f^{(1)}(\tau^{(j)}) > d, \text{ for all } j \geq 2. \tag{24}$$

By Remark 12.9 we have that $f^{(j)}(\tau^{(j)}) = f^{(j-1)}(\tau^{(j)})$ for all $j \geq 2$. Proposition 12.11 implies then that $m^{(j)} = m^{(j-1)}$ for all $j \geq 2$.

Claim 12.13. *There exists a strictly increasing sequence $(i_j)_{j \geq 1}$ of integers ≥ 0 such that $\tau^{(i_j)} \neq \tau^{(i_j+1)}$, that is, the inclusion $\tau^{(i_j)} \supset \tau^{(i_j+1)}$ is strict, for $j \geq 1$.*

Proof of the claim. Assume that the claim does not hold. This implies that $\tau^{(j)} = \tau^{(j-1)}$ for all $j \geq 1$. By Proposition 12.6 the blowing up of the ideal $\mathcal{J}^{(j)}_{\tau^{(j-1)}}$ of $T^{\Gamma^{(j)}_{\tau^{(j-1)}}}$ is a finite morphism, dominated by the normalization of T^{Γ}, for all $j \geq 1$. It follows that for $j \gg 0$ the variety $T^{\Gamma^{(j)}_{\tau^{(j-1)}}}$ is normal. By Proposition 12.5, this variety is also smooth. By Lemma 12.10 it follows that $f^{(j)}(\tau^{(j)}) = d$ for $j \gg 0$. This is a contradiction with (24). □

Let us fix a representation for $m = m^{(1)}$ in terms of the generators of Γ:

$$m = \gamma_1 + \cdots + \gamma_d \text{ with } \gamma_1 \wedge \cdots \wedge \gamma_d \neq 0,$$

(up to an eventual relabelling of the generators $\{\gamma_i\}_{i=1}^{r}$ of the semigroup Γ).

By Claim 12.13 we can suppose without loss of generality that $i_1 = 0$, that is, the Newton polyhedron $\mathcal{N}_\sigma(\mathcal{J})$ has at least two different vertices m and n.

Lemma 12.14. *Given one of the γ_j which appear in the decomposition of m, say γ_d, for any $j \geq 0$ the vector $n_j := n - j\gamma_d$ verifies that*

$$n_j \in \mathcal{J}^{(j+1)}_{\tau^{(j)}} \text{ and } (n_j - m) \wedge \gamma_1 \wedge \cdots \wedge \gamma_{d-1} \neq 0.$$

Proof We prove the assertion by induction on j. Notice that for $j = 0$ the vector $n_0 = n$ belongs to $\mathcal{J}$ by hypothesis. We suppose by induction that $n_l \in \mathcal{J}^{(l+1)}_{\tau^{(l)}}$, $1 \leq l \leq j$.

We prove first that:

$$(n_l - m) \wedge \gamma_1 \wedge \cdots \wedge \gamma_{d-1} \neq 0, \text{ for } 0 \leq l \leq j. \tag{25}$$

Assume on the contrary that (25) does not hold for some $0 \leq l \leq j$. After relabelling the vectors $\gamma_1, \dots, \gamma_{d-1}$ if necessary, we have an expansion of the form:

$$n_l - m = a_1\gamma_1 + \cdots + a_h\gamma_h \text{ with } h \leq d-1, \tag{26}$$

and in addition the coefficients of (26) are non-zero rational numbers which are not of the same sign, that is,

$$\begin{cases} a_i > 0 & \text{for } i = 1, \dots, s \\ a_i < 0 & \text{for } i = s+1, \dots, h. \end{cases}$$

Indeed, if all coefficients a_i in (26) are ≥ 0 we obtain that

$$n = m + a_1\gamma_1 + \cdots + a_h\gamma_h + l\gamma_d \subset m + \check{\sigma},$$

contradicting that $n \neq m$ is a vertex of $\mathcal{N}_\sigma(\mathcal{J})$. In particular, we have that $n_l \neq m$. Similarly, if all the coefficients a_i are smaller than zero we get that

$$m = n_l - a_1\gamma_1 - \cdots - a_h\gamma_h \subset n_l + \check{\sigma} \subset n_l + \check{\tau}^{(l)}.$$

This implies that m is not a vertex of the Newton polyhedron $\mathcal{N}_{\tau^{(l)}}(\mathcal{J}_{\tau^{(l)}}^{(l+1)})$, since $n_l \in \mathcal{J}_{\tau^{(l)}}^{(l+1)}$ and $n_l \neq m$.

If $\alpha \in \mathbf{R}$ we denote by $\lceil \alpha \rceil$ the smallest integer p such that $\alpha \leq p$.

Claim 12.15. *If $q := \sum_{i=1}^{s} \lceil a_i \rceil$, $0 \leq p \leq q$ and b_i are integers such that $0 \leq b_i \leq \lceil a_i \rceil$, $i = 1, \dots, s$ and $\sum_{i=1}^{s} b_i = p$ then the vector $\beta_{l,p} := n_l - \sum_{i=1}^{s} b_i\gamma_i$ belongs to $\mathcal{J}_{\tau^{(l+p)}}^{(l+p+1)}$.*

Proof of the claim. We prove the assertion by induction on p. For $p = 0$ we have $\beta_{l,0} = n_l$ hence the assertion holds by assumption. Suppose that $\beta_{l,p} \in \mathcal{J}_{\tau^{(l+p)}}^{(l+p+1)}$ for $0 \leq p < q$. The vector

$$\beta_{l,p} - m = n_l - m - \sum_{i=1}^{s} b_i\gamma_i = \sum_{i=1}^{s}(a_i - b_i)\gamma_i + \sum_{i=s+1}^{h} a_i\gamma_i$$

belongs to $\Gamma_{\tau^{(l+p+1)}}^{(l+p+2)}$. Since $p < q$ there is a strictly positive coefficient in this expansion of $\beta_{l,p} - m$, say $a_1 - b_1$, for instance. We get $(\beta_{l,p} - m) \wedge \gamma_2 \wedge \cdots \wedge \gamma_d \neq 0$, hence the vector

$$\beta_{l,p+1} := \beta_{l,p} - m + \gamma_2 + \cdots + \gamma_d = \beta_{l,p} - \gamma_1$$

belongs to $\mathcal{J}_{\tau^{(l+p+1)}}^{(l+p+2)}$. □

By Claim 12.15 the expansion

$$\beta_{l,q} - m = \sum_{i=1}^{s}(a_i - \lceil a_i \rceil)\gamma_i + \sum_{i=s+1}^{h} a_i\gamma_i$$

has only coefficients ≤ 0 and $\beta_{l,q} \in \mathcal{J}_{\tau^{(l+q+1)}}^{(l+q+2)}$. We get also that $m \neq \beta_{l,q}$ since the coefficients $a_{s+1}, \dots, a_h$ are non-zero and the vectors $\gamma_1, \dots, \gamma_h$ are linearly independent. We deduce from this that

$$m = \beta_{l,q} - \sum_{i=1}^{s}(a_i - \lceil a_i \rceil)\gamma_i - \sum_{i=s+1}^{h} a_i\gamma_i \in \beta_{l,q} + \check{\sigma} \subset \beta_{l,q} + \check{\tau}^{(j+q+1)}.$$

This contradicts the assumption, m being a vertex of the Newton polyhedron of $\mathcal{J}_{\tau^{(l+q+1)}}^{(l+q+2)}$.

Finally, we have proven that (25) holds hence

$$n_{j+1} = (n_j - m) + \gamma_1 + \cdots + \gamma_{d-1} \in \mathcal{J}_{\tau^{(j+1)}}^{(j+2)}.$$

This concludes the induction in the proof of Lemma 12.14. □

The cone

$$\tau^{(\infty)} = \bigcap_{l\geq 1} \tau^{(l)} = \bigcap_{j\geq 1} \tau^{(i_j)}$$

is a closed convex subset of σ different from 0. A vector $0 \neq w \in \tau^{(\infty)}$ defines a *monomial valuation* ω of the field of fractions of $k[t^\Gamma]$, which verifies that if $0 \neq \sum a_\gamma t^\gamma \in k[t^\Gamma]$ then $\omega(\sum a_\gamma t^\gamma) = \min_{a_\gamma \neq 0} \langle w, \gamma\rangle$. By definition this valuation is non-negative in the subrings $k[t^{\Gamma^{(j)}_{\tau^{(j-1)}}}]$ for all $j \geq 1$. Notice that the vector $w \in N_{\mathbf{R}}$ is not necessarily an element of $N_{\mathbf{Q}}$ and it may lie in a face of σ (different from 0). We remark that for all $j \geq 1$ we have that $\min\{\langle w, \gamma\rangle \mid \gamma \in \mathcal{J}^{(j)}_{\tau^{(j-1)}}\} = \langle w, m\rangle$ since w takes non negative values on $\mathcal{J}^{(j+1)}_{\tau^{(j)}}$ which contains the set $\{\gamma - m \mid \gamma \in \mathcal{J}^{(j)}_{\tau^{(j-1)}}\}$.

Since $\gamma_1 \dots, \gamma_d$ span $M_{\mathbf{R}}$ at least one of the vectors γ_i verifies that $\langle w, \gamma_i\rangle \neq 0$. Suppose for instance that $\langle w, \gamma_d\rangle > 0$.

By Lemma 12.14, for any integer $j \geq 0$ the vector $n_j = n - j\gamma_d$ belongs to $\mathcal{J}^{(j+1)}_{\tau^{(j)}} \subset \Gamma^{(j+1)}_{\tau^{(j)}}$. This implies that

$$\omega(t^{n_j}) = \langle w, n_j\rangle = \langle w, n\rangle - j\langle w, \gamma_d\rangle$$

becomes strictly negative for j large enough. This is a contradiction since $t^{n_j} \in k[t^{\Gamma^{(j+1)}_{\tau^{(j)}}}]$ and the valuation ω is non negative on the ring $k[t^{\Gamma^{(j+1)}_{\tau^{(j)}}}]$. □

Corollary 12.16. *With the previous notations, given any sequence of the form (23) if T^Γ is not smooth is not possible that $m^{(1)} = m^{(j)}$ for all $j \geq 2$.*

This is now a consequence of Proposition 12.11.

Definition 12.17.

i. If $0 \neq \eta$ is a cone we denote by ν_η the sum of the primitive vectors, for the lattice N in the edges of the cone η.
ii. A cone $\eta \subset \sigma$ is *stable* if there is an integer $I \geq 1$ such that $\eta \in \Sigma^{(j)}$ for all $j \geq I$. The *stability problem* for the toric variety T^Γ consists of determining if the sequence of fans $(\Sigma^{(j)})_{j\geq 0}$ stabilizes, that is $\Sigma^{(j)} = \Sigma^{(j+1)}$ for $j \gg 0$.
iii. If $\theta \in \Sigma^{(l)}$ the *stability problem* for the cone θ consists of determining if the sequence of fans, $\{\theta^{(j)} \in \Sigma^{(j)} \mid \theta^{(j)} \subset \theta\}$ $j \geq l$, stabilizes.

For instance, if $\rho \in \Sigma^{(j)}$ is of dimension one then ρ is stable.

The solution of the stability problem is the essential part of our proof that the iteration of Semple Nash modifications eventually resolves singularities of toric varieties.

Lemma 12.18. *If $\theta \in \Sigma^{(j-1)}$ is a cone of codimension > 0 and $M(\theta, \Gamma^{(j)}_\theta) = M(\theta)$ (see Notation 4.9), then the stability problem for the cone $\theta \subset N_{\mathbf{R}}$ is equivalent to a stability problem for the cone θ, viewed in $(N_\theta)_{\mathbf{R}}$, with respect to the sequence of iterated Semple-Nash modifications of another toric variety of dimension equal to* $\dim\theta$.

Proof. The sublattice $M(\theta) = M \cap \theta^\perp$ of M is obviously saturated, so that it is a direct summand of M. Let us consider a sublattice M' of M such that $M = M(\theta) \oplus M'$. Such a sublattice M' is spanned by vectors $v_{q_0+1}, \dots, v_d$, completing a basis $v_1, \dots, v_{q_0}$ of $M(\theta)$ to a basis of M. Any $\gamma \in M$ can be written in a unique way as $\gamma = \alpha_1(\gamma) + \alpha_2(\gamma)$ with $\alpha_1(\gamma) \in M(\theta)$ and $\alpha_2(\gamma) \in M'$. The restriction $\beta\colon M' \mapsto M/M(\theta)$ of the canonical map $M \to M/M(\theta)$ to the sublattice $M' \subset M$ is an isomorphism. The image $\tilde{\Gamma}^{(j)}_\theta$ of $\Gamma^{(j)}_\theta$ by the canonical map $M \mapsto M/M(\theta)$ is a semigroup of finite type, generates the lattice $M/M(\theta)$ and spans a strictly convex cone $\mathbf{R}_{\geq 0}\tilde{\Gamma}^{(j)}_\theta$.

The semigroup $\Gamma_\theta^{(j)} \cap \theta^\perp$, which is the minimal face of $\Gamma_\theta^{(j)}$, is a rank q_0 lattice by Lemma 3.9. Using the equality $\Gamma_\theta^{(j)} \cap \theta^\perp = M(\theta)$ one checks directly that the map

$$\Gamma_\theta^{(j)} \longrightarrow M(\theta) \times \tilde{\Gamma}_\theta^{(j)}, \quad \gamma \mapsto (\alpha_1(\gamma), \beta \circ \alpha_2(\gamma)) \tag{27}$$

is a semigroup isomorphism. The map (27) determines an isomorphism $T^{\Gamma_\theta^{(j)}} \xrightarrow{\simeq} \mathrm{orb}(\theta, \Gamma_\theta^{(j)}) \times \tilde{\Gamma}_\theta^{(j)}$. In particular, the variety $T^{\Gamma_\theta^{(j)}}$ is smooth if and only if $T^{\tilde{\Gamma}_\theta^{(j)}}$ is smooth.

According to Remark 10.3 and Lemma 11.5 the blowing up of logarithmic jacobian ideals commutes with the splitting defined by (27). The assertion follows from this since the cone θ, viewed in the $\mathbf{R}$-linear subspace $(N_\theta)_\mathbf{R}$ it spans in $N_\mathbf{R}$, is the dual cone of $\mathbf{R}_{\geq 0}\tilde{\Gamma}_\theta^{(j)}$. ☐

Remark 12.19. Geometrically, we see that the semigroup $\tilde{\Gamma}_\theta^{(j)}$ in Lemma 12.18 corresponds to the toric variety of dimension $\dim\theta$ which is a transverse linear section of $T^{\Gamma_\theta^{(j)}}$ at the point $(1,\dots,1)$ of the orbit corresponding to θ.

Proposition 12.20. *If θ is a stable cone then there is an integer $I \geq 1$ such that the variety $T^{\Gamma_\theta^{(j)}}$ is smooth for all $j \geq I$. If $T^{\Gamma_\theta^{(j)}}$ is smooth the cone θ is stable.*

Proof. By Lemma 12.18 we can assume that $\mathrm{codim}\theta = 0$. The blowing up of the ideal $\mathcal{J}_\theta^{(j)}$ of $T^{\Gamma_\theta^{(j)}}$ is a finite morphism, dominated for all $j \gg 0$ by the normalization of $T^{\Gamma_\theta^{(j)}}$, which is equal to that of $T^{\Gamma_\theta^{(1)}}$ (see Proposition 12.6). It follows that for $j \gg 0$ the map $T^{\Gamma_\theta^{(j+1)}} \to T^{\Gamma_\theta^{(j)}}$ is an isomorphism. By Proposition 12.5, this variety is smooth. The converse follows directly from the definitions. ☐

Proposition 12.21. *Given $0 \neq \nu \in \sigma \cap N$, for any $j \geq 1$ there exists a unique cone $\theta^{(j)}$ such that $\nu \in \mathrm{int}(\theta^{(j)})$. Then for $j \gg 0$ we have:*

i. *If $q_0 = \mathrm{codim}\theta^{(j-1)}$ and if $m^{(j)} \in \mathcal{J}^{(j)}_{\theta^{(j-1)}}$ is such that $\mathrm{ord}_{\mathcal{J}^{(j)}}(\nu) = \langle \nu, m^{(j)} \rangle$, then for any representation*

$$m^{(j)} = \gamma_1^{(j)} + \dots + \gamma_d^{(j)} \tag{28}$$

as a sum of linearly independent elements in the semigroup $\Gamma^{(j)}_{\theta^{(j-1)}}$, there are exactly q_0 of them in $M(\theta^{(j-1)})$.

ii. *$M(\theta^{(j-1)}, \Gamma^{(j)}_{\theta^{(j-1)}}) = M(\theta^{(j-1)}) = M(\theta^{(j)})$ (cf. Notation 4.9).*

Moreover, the sequence of cones $(\theta^{(j)})$ stabilizes.

Proof. Notice that the vector ν is in a subdivision of $\theta^{(j)}$ induced by $\Sigma^{(j+1)}$, so that we have $\theta^{(j+1)} \subseteq \theta^{(j)}$. The sequence $(\mathrm{codim}_{N_\mathbf{R}}\, \theta^{(j)})_j$ is increasing, thus there exists an integer $0 \leq q_0 \leq d-1$ such that $\mathrm{codim}\, \theta^{(j)} = q_0$, for $j \gg 0$.

Since $\nu \in \mathrm{int}(\theta^{(j)}) \cap N$, we have the equalities

$$\Gamma^{(j)}_{\theta^{(j-1)}} \cap (\theta^{(j-1)})^\perp = \Gamma^{(j)}_{\theta^{(j-1)}} \cap \nu^\perp = M(\theta^{(j-1)}, \Gamma_\theta^{(j)}). \tag{29}$$

The lattice $M(\theta^{(j-1)}, \Gamma^{(j)}_{\theta^{(j-1)}})$ is a sublattice of finite index $i(\theta^{(j-1)}, \Gamma^{(j)}_{\theta^{(j-1)}})$ of $M(\theta^{(j-1)})$ (see Notation 4.9 and Lemma 3.9). By definition $M(\theta^{(j-1)})$ is a saturated subsemigroup of M, and it is also a rank q_0 lattice. By (19) we deduce that $M(\theta^{(j-1)}) = M(\theta^{(j)})$ and $M(\theta^{(j-1)}, \Gamma_\theta^{(j)}) \subset M(\theta^{(j)}, \Gamma_\theta^{(j+1)})$. Then, the sequence of indices $(i(\theta^{(j-1)}, \Gamma^{(j)}_{\theta^{(j-1)}}))_j$ stabilizes, hence $M(\theta^{(j-1)}, \Gamma_\theta^{(j)}) = M(\theta^{(j)}, \Gamma_\theta^{(j+1)})$ for $j \gg 0$. The lattice $M(\theta^{(j-1)}, \Gamma_\theta^{(j)})$ is *a priori* a sublattice of finite index of $M(\theta^{(j-1)})$.

We deal first with the proof of (i). Up to relabelling the vectors we can assume that those $\gamma_i^{(j)}$ appearing in (28), which belong to $M(\theta^{(j-1)})$ are $\gamma_1^{(j)} \dots, \gamma_s^{(j)}$ for $0 \leq s \leq q_0$. By (29) these vectors belong to $M(\theta^{(j-1)}, \Gamma^{(j)}_{\theta^{(j-1)}})$. Suppose that $s \neq q_0$. Since, the images of the $\gamma_i^{(j)}$, $i = s+1, \dots, d$, generate the rank $d - q_0$ lattice $M/M(\theta^{(j-1)})$, we get that $d - q_0$ of them, say for $i = q_0 + 1, \dots, d$, are linearly independent modulo $M(\theta^{(j-1)})$. We can find vectors $\tilde{\gamma}^{(j)}_{s+1}, \dots, \tilde{\gamma}^{(j)}_{q_0} \in \Gamma^{(j)}_{\theta^{(j-1)}} \cap \nu^\perp$, such that $\gamma_1^{(j)} \wedge \dots \wedge \gamma_s^{(j)} \wedge \tilde{\gamma}^{(j)}_{s+1} \wedge \dots \wedge \tilde{\gamma}^{(j)}_{q_0} \neq 0$. Then the vector

$$m' := \gamma_1^{(j)} + \cdots + \gamma_s^{(j)} + \tilde{\gamma}^{(j)}_{s+1} + \cdots + \tilde{\gamma}^{(j)}_{q_0} + \gamma^{(j)}_{q_0+1} + \cdots + \gamma^{(j)}_d$$

verifies that $m' \in \mathcal{J}^{(j-1)}_{\theta^{(j-1)}}$. Since $0 = \langle \nu, \tilde{\gamma}^{(j)}_j \rangle < \langle \nu, \tilde{\gamma}^{(j)}_j \rangle$, for $s+1 \leq j \leq q_0$ we would have $\langle \nu, m' \rangle < \langle \nu, m^{(j)} \rangle$, a contradiction.

Suppose that (ii) does not hold. Then, since $\Gamma^{(j)}_{\theta^{(j-1)}}$ generates the lattice M, there exist $\gamma, \gamma' \in \Gamma^{(j)}_{\theta^{(j-1)}}$ such that $\gamma - \gamma' \in M(\theta^{(j-1)}) \setminus M(\theta^{(j-1)}, \Gamma^{(j)}_{\theta^{(j-1)}})$. In view of (29) we get $0 \neq \langle \nu, \gamma \rangle = \langle \nu, \gamma' \rangle$. Since $\gamma^{(j)}_{q_0+1}, \dots, \gamma^{(j)}_d$ define linearly independent vectors in the lattice $M/M(\theta^{(j-1)})$ which is of rank $d - q_0$, there exists an integer $q_0 + 1 \leq i_0 \leq d$ such that γ (resp. γ') together with $\gamma^{(j)}_{q_0+1}, \dots, \gamma^{(j)}_{i_0-1}, \gamma^{(j)}_{i_0+1}, \dots, \gamma^{(j)}_d$ are linearly independent modulo $M(\theta^{(j-1)})$. Suppose without loss of generality that $i_0 = d$. Then the vectors $\tilde{\gamma} := \gamma + \sum_{i=1}^{d-1} \gamma_i^{(j)}$ and $\tilde{\gamma}' := \gamma' + \sum_{i=1}^{d-1} \gamma_i^{(j)}$ belong to $\mathcal{J}^{(j)}_{\theta^{(j-1)}}$, hence $\tilde{\gamma} - m^{(j)} = \gamma - \gamma_d^{(j)}$ and $\tilde{\gamma}' - m^{(j)} = \gamma' - \gamma_d^{(j)}$ are both elements of $\Gamma^{(j+1)}_{\theta^{(j)}}$. Since $\langle \nu, \gamma_d^{(j)} \rangle > 0$, it follows that $\tilde{\gamma} - \tilde{\gamma}' = \gamma - \gamma'$ and $\langle \nu, \tilde{\gamma} \rangle = \langle \nu, \tilde{\gamma}' \rangle < \langle \nu, \gamma \rangle = \langle \nu, \gamma' \rangle$. By repeating this construction, since $\nu \in \sigma \cap N$, in a finite number of steps we reduce to the case when $\langle \nu, \gamma \rangle = \langle \nu, \gamma' \rangle = 0$. As we remarked before this implies that $\gamma - \gamma' \in M(\theta^{(j-1)}, \Gamma^{(j)}_{\theta^{(j-1)}})$, a contradiction.

By Lemma 12.18 and ii. we can assume that $q_0 = 0$. By (20) the sequence of positive integers $(\langle \nu, m^{(j)} \rangle)_{j \geq 1}$ is decreasing, hence it stabilizes. Suppose that the sequence $(\theta^{(j)})$ does not stabilize. By Proposition 12.20 this implies that the toric variety $T^{\Gamma^{(j)}_{\theta^{(j-1)}}}$ is not smooth for any $j \geq 1$.

Let us fix an integer $j_1 \gg 0$. By Proposition 12.12 there exists a smallest integer $j_2 > j_1$ such that $m^{(j_2)} \neq m^{(j_1)}$. We consider a representation of $m^{(j_2)} \in \mathcal{J}^{(j_2)}_{\theta^{(j_2-1)}}$ of the form (28). The vector $m^{(j_1)} - m^{(j_2)}$ belongs to the semigroup $\Gamma^{(j_2+1)}_{\theta^{(j_2)}}$ since $m^{(j_1)} \in \mathcal{J}^{(j_2)}_{\theta^{(j_2-1)}}$ by (19). We obtain that $\langle \nu, m^{(j_1)} - m^{(j_2)} \rangle = 0$, hence $m^{(j_1)} - m^{(j_2)} \in \Gamma^{(j_2+1)}_{\theta^{(j_2)}} \cap \nu^\perp$. Since $\nu \in \mathrm{int} \theta^{(j_2)}$ and $\dim \theta^{(j_2)} = d$ we deduce also that $\Gamma^{(j_2+1)}_{\theta^{(j_2)}} \cap \nu^\perp = \{0\}$, a contradiction, which ends the proof of Proposition 12.21. □

If $\eta \subset N_{\mathbf{R}}$ is a rational convex polyhedral cone the duality between the lattices N and M induces a duality between the lattices N_η and $M_\eta = M/M(\eta)$ and also a duality between $N(\eta) = N/N_\eta$ and $M(\eta) = M \cap \eta^\perp$ (cf. Notations 4.9).

If $0 \neq \eta$ is a stable cone one can consider for $j \gg 0$ the orbit closure $T^{\Gamma^{(j)}}_{\Sigma^{(j-1)}}(\eta)$ associated to $\eta \in \Sigma^{(j-1)}$. By convenience we recall the notations to describe this toric variety in this case. See Lemma 4.10 and Notations 4.9.

Notation 12.22. If $\eta \in \Sigma^{(j-1)}$ is a stable cone the variety $T^{\Gamma^{(j)}}_{\Sigma^{(j-1)}}(\eta)$ is covered by charts defined by the semigroups $\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^\perp$, for $\eta \leq \theta^{(j-1)}$ and $\theta^{(j-1)} \in \Sigma^{(j-1)}$. Notice that the semigroup $\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^\perp$ is a face of $\Gamma^{(j)}_{\theta^{(j-1)}}$, and it spans the lattice $M(\eta, \Gamma^{(j)}_\eta) = \Gamma^{(j)}_\eta \cap \eta^\perp$ by Lemma 3.9. By Proposition 12.21 ii. this lattice is equal to $M(\eta)$ if $j \gg 0$. The fan

$\Sigma^{(j-1)}(\eta)$ consists of the images $\theta^{(j-1)}(\eta)$ of cones $\theta^{(j-1)} \in \Sigma^{(j-1)}$ in $N(\eta)_{\mathbf{R}}$. The cone $\theta^{(j-1)}(\eta)$ is the dual cone of $\mathbf{R}_{\geq 0}(\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^{\perp})$. We denote by $\mathcal{J}^{(j)}_{\theta^{(j-1)}}(\eta)$ the logarithmic jacobian ideal of $k[t^{\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^{\perp}}]$.

The following technical lemma will be useful.

Lemma 12.23. *Let $0 \neq \eta$ be a stable cone of codimension $d_0 < d$. We denote by*

$$\pi : M_{\mathbf{R}} \to M_{\mathbf{R}}/M(\eta)_{\mathbf{R}}, \quad \alpha \mapsto \tilde{\alpha}$$

the canonical projection. If $(\theta^{(j)})_j$ is a sequence such that

$$\theta^{(j)} \in \Sigma^{(j)},\ \theta^{(j)} \supset \theta^{(j+1)} \ \text{ and } \ \eta \leq \theta^{(j)}, \tag{30}$$

then for $j \gg 0$ we have:

i. $\pi(\Gamma^{(j)}_{\theta^{(j-1)}}) = \pi(\Gamma^{(j+1)}_{\theta^{(j)}})$ *and the semigroup* $\tilde{\Gamma}_\eta := \pi(\Gamma^{(j)}_{\theta^{(j-1)}})$ *is generated by a basis* $\tilde{e}_{d_0+1}, \dots, \tilde{e}_d$ *of* $M/M(\eta)$.

ii. *If $m^{(j)}$ is the vertex of the polyhedron $\mathcal{N}(\mathcal{J}^{(j)}_{\theta^{(j-1)}})$ such that*

$$\mathrm{ord}_{\mathcal{J}^{(j)}}(\nu) = \langle \nu, m^{(j)} \rangle, \quad \forall \nu \in \theta^{(j)}, \tag{31}$$

then for any representation of the form (28) of $m^{(j)}$ as a sum of linearly independent vectors in $\Gamma^{(j)}_{\theta^{(j-1)}}$ then exactly d_0 of the $\gamma^{(j)}_i$ belong to $\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^{\perp}$, say for $i = 1, \dots, d_0$, while $\langle \nu_\eta, \gamma^{(j)}_i \rangle = 1$ for $i = d_0 + 1, \dots, d$.

iii. *The vector $m^{(j)}$ belongs to the face $\mathcal{F}^{(j)}_{\theta^{(j-1)}}$ of $\mathcal{N}(\mathcal{J}^{(j)}_{\theta^{(j-1)}})$ determined by ν_η (cf. Definition 12.17). The face $\mathcal{F}^{(j)}_{\theta^{(j-1)}}$ is the Minkowski sum*

$$\mathcal{N}(\mathcal{J}^{(j)}_{\theta^{(j-1)}}(\eta)) + \mathcal{P}^{(j)}_{\theta^{(j-1)}},$$

where $\mathcal{P}^{(j)}_{\theta^{(j-1)}}$ is the convex hull of the set $\bigcup \delta^{(j)}_{i_{d_0+1}} + \cdots + \delta^{(j)}_{i_d} + (\check{\theta}^{(j-1)} \cap \eta^{\perp})$, for $\delta^{(j)}_{i_{d_0+1}}, \dots, \delta^{(j)}_{i_d} \in \Gamma^{(j)}_{\theta^{(j-1)}}$ such that $\delta^{(j)}_{i_{d_0+1}} \wedge \cdots \wedge \delta^{(j)}_{i_d} \neq 0$, and $\langle \nu_\eta, \delta^{(j)}_{i_l} \rangle = 1$ for $l = d_0 + 1, \dots, d$.

Proof. Since η is a stable cone we get that $M(\eta) = M(\eta, \Gamma^{(j)}_\eta)$, by applying Proposition 12.21 to the constant sequence of cones $\eta_j := \eta$. By Lemma 12.18 the cone $\eta \subset (N_\eta)_{\mathbf{R}}$ is a stable cone for the semigroup $\pi(\Gamma^{(j)}_\eta)$ for $j \geq j_0 \gg 0$. By Proposition 12.20 the sequence of semigroups $(\pi(\Gamma^{(j)}_\eta))_{j \geq j_0}$ stabilizes and $\pi(\Gamma^{(j)}_\eta)$ is generated by a basis of $M/M(\eta)$, for $j \gg 0$. By Lemma 3.9 we have that $\Gamma^{(j)}_\eta = \Gamma^{(j)}_{\theta^{(j-1)}} + \mathbf{Z}_{\geq 0}(-u_j)$, for any $u_j \in \Gamma^{(j)}_{\theta^{(j-1)}}$ which belongs to $\mathrm{int}(\check{\theta}^{(j-1)} \cap \eta^{\perp})$. We deduce that $\pi(\Gamma^{(j)}_{\theta^{(j-1)}}) = \pi(\Gamma^{(j)}_\eta)$ for $j \gg 0$.

Since $\eta \leq \theta^{(j)}$ we get from (31) that $\mathrm{ord}_{\mathcal{J}^{(j)}}(\nu_\eta) = \langle \nu_\eta, m^{(j)} \rangle$. This implies that $m^{(j)}$ belongs to $\mathcal{F}^{(j)}_{\theta^{(j-1)}}$. Notice also that $m^{(j)}$ belongs to $\mathcal{J}^{(j)}_\eta$. By Proposition 12.21 i., for any representation of $m^{(j)}$ of the form (28), d_0 of the $\gamma^{(j)}_i$ belong to $\eta^{\perp}$, say for $i = 1, \dots, d_0$. By Lemma 11.5 the vectors $\pi(\gamma^{(j)}_i) = \tilde{\gamma}^{(j)}$, $i = d_0 + 1, \dots, d$ are linearly independent elements of Γ_η, such that $\tilde{m}^{(j)} = \sum_i \tilde{\gamma}^{(j)}_i$ belongs to the logarithmic jacobian ideal $\tilde{\mathcal{J}}_\eta$ of $k[t^{\tilde{\Gamma}_\eta}]$. We know that $\langle \nu, \gamma \rangle = \langle \nu, \tilde{m} \rangle$ for any $m \in M$ and $\nu \in \eta$, thus the vector $\tilde{m}^{(j)}$ is in the face of $\mathcal{N}(\tilde{\mathcal{J}}_\eta)$ determined by ν_η. By i. the semigroup $\tilde{\Gamma}_\eta$ is regular, hence we deduce that $\tilde{m}^{(j)} = \sum_{i=d_0+1}^{d} \tilde{e}_i$ and, up to relabelling, $\tilde{e}_i = \tilde{\gamma}^{(j)}_i$, $i = d_0 + 1, \dots, d$. This ends the proof of ii. and also shows that $m^{(j)} \in \mathcal{N}(\mathcal{J}^{(j)}_{\theta^{(j-1)}}(\eta)) + \mathcal{P}^{(j)}_{\theta^{(j-1)}}$.

We have also shown the equalities

$$\mathrm{ord}_{\mathcal{J}^{(j)}}(\nu_\eta) = \langle \nu_\eta, m^{(j)} \rangle = \langle \nu_\eta, \tilde{m}^{(j)} \rangle = d - d_0.$$

Finally, we remark that the argument given above applies more generally for any vector $m \in \mathcal{J}^{(j)}_{\theta^{(j-1)}}$ in the face $\mathcal{F}^{(j)}_{\theta^{(j-1)}}$. This implies that iii. holds. □

Proposition 12.24. *If $0 \neq \eta$ is a stable cone and if $(\theta^{(j)})_j$ is a sequence of cones of the form (30) such that $\theta^{(j)}$ contains η as a face of codimension one, then*

$$M(\theta^{(j-1)}, \Gamma^{(j)}_{\theta^{(j-1)}}) = M(\theta^{(j-1)}) \text{ for } j \gg 0, \tag{32}$$

and the sequence of cones $(\theta^{(j)})_{j \geq I}$ stabilizes.

Proof. We denote by q_0 the integer $\mathrm{codim}_{N_{\mathbf{R}}} \theta^{(j)}$ for $j \gg 0$ and by d_0 the codimension of η. Notice that $0 \leq q_0 < d - 1$ since $0 \neq \eta \leq \theta^{(j)}$.

We deal first with the proof of (32). By the argument given in the proof of Proposition 12.21 we get $M(\theta^{(j-1)}) = M(\theta^{(j)})$ and $M(\theta^{(j-1)}, \Gamma^{(j)}_{\theta^{(j-1)}}) = M(\theta^{(j)}, \Gamma^{(j+1)}_{\theta^{(j)}})$ for $j \gg 0$. We recall that for $j \gg 0$ the lattice spanned by the face $\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^{\perp}$ is equal to $M(\eta)$ (see Notations 12.22).

If (32) were not true then there exists $\gamma, \gamma' \in \Gamma^{(j)}_{\theta^{(j-1)}}$ such that $\gamma - \gamma' \in M(\theta^{(j-1)}) \setminus M(\theta^{(j-1)}, \Gamma^{(j)}_{\theta^{(j-1)}})$. Notice then that $\langle \nu_\eta, \gamma \rangle = \langle \nu_\eta, \gamma' \rangle$ since $M(\theta^{(j-1)}) \subset M(\eta)$ by duality.

By Lemma 12.23, if $m^{(j)} \in \mathcal{J}^{(j)}_{\theta^{(j-1)}}$ is such that $\mathrm{ord}_{\mathcal{J}^{(j)}}(\nu) = \langle \nu, m^{(j)} \rangle$ for any $\nu \in \theta^{(j)}$ then for any representation of $m^{(j)}$ of the form (28), d_0 of the $\gamma_i^{(j)}$ belong to $M(\eta)$.

By applying the argument of Proposition 12.21 we can assume, replacing j by a bigger number, that $\langle \nu_\eta, \gamma \rangle = 0$, that is γ and γ' belong to the face $\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^{\perp}$.

By hypothesis $\eta \leq \theta^{(j-1)}$ is a face of codimension one. The image of $\check{\theta}^{(j-1)} \cap \eta^{\perp}$ in $M(\eta)_{\mathbf{R}}/M(\theta^{(j-1)})_{\mathbf{R}}$ is the dual cone of the image $\bar{\theta}^{(j-1)}$ of $\theta^{(j-1)}$ in $(N_{\theta^{(j-1)}})_{\mathbf{R}}/(N_\eta)_{\mathbf{R}}$. Notice that the lattice $N_{\theta^{(j-1)}}/N_\eta$ and the cone $\bar{\theta}^{(j-1)}$ are independent of j for $j \gg 0$. We denote by $\bar{\nu}$ the generator of the semigroup $\bar{\theta}^{(j-1)} \cap (N_{\theta^{(j-1)}}/N_\eta)$ and by $\bar{\alpha}$ the class of $\alpha \in M(\eta)$ modulo $M(\theta^{(j-1)})$.

Among those $\gamma_i^{(j)}$ which belong to $\eta^{\perp}$ there exists at least one, say $\gamma_d^{(j)}$, which does not belong to $M(\theta^{(j-1)})$, hence $\langle \bar{\nu}, \bar{\gamma}_d^{(j)} \rangle \neq 0$. Notice also that $0 \neq \langle \bar{\nu}, \bar{\gamma} \rangle = \langle \bar{\nu}, \bar{\gamma}' \rangle$. Then, we apply the same argument as in the proof of Proposition 12.21 i. to get that $\gamma - \gamma_d^{(j)}$ and $\gamma' - \gamma_d^{(j)}$ belong to $\Gamma^{(j+1)}_{\theta^{(j)}} \cap \eta^{\perp}$ and $\langle \bar{\nu}, \bar{\gamma} - \bar{\gamma}_d^{(j)} \rangle < \langle \bar{\nu}, \bar{\gamma} \rangle$. By iterating this procedure, replacing γ and γ' by $\gamma - \gamma_d^{(j)}$ and $\gamma - \gamma_d^{(j)}$, respectively, we reduce to the case $0 = \langle \bar{\nu}, \bar{\gamma} \rangle = \langle \bar{\nu}, \bar{\gamma}' \rangle$. But this implies that γ, γ' belong to $M(\theta^{(j-1)}, \Gamma^{(j)}_{\theta^{(j-1)}})$, contradicting the hypothesis. This ends the proof of the equality (32).

We prove now that the sequence $(\theta^{(j)})$ stabilizes. Since (32) holds, by Lemma 12.18 we can assume that $q_0 = 0$. Then, by hypothesis the lattice $M(\eta)$ is of rank one. The cone $\check{\theta}^{(j-1)} \cap \eta^{\perp} \subset M(\eta)_{\mathbf{R}}$ is a one dimensional face of $\check{\theta}^{(j-1)}$ for all $j \gg 0$. Since $\check{\theta}^{(j-1)} \subset \check{\theta}^{(j)}$ we get that the cone $\check{\vartheta} := \check{\theta}^{(j-1)} \cap \eta^{\perp}$ is independent of j, for $j \gg 0$.

With notations of Lemma 12.23, the Minkowski sum $\pi^{-1}(\tilde{e}_i) \cap \Gamma^{(j)}_{\theta^{(j-1)}} + \check{\vartheta}$ is an affine one dimensional cone with only one vertex $\gamma_i^{(j)}$, which belongs to $\Gamma^{(j)}_{\theta^{(j-1)}}$, for $i = 2, \dots, d$. We denote by $\gamma_1^{(j)}$ the smallest generator of the rank one semigroup $\Gamma^{(j)}_{\theta^{(j-1)}} \cap \check{\vartheta}$. Notice that $\gamma_1^{(j)}$ is independent of j for $j \gg 0$.

By Lemma 12.23 the vector $m^{(j)} = \gamma_1^{(j)} + \cdots + \gamma_d^{(j)} \in \mathcal{J}_{\theta(j-1)}^{(j)}$ is the unique vertex of the face $\mathcal{F}_{\theta(j-1)}^{(j)} = m^{(j)} + \check{\vartheta}$. Since $\nu_\eta \in \theta^{(j)}$ the cone $\theta^{(j)}$ is dual to the cone spanned by $\{\gamma - m^{(j)} \mid \gamma \in \mathcal{J}_{\theta(j-1)}^{(j)}\}$.

Claim 12.25. *The semigroup $\Gamma_{\theta(j-1)}^{(j)} \cap \eta^\perp$ is equal to $M(\eta)$, for $j \gg 0$.*

Proof of the claim. By Proposition 12.21 the semigroup $\Gamma_{\theta(j-1)}^{(j)} \cap \eta^\perp = \Gamma_{\theta(j-1)}^{(j)} \cap \check{\vartheta}$ generates the group $M(\eta)$, for $j \gg 0$. If it has only one generator the result follows directly. Assume that it has at least two generators. Let us denote by $\delta > \gamma_1^{(j)}$ the second element by order of size. The element $n^{(j)} = \gamma_2^{(j)} + \cdots + \gamma_1^{(j)} + \delta$ belongs to $\mathcal{J}_{\theta(j-1)}^{(j)}$ so that $n^{(j)} - m^{(j)} = \delta - \gamma_1^{(j)}$ is in $\Gamma_{\theta(j)}^{(j+1)}$. We see that after finitely many steps the smallest generator of our semigroup has decreased, so in the end we reach the generator of the regular semigroup $\check{\vartheta} \cap M$, which proves the result. This is similar to the resolution process for one dimensional affine toric varieties by Semple-Nash modifications. ☐

By Lemma 12.23 and Claim 12.25 the elements $\gamma_1^{(j)}, \ldots, \gamma_d^{(j)}$ form a basis of M.

The expansion of $\gamma \in \Gamma_{\theta(j-1)}^{(j)}$ in terms of the basis $\gamma_1^{(j)}, \ldots, \gamma_d^{(j)}$ is of the form

$$\gamma = a_1 \gamma_1^{(j)} + \cdots + a_d \gamma_d^{(j)}, \tag{33}$$

where $a_i \in \mathbf{Z}_{\geq 0}$ and $a_d \in \mathbf{Z}$. If γ is of the form (33) then we have $\langle \nu_\eta, \gamma \rangle = \sum_{i=2}^d a_i$. In particular, we get that $\mathrm{ord}_{\mathcal{J}^{(j)}}(\nu_\eta) = \langle \nu_\eta, m^{(j)} \rangle = d - 1$.

We denote by $G^{(j)}$ the minimal generating system of the semigroup $\Gamma_{\theta(j-1)}^{(j)}$, and by $g^{(j)}$ the maximum of ν_η on the set $G^{(j)}$.

Notice that the elements $\gamma_1^{(j)}, \ldots, \gamma_d^{(j)}$ belong to $G^{(j)}$ by our assumptions. If a generator $\gamma \in \Gamma^{(j)}$ is different from $\gamma_1^{(j)}, \ldots, \gamma_d^{(j)}$ then the coefficient a_1 in (33) is < 0, and $\langle \nu_\eta, \gamma \rangle \geq 2$. Otherwise γ would be in the semigroup generated by the $\gamma_i^{(j)}$, contradicting the minimality of the generating system $G^{(j)}$. We deduce that the equality $g^{(j)} = 1$ implies that $G^{(j)} = \{\gamma_i^{(j)}\}_{i=1}^d$, hence in this case the semigroup $\Gamma_{\theta(j-1)}^{(j)}$ is regular.

Claim 12.26. *If $g^{(j)} > 1$ there exists an integer $t_0 \geq 1$ such that $g^{(j+t_0)} < g^{(j)}$.*

Proof of the claim. If $\gamma \in G^{(j)}$ with $\langle \nu_\eta, \gamma \rangle \geq 2$ is of the form (33) and if $a_l \neq 0$ then $\gamma + \sum_{i=1,\ldots,d}^{i \neq l} \gamma_i^{(j)}$ belongs to $\mathcal{J}_{\theta(j-1)}^{(j)}$ hence $\gamma - \gamma_l^{(j)} = \gamma + \sum_{i=1,\ldots,d}^{i \neq l} \gamma_i^{(j)} - m^{(j)}$ belongs to the semigroup $\Gamma_{\theta(j)}^{(j+1)}$ by definition. We call these elements *followers* of γ. Notice that $\gamma - \gamma_1^{(j)}$ is always a follower of γ with $\langle \nu_\eta, \gamma \rangle = \langle \nu_\eta, \gamma - \gamma_1^{(j)} \rangle$, while $\langle \nu_\eta, \gamma - \gamma_l^{(j)} \rangle < \langle \nu_\eta, \gamma \rangle$ if $l \neq 1$.

We show first that the semigroup $\Gamma_{\theta(j)}^{(j+1)}$ is generated by $\gamma_1^{(j)}, \ldots, \gamma_d^{(j)}$ and the followers of γ, for γ in $G^{(j)}$ with $\langle \nu_\eta, \gamma \rangle \geq 2$. Let $\gamma'_1, \ldots, \gamma'_d$ be linearly independent elements in $G^{(j)}$. Up to relabelling we can assume that $\gamma'_i = \gamma_i^{(j)}$ for $i = s+1, \ldots, d$, and $\gamma'_i \notin \{\gamma_1^{(j)}, \ldots, \gamma_s^{(j)}\}$, in particular $\langle \nu_\eta, \gamma'_i \rangle \geq 2$, for $i = 1, \ldots, s$. Then, there is a permutation $(l_1, \ldots, l_s)$ of $(1, \ldots, s)$ such that the i-th coeeffient of the expansion of γ_{l_i} in terms of the basis $\gamma_1^{(j)}, \ldots, \gamma_d^{(j)}$ is non zero, for $i = 1, \ldots, s$ (otherwise we would get $\gamma'_1 \wedge \ldots \wedge \gamma'_s = 0$, which is contrary to the assumption). We deduce that $\gamma_{l_i} - \gamma_i$ are followers of γ'_{l_i} for $i = 1, \ldots, s$, and the element $\sum_{i=1}^d \gamma'_i - m^{(j)} = \sum_{i=1}^s (\gamma'_{l_i} - \gamma_i^{(j)})$ of $\Gamma_{\theta(j)}^{(j+1)}$ is in the semigroup generated by $\gamma_1^{(j),} \ldots, \gamma_d^{(j)}$ and the followers of γ, for γ in $G^{(j)}$ with $\langle \nu_\eta, \gamma \rangle \geq 2$ as stated.

We deduce also that $m^{(j)} \neq m^{(j+1)}$ if and only if there is an element $\gamma \in G^{(j)}$ with $\langle \nu_\eta, \gamma \rangle = 2$. Indeed, if $\langle \nu_\eta, \gamma \rangle = 2$ then there is $2 \leq i \leq d$ such that $\gamma - \gamma_i^{(j-1)}$ is a follower of γ with $\langle \nu_\eta, \gamma - \gamma_i^{(j-1)} \rangle = 1$. This implies that $\gamma_i^{(j)}$ is in the semigroup generated by $\gamma - \gamma_i^{(j)}$ and $\gamma_d^{(j)}$, thus $\gamma_i^{(j)}$ does not belong to $G^{(j+1)}$ hence $\gamma_i^{(j)} \neq \gamma_i^{(j+1)}$ and $m^{(j)} \neq m^{(j+1)}$. The converse is deduced similarly.

Assume that $m^{(j)} = m^{(j+1)} = \cdots = m^{(j+t_0-1)}$ for some $t_0 \geq 1$. Let $\gamma \in G^{(j)}$ be such that $\langle \nu_\eta, \gamma \rangle = g^{(j)}$. If γ is of the form (33) the followers of γ after at most t_0 iterations are $\gamma_\alpha = \sum_{i=1}^d (a_i - \alpha_i)\gamma_i^{(j)}$, where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{Z}_{\geq 0}^d$ verify that $\sum_{i=1}^d \alpha_i \leq t_0$, $\alpha_i \leq a_i$ for $i = 2, \ldots, d$ and $\langle \nu_\eta, \gamma_\alpha \rangle = \sum_{i=2}^d (a_i - \alpha_i) \geq 1$. Those γ_α with $\langle \nu_\eta, \gamma_\alpha \rangle = \langle \nu_\eta, \gamma \rangle$ are precisely $\gamma - l\gamma_1^{(j)}$ for $0 \leq l \leq t_0$. The elements $\gamma - l\gamma_1^{(j)}$, $l = 0, \ldots, t_0 - 1$ do not belong to $G^{(j+t_0)}$ since they are in the semigroup generated by $\gamma_1^{(j)}$ and $\gamma - t_0\gamma_1^{(j)}$.

Assume that $t_0 > 1$ is the smallest integer such that $m^{(j)} \neq m^{(j+t_0)}$. Notice that $t_0 \leq g^{(j)} - 1$, otherwise we would get a follower γ_α of γ with $\langle \nu_\eta, \gamma_\alpha \rangle = 1$, which is necessarily different from the $\gamma_i^{(j)}$, $i = 2, \ldots, d$, and then $m^{(j)} \neq m^{(j+g^{(j)}-1)}$, a contradiction.

There exists $0 \neq (p_2, \ldots, p_d) \in \mathbf{Z}_{\geq 0}^{d-1}$ such that

$$\gamma_i^{(j+t_0)} = \gamma_i^{(j)} - p_i\gamma_1^{(j)},\ i = 2, \ldots, d, \quad \text{and} \quad \gamma_1^{(j+t_0)} = \gamma_1^{(j)}.$$

We deduce the expansion

$$\gamma_\alpha = \sum_{i=2}^d (a_i - \alpha_i)\gamma_i^{(j+t_0)} + (a_1 - \alpha_1 + \sum_{i=2}^d p_i(a_i - \alpha_i))\gamma_1^{(j+t_0)}. \tag{34}$$

If $a_ip_i = 0$ for $i = 2, \ldots, d$ we iterate this procedure replacing γ by $\gamma - t_0\gamma_1^{(j)}$. In at most $\langle \nu_\eta, \gamma \rangle - 1$ steps we get to the situation where at least one of the a_ip_i is non zero, say $a_dp_d \neq 0$ for simplicity. We prove that $\alpha_0 = (t_0 - 1, 0, \ldots, 0, 1)$ defines a follower of γ such that $\gamma - t_0\gamma_1^{(j)}$ belongs to the semigroup generated by γ_{α_0} and $\gamma_i^{(j+t_0)}$, $i = 1, \ldots, d$, in particular $\gamma - t_0\gamma_1^{(j)}$ does not belong to $G^{(j+t_0)}$. We check this assertion by verifying that the coefficient of the term $\gamma_i^{(j+t_0)}$ in the expansion (34) of γ_{α_0} is less than or equal to the corresponding coefficient in the expansion (34) of $\gamma - t_0\gamma_1^{(j)}$ for $i = 1, \ldots, d$. If $i = d$ we get a strict inequality. The inequality is trivial for $i = 2, \ldots, d-1$. If $i = 1$ we have to show the inequality:

$$a_1 - (t_0 - 1) - p_d + \sum_{i=2}^d p_ia_i \leq a_1 - t_0 + \sum_{i=2}^d p_ia_i.$$

This inequality is equivalent to $p_d \geq 1$, which holds since $a_dp_d \neq 0$.

Since there is a finite number of $\gamma \in G^{(j)}$ with $\langle \nu_\eta, \gamma \rangle = g^{(j)}$ there exist an integer $t_2 \geq 0$ such that $g^{(j+t_2)} < g^{(j)}$ as claimed. ☐

Using this claim and induction there exists an integer $t_1 \geq 1$ such that $g^{(j+t_1)} = 1$ hence the semigroup $\Gamma_{\theta^{(j-1)}}^{(j)}$ is regular. This ends the proof of Proposition 12.24. ☐

We say that the nested sequence of cones $(\theta^{(j)})_j$ is *distinguished* if there exist a stable cone $0 \neq \eta$ and an integer I such that for any $j \geq I$ there exists a sequence of faces of $\theta^{(j)}$

$$\eta = \zeta_0^{(j)} \leq \zeta_1^{(j)} \leq \cdots \leq \zeta_{l_0}^{(j)} = \theta^{(j)} \tag{35}$$

such that $\dim \zeta_i^{(j)} = \dim \eta + i$ and $\zeta_i^{(j)} \supset \zeta_i^{(j+1)}$ for $i = 0, \ldots, l_0$ for $l_0 \leq \operatorname{codim}\eta$.

Proposition 12.27. *If the sequence $(\theta^{(j)})_j$ is distinguished then it stabilizes.*

Proof. Let $(\theta^{(j)})_j$ be a distinguished sequence as above. We consider the sequence of faces $(\zeta_1^{(j)})_j$. Since η is a face of codimension one of $\zeta_1^{(j)}$ we get that the sequence $(\zeta_1^{(j)})_j$ stabilizes by Proposition 12.24. We proceed replacing $\eta = \zeta_0^{(j)}$ and $\zeta_1^{(j)}$ by $\zeta_1^{(j)}$ and $\zeta_2^{(j)}$ respectively, in the previous argument, and then the result follows by induction on the length l_0 of the sequence (35). ☐

Proposition 12.28. *The following assertions are equivalent:*

(i) *The sequence of fans* $(\Sigma^{(j)})_{j\geq 1}$ *stabilizes.*

(ii) *If* $0 \neq \nu \in \sigma \cap N$ *then there exists an integer* $l = l(\nu) \geq 1$ *such that if* $\nu \in \tau^{(l)} \in \Sigma^{(l)}(d)$ *then* $\tau^{(l)}$ *is a stable cone.*

(iii) *If* $0 \neq \eta \in \Sigma^{(j)}$, $j \geq I$ *is a stable cone then the sequence of fans* $(\Sigma^{(j)}(\eta))_{j\geq I}$ *stabilizes.*

Proof. $(ii) \Rightarrow (i)$. Let us fix a vector $m \in M$ in the interior of the cone $\check{\sigma}$ and the convex polytope $\Theta := \{\nu \in \sigma \mid \langle \nu, m\rangle = 1\}$ which is rational for the lattice $N \cap m^{\perp}$. If $0 \neq \nu \in \sigma$ the ray $\nu \mathbf{R}_{\geq 0}$ intersects the polytope Θ exactly at one point $\xi(\nu)$.

For $0 \neq \nu \in \sigma \cap N$ we denote by $l(\nu)$ the integer verifying the hypothesis. The set

$$U_\nu := \operatorname{int}(\bigcup_{\nu\in\tau, \tau\in\Sigma^{(l(\nu))}} \tau) \cap \Theta$$

is an open subset of Θ which contains $\xi(\nu)$. It follows that $\{U_\nu \mid 0 \neq \nu \in \sigma \cap N\}$ is an open covering of the compact set Θ; there exists a finite subcovering $\{U_{\nu_1}, \dots, U_{\nu_s}\}$.

If $l = \max_{i=1,\dots,s}\{l(\nu_i)\}$ then any cone $\tau \in \Sigma^{(l)}(d)$ is stable hence the sequence $(\Sigma^{(j)})$ stabilizes.

$(i) \Rightarrow (iii)$. It is trivial from the definitions.

$(iii) \Rightarrow (ii)$. If $0 \neq \nu \in \sigma \cap N$ and if $\theta_j \in \Sigma^{(j)}$ is such that $\nu \in \operatorname{int}\theta^{(j)}$ then the sequence $(\theta^{(j)})$ stabilizes by Proposition 12.21. We denote by ϑ the cone $\theta^{(j)}$, $j \gg 1$. If the codimension q_0 of ϑ is zero then we have finished.

Suppose that $q_0 > 0$. Let $\tau^{(j)} \in \Sigma^{(j)}(d)$ be a cone containg ϑ as a face. If $j \gg 0$ the cone $\tau^{(j)}(\vartheta)$ is stable, since the sequence $\Sigma^{(j)}(\vartheta)$ stabilizes by hypothesis. Then there is a unique cone $\tau^{(j+1)} \in \Sigma^{(j+1)}$ with

$$\vartheta \leq \tau^{(j+1)},\ \tau^{(j)} \supset \tau^{(j+1)} \text{ and } \tau^{(j)}(\vartheta) = \tau^{(j+1)}(\vartheta).$$

For $j \gg 0$ we fix a chain of faces of $\tau^{(j)}(\vartheta)$

$$0 = \zeta_0(\vartheta) \leq \zeta_1(\vartheta) \leq \cdots \leq \zeta_{q_0}(\vartheta) = \tau^{(j)}(\vartheta)$$

with $\dim \zeta_i(\vartheta) = i$ we get a chain of faces

$$\eta = \zeta_0^{(j)} \leq \zeta_1^{(j)} \leq \cdots \leq \zeta_{q_0}^{(j)} = \tau^{(j)}$$

such that $\zeta_i^{(j)} \supset \zeta_i^{(j+1)}$, $\zeta_i^{(j)}(\vartheta) = \zeta_i(\vartheta)$, and the cone $\zeta_i^{(j)}$ is a face of codimension one of $\zeta_{i+1}^{(j)}$. That is, the sequence $(\tau^{(j)})$ is distinguished. By Proposition 12.27 the sequence $\tau^{(j)}$ stabilizes. Since there are finitely many q_0 dimensional faces in the fan $\Sigma^{(j)}(\vartheta)$ it follows that for $j \gg 0$ the cone ϑ, and hence the vector ν, is contained in the interior of the union of stable cones $\bigcup_{\tau^{(j)}(\vartheta)\ \in \Sigma^{(j)}(\vartheta)} \tau^{(j)}$. ☐

Remark 12.29. It is interesting to verify that propositions 12.28, 12.24 and 12.21 imply resolution by logarithmic jacobian blowing-ups in the two-dimensional case.

Proposition 12.30. *Let* $\eta \in \Sigma^{(j)}$, $j \geq I$, *be a stable cone of codimension* $q_0 > 0$. *The following assertions are equivalent.*

(i) *The sequence of fans* $(\Sigma^{(j)}(\eta))_{j\geq I}$ *stabilizes.*

(ii) *If* $0 \neq w \in |\Sigma^{(j)}(\eta)| \cap N(\eta)$ *then there exists an integer* $l = l(w) \geq I$ *such that if* $w \in \tau^{(l)}(\eta)$, $\tau^{(l)}(\eta) \in \Sigma^{(l)}(\eta)$ *then* $\tau^{(l)}$ *is stable.*

Proof. The proof of $(ii) \Rightarrow (i)$is analogous to the implication $(ii) \Rightarrow (i)$ in Proposition 12.11. The implication $(i) \Rightarrow (ii)$ is trivial. □

Remark 12.31. If $0 \neq \eta$ is a stable cone, for $j \gg 0$, the blowing up of the logarithmic jacobian ideal induces a proper toric modification $\xi_\eta^{(j)} : T_{\Sigma^{(j+1)}}^{\Gamma^{(j+1)}}(\eta) \to T_{\Sigma^{(j)}}^{\Gamma^{(j)}}(\eta)$. As we explain below, the map $\xi_\eta^{(j)}$ is not necessarily equal to the blowing up of the logarithmic jacobian ideal of $T_{\Sigma^{(j)}}^{\Gamma^{(j)}}(\eta)$.

Proposition 12.32. *Given a stable cone* η *of codimension* d_0*, a vector* $0 \neq w \in N(\eta)$ *and a sequence of cones* $\theta^{(j)}(\eta) \in \Sigma^{(j)}(\eta)$ *such that* $\theta^{(j)}(\eta) \supset \theta^{(j+1)}(\eta)$ *for* $j \gg 0$*. Then, if* $w \in \mathrm{int}(\theta^{(j)}(\eta))$ *for* $j \gg 0$*, we have that*

$$M(\theta^{(j-1)}(\eta), \Gamma_{\theta^{(j-1)}}^{(j)} \cap \eta^\perp) = M(\theta^{(j-1)}(\eta)) \textit{ for } j \gg 0, \tag{36}$$

and the sequence $\theta^{(j)}(\eta)$ *stabilizes.*

Proof. We use notations 12.22. We denote by q_0 the codimension of $\theta^{(j)}(\eta)$ for $j \gg 0$. If $\theta^{(j-1)} \in \Sigma^{(j-1)}$ is a cone such that $\eta \leq \theta^{(j-1)}$ and its image in $N(\eta)$ is equal to $\theta^{(j-1)}(\eta)$ then by definition we get $M(\theta^{(j-1)}(\eta), \Gamma_{\theta^{(j-1)}}^{(j)} \cap \eta^\perp) = M(\theta^{(j-1)}, \Gamma_{\theta^{(j-1)}}^{(j)} \cap \theta^{(j-1)\perp}) \subset M(\eta, \Gamma_\eta^{(j)})$ and $M(\theta^{(j-1)}(\eta)) = M(\theta^{(j-1)}) \subset M(\eta)$.

We obtain that $M(\theta^{(j-1)}(\eta), \Gamma_{\theta^{(j-1)}}^{(j)} \cap \eta^\perp) = M(\theta^{(j)}(\eta), \Gamma_{\theta^{(j)}}^{(j+1)} \cap \eta^\perp)$ and $M(\theta^{(j-1)}(\eta)) = M(\theta^{(j)}(\eta))$ for $j \gg 0$. Then, the proof of formula (36) follows similarly as that of (32) and of Proposition 12.21 ii.

We deduce from (36) and Lemma 12.23 that it is enough to prove the result in the case $\mathrm{codim}\theta^{(j)}(\eta) = 0$. In this case, the cone $\theta^{(j-1)}(\eta)$ is the image of a cone $\theta^{(j-1)} \in \Sigma^{(j)}(d)$ by the projection $\pi \colon N_{\mathbf{R}} \to N(\eta)_{\mathbf{R}}$.

The charts of the blowing up of the logarithmic jacobian ideal of $T_{\theta^{(j-1)}}^{\Gamma^{(j)}}$ which are defined by those d dimensional cones which contain η, are in bijection with the vertices of the face $\mathcal{F}_{\theta^{(j-1)}}^{(j)}$ of the Newton polyhedron of the logarithmic jacobian ideal $\mathcal{J}_{\theta^{(j-1)}}^{(j)}$ determined by the vector ν_η (see Lemma 12.23).

One of these charts $T_{\theta^{(j)}}^{\Gamma^{(j+1)}}$, corresponding to a vertex $m^{(j)}$ of $\mathcal{F}_{\theta^{(j-1)}}^{(j)}$ is such that the image of the cone $\theta^{(j)}$ in $N(\eta)$ is equal to the cone $\theta^{(j)}(\eta)$ in the sequence. By hypothesis $w \in N(\eta)$ is in the interior of $\theta^{(j)}(\eta)$, hence the face of $\mathcal{F}_{\theta^{(j-1)}}^{(j)}$ supported by the vector w is equal to the vertex $m^{(j)}$.

By Lemma 12.23 we get that $m^{(j)} = \bar{m}^{(j)} + m'^{(j)}$ where $m^{(j)}$ is a vertex of the Newton polyhedron of the logarithmic jacobian ideal $\mathcal{J}_{\theta^{(j-1)}}^{(j)}(\eta)$ of $k[t^{\Gamma_{\theta^{(j-1)}}^{(j)}}]$ and $m'^{(j)}$ is a vertex of $\mathcal{P}_{\theta^{(j-1)}}^{(j)}$. We choose representations

$$\bar{m}^{(j)} = \gamma_1^{(j)} + \cdots + \gamma_{d_0}^{(j)} \text{ and } m'^{(j)} = \gamma_{d_0+1}^{(j)} + \cdots + \gamma_d^{(j)}, \tag{37}$$

as sum of linearly independent elements of $\Gamma_{\theta^{(j-1)}}^{(j)}$.

Remark that the chart associated to $\bar{m}^{(j)}$ of the blowing up of the logarithmic jacobian ideal $\mathcal{J}_{\theta^{(j-1)}}^{(j)}(\eta)$ of $k[t^{\Gamma_{\theta^{(j-1)}}^{(j)}}]$ is defined by the semigroup $S_{j,w}^{(j+1)}$ generated by $\Gamma_{\theta^{(j-1)}}^{(j)} \cap \eta^\perp$ and vectors of the form $\gamma - \bar{m}^{(j)}$ for γ in $\mathcal{J}_{\theta^{(j-1)}}^{(j)}(\eta)$. The inclusion

$$S_{j,w}^{(j+1)} \subset \Gamma_{\theta^{(j)}}^{(j+1)} \cap \eta^\perp, \tag{38}$$

may be strict (cf. Remark 12.31). The reason is the following. If $\gamma \in \Gamma^{(j)}_{\theta^{(j-1)}}$ and if $\gamma \wedge \gamma^{(j)}_1 \wedge \cdots \wedge \gamma^{(j)}_{i-1} \wedge \gamma^{(j)}_{i+1} \wedge \cdots \wedge \gamma^{(j)}_d \neq 0$ then $\delta := \gamma + m^{(j)} - \gamma^{(j)}_i$ belongs to $\mathcal{J}^{(j)}_{\theta^{(j-1)}}$ and $\delta - m^{(j)} = \gamma - \gamma^{(j)}_i \in \Gamma^{(j+1)}_{\theta^{(j)}}$. If in addition, $\langle \nu_\eta, \gamma\rangle = \langle \nu_\eta, \gamma^{(j)}_i\rangle$ we obtain that $\delta - m^{(j)}$ belongs to $\Gamma^{(j+1)}_{\theta^{(j)}} \cap \eta^\perp$, even if $\gamma \notin \eta^\perp$.

Notice that the sequence $(\langle w, \bar{m}^{(j)}\rangle)$ becomes stationary for $j \gg 0$, since w belongs to the interior of $\theta^{(j-1)}(\eta)$.

If the sequence $(\theta^{(j)}(\eta))$ does not stabilize then by Proposition 12.20 the toric variety $T^{\Gamma^{(j)}_{\theta^{(j-1)}}}$ is not smooth for any integer $j \geq 1$. Let us fix some $j_1 \gg 0$. By Proposition 12.12, if the toric variety $T^{\Gamma^{(j_1)}_{\theta^{(j_1-1)}}}$ is not smooth then there is a smallest $j_2 > j_1$ such that $m^{(j_1)} \neq m^{(j_2)}$. By (19) we get that $m^{(j_1)} - m^{(j_2)} \in \Gamma^{(j_2+1)}_{\theta^{(j_2)}}$, which can be written

$$m^{(j_1)} - m^{(j_2)} = (\bar{m}^{(j_1)} - \bar{m}^{(j_2)}) + ({m'}^{(j_1)} - {m'}^{(j_2)}),$$

belongs to the face $\Gamma^{(j_2+1)}_{\theta^{(j_2)}} \cap \eta^\perp$ of $\Gamma^{(j_2+1)}_{\theta^{(j_2)}}$. It follows that both terms $\bar{m}^{(j_1)} - \bar{m}^{(j_2)}$ and ${m'}^{(j_1)} - {m'}^{(j_2)}$ belong to the face $\Gamma^{(j_2+1)}_{\theta^{(j_2)}} \cap \eta^\perp$, and at least one of them is non zero.

If $\bar{m}^{(j_1)} - \bar{m}^{(j_2)} \neq 0$ then we get that $\langle w, \bar{m}^{(j_1)} - \bar{m}^{(j_2)}\rangle = 0$. This implies that $\bar{m}^{(j_1)} - \bar{m}^{(j_2)} \in (\Gamma^{(j_2+1)}_{\theta^{(j_2)}} \cap \eta^\perp) \cap w^\perp$, but since $w \in \mathrm{int}(\theta^{(j_2)}(\eta))$ the face $\Gamma^{(j_2+1)}_{\theta^{(j_2)}} \cap \eta^\perp \cap w^\perp$ is reduced to zero, a contradiction.

We deduce that $\bar{m}^{(j)} = \bar{m}^{(j_1)}$ for all $j \geq j_1$. We distinguish two cases:

If $\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^\perp$ is a regular semigroup for all $j \geq j_1$ then it follows that $\gamma^{(j)}_i$, $i = 1, \dots, d_0$ is a basis of $\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^\perp$. Since $\bar{m}^{(j)} = \bar{m}^{(j+1)}$ and $\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^\perp \subset \Gamma^{(j+1)}_{\theta^{(j)}} \cap \eta^\perp$ the equality $\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^\perp = \Gamma^{(j+1)}_{\theta^{(j)}} \cap \eta^\perp$ holds, for all $j \geq j_1$. That means that the sequence $\theta^{(j)}(\eta)$ stabilizes, contradicting the assumption.

If $\Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^\perp$ is not regular for some $j_0 \geq j_1$ then we define inductively a sequence of semigroups $S^{(j)}_{j_0,w}$, $j > j_0$ such that $S^{(j)}_{j_0,w} \subset \Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^\perp$ and $\bar{m}^{(j_1)}$ is the vertex supported by w of the Newton polyhedron of the logarithmic jacobian ideal of $k[t^{S^{(j)}_{j_0,w}}]$. For $j = j_0 + 1$ the assertion holds for $S^{(j_0+1)}_{j_0,w}$ Assume then by induction the assertion for $j-1$ and define $S^{(j)}_{j_0,w}$ the semigroup generated by $S^{(j-1)}_{j_0,w}$ and $\gamma - \bar{m}^{(j_1)}$, with γ running through the set defining the logarithmic jacobian ideal of $k[t^{S^{(j-1)}_{j_0,w}}]$. We deduce similarly as (38) that the inclusion $S^{(j)}_{j_0,w} \subset \Gamma^{(j)}_{\theta^{(j-1)}} \cap \eta^\perp$ holds. Since the vector w supports the vertex $\bar{m}^{(j_1)}$ of the polyhedron $\mathcal{N}(\mathcal{J}^{(j)}_{\theta^{(j-1)}}(\eta))$ it follows that the face supported by w of the Newton polyhedron of the logarithmic jacobian ideal of $k[t^{S^{(j)}_{j_0,w}}]$ is a vertex equal to $\bar{m}^{(j_1)}$ and this holds by induction for all $j > j_1$. This contradiction with Corollary 12.16 ends the proof of Proposition 12.32. ☐

Proposition 12.33. *Let $\eta \in \Sigma^{(j)}$, $j \geq I$, be a stable cone. Then there exists an integer $l(\eta) \geq I$ such that the sequence of fans $(\Sigma^{(j)}(\eta))_{j \geq I}$ is constant for $j \geq l(\eta)$.*

Proof. We prove the it by induction on the codimension q_0 of η. There is nothing to prove if $q_0 = 0$. If $q_0 = 1$ the fan $\Sigma^{(j)}(\eta)$ has at most two cones which are one dimensional and then the assertion follows by Claim 12.25.

We suppose the assertion true for stable cones of codimension less than $q_0 \leq d-1$ and we prove it for an stable cone η of codimension q_0.

We show that if $0 \neq w \in N(\eta)$ and if $\tau^{(j)}(\eta) \in \Sigma^{(j)}(\eta)$ is a cone of dimension q_0 such that $w \in \tau^{(j)}(\eta)$ and $\tau^{(j)}(\eta) \supset \tau^{(j+1)}(\eta)$ for all $j \gg I$, then the sequence of cones $(\tau^{(j)}(\eta))$ stabilizes. Then we prove that $\Sigma^{(j)}(\eta)$ stabilizes.

Let $\theta^{(j)}(\eta)$ be the face of $\tau^{(j)}(\eta)$ such that $w \in \mathrm{int}(\theta^{(j)}(\eta))$. By Proposition 12.32 the sequence $\theta^{(j)}(\eta)$ stabilizes for $j \gg 0$. We denote by $\vartheta(\eta)$ the cone $\theta^{(j)}(\eta)$ for $j \gg 0$. We distinguish two cases.

If $\dim \vartheta(\eta) = q_0$ then we have finished.

If $1 \leq \dim \vartheta(\eta) < q_0$ then we consider a sequence $\tau^{(j)} \in \Sigma^{(j)}$ such that $\eta \leq \tau^{(j)} \in \Sigma^{(j)}$, $\tau^{(j)} \subset \tau^{(j+1)}$ and the image of $\tau^{(j)}$ in $N(\eta)$ is $\tau^{(j)}(\eta)$. We denote also by $\theta^{(j)}$ the unique face of $\tau^{(j)}$ such that $\theta^{(j)}(\eta) = \vartheta(\eta)$ for $j \gg 0$. By the same construction as in the proof of implication $(iii) \Rightarrow (ii)$ in Proposition 12.28 we get that the sequence $\theta^{(j)}$ is distinguished for $j \gg 0$ hence it stabilizes by Proposition 12.27. We denote by ϑ the the cone $\theta^{(j)}$ for $j \gg 0$. By the induction hypothesis we get that the sequence $(\Sigma^{(j)}(\vartheta))_j$ stabilizes for $j \gg 0$, since ϑ is of codimension less than q_0.

Notice that for $j \gg 0$ we have that $\Sigma^{(j)}(\vartheta) = \Sigma^{(j)}(\eta)(\vartheta(\eta))$; this implies that the set $\{\tau^{(j)}(\eta) \in \Sigma^{(j)}(\eta) \mid \vartheta(\eta) \leq \tau^{(j)}(\eta)\}$ is stabilized for $j \gg 0$.

This shows that assertion (ii) in Proposition 12.30 holds hence also assertion (i) as required. □

Theorem 12.34. *Any toric variety has a canonical resolution of singularities by iteration of blowing ups of the sheaf of logarithmic jacobian ideals.*

Proof. It is enough to prove it in the affine case. We can assume in addition (see Remark 10.3) that the affine toric variety has no torus factors, i.e., it is defined by a semigroup Γ such that the cone $\check{\sigma} = \mathbf{R}_{\geq 0}\Gamma$ is *strictly convex* and of dimension $d = \dim M_{\mathbf{R}}$.

By Propositions 12.28 and 12.33 we get that there exists an integer $l \geq 1$ such that the sequence of fans $(\Sigma^{(j)})$ stabilizes for $j \geq I$. Then applying Proposition 12.20 to the finitely many cones of $\Sigma^{(I)}$ we deduce that there exists an integer $l' \geq I$ such that the iterated blow up of logarithmic jacobian ideals $T_{\Sigma^{(l')}}^{\Gamma^{(l')}}$ is smooth. □

Remark 12.35. The recent paper [1] suggests that it would be interesting to develop an approach from a computational viewpoint to the iteration of Semple-Nash modification.

Acknowledgements. We are grateful to Monique Lejeune-Jalabert for introducing us to the reference [20], to Ezra Miller for bringing to our attention the work of H.M. Thompson, to Michael Thaddeus for his perspicacious and useful remarks on a previous version of this work, and to the *Institut Mathématique de Jussieu* and the *Dpto. Álgebra, Universidad Complutense de Madrid* for their hospitality.

References

1. A. Atanasov, C. Lopez, A. Perry, N. Proudfoot, and M. Thaddeus, *Resolving toric varieties with Nash blow-ups*, ArXiv math.AG/09105028 (2009).
2. N. Bourbaki, *Éléments de mathématique. VII. Première partie: Les structures fondamentales de l'analyse. Livre II: Algèbre. Chapitre III: Algèbre multilinéaire*, Actualités Sci. Ind., no. 1044, Hermann et Cie., Paris, 1948.
3. V. I. Danilov, *The geometry of toric varieties*, Uspekhi Mat. Nauk **33** (1978), no. 2(200), 85–134, 247.
4. D. Eisenbud and B. Sturmfels, *Binomial ideals*, Duke Math. J. **84** (1996), no. 1, 1–45.
5. G. Ewald, *Combinatorial convexity and algebraic geometry*, Graduate Texts in Mathematics, vol. 168, Springer-Verlag, New York, 1996.
6. W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993, The William H. Roever Lectures in Geometry.
7. I. M. Gel′fand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston Inc., Boston, MA, 1994.

8. P. D. González Pérez and B. Teissier, *Embedded resolutions of not necessarily normal affine toric varieties*, C.R. Math. Acad. Sci. Paris **334** (2002), no. 5, 379–382.
9. G. Gonzalez Sprinberg, *Éventails en dimension 2 et transformé de Nash*, Publications du Centre de Mathématiques de l'E.N.S., 45, rue d'Ulm, Paris 5e (1977), 1–68.
10. ______, *Transformé de Nash et éventail de dimension* 2, C. R. Acad. Sci. Paris Sér. A-B **284** (1977), no. 1, A69–A71.
11. ______, *Résolution de Nash des points doubles rationnels*, Ann. Inst. Fourier (Grenoble) **32** (1982), no. 2, x, 111–178.
12. R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.
13. H. Hironaka, *On Nash blowing-up*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Mass., 1983, pp. 103–111.
14. K. Kaveh and A. G. Khovanskii, *Newton convex bodies, semigroups of integral points, graded algebras and intersection theory*, ArXiv math.AG/09043350 (2009).
15. G. Kempf, F. F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin, 1973.
16. M. Lejeune-Jalabert and A. J. Reguera, *The Denef-Loeser series for toric surface singularities*, Proceedings of the International Conference on Algebraic Geometry and Singularities (Spanish) (Sevilla, 2001), Rev. Mat. Iberoamericana vol. 19, 2003, pp. 581–612.
17. A. Nobile, *Some properties of the Nash blowing-up*, Pacific J. Math. **60** (1975), no. 1, 297–305.
18. T. Oda, *Torus embeddings and applications*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, vol. 57, Tata Institute of Fundamental Research, Bombay, 1978, Based on joint work with Katsuya Miyake.
19. ______, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 15, Springer-Verlag, Berlin, 1988.
20. J. G. Semple, *Some investigations in the geometry of curve and surface elements*, Proc. London Math. Soc. (3) **4** (1954), 24–49.
21. M. Spivakovsky, *Sandwiched singularities and desingularization of surfaces by normalized Nash transformations*, Ann. of Math. (2) **131** (1990), no. 3, 411–491.
22. B. Sturmfels, *Gröbner bases and convex polytopes*, University Lecture Series, vol. 8, American Mathematical Society, Providence, RI, 1996.
23. H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28.
24. B. Teissier, *The hunting of invariants in the geometry of discriminants*, Real and complex singularities, Oslo 1976, Sijthoff and Noordhoff International publishers, Alphen aan den Rijn, 1977, pp. 565–677.
25. ______, *Valuations, deformations, and toric geometry*, Valuation theory and its applications, Vol. II (Saskatoon, SK, 1999), Fields Inst. Commun., vol. 33, Amer. Math. Soc., Providence, RI, 2003, pp. 361–459.
26. H. M Thompson, *Fan is to monoid as scheme is to ring: a generalization of the notion of a fan*, ArXiv math.AG/0306221.
27. ______, *Comments on toric varieties*, ArXiv math.AG/0310336.
28. ______, *Toric singularities revisited*, J. Algebra 299 (2006), no. 2, 503–534.
29. H. Whitney, *Tangents to an analytic variety*, Ann. of Math. (2) 81 1965 496–549.

Instituto de Ciencias Matemáticas-CSIC-UAM-UC3M-UCM. Dpto. de Algebra. Facultad de Ciencias Matemáticas. Universidad Complutense de Madrid. Plaza de las Ciencias 3. 28040. Madrid. Spain.

E-mail address: pgonzalez@mat.ucm.es

Institut Mathématique de Jussieu, UMR 7586 du CNRS, 175 Rue du Chevaleret, 75013 Paris, France

E-mail address: teissier@math.jussieu.fr