Extending a valuation centered in a local domain to the formal completion.

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1 Introduction

All the rings in this paper will be commutative with 1.

Let (R, m, k) be a local noetherian domain with field of fractions K and R_ν a valuation ring, dominating R (not necessarily birationally). Let $\nu|_K : K^* \twoheadrightarrow \Gamma$ be the restriction of ν to K; by definition, $\nu|_K$ is centered at R. Let $\hat{R}$ denote the m-adic completion of R (which, of course, need not in general be a domain). In the applications of valuation theory to commutative algebra and the study of singularities, one is often induced to replace R by its m-adic completion $\hat{R}$ and ν by a suitable extension $\hat{\nu}$ to $\frac{\hat{R}}{P}$ for a suitably chosen prime ideal P, such that $P \cap R = (0)$ (below, we will mention two specific applications we have in mind). It is well known and not hard to prove that such extensions $\hat{\nu}$ exist for some minimal prime ideals P of $\hat{R}$. In general, such a $\hat{\nu}$ is far from being unique. The purpose of this paper is to give, assuming that R is excellent, a systematic description of all such extensions $\hat{\nu}$ and to identify certain classes of extensions which are of particular interest for applications. In fact, the only assumption about R we ever use in this paper is a weaker and more natural condition than excellence, called the G condition, but we chose to talk about excellent rings since this terminology seems to be more familiar to most people. For the reader's convenience, the definitions of excellent and G-rings are recalled in the Appendix.

Under this assumption, we show that **an extension to the completion $\hat{R}$ of a valuation ν of rank r has rank at most $2r$ and we give descriptions of the valuations with which**

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it is composed. In particular we give criteria for the uniqueness of the extension if certain simple data on these composed valuations are fixed. Another consequence of our work (see Theorem 1.1 below) is the following result, announced in [17]:
Given an excellent local ring R and a valuation ν of R positive on its maximal ideal m, there exists a prime ideal H of the m-adic completion $\hat{R}$ such that $H\bigcap R=(0)$ and an extension of ν to $\frac{\hat{R}}{H}$ which has the same value group.

When studying the extensions of ν to the completion of R, one is led to the study of its extensions to the henselization $\tilde{R}$ of R as a natural first step. This, in turn, leads to the study of extensions of ν to finitely generated local strictly étale extensions R^e of R. We therefore start out by letting $\sigma:R\to R^\dagger$ denote one of the three operations of completion, (strict) henselization, or a finitely generated local strictly étale extension:

$$R^\dagger = \hat{R} \quad \text{or} \tag{1}$$
$$R^\dagger = \tilde{R} \quad \text{or} \tag{2}$$
$$R^\dagger = R^e. \tag{3}$$

The ring $R^\dagger$ is local; let $m^\dagger$ denote its maximal ideal. The homomorphisms $R\to\tilde{R}$ and $R\to R^e$ are regular for any ring R; by definition, if R is an excellent ring then the completion homomorphism is regular (in fact, regularity of the completion homomorphism is precisely the defining property of G-rings; see the Appendix for the definition of regular homomorphism).

Let r denote the (real) rank of ν. Let $(0)=\Delta_r\subsetneqq\Delta_{r-1}\subsetneqq\cdots\subsetneqq\Delta_0=\Gamma$ be the isolated subgroups of Γ and $P_0=(0)\subsetneqq P_1\subseteq\cdots\subseteq P_r=m$ the prime valuation ideals of R, which need not, in general, be distinct. In this paper, we will assume that R is excellent. Under this assumption, we will canonically associate to ν a chain $H_1\subset H_3\subset\cdots\subset H_{2r+1}=mR^\dagger$ of ideals of $R^\dagger$, numbered by odd integers from 1 to $2r+1$, such that $H_{2\ell+1}\cap R=P_\ell$ for $0\le\ell\le r$. We will show that all the ideals $H_{2\ell+1}$ are prime. We will define $H_{2\ell}$ to be the unique minimal prime ideal of $P_\ell R^\dagger$, contained in $H_{2\ell+1}$ (that such a minimal prime is unique follows from the regularity of the homomorphism σ).

We will thus obtain, in the cases (1)–(3), a chain of $2r+1$ prime ideals

$$H_0\subset H_1\subset\cdots\subset H_{2r}=H_{2r+1}=mR^\dagger,$$

satisfying $H_{2\ell}\cap R=H_{2\ell+1}\cap R=P_\ell$ and such that $H_{2\ell}$ is a minimal prime of $P_\ell R^\dagger$ for $0\le\ell\le r$. Moreover, if $R^\dagger=\tilde{R}$, then $H_{2\ell}=H_{2\ell+1}$. We call H_i the **i-th implicit prime ideal** of $R^\dagger$, associated to R and ν. The ideals H_i behave well under local blowing ups along ν (that is, birational local homomorphisms $R\to R'$ such that ν is centered in R'), and more generally under *ν-extensions* of R defined below in subsection 1.1. This means that given any local blowing up along ν or ν-extension $R\to R'$, the i-th implicit prime ideal H'_i of $R'^\dagger$ has the property that $H'_i\cap R^\dagger=H_i$. This intersection has a meaning in view of Lemma 1.1 below.

For a prime ideal P in a ring R, $\kappa(P)$ will denote the residue field $\frac{R_P}{PR_P}$.

Let $(0)\subsetneqq\mathbf{m}_1\subsetneqq\cdots\subsetneqq\mathbf{m}_{r-1}\subsetneqq\mathbf{m}_r=\mathbf{m}_\nu$ be the prime ideals of the valuation ring R_ν. By definitions, our valuation ν is a composition of r rank one valuations $\nu=\nu_1\circ\nu_2\cdots\circ\nu_r$, where ν_ℓ is a valuation of the field $\kappa(\mathbf{m}_{\ell-1})$, centered at $\frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}}$ (see [19], Chapter VI, §10, p. 43 for the definition of composition of valuations; more information and a simple example of composition is given below in subsection 1.1, where we interpret each $\mathbf{m}_\ell$ as the limit of a tree of ideals).

If $R^\dagger=\tilde{R}$, we will prove that there is a unique extension $\tilde{\nu}$ of ν to $\frac{R^\dagger}{H_0}$. If $R^\dagger=\hat{R}$, the situation is more complicated. First, we need to discuss the behaviour of our constructions under ν-extensions.

1.1 Local blowings up and trees.

We consider *extensions* $R \to R'$ of local rings, that is, injective morphisms such that R' is an R-algebra essentially of finite type and $m' \cap R = m$. In this paper we consider only extensions with respect to ν; that is, both R and R' are contained in a fixed valuation ring R_ν. Such extensions form a direct system $\{R'\}$. We will consider many direct systems of rings and of ideals indexed by $\{R'\}$; direct limits will always be taken with respect to the direct system $\{R'\}$. Unless otherwise specified, we will assume that

$$\lim_{\overrightarrow{R'}} R' = R_\nu. \tag{4}$$

Remark that by the fundamental properties of valuation rings ([20], §VI.), assuming the equality (4) is equivalent to assuming that $\lim\limits_{\overrightarrow{R'}} K' = K_\nu$, where K' stands for the field of fractions of R' and K_ν for that of R_ν and that $\lim\limits_{\overrightarrow{R'}} R'$ is a valuation ring.

Definition 1.1 *A* ***tree*** *of R'-algebras is a direct system $\{S'\}$ of rings, indexed by the directed set $\{R'\}$, where S' is an R'-algebra. Note that the maps are not necessarily injective. A morphism $\{S'\} \to \{T'\}$ of trees is the datum of a map of R'-algebras $S' \to T'$ for each R' commuting with the tree morphisms for each map $R' \to R''$.*

Lemma 1.1 *Let $R \to R'$ be an extension of local rings. We have:*
1) The ideal $N := m^\dagger \otimes_R 1 + 1 \otimes_R m'$ is maximal in the R-algebra $R^\dagger \otimes_R R'$.
2) The natural map of completions (resp. henselizations) $R^\dagger \to R'^\dagger$ is injective.

Proof: 1) follows from that fact that $R^\dagger/m^\dagger = R/m$. The proof of 2) relies on a construction which we shall use often: the map $R^\dagger \to R'^\dagger$ can be factored as

$$R^\dagger \to \left(R^\dagger \otimes_R R'\right)_N \to R'^\dagger, \tag{5}$$

where the first map sends x to $x \otimes 1$ and the second is determined by $x \otimes x' \mapsto \hat{b}(x).c(x')$ where $\hat{b}$ is the natural map $R^\dagger \to R'^\dagger$ and c is the canonical map $R' \to R'^\dagger$. The first map is injective because $R^\dagger$ is a flat R-algebra and it is obtained by tensoring the injection $R \to R'$ by the R-algebra $R^\dagger$; furthermore, elements of $R^\dagger$ whose image in $R^\dagger \otimes_R R'$ lie outside of N are precisely units of $R^\dagger$, hence they are not zero divisors in $R^\dagger \otimes_R R'$ and $R^\dagger$ injects in every localization of $R^\dagger \otimes_R R'$.
Since $m' \cap R = m$, we see that the inverse image by the natural map of R'-algebras

$$\iota \colon R' \to (R^\dagger \otimes_R R')_N,$$

defined by $x' \mapsto 1 \otimes_R x'$, of the maximal ideal $M = (m^\dagger \otimes_R 1 + 1 \otimes_R m')(R^\dagger \otimes_R R')_N$ of $(R^\dagger \otimes_R R')_N$ is the ideal m' and that ι induces a natural isomorphism $\frac{R'}{m'^i} \xrightarrow{\sim} \frac{(R^\dagger \otimes_R R')_N}{M^i}$ for each i. From this it follows by the universal properties of completion and henselization that the second map in the sequence (5) is the completion (resp. the henselization inside the completion) of $R^\dagger \otimes_R R'$ with respect to the ideal M. It is therefore also injective.

Definition 1.2 *Let $\{S'\}$ be a tree of R'-algebras. For each S', let I' be an ideal of S'. We say that $\{I'\}$ is a tree of ideals if for any arrow $b_{S'S''} \colon S' \to S''$ in our direct system, we have $b^{-1}_{S'S''} I'' = I'$. We have the obvious notion of inclusion of trees of ideals. In particular, we may speak about chains of trees of ideals.*

Examples. The maximal ideals of the local rings of our system $\{R'\}$ form a tree of ideals. For a prime ideal P in a ring R, $\kappa(P)$ will denote the residue field $\frac{R_P}{PR_P}$.

For any non-negative element $\beta \in \Gamma$, the valuation ideals $\mathcal{P}'_\beta \subset R'$ of value β form a tree of ideals of $\{R'\}$. Similarly, the i-th prime valuation ideals $P'_i \subset R'$ form a tree. If $rk\ \nu = r$, the prime valuation ideals P'_i give rise to a chain

$$P'_0 = (0) \subsetneqq P'_1 \subseteq \cdots \subseteq P'_r = m' \tag{6}$$

of trees of prime ideals of $\{R'\}$.

We discuss this last example in a little more detail and generality in order to emphasize our point of view, crucial throughout this paper: the data of a composite valuation is equivalent to the data of its components. Namely, suppose we are given a chain of trees of ideals as in (6), where we relax our assumptions of the P'_i as follows. We no longer assume that the chain (6) is maximal, nor that $P'_i \subsetneqq P'_{i+1}$, even for R' sufficiently large; in particular, for the purposes of this example we momentarily drop the assumption that $rk\ \nu = r$. We will still assume, however, that $P'_0 = (0)$ and that $P'_r = m'$.

Taking the limit in (6), we obtain a chain

$$(0) = \mathbf{m}_0 \subsetneqq \mathbf{m}_1 \subseteqq \cdots \subseteqq \mathbf{m}_r = \mathbf{m}_\nu \tag{7}$$

of prime ideals of the valuation ring R_ν.

Similarly, for each $1 \le \ell \le r$ one has the equality

$$\lim_{\overrightarrow{R'}} \frac{R'}{P'_\ell} = \frac{R_\nu}{\mathbf{m}_\ell}.$$

Then **specifying the valuation ν is equivalent to specifying valuations $\nu_0, \nu_1, \ldots, \nu_r$, where ν_0 is the trivial valuation of K and, for $1 \le \ell \le r$, ν_ℓ is a valuation of the residue field $k_{\nu_{\ell-1}} = \kappa(\mathbf{m}_{\ell-1})$, centered at the local ring $\lim\limits_{\longrightarrow} \frac{R'_{P'_\ell}}{P'_{\ell-1}R'_{P'_\ell}} = \frac{(R_\nu)_{\mathbf{m}_\ell}}{\mathbf{m}_{\ell-1}}$ and taking its values in the totally ordered group $\frac{\Delta_{\ell-1}}{\Delta_\ell}$.**

The relationship between ν and the ν_ℓ is that ν is the composition

$$\nu = \nu_1 \circ \nu_2 \circ \cdots \circ \nu_r. \tag{8}$$

For example, the datum of the valuation ν, or of its valuation ring R_ν, is equivalent to the datum of the valuation ring $\frac{R_\nu}{\mathbf{m}_{r-1}} \subset \frac{(R_\nu)_{\mathbf{m}_{r-1}}}{\mathbf{m}_{r-1}(R_\nu)_{\mathbf{m}_{r-1}}} = \kappa(\mathbf{m}_{r-1})$ of the valuation ν_r of the field $\kappa(\mathbf{m}_{r-1})$ and the valuation ring $(R_\nu)_{\mathbf{m}_{r-1}}$. If we assume, in addition, that for R sufficiently large the chain (6) (equivalently, (7)) is a maximal chain of distinct prime ideals then $rk\ \nu = r$ and $rk\ \nu_\ell = 1$ for each ℓ.

Remark 1.1 *Another way to describe the same property of valuations is that, given a prime ideal H of the local integral domain R one builds all valuations centered in R having H as one of the P_ℓ by choosing a valuation of R centered at H, say ν_1, so that $\mathbf{m}_{\nu_1} \cap R = H$ and choosing a valuation subring $\overline{R}_{\overline{\nu}}$ of the field $\frac{R_{\nu_1}}{\mathbf{m}_{\nu_1}}$ centered at R/H. Then $\nu = \nu_1 \circ \overline{\nu}$.*
Remark that choosing a valuation of R/H determines a valuation of its field of fractions $\kappa(H)$, which is in general much smaller than $\frac{R_{\nu_1}}{\mathbf{m}_{\nu_1}}$. Given a valuation of R with center H, in order to determine a valuation of R with center m inducing on R/H a given valuation μ we must choose an extension $\overline{\nu}$ of μ to $\frac{R_{\nu_1}}{\mathbf{m}_{\nu_1}}$, and there are in general many possibilities. Similarly,

This will be used in the sequel. In particular, it will be applied to the case where a valuation ν of R extends uniquely to a valuation $\hat{\nu}$ of $\frac{\hat{R}}{H}$ for some prime H of $\hat{R}$. Assuming that $\hat{R}$ is an integral domain, this determines a unique valuation of $\hat{R}$ only if the height ht H *is at most one. In all other cases the dimension of $\hat{R}_H$ is at least* 2 *and we have infinitely many valuations with which to compose $\hat{\nu}$. This is the source of the height conditions we shall see in §6.*

Example. Let k_0 be a field and $K = k_0((u,v))$ the field of fractions of the complete local ring $R = k_0[[u,v]]$. Let $\Gamma = \mathbf{Z}^2$ with lexicographical ordering. The isolated subgroups of Γ are $(0) \subsetneqq (0) \oplus \mathbf{Z} \subsetneqq \mathbf{Z}^2$. Consider the valuation $\nu : K^* \to \mathbf{Z}^2$, centered at R, given by

$$\begin{aligned} \nu(v) &= (0,1) && (9)\\ \nu(u) &= (1,0) && (10)\\ \nu(c) &= 0 \quad \text{for any } c \in k_0^*. && (11)\end{aligned}$$

This information determines ν completely; namely, for any power series

$$f = \sum_{\alpha,\beta} c_{\alpha\beta} u^\alpha v^\beta \in k_0[[u,v]],$$

we have

$$\nu(f) = \min\{(\alpha,\beta) \mid c_{\alpha\beta} \neq 0\}.$$

We have $rk\ \nu = rat.rk\ \nu = 2$. Let $\Delta = (0) \oplus \mathbf{Z}$. Let Γ_+ denote the semigroup of all the non-negative elements of Γ. Let $k_0[[\Gamma_+]]$ denote the R-algebra of power series $\sum c_{\alpha,\beta} u^\alpha v^\beta$ where $c_{\alpha,\beta} \in k_0$ and the exponents (α,β) form a well ordered subset of Γ_+. By classical results (see [7], [8]), it is a valuation ring with maximal ideal generated by all the monomials $u^\alpha v^\beta$, where $(\alpha,\beta) > (0,0)$ (in other words, either $\alpha > 0, \beta \in \mathbf{Z}$ or $\alpha = 0, \beta > 0$). Then $R_\nu = k_0[[\Gamma_+]] \bigcap k_0((u,v))$ is a valuation ring of K, and contains $k[[u,v]]$; it is the valuation ring of the valuation ν. The prime ideal $\mathbf{m}_1$ is the ideal of R_ν generated by all the uv^β, $\beta \in \mathbf{Z}$.

The valuation ν_1 is the discrete rank 1 valuation of K with valuation ring

$$(R_\nu)_{\mathbf{m}_1} = k_0[[u,v]]_{(u)}$$

and ν_2 is the discrete rank 1 valuation of $k_0((v))$ with valuation ring $\frac{R_\nu}{\mathbf{m}_1} \cong k_0[[v]]$.

Example. To give a more interesting example, let k_0 be a field of characteristic zero and $K = k_0(x,y,z)$ a purely transcendental extension of k_0 of degree 3. Let $k = \bigcup_{j=1}^{\infty} k_0\left(x^{\frac{1}{j}}\right)$. Let $\Gamma = \mathbf{Z}^2$ with lexicographical ordering and Δ the non-trivial isolated subgroup of Γ, as above. Let u,v be new variables let $\nu : k((u,v)) \to \Gamma$ be the valuation of the previous example. Consider the map $\iota : k_0[x,y,z] \to k[[u,v]]$ which sends x to u, y to v and z to $\sum_{j=1}^{\infty} x^{\frac{1}{j}} v^j$. Consider the restriction of ν to K; by abuse of notation, we will denote this valuation also by ν and its valuation ring in K by R_ν. The valuation $\nu : K^* \to \Gamma$ is centered at the local ring $R = k_0[x,y,z]_{(x,y,z)}$; we have

$$\begin{aligned} \nu(x) &= (0,1) && (12)\\ \nu(y) &= (1,0), && (13)\\ \nu(z) &= (1,1). && (14)\end{aligned}$$

It is not hard to show that for each j, there exists a local blowing up $R \to R'$ of R such that, in the notation of (6), we have $\kappa(P_1') = k_0\left(x^{\frac{1}{j!}}\right)$ and that $\kappa(\mathbf{m}_1) = \lim\limits_{j\to\infty} \kappa(P_1') = k$. The first one is the blowing up of the ideal $(y,z)R$, localized at the point $y=0, z/y=x$. Then one blows up the ideal $(z/y-x, y)$, and so on. Here $\nu=\nu_1\circ\nu_2$, where ν_1 is the restriction to K of the v-adic valuation under the inclusion of fields deduced from the inclusion of rings

$$k_0[[x,y,z]]_{(y,z)} \hookrightarrow k\left[\left[v^{\mathbf{Q}_+}\right]\right]$$

which sends x to x, y to v and z to $\sum\limits_{j=1}^{\infty} x^{\frac{1}{j}}v^j$. Recall that the ring on the right is made of power series with non negative rational exponents whose set of exponents is well ordered. We have $k_{\nu_1}=k$, and ν_2 is nothing but the x-adic valuation of k.

Remark 1.2 *The point of the last example is to show that, given a composed valuation as in (8), ν_ℓ is a valuation of the field $k_{\nu_{\ell-1}}$, which may* ***properly*** *contain $\kappa(P'_{\ell-1})$ for* ***every*** *$R'\in\mathcal{T}$. This fact will be a source of some complication later on and we prefer to draw attention to it from the beginning.*

Coming back to the implicit prime ideals, we will see that the implicit prime ideals H'_i form a tree of ideals of $R^\dagger$.

We will show that if ν extends to a valuation of $\hat{\nu}$ centered at $\frac{\hat{R}}{P}$ with $P\cap R=(0)$ then the prime P must contain the minimal prime H_0 of $\hat{R}$. We will then show that specifying an extension $\hat{\nu}$ of ν as above is equivalent to specifying a chain of prime valuation ideals

$$\tilde{H}'_0\subset\tilde{H}'_1\subset\cdots\subset\tilde{H}'_{2r}=m'\hat{R}' \tag{15}$$

of $\hat{R}'$ such that $H'_\ell\subset\tilde{H}'_\ell$ for all $\ell\in\{0,\dots,2r\}$, and valuations $\hat{\nu}_1,\hat{\nu}_2,\dots,\hat{\nu}_{2r}$, where $\hat{\nu}_i$ is a valuation of the field $k_{\hat{\nu}_{i-1}}$ (the residue field of the valuation ring $R_{\hat{\nu}_{i-1}}$), arbitrary when i is odd and satisfying certain conditions, coming from the valuation $\nu_{\frac{i}{2}}$, when i is even.

The prime ideals H_i are defined as follows.
Recall that given a valued ring (R,ν) with value group Γ, that is a subring $R\subseteq R_\nu$ of the valuation ring R_ν of a valuation with value group Γ, one defines for each $\beta\in\Gamma$ the valuation ideals of R associated to β:

$$\begin{aligned}\mathcal{P}_\beta(R)= &\quad \{x\in R/\nu(x)\geq\beta\}\\ \mathcal{P}^+_\beta(R)= &\quad \{x\in R/\nu(x)>\beta\}\end{aligned}$$

and the associated graded ring

$$\mathrm{gr}_\nu R=\bigoplus_{\beta\in\Gamma}\frac{\mathcal{P}_\beta(R)}{\mathcal{P}^+_\beta(R)}=\bigoplus_{\beta\in\Gamma_+}\frac{\mathcal{P}_\beta(R)}{\mathcal{P}^+_\beta(R)}.$$

The second equality comes from the fact that if $\beta\in\Gamma_-\setminus\{0\}$, we have $\mathcal{P}^+_\beta(R)=\mathcal{P}_\beta(R)=R$. If $R\to R'$ is an extension of local rings such that $R\subset R'\subset R_\nu$ and $m_\nu\cap R'=m'$, we will write $\mathcal{P}'_\beta$ for $\mathcal{P}_\beta(R')$.

Fix a valuation ring R_ν dominating R, and a tree $\mathcal{T}=\{R'\}$ of nœtherian local R-subalgebras of R_ν, having the following properties: for each ring $R'\in\mathcal{T}$, all the birational ν-extensions of R' belong to $\mathcal{T}$. Moreover, we assume that the field of fractions of R_ν equals $\varinjlim\limits_{R'} K'$, where K' is the field of fractions of R'. The tree $\mathcal{T}$ will stay constant throughout this paper. In the special case when R happens to have the same field of fractions as R_ν, we may take $\mathcal{T}$ to be the tree of all the birational ν-extensions of R.

Notation. For a ring $R' \in \mathcal{T}$, we shall denote by $\mathcal{T}(R')$ the subtree of $\mathcal{T}$ consisting of all the ν-extensions R'' of R'.

We now define

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \left(\left(\lim_{\overrightarrow{R'}} \mathcal{P}'_\beta R'^\dagger \right) \bigcap R^\dagger \right), \ 0 \le \ell \le r-1 \tag{16}$$

(in the beginning of next section we provide some motivation for this definition and give several elementary examples of H'_i and $\tilde{H}'_i$).

The questions answered in this paper originally arose from our work on the Local Uniformization Theorem, where passage to completion is required in both the approaches of [16] and [17]. In [17], one really needs to pass to completion for valuations of arbitrary rank. One of the main applications of the theory of implicit prime ideals, proved in the present paper, is the following result, announced in [17]. Let

$$\Gamma \hookrightarrow \hat{\Gamma} \tag{17}$$

be an extension of ordered groups of the same rank. Let

$$(0) = \Delta_r \subsetneqq \Delta_{r-1} \subsetneqq \dots \subsetneqq \Delta_0 = \Gamma \tag{18}$$

be the isolated subgroups of Γ and

$$(0) = \hat{\Delta}_r \subsetneqq \hat{\Delta}_{r-1} \subsetneqq \dots \subsetneqq \hat{\Delta}_0 = \hat{\Gamma}$$

the isolated subgroups of $\hat{\Gamma}$, so that the inclusion (17) induces inclusions

$$\Delta_\ell \hookrightarrow \hat{\Delta}_\ell \quad \text{and} \tag{19}$$

$$\frac{\Delta_\ell}{\Delta_{\ell+1}} \hookrightarrow \frac{\hat{\Delta}_\ell}{\hat{\Delta}_{\ell+1}}. \tag{20}$$

Let $G \hookrightarrow \hat{G}$ be an extension of graded algebras without zero divisors, such that G is graded by Γ_+ and $\hat{G}$ by $\hat{\Gamma}_+$. The graded algebra G is endowed with a natural valuation with value group Γ and similarly for $\hat{G}$ and $\hat{\Gamma}$. This natural valuation will be denoted by *ord*.

Definition 1.3 *We say that the extension $G \hookrightarrow \hat{G}$ is* ***scalewise birational*** *if for any $x \in \hat{G}$ and $\ell \in \{1, \dots, r\}$ such that ord $x \in \hat{\Delta}_\ell$ there exists $y \in G$ such that ord $y \in \Delta_\ell$ and $xy \in G$.*

Of course, scalewise birational implies birational and also that $\hat{\Gamma} = \Gamma$.
While the first main result of this paper is the primality of the implicit ideals associated to a valuation, and the subsequent description of the extensions of the valuation to the completion, the second one is the following:

Theorem 1.1 *Assume that* $\dim\ R' = \dim\ R$ *for all $R' \in \mathcal{T}$. Then there exists a tree of prime ideals H' of $\hat{R}'$ with $H' \cap R' = (0)$ and a valuation $\hat{\nu}$, centered at $\lim\limits_{\to} \frac{\hat{R}'}{H'}$ and having the following property:*
For any $R' \in \mathcal{T}$ the graded algebra $gr_{\hat{\nu}} \frac{\hat{R}'}{H'}$ is a scalewise birational extension of $gr_\nu R'$.

The example given in remark 5.20, 4) of [17] shows that the morphism of associated graded rings is not an isomorphism in general.

The approach to the Local Uniformization Theorem taken in [16] is to reduce the problem to the case of rank 1 valuations. The theory of implicit prime ideals is much simpler for valuations of rank 1 and takes only a few pages in Section 2.

Another recent application of these results on extending valuations to the formal completion is the work [10], [11] on the Pierce-Birkhoff conjecture. This conjecture was reduced by J. Madden to studying the separating ideal $<\alpha,\beta>$ of two points α and β in the real spectrum Sper A of A (of course A must be of characteristic 0 in order for Sper A to be non-empty). It is easy to reduce the problem to the case when A is local. However, for various technical reasons one is induced to pass to the formal completion $\hat{A}$ of A. At this point, we need to know that there exist points $\hat{\alpha}$ and $\hat{\beta}$ in the real spectrum of $\hat{A}$ such that $<\hat{\alpha},\hat{\beta}>\cap A=<\alpha,\beta>$. This fact is deduced readily from Theorem 1.1.

The paper is organized as follows. In §3 we define the odd-numbered implicit ideals $H_{2\ell+1}$ and prove that $H_{2\ell+1}\cap R=P_\ell$. We observe that by their very definition, the ideals $H_{2\ell+1}$ behave well under ν-extensions; they form a tree. Proving that $H_{2\ell+1}$ is indeed prime is postponed until later sections; it will be proved gradually in §4–§8. In the beginning of §3 we will explain in more detail the respective roles played by the odd-numbered and the even-numbered implicit ideals, give several examples (among other things, to motivate the need for taking the limit with respect to R' in (16)) and say one or two words about the techniques used to prove our results.

In §4 we prove the primality of the implicit prime ideals assuming a certain technical condition, called **stability**, about the tree $\mathcal{T}$ and the operation †. It follows from the noetherianity of $R^\dagger$ that there exists a specific R' for which the limit in (16) is attained. One of the main points of §4 is to prove properties of stable rings which guarantee that this limit is attained whenever R' is stable. We then use the excellence of R to define the even-numbered implicit prime ideals: for $i=2\ell$ the ideal $H_{2\ell}$ is defined to be the unique minimal prime of $P_\ell R^\dagger$, contained in $H_{2\ell+1}$ (in the case $R^\dagger=\hat{R}$ it is the excellence of R which implies the uniqueness of such a minimal prime). We have

$$H_{2\ell}\cap R=P_\ell$$

for $\ell\in\{0,\dots,r\}$. The results of §4 apply equally well to completions, henselizations and other local étale extensions; to complete the proof of the primality of the implicit ideals in various contexts such as henselization or completion, it remains to show the existence of stable ν-extensions in the corresponding context.

In §5 we describe the set of extensions $\nu^\dagger$ of ν to $\lim\limits_{\overrightarrow{R'}}\frac{R'^\dagger}{P'}$, where P' is a tree of prime ideals of $R'^\dagger$ such that $P'\cap R'=(0)$. We show (Theorem 5.1) that specifying such a valuation $\nu^\dagger$ is equivalent to specifying the following data:

(1) a chain (15) of trees of prime ideals $\tilde{H}'_i$ of $R'^\dagger$ (where $\tilde{H}'_0=P'$), such that $H'_i\subset\tilde{H}'_i$ for each i and each $R'\in\mathcal{T}$, satisfying one additional condition (we will refer to the chain (15) as the chain of trees of ideals, **determined by** the extension $\nu^\dagger$)

(2) a valuation $\nu^\dagger_i$ of the residue field $k_{\nu^\dagger_{i-1}}$ of $\nu^\dagger_{i-1}$, whose restriction to the field $\lim\limits_{\overrightarrow{R'}}\kappa(\tilde{H}'_{i-1})$ is centered at the local ring $\lim\limits_{\overrightarrow{R'}}\frac{R'^\dagger_{\tilde{H}'_i}}{\tilde{H}'_{i-1}R'^\dagger_{\tilde{H}'_i}}$.

If $i=2\ell$ is even, the valuation $\nu^\dagger_i$ must be of rank 1 and its restriction to $\kappa(\mathbf{m}_{\ell-1})$ must coincide with ν_ℓ.

Notice the recursive nature of this description of $\nu^\dagger$: in order to describe $\nu^\dagger_i$ we must know $\nu^\dagger_{i-1}$ in order to talk about its residue field $k_{\nu^\dagger_{i-1}}$.

In §6 we address the question of uniqueness of $\nu^\dagger$. We describe several classes of extensions $\nu^\dagger$ which are particularly useful for the applications: **minimal** and **evenly minimal** extensions, and also those $\nu^\dagger$ for which, denoting by ht I the height of an ideal, we have

$$\text{ht } \tilde{H}'_{2\ell+1} - \text{ht } \tilde{H}'_{2_\ell} \leq 1 \quad \text{ for } 0 \leq \ell \leq r; \tag{21}$$

in fact, the special case of (21) which is of most interest for the applications is

$$\tilde{H}'_{2\ell} = \tilde{H}'_{2\ell+1} \quad \text{ for } 1 \leq \ell \leq r. \tag{22}$$

We prove some necessary and some sufficient conditions under which an extension $\nu^\dagger$ whose corresponding ideals $\tilde{H}'_i$ satisfy (22) is uniquely determined by the ideals $\tilde{H}'_i$. We also give sufficient conditions for the graded algebra $gr_\nu R'$ to be scalewise birational to $gr_{\hat{\nu}} \hat{R}'$ for each $R' \in \mathcal{T}$. These sufficient conditions are used in §9 to prove Theorem 1.1.

In §7 we show the existence of ν-extensions in $\mathcal{T}$, stable for henselization, thus reducing the proof of the primality of $H_{2\ell+1}$ to the results of §4. We study the extension of ν to $\tilde{R}$ modulo its first prime ideal and prove that such an extension is unique.

In §8 we use the results of §7 to prove the existence of ν-extensions in $\mathcal{T}$, stable for completion. Combined with the results of §4 this proves that $H_{2\ell+1}$ are prime for the completion.

In §9 we construct a chain of trees (15) of prime ideals of $\hat{R}'$ satisfying (22) and a corresponding valuation $\hat{\nu}$ which satisfies the conclusion of Theorem 1.1. We also prove another necessary condition for the uniqueness of $\hat{\nu}$.

We would like to acknowledge the paper [4] by Bill Heinzer and Judith Sally which inspired one of the authors to continue thinking about this subject.

2 Extending a valuation of rank one centered in a local domain to its formal completion.

Let (R, M, k) be a local noetherian ring, $\mathcal{P}_\infty$ a minimal prime of R, $K := R_{\mathcal{P}_\infty}$, and $\nu : K \to \Gamma_+ \cup \{\infty\}$ a valuation, centered at R (that is, non-negative on R and positive on M). We shall sometimes say for short that $\mathcal{P}_\infty$ is the support of ν.

Let $\hat{R}$ denote the formal completion of R. It is convenient to extend ν to a valuation centered at $\frac{\hat{R}}{H}$, where H is a prime ideal of $\hat{R}$ such that $H \cap R = \mathcal{P}_\infty$. In this section, we will assume that ν is of rank one, so that the value group Φ is archimedian. We will explicitly describe a prime ideal H of $\hat{R}$, canonically associated to ν, such that $H \cap R = \mathcal{P}_\infty$ and such that ν has a unique extension $\hat{\nu}$ to $\frac{\hat{R}}{H}$.

Let $\Phi = \nu(R \setminus \mathcal{P}_\infty)$, let $\mathcal{P}_\beta$ denote the ν-ideal of R of value β and $\mathcal{P}^+_\beta$ the greatest ν-ideal, properly contained in $\mathcal{P}_\beta$. We now define the main object of study of this section. Let

$$H := \bigcap_{\beta \in \Phi} (\mathcal{P}_\beta \hat{R}). \tag{23}$$

Remark 2.1 *Since R is noetherian, we have $\nu(M) > 0$ and since the ordered group Φ is archimedian, for every $\beta \in \Phi$ there exists $n \in \mathbf{N}$ such that $M^n \subset \mathcal{P}_\beta$. In other words, the M-adic topology on R is finer than (or equal to) the ν-adic topology. Therefore an element $x \in \hat{R}$ lies in $\mathcal{P}_\beta \hat{R} \iff$ there exists a Cauchy sequence $\{x_n\} \subset R$ in the M-adic topology, converging to x, such that $\nu(x_n) \geq \beta$ for all $n \iff$ for every Cauchy sequence $\{x_n\} \subset R$, converging to x, $\nu(x_n) \geq \beta$ for all $n \gg 0$. By the same token, $x \in H \iff$ there exists a Cauchy sequence $\{x_n\} \subset R$, converging to x, such that $\lim\limits_{n\to\infty} \nu(x_n) = \infty \iff$ for every Cauchy sequence $\{x_n\} \subset R$, converging to x, $\lim\limits_{n\to\infty} \nu(x_n) = \infty$.*

Example. Let $R = k[u,v]_{(u,v)}$. Then $\hat{R} = k[[u,v]]$. Consider an element $w = u - \sum_{i=1}^{\infty} c_i v^i \in \hat{R}$, where $c_i \in k^*$ for all $i \in \mathbf{N}$, such that w is transcendental over $k(u,v)$. Consider the injective map $\iota : k[u,v]_{(u,v)} \to k[[t]]$ which sends v to t and u to $\sum_{i=1}^{\infty} c_i t^i$. Let ν be the valuation induced from the t-adic valuation of $k[[t]]$ via ι. The value group of ν is $\mathbf{Z}$ and $\Phi = \mathbf{N}_0$. For each $\beta \in \mathbf{N}$, $P_\beta = \left(v^\beta, u - \sum_{i=1}^{\beta-1} c_i v^i\right)$. Thus $H = (w)$.

We come back to the general theory. Since the formal completion homomorphism $R \to \hat{R}$ is faithfully flat,

$$\mathcal{P}_\beta \hat{R} \cap R = \mathcal{P}_\beta \quad \text{for all } \beta \in \Phi. \tag{24}$$

Taking the intersection over all $\beta \in \Phi$, we obtain

$$H \cap R = \left(\bigcap_{\beta \in \Phi} \left(\mathcal{P}_\beta \hat{R} \right) \right) \cap R = \bigcap_{\beta \in \Phi} \mathcal{P}_\beta = \mathcal{P}_\infty, \tag{25}$$

In other words, we have a natural inclusion $\frac{R}{\mathcal{P}_\infty} \hookrightarrow \frac{\hat{R}}{H}$.

Theorem 2.1 *1. H is a prime ideal of $\hat{R}$.*

2. ν extends uniquely to a valuation $\hat{\nu}$, centered at $\frac{\hat{R}}{H}$.

Proof: Let $\bar{x} \in \frac{\hat{R}}{H} \setminus \{0\}$. Pick in $\hat{R}$ a representative x of $\bar{x}$, so that $\bar{x} = x \bmod H$. Since $x \notin H$, we have $x \notin \mathcal{P}_\alpha \hat{R}$ for some $\alpha \in \Phi$.

Lemma 2.1 *Let ν be a valuation of rank one centered in a local noetherian ring (R, M, k) . Let $\mathcal{P}_\infty$ denote the support of ν and let*

$$\Phi = \nu(R \setminus \mathcal{P}_\infty) \subset \Gamma.$$

Then Φ contains no infinite bounded sequences.

Proof: An infinite ascending sequence $\alpha_1 < \alpha_2 < \dots$ in Φ, bounded above by an element $\beta \in \Phi$, would give rise to an infinite descending chain of ideals in $\frac{R}{\mathcal{P}_\beta}$. Thus it is sufficient to prove that $\frac{R}{\mathcal{P}_\beta}$ has finite length.

Let $\delta := \nu(M) \equiv \min(\Phi \setminus \{0\})$. Since Φ is archimedian, there exists $n \in \mathbf{N}$ such that $\beta \le n\delta$. Then $M^n \subset \mathcal{P}_\beta$, so that there is a surjective map $\frac{R}{M^n} \twoheadrightarrow \frac{R}{\mathcal{P}_\beta}$. Thus $\frac{R}{\mathcal{P}_\beta}$ has finite length, as desired. $\square$

By Lemma 2.1, the set $\{\beta \in \Phi \mid \beta < \alpha\}$ is finite. Hence there exists a unique $\beta \in \Phi$ such that

$$x \in \mathcal{P}_\beta \hat{R} \setminus \mathcal{P}_\beta^+ \hat{R}. \tag{26}$$

Note that β depends only on $\bar{x}$, but not on the choice of the representative x. Define the function $\hat{\nu} : \frac{\hat{R}}{H} \setminus \{0\} \to \Phi$ by

$$\hat{\nu}(\bar{x}) = \beta. \tag{27}$$

By (24), if $x \in R \setminus \{0\}$ then

$$\hat{\nu}(x) = \nu(x). \tag{28}$$

It is obvious that

$$\hat{\nu}(x+y) \geq \min\{\hat{\nu}(x), \hat{\nu}(y)\} \tag{29}$$

$$\hat{\nu}(xy) \geq \hat{\nu}(x) + \hat{\nu}(y) \tag{30}$$

for all $x, y \in \frac{\hat{R}}{H}$. The point of the next lemma is to show that $\frac{\hat{R}}{H}$ is a domain and that $\hat{\nu}$ is, in fact, a valuation (i.e. that the inequality (30) is, in fact, an equality).

Lemma 2.2 *For any non-zero $\bar{x}, \bar{y} \in \frac{\hat{R}}{H}$, we have $\bar{x}\bar{y} \neq 0$ and $\hat{\nu}(\bar{x}\bar{y}) = \hat{\nu}(\bar{x}) + \hat{\nu}(\bar{y})$.*

Proof: Let $\alpha = \hat{\nu}(\bar{x})$, $\beta = \hat{\nu}(\bar{y})$. Let x and y be representatives in $\hat{R}$ of $\bar{x}$ and $\bar{y}$, respectively. We have $M\mathcal{P}_\alpha \subset \mathcal{P}_\alpha^+$, so that

$$\frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+} \cong \frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+ + M\mathcal{P}_\alpha} \cong \frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+} \otimes_R k \cong \frac{\mathcal{P}_\alpha}{\mathcal{P}_\alpha^+} \otimes_R \frac{\hat{R}}{M\hat{R}} \cong \frac{\mathcal{P}_\alpha \hat{R}}{(\mathcal{P}_\alpha^+ + M\mathcal{P}_\alpha)\hat{R}} \cong \frac{\mathcal{P}_\alpha \hat{R}}{\mathcal{P}_\alpha^+ \hat{R}}, \tag{31}$$

and similarly for β. By (31) there exist $z \in \mathcal{P}_\alpha$, $w \in \mathcal{P}_\beta$, such that $z \equiv x \mod \mathcal{P}_\alpha^+ \hat{R}$ and $w \equiv y \mod \mathcal{P}_\beta^+ \hat{R}$. Then

$$xy \equiv zw \mod \mathcal{P}_{\alpha+\beta}^+ \hat{R}. \tag{32}$$

Since ν is a valuation, $\nu(zw) = \alpha + \beta$, so that $zw \in \mathcal{P}_{\alpha+\beta} \setminus \mathcal{P}_{\alpha+\beta}^+$. By (24) and (32), this proves that $xy \in \mathcal{P}_{\alpha+\beta}\hat{R} \setminus \mathcal{P}_{\alpha+\beta}^+ \hat{R}$. Thus $xy \notin H$ (hence $\bar{x}\bar{y} \neq 0$ in $\frac{\hat{R}}{H}$) and $\hat{\nu}(\bar{x}\bar{y}) = \alpha + \beta$, as desired. □

By lemma 2.2, H is a prime ideal of $\hat{R}$. By (29) and lemma 2.2, $\hat{\nu}$ is a valuation, centered at $\frac{\hat{R}}{H}$. To complete the proof of theorem 2.1, it remains to prove the uniqueness of $\hat{\nu}$. Let x, $\bar{x}$, the element $\alpha \in \Phi$ and

$$z \in \mathcal{P}_\alpha \setminus \mathcal{P}_\alpha^+ \tag{33}$$

be as in the proof of lemma 2.2. Then there exist

$$\begin{array}{ll} u_1, \dots, u_n & \in \mathcal{P}_\alpha^+ \text{ and} \\ v_1, \dots, v_n & \in \hat{R} \end{array} \tag{34}$$

such that $x = z + \sum\limits_{i=1}^{n} u_i v_i$. Letting $\bar{v}_i := v_i \mod H$, we obtain $\bar{x} = \bar{z} + \sum\limits_{i=1}^{n} \bar{u}_i \bar{v}_i$ in $\frac{\hat{R}}{H}$. Therefore, by (33)–(34), for any extension of ν to a valuation $\tilde{\nu}$, centered at $\frac{\hat{R}}{H}$, we have

$$\tilde{\nu}(x) = \alpha = \hat{\nu}(x), \tag{35}$$

as desired. This completes the proof of theorem 2.1. □

Definition 2.1 *The ideal H is called the* **implicit prime ideal** *of $\hat{R}$, associated to ν. When dealing with more than one ring at a time, we will sometimes write $H(R, \nu)$ for H.*

More generally, let ν be a valuation centered at R, not necessarily of rank one. In any case, we may write ν as a composition $\nu = \mu_2 \circ \mu_1$, where μ_2 is centered at a non-maximal prime ideal P of R and $\mu_1 \Big|_{\frac{R}{P}}$ is of rank one. The valuation $\mu_1 \Big|_{\frac{R}{P}}$ is centered at $\frac{R}{P}$. We define the **implicit prime ideal** of R with respect to ν, denoted $H(R, \nu)$, to be the inverse image in $\hat{R}$ of the implicit prime ideal of $\frac{\hat{R}}{P}$ with respect to $\mu_1 \Big|_{\frac{R}{P}}$. For the rest of this section, we will continue to assume that ν is of rank one.

Remark 2.2 *By (31), we have the following natural isomorphisms of graded algebras:*

$$\begin{aligned} gr_\nu R &\cong gr_{\hat\nu}\frac{\hat R}{H} \\ G_\nu &\cong G_{\hat\nu}. \end{aligned}$$

Corollary 2.1 *Assume that R is a reduced G-ring. Then $\hat R_H$ is a regular local ring.*

Proof: Let $K = R_{\mathcal{P}_\infty} = \kappa(\mathcal{P}_\infty)$ (here we are using that R is reduced and that $\mathcal{P}_\infty$ is a minimal prime of R). By definition of G-ring, the map $R \to \hat R$ is a regular homomorphism. Then by (25) $\hat R_H$ is geometrically regular over K, hence regular. $\square$

We will now study the behaviour of H under local blowings up of R with respect to ν and, more generally, under local homomorphisms. Let $\pi : (R, M) \to (R', M')$ be a local homomorphism of local noetherian rings. Assume that there is a minimal prime $\mathcal{P}'_\infty$ of R' such that $\mathcal{P}_\infty = \mathcal{P}'_\infty \cap R$ and that ν extends to a rank one valuation $\nu' : R' \to \Gamma' \cup \{\infty\}$, where $\Gamma' \supset \Gamma$. The homomorphism π induces a local homomorphism $\hat\pi : \hat R \to \hat R'$ of formal completions. Let $\Phi' = \nu'(R' \setminus \{0\})$. For $\beta \in \Phi'$, let $\mathcal{P}'_\beta$ denote the ν'-ideal of $R_{\nu'}$ of value β, as above. Let $H' = H(R', \nu')$.

Lemma 2.3 *Let $\beta \in \Phi$. Then*

$$\left(\mathcal{P}'_\beta \hat R'\right) \cap \hat R = \mathcal{P}_\beta \hat R. \tag{36}$$

Proof: Since by assumption ν' extends ν we have $\mathcal{P}'_\beta \cap R = \mathcal{P}_\beta$ and the inclusion

$$\left(\mathcal{P}'_\beta \hat R'\right) \cap \hat R \supseteq \mathcal{P}_\beta \hat R. \tag{37}$$

We will now prove the opposite inclusion. Take an element $x \in \left(\mathcal{P}'_\beta \hat R'\right) \cap \hat R$. Let $\{x_n\} \subset R$ be a Cauchy sequence in the M-adic topology, converging to x. Then $\{\pi(x_n)\}$ converge to $\hat\pi(x)$ in the M'-adic topology of $\hat R'$. Applying remark 2.1 to R', we obtain

$$\nu(x_n) \equiv \nu'(\pi(x_n)) \geq \beta \quad \text{for } n \gg 0. \tag{38}$$

By (38) and Remark 2.2, applied to R, we have $x \in \mathcal{P}_\beta \hat R$. This proves the opposite inclusion in (37), as desired. $\square$

Corollary 2.2 *We have*

$$H' \cap \hat R = H.$$

Proof: Since ν' is of rank one, Φ is cofinal in Φ'. Now the Corollary follows by taking the intersection over all $\beta \in \Phi$ in (36). $\square$

Let J be a non-zero ideal of R and let $R \to R'$ be the local blowing up along J with respect to ν. Assume that ν remains of rank one in R', that is, $J \not\subset \sqrt{\mathcal{P}_\infty}$. Take an element $f \in J$, such that $\nu(f) = \nu(J)$. By the **strict transform** of H in $\hat R'$ we will mean the ideal

$$H^{\text{str}} := \bigcup_{i=1}^{\infty}\left(\left(H\hat R'\right) : f^i\right) \equiv \left(H\hat R'_f\right) \cap \hat R'.$$

By our assumptions, the element f is neither 0 nor a zero divisor in R' and if g is another element of J such that $\nu(g) = \nu(J)$ then $\nu\left(\frac{f}{g}\right) = 0$, so that $\frac{f}{g}$ is a unit in R'. Thus the definition of strict transform is independent of the choice of f.

Corollary 2.3 $H^{str} \subset H'$.

Proof: Since $H\hat{R}' \subset H'$, we have $H^{\rm str} = \left(H\hat{R}'_f\right) \cap \hat{R}' \subset \left(H'\hat{R}'_f\right) \cap \hat{R}' = H'$, where the last equality holds because H' is a prime ideal of $\hat{R}'$, not containing f. $\square$

Using Zariski's Main Theorem, it can be proved that $H^{\rm str}$ is prime. Since this fact is not used in the sequel, we omit the proof.

Corollary 2.4 *Let the notation and assumptions be as in corollary 2.3. Then*

$$\text{ht } H' \geq \text{ht } H. \tag{39}$$

In particular,

$$\dim \frac{\hat{R}'}{H'} \leq \dim \frac{\hat{R}}{H}. \tag{40}$$

Proof: Let $\bar{R} := \left(\hat{R} \otimes_R R'\right)_{M'\hat{R}' \cap (\hat{R} \otimes_R R')}$. Let ϕ denote the natural local homomorphism $\bar{R} \to \hat{R}'$. Let $\bar{H} := H' \cap \bar{R}$. Now, take $f \in J$ such that $\nu(f) = \nu(J)$. Then $f \notin H'$ and, in particular, $f \notin \bar{H}$. Since $R'_f \cong R_f$, we have $\hat{R}_f = \bar{R}_f$. In view of corollary 2.2, we obtain $H\hat{R}_f \cong \bar{H}\bar{R}_f$, so

$$\text{ht } H = \text{ht } \bar{H}. \tag{41}$$

Now, $\bar{R}$ is a local noetherian ring, whose formal completion is $\hat{R}'$. Hence ϕ is faithfully flat and therefore satisfies the going down theorem. Thus we have ht $H' \geq$ ht $\bar{H}$. Combined with (41), this proves (39). As for the last statement of the Corollary, it follows from the well known fact that dimension does not increase under blowing up ([15], Lemma 2.2): we have $\dim R' \leq \dim R$, hence

$$\dim \hat{R}' = \dim R' \leq \dim R = \dim \hat{R},$$

and (40) follows from (39) and from the fact that complete local rings are catenarian. $\square$

It may well happen that the containment of corollary 2.3 and the inequality in (39) are strict. The possibility of strict containement in corollary 2.3 is related to the existence of subanalytic functions, which are not analytic. We illustrate this statement by an example in which $H^{\rm str} \subsetneqq H'$ and ht $H <$ ht H'.

Example. Let k be a field and let

$$\begin{aligned} R &= k[x,y,z]_{(x,y,z)}, \\ R' &= k[x',y',z']_{(x',y',z')}, \end{aligned}$$

where $x' = x$, $y' = \frac{y}{x}$ and $z' = z$. We have $K = k(x,y,z)$, $\hat{R} = k[[x,y,z]]$, $\hat{R}' = k[[x',y',z']]$. Let t_1, t_2 be auxiliary variables and let $\sum\limits_{i=1}^{\infty} c_i t_1^i$ (with $c_i \in k$) be an element of $k[[t_1]]$, transcendental over $k(t_1)$. Let θ denote the valuation, centered at $k[[t_1,t_2]]$, defined by $\theta(t_1) = 1$, $\theta(t_2) = \sqrt{2}$ (the value group of θ is the additive subgroup of $\mathbf{R}$, generated by 1 and $\sqrt{2}$). Let $\iota : R' \hookrightarrow k[[t_1,t_2]]$ denote the injective map defined by $\iota(x') = t_2$, $\iota(y') = t_1$, $\iota(z') = \sum\limits_{i=1}^{\infty} c_i t_1^i$. Let ν denote the restriction of θ to K, where we view K as a subfield of $k((t_1,t_2))$ via ι. Let $\Phi = \nu(R \setminus \{0\})$; $\Phi' = \nu(R' \setminus \{0\})$. For $\beta \in \Phi'$, P'_β is generated by all the monomials of the form $x'^\alpha y'^\gamma$ such that $\sqrt{2}\alpha + \gamma \geq \beta$, together with $z' - \sum\limits_{j=1}^{i} c_j y'^j$, where i is the greatest non-negative integer such that $i < \beta$.

Let $w' := z' - \sum_{i=1}^{\infty} c_i y'^i$. Then $H' = (w')$, but $H = H' \cap \hat{R} = (0)$, so that $H^{\text{str}} = (0) \subsetneqq H'$ and ht $H = 0 < 1 =$ ht H'.

Remark 2.3 *Having extended in a unique manner the valuation ν to a valuation $\hat{\nu}_+$ of $\frac{\hat{R}}{H}$, we see that if R is a G-ring, by Corollary 2.1 there is a unique minimal prime $\hat{\mathcal{P}}_\infty$ of $\hat{R}$ contained in H, corresponding to the ideal (0) in $\hat{R}_H$. Since $H \cap R = (0)$, we have the equality $\hat{\mathcal{P}}_\infty \cap R = (0)$. Choosing a valuation μ of the fraction field of $\frac{\hat{R}_H}{\hat{\mathcal{P}}_\infty}$ centered at $\frac{\hat{R}_H}{\hat{\mathcal{P}}_\infty}$ and with value group Ψ produces a composed valuation $\hat{\nu}_+ \circ \mu$ on $\frac{\hat{R}}{\hat{\mathcal{P}}_\infty}$ with value group $\Psi \bigoplus \Gamma$ ordered lexicographically, as follows:*
Given $x \in \frac{\hat{R}}{\hat{\mathcal{P}}_\infty}$, let $\psi = \mu(x)$ and blow up in R the ideal $\mathcal{P}_\psi$ along our original valuation, obtaining a local ring R'. According to what we have seen so far in this section, in its completion $\hat{R}'$ we can write $x = ye$ with $\mu(e) = \psi$ and $y \in \hat{R}' \setminus H'$. The valuation ν on R' extends uniquely to a valuation of $\frac{R'}{H'}$, which we may still denote by $\overline{\nu}_+$ because it induces $\overline{\nu}_+$ on $\frac{\hat{R}}{H}$. Let us consider the image $\overline{y}$ of y in $\frac{R'}{H'}$. Setting $\hat{\nu}_+ \circ \mu(x) = \psi \bigoplus \hat{\nu}_+(\overline{y}) \in \Psi \bigoplus \Gamma$ determines a valuation of $\frac{\hat{R}_H}{\hat{\mathcal{P}}_\infty}$ as required.
In the sequel we shall reduce to the case where $\hat{R}$ is an integral domain, so that $\hat{\mathcal{P}}_\infty = (0)$ and we well have constrcted a valuation of $\hat{R}$.

3 Definition and first properties of implicit ideals.

Let the notation be as above. Before plunging into technical details, we would like to give a brief and informal overview of our constructions and the motivation for them. Above we recalled the well known fact that if $rk\ \nu = r$ then for every ν-extension $R \to R'$ the valuation ν canonically determines a flag (6) of r subschemes of Spec R'. This paper shows the existence of subschemes of Spec $\hat{R}$, determined by ν, which are equally canonical and which become explicit only after completion. To see what they are, first of all note that the ideal $P'_l \hat{R}'$, for $R' \in \mathcal{T}$ and $0 \le \ell \le r-1$, need not in general be prime (although it is prime whenever R' is henselian). Another way of saying the same thing is that the ring $\frac{\hat{R}'}{P'_\ell}$ need not be analytically irreducible in general. However, we will see in §8 (resp. §7) that the valuation ν picks out in a canonical way one of the minimal primes of $P'_l \hat{R}'$ (resp. $P'_l \tilde{R}'$). We call this minimal prime $H'_{2\ell}$ for reasons which will become apparent later. By the flatness of completion (resp. henselization), we have $H'_{2\ell} \cap R' = P'_\ell$. We will show that the ideals $H'_{2\ell}$ form a tree.

Let

$$(0) = \Delta_r \subsetneqq \Delta_{r-1} \subsetneqq \dots \subsetneqq \Delta_0 = \Gamma \tag{42}$$

be the isolated subgroups of Γ. There are other ideals of $\hat{R}$, apart from the $H_{2\ell}$, canonically associated to ν, whose intersection with R equals P_ℓ, for example, the ideal $\bigcap_{\beta \in \Delta_\ell} \mathcal{P}_\beta \hat{R}$. The same is true of the even larger ideal

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \left(\left(\lim_{\overrightarrow{R'}} \mathcal{P}'_\beta \hat{R}' \right) \bigcap \hat{R} \right), \tag{43}$$

(that $H_{2\ell+1} \cap R = P_\ell$ is easy to see and will be shown later in this section, in Proposition 3.1). While the examples below show that the ideal $\bigcap_{\beta \in \Delta_\ell} \mathcal{P}_\beta \hat{R}$ need not, in general, be prime, the ideal $H_{2\ell+1}$ always is (this is one of the two main theorems of this paper; it will be proved in

§8). The ideal $H_{2\ell+1}$ contains $H_{2\ell}$ but is not, in general equal to it. To summarize, we will show that the valuation ν picks out in a canonical way a generic point $H_{2\ell}$ of the formal fiber over P_ℓ and also another point $H_{2\ell+1}$ in the formal fiber, which is a specialization of $H_{2\ell}$.

The main technique used to prove these results is to to analyze the set of zero divisors of $\frac{R'^\dagger}{P'_\ell R'^\dagger}$ (where $R^\dagger$ stands for either $\hat{R}$, $\tilde{R}$, or a finite type étale extension R^e of R), as follows. We show that the reducibility of $\frac{R'^\dagger}{P'_\ell R'^\dagger}$ is related to the existence of non-trivial algebraic extensions of $\kappa(P_\ell)$ inside $\kappa(P_\ell)\otimes_R R^\dagger$. More precisely, in the next section we define R to be **stable** if $\frac{R^\dagger}{P_{\ell+1}R^\dagger}$ is a domain and there does not exist a non-trivial algebraic extension of $\kappa(P_{\ell+1})$ which embeds both into $\kappa(P_{\ell+1})\otimes_R R^\dagger$ and into $\kappa(P'_{\ell+1})$ for some $R'\in\mathcal{T}$. We show that if R is stable then $\frac{R'^\dagger}{P'_{\ell+1}R'^\dagger}$ is a domain for all $R'\in\mathcal{T}$. For $\overline{\beta}\in\frac{\Delta_\ell}{\Delta_{\ell+1}}$, let

$$\mathcal{P}_{\overline{\beta}}=\{x\in R\mid \nu(x)\mod \Delta_{\ell+1}\geq\overline{\beta}\}\tag{44}$$

If Φ denotes the semigroup $\nu(R\setminus\{0\})\subset\Gamma$, which is well ordered since R is noetherian (see [Z], appendix 4, Proposition 2), and

$$\beta(\ell)=\min\{\gamma\in\Phi\mid \beta-\gamma\in\Delta_{\ell+1}\}$$

then $\mathcal{P}_{\overline{\beta}}=\mathcal{P}_{\beta(l)}$.
We have the inclusions

$$P_\ell\subset\mathcal{P}_{\overline{\beta}}\subset P_{\ell+1},$$

and $\mathcal{P}_{\overline{\beta}}$ is the inverse image in R by the canonical map $R\to\frac{R}{P_\ell}$ of a valuation ideal $\overline{\mathcal{P}}_{\overline{\beta}}\subset\frac{R}{P_\ell}$ for the rank one valuation $\frac{R}{P_\ell}\setminus\{0\}\to\frac{\Delta_\ell}{\Delta_{\ell+1}}$ induced by $\nu_{\ell+1}$.

We deduce from the above that if R is stable then for each $\overline{\beta}\in\frac{\Delta_\ell}{\Delta_{\ell+1}}$ and each ν-extension $R\to R'$ we have $\mathcal{P}'_{\overline{\beta}}R'^\dagger\cap R^\dagger=\mathcal{P}_{\overline{\beta}}R^\dagger$, which gives us a very good control of the limit in the definition of $H_{2\ell+1}$ and of the ν-extensions R' for which the limit is attained.

We then show, separately in the cases when $R^\dagger=\tilde{R}$ (§7) and $R^\dagger=\hat{R}$ (§8), that there always exists a stable ν-extension $R'\in\mathcal{T}$.

We are now ready to go into details, after giving several examples of implicit ideals and the phenomena discussed above.

Let $0\leq\ell\leq r$. We define our main object of study, the $(2\ell+1)$-st implicit prime ideal $H_{2\ell+1}\subset R^\dagger$, by

$$H_{2\ell+1}=\bigcap_{\beta\in\Delta_\ell}\left(\left(\lim_{\overrightarrow{R'}}\mathcal{P}'_\beta R'^\dagger\right)\bigcap R^\dagger\right),\tag{45}$$

where R' ranges over $\mathcal{T}$. As usual, we think of (45) as a tree equation: if we replace R by any other $R''\in\mathcal{T}$ in (45), it defines the corresponding ideal $H''_\ell\subset\hat{R}''$. Note that for $\ell=r$ (45) reduces to

$$H_{2r+1}=mR^\dagger.$$

We start by giving several examples of the ideals H'_i (and also of $\tilde{H}'_i$, which will appear a little later in the paper).

Example 3.1. Let $R=k[x,y,z]_{(x,y,z)}$. Let ν be the valuation with value group $\Gamma=\mathbf{Z}^2_{lex}$, defined as follows. Take a transcendental power series $\sum\limits_{j=1}^{\infty}c_ju^j$ in a variable u over k. Consider the homomorphism $R \hookrightarrow k[[u,v]]$ which sends x to v, y to u and z to $\sum\limits_{j=1}^{\infty} c_j u^j$. Consider the valuation ν, centered at $k[[u,v]]$, defined by $\nu(v) = (0,1)$ and $\nu(u) = (1,0)$; its restriction to R will also be denoted by ν, by abuse of notation. Let R_ν denote the valuation ring of ν in $k(x,y,z)$ and let $\mathcal{T}$ be the tree consisting of all the local rings R' essentially of finite type over R, birationally dominated by R_ν. Let $^\dagger = \hat{}$ denote the operation of formal completion. Given $\beta = (a,b) \in \mathbf{Z}^2_{lex}$, we have $\mathcal{P}_\beta = x^b \left(y^a, z - c_1 y - \dots - c_{a-1} y^{a-1}\right)$. The first isolated subgroup $\Delta_1 = (0) \oplus \mathbf{Z}$. Then $\bigcap\limits_{\beta \in (0)\oplus\mathbf{Z}} \left(\mathcal{P}_\beta \hat{R}\right) = (y,z)$ and $\bigcap\limits_{\beta \in \Gamma = \Delta_0} \left(\mathcal{P}_\beta \hat{R}\right) = \left(z - \sum\limits_{j=1}^{\infty} c_j y^j\right)$. It is not hard to show that for any $R' \in \mathcal{T}$ we have $H'_1 = \left(z - \sum\limits_{j=1}^{\infty} c_j y^j\right) \hat{R}'$ and that $H_3 = (y,z)\hat{R}$. It will follow from the general theory developed in §6 that ν admits a unique extension $\hat{\nu}$ to $\lim\limits_{\overrightarrow{R'}} \hat{R}'$. This extension has value group $\hat{\Gamma} = \mathbf{Z}^3_{lex}$ and is defined by $\hat{\nu}(x) = (0,0,1)$, $\hat{\nu}(y) = (0,1,0)$ and $\hat{\nu}\left(z - \sum\limits_{j=1}^{\infty} c_j y^j\right) = (1,0,0)$. For each $R' \in \mathcal{T}$ the ideal H'_1 is the prime valuation ideal corresponding to the isolated subgroup $(0) \oplus \mathbf{Z}^2_{lex}$ of $\hat{\Gamma}$ (that is, the ideal whose elements have values outside of $(0) \oplus \mathbf{Z}^2_{lex}$) while H'_3 is the prime valuation ideal corresponding to the isolated subgroup $(0) \oplus (0) \oplus \mathbf{Z}$.

Example 3.2. Let $R = k[x,y,z]_{(x,y,z)}$, $\Gamma = \mathbf{Z}^2_{lex}$, the power series $\sum\limits_{j=1}^{\infty} c_j u^j$ and the operation $^\dagger = \hat{}$ be as in the previous example. This time, let ν be defined as follows. Consider the homomorphism $R \hookrightarrow k[[u,v]]$ which sends x to u, y to $\sum\limits_{j=1}^{\infty} c_j u^j$ and z to v. Consider the valuation ν, centered at $k[[u,v]]$, defined by $\nu(v) = (1,0)$ and $\nu(u) = (0,1)$; its restriction to R will be also denoted by ν. Let R_ν denote the valuation ring of ν in $k(x,y,z)$ and let $\mathcal{T}$ be the tree consisting of all the local rings R' essentially of finite type over R, birationally dominated by R_ν. Given $\beta = (a,b) \in \mathbf{Z}^2_{lex}$, we have $\mathcal{P}_\beta = z^a \left(x^b, y - c_1 x - \dots - c_{b-1} x^{b-1}\right)$. The first isolated subgroup $\Delta_1 = (0) \oplus \mathbf{Z}$. Then $\bigcap\limits_{\beta \in (0)\oplus\mathbf{Z}} \left(\mathcal{P}_\beta \hat{R}\right) = \left(y - \sum\limits_{j=1}^{\infty} c_j x^j, z\right)$ and $\bigcap\limits_{\beta \in \Gamma = \Delta_0} \left(\mathcal{P}_\beta \hat{R}\right) = (0)$. It is not hard to show that for any $R' \in \mathcal{T}$ we have $H'_1 = (0)$ and that $H_3 = (y - \sum\limits_{j=1}^{\infty} c_j x^j, z)\hat{R}'$. In this case, the extension $\hat{\nu}$ to $\lim\limits_{\overrightarrow{R'}} \hat{R}'$ is not unique. Indeed, one possible extension $\hat{\nu}^{(1)}$ has value group $\hat{\Gamma} = \mathbf{Z}^3_{lex}$ and is defined by $\hat{\nu}^{(1)}(x) = (0,0,1)$, $\hat{\nu}^{(1)}\left(y - \sum\limits_{j=1}^{\infty} c_j x^j\right) = (0,1,0)$ and $\hat{\nu}^{(1)}(z) = (1,0,0)$. In this case, for any $R' \in \mathcal{T}$ the ideal H'_3 is the prime valuation ideal corresponding to the isolated subgroup $(0) \oplus (0) \oplus \mathbf{Z}$ of $\hat{\Gamma}$.

Another extension $\hat{\nu}^{(2)}$ of ν is defined by $\hat{\nu}^{(2)}(x) = (0,0,1)$, $\hat{\nu}^{(2)}\left(y - \sum\limits_{j=1}^{\infty} c_j x^j\right) = (1,0,0)$ and $\hat{\nu}^{(2)}(z) = (0,1,0)$. In this case, the tree of ideals corresponding to the isolated subgroup $(0) \oplus (0) \oplus \mathbf{Z}$ is H'_3 (exactly the same as for $\hat{\nu}^{(1)}$) while that corresponding to $(0) \oplus \mathbf{Z}^2_{lex}$ is $\tilde{H}'_1 = \left(y - \sum\limits_{j=1}^{\infty} c_j x^j\right)$. The tree $\tilde{H}'_1$ of prime $\hat{\nu}^{(2)}$-ideals determines the extension $\hat{\nu}^{(2)}$ completely.

The following two examples illustrate the need for taking the limit over the tree $\mathcal{T}$.

Example 3.3. Let us consider the local domain $S = \frac{k[x,y]_{(x,y)}}{(y^2-x^2-x^3)}$. There are two distinct valuations centered in (x, y). Let $a_i \in k,\ i \geq 2$ be such that

$$\left(y + x + \sum_{i\geq 2} a_i x^i\right)\left(y - x - \sum_{i\geq 2} a_i x^i\right) = y^2 - x^2 - x^3.$$

We shall denote by ν_+ the rank one discrete valuation defined by

$$\nu_+(x) = \nu_+(y) = 1,$$

$$\nu_+(y + x) = 2,$$

$$\nu_+\left(y + x + \sum_{i\geq 2}^{b-1} a_i x^i\right) = b.$$

Now let $R = \frac{k[x,y,z]_{(x,y,z)}}{(y^2-x^2-x^3)}$. Let $\Gamma = \mathbf{Z}^2$ with the lexicographical ordering. Let ν be the composite valuation of the (z)-adic one with ν_+, centered in $\frac{R}{(z)}$. The point of this example is to show that

$$H^*_{2\ell+1} = \bigcap_{\beta\in\Delta_\ell} \mathcal{P}_\beta \hat{R}$$

does not work as the definition of the $(2\ell + 1)$-st implicit prime ideal because the resulting ideal $H^*_{2\ell+1}$ is not prime. Indeed, as $\mathcal{P}_{(a,0)} = (z^a)$, we have

$$H^*_1 = \bigcap_{(a,b)\in\mathbf{Z}^2} \mathcal{P}_{(a,b)}\hat{R} = (0).$$

Let $f = y + x + \sum\limits_{i\geq 2} a_i x^i, g = y - x - \sum\limits_{i\geq 2} a_i x^i \in \hat{R}$. Clearly $f, g \notin H^*_1 = (0)$, but $f \cdot g = (0)$, so the ideal H^*_1 is not prime.

One might be tempted (as we were) to correct this problem by localizing at $H^*_{2\ell+3}$. Indeed, if we take the new definition of $H^*_{2\ell+1}$ to be, recursively in the descending order of ℓ,

$$H^*_{2\ell+1} = \left(\bigcap_{\beta\in\Delta_\ell} \mathcal{P}_\beta \hat{R}_{H^*_{2\ell+3}}\right) \cap \hat{R}, \tag{46}$$

then in the present example the resulting ideals $H^*_3 = (z, f)$ and $H^*_1 = (f)$ are prime. However, the next example shows that the definition (46) also does not, in general, give rise to prime ideals.

Example 3.4. Let $R = \frac{k[x,y,z]_{(x,y,z)}}{(z^2-y^2(1+x))}$. Let $\Gamma = \mathbf{Z}^2$ with the lexicographical ordering. Let t be an independent variable and let ν be the valuation, centered in R, induced by the t-adic valuation of $k\left[\left[t^\Gamma\right]\right]$ under the injective homomorphism $\iota : R \hookrightarrow k\left[\left[t^\Gamma\right]\right]$, defined by $\iota(x) = t^{(0,1)}$, $\iota(y) = t^{(1,0)}$ and $\iota(z) = t^{(1,0)}\sqrt{1 + t^{(0,1)}}$. The prime ν-ideals of R are $(0) \subsetneqq P_1 \subsetneqq m$, with $P_1 = (y, z)$. We have $\bigcap\limits_{\beta\in\Delta_1} \mathcal{P}_\beta \hat{R} = (y, z)\hat{R} = P_1\hat{R}$ and $\bigcap\limits_{\beta\in\Gamma} \mathcal{P}_\beta \hat{R}_{(y,z)} = \bigcap\limits_{\beta\in\Gamma} \mathcal{P}_\beta \hat{R} = (0)$. Note that the ideal (0) is not prime in $\hat{R}$. Now, let $R' = R\left[\frac{z}{y}\right]_{m'}$, where $m' = \left(x, y, \frac{z}{y} - 1\right)$ is the center of ν in $R\left[\frac{z}{y}\right]$.

We have $z-y\sqrt{1+x}\in\hat R\setminus\mathcal{P}_{(2,0)}\hat R$. On the other hand, $z-y\sqrt{1+x}=y\left(\frac{z}{y}-\sqrt{1+x}\right)=0$ in $\hat R'$; in particular, $z-y\sqrt{1+x}\in\bigcap\limits_{\beta\in\Gamma}\mathcal{P}'_\beta\hat R'$. Thus this example also shows that the ideals $\mathcal{P}_\beta\hat R$, $\bigcap\limits_{\beta\in\Delta_\ell}\mathcal{P}_\beta\hat R$ and $\bigcap\limits_{\beta\in\Delta_\ell}\mathcal{P}_\beta\hat R_{H_{2\ell+3}}$ do not behave well under blowing up.

Note that both Examples 3.3 and 3.4 occur not only for the completion $\hat R$ but also for the henselization $\tilde R$.

We come back to the general theory of implicit ideals.

Proposition 3.1 *We have $H_{2\ell+1}\cap R=P_\ell$.*

Proof: Recall that $P_\ell=\{x\in R/\ \nu(x)\notin\Delta_\ell\}$. If $x\in P_\ell$ then, since Δ_ℓ is an isolated subgroup, we have $x\in\mathcal{P}_\beta$ for all $\beta\in\Delta_\ell$. The same inclusion holds for the same reason in all extensions $R'\subset R_\nu$ of R, and this implies the inclusion $P_\ell\subseteq H_{2\ell+1}\cap R$. Now let x be in $H_{2\ell+1}\cap R$ and assume $x\notin P_\ell$. Then there is a $\beta\in\Delta_\ell$ such that $x\notin\mathcal{P}_\beta$. By faithful flatness of $R^\dagger$ over R we have $\mathcal{P}_\beta R^\dagger\cap R=\mathcal{P}_\beta$. This implies that $x\notin\mathcal{P}_\beta R^\dagger$, and the same argument holds in all the extensions $R'\in\mathcal{T}$, so x cannot be in $H_{2\ell+1}\cap R$. This contradiction shows the desired equality. $\square$

Proposition 3.2 *The ideals $H'_{2\ell+1}$ behave well under ν-extensions $R\to R'$ in $\mathcal{T}$. In other words, let $R\to R'$ be a ν-extension in $\mathcal{T}$ and let $H'_{2\ell+1}$ denote the $(2\ell+1)$-st implicit prime ideal of $\hat R'$. Then $H_{2\ell+1}=H'_{2\ell+1}\cap R^\dagger$.*

Proof: Immediate from the definitions. $\square$

To study the ideals $H_{2\ell+1}$, we need to understand more explicitly the nature of the limit appearing in (45). To study the relationship between the ideals $P_\beta R^\dagger$ and $P'_\beta R'^\dagger\bigcap R^\dagger$, it is useful to factor the natural map $R^\dagger\to R'^\dagger$ as $R^\dagger\to(R^\dagger\otimes_R R')_{M'}\overset{\phi}{\to}R'^\dagger$ as we did in the proof of Lemma 1.1. In general, the ring $R^\dagger\otimes_R R'$ is not local (see the above examples), but it has one distinguished maximal ideal M', namely, the ideal generated by $mR^\dagger\otimes 1$ and $1\otimes m'$, where m' denotes the maximal ideal of R'. The map ϕ factors through the local ring $\left(R^\dagger\otimes_R R'\right)_{M'}$ and the resulting map $\left(R^\dagger\otimes_R R'\right)_{M'}\to R'^\dagger$ is either the formal completion or the henselization; in either case, it is faithfully flat. Thus $P'_\beta R'^\dagger\cap\left(R^\dagger\otimes_R R'\right)_{M'}=P'_\beta\left(R^\dagger\otimes_R R'\right)_{M'}$. This shows that we may replace $R'^\dagger$ by $\left(R^\dagger\otimes_R R'\right)_{M'}$ in (45) without affecting the result, that is,

$$H_{2\ell+1}=\bigcap_{\beta\in\Delta_\ell}\left(\left(\lim_{\overrightarrow{R'}}\mathcal{P}'_\beta\left(R^\dagger\otimes_R R'\right)_{M'}\right)\bigcap R^\dagger\right). \tag{47}$$

From now on, we will use (47) as our working definition of the implicit prime ideals. One advantage of the expression (47) is that it makes sense in a situation more general than the completion and the henselization. Namely, to study the case of the henselization $\tilde R$, we will need to consider local étale extensions R^e of R, which are contained in $\tilde R$ (particularly, those which are essentially of finite type). The definition (47) of the implicit prime ideals makes sense also in that case.

4 Stable rings and primality of their implicit ideals.

Let the notation be as in the preceding sections. As usual, $R^\dagger$ will denote one of $\hat R$, $\tilde R$ or R^e (a local étale ν-extension essentially of finite type). Take an $R'\in\mathcal{T}$ and $\overline{\beta}\in\frac{\Delta_\ell}{\Delta_{\ell+1}}$. We have the obvious inclusion of ideals

$$\mathcal{P}_{\overline{\beta}}R^\dagger \subset \mathcal{P}_{\overline{\beta}}R'^\dagger \cap R^\dagger \tag{48}$$

(where $\mathcal{P}_{\overline{\beta}}$ is defined in (44)). A useful subtree of $\mathcal{T}$ is formed by the ℓ-stable rings, which we now define. An important property of stable rings, proved below, is that the inclusion (48) is an equality whenever R' is stable.

Definition 4.1 *A ring $R' \in \mathcal{T}(R)$ is said to be ℓ-**stable** if the following two conditions hold:*

(1) the ring

$$\kappa\left(P'_\ell\right) \otimes_R \left(R' \otimes_R R^\dagger\right)_{M'} \tag{49}$$

is an integral domain and

(2) there do not exist an $R'' \in \mathcal{T}(R')$ and a non-trivial algebraic extension L of $\kappa(P'_\ell)$ which embeds both into $\kappa\left(P'_\ell\right) \otimes_R \left(R' \otimes_R R^\dagger\right)_{M'}$ and $\kappa(P''_\ell)$.

*We say that R is **stable** if it is ℓ-stable for each $\ell \in \{0, \dots, r\}$.*

Remark 4.1 *(1) Rings of the form (49) will be a basic object of study in this paper. Another way of looking at the same ring, which we will often use, comes from interchanging the order of tensor product and localization. Namely, let T' denote the image of the multiplicative system $\left(R' \otimes_R R^\dagger\right) \setminus M'$ under the natural map $R' \otimes_R R^\dagger \to \kappa\left(P'_\ell\right) \otimes_R R^\dagger$. Then the ring (49) equals the localization $(T')^{-1}\left(\kappa\left(P'_\ell\right) \otimes_R R^\dagger\right)$.*

(2) In the special case $R' = R$ in Definition 4.1, we have

$$\kappa\left(P'_\ell\right) \otimes_R \left(R' \otimes_R R^\dagger\right)_{M'} = \kappa\left(P_\ell\right) \otimes_R R^\dagger.$$

If, moreover, $\frac{R}{P_\ell}$ is analytically irreducible then the hypothesis that $\kappa\left(P_\ell\right) \otimes_R R^\dagger$ is a domain holds automatically; in fact, this hypothesis is equivalent to analytic irreducibility of $\frac{R}{P_\ell}$ if $R^\dagger = \hat{R}$ or $R^\dagger = \tilde{R}$.

In this section we study ℓ-stable rings. We prove that if R is ℓ-stable then so is any $R' \in \mathcal{T}(R)$ (justifying the name "stable"). The main result of this section, Theorem 4.1, says that if R is stable then the implicit ideal $H'_{2\ell+1}$ is prime for each $\ell \in \{0, \dots, r\}$ and each $R' \in \mathcal{T}(R)$.

Remark 4.2 *In the next two sections we will show that there exist stable rings $R' \in \mathcal{T}$ for both $R^\dagger = \hat{R}$ and $R^\dagger = R^e$. However, the proof of this is different depending on whether we are dealing with completion or with an étale extension, and will be carried out separately in two separate sections: one devoted to henselization, the other to completion.*

Proposition 4.1 *Fix an integer ℓ, $0 \le \ell \le r$. Assume that R' is ℓ-stable and let $R'' \in \mathcal{T}(R')$. Then R'' is ℓ-stable.*

Proof: We have to show that (1) and (2) of Definition 4.1 for R' imply (1) and (2) of Definition 4.1 for R''. The ring

$$\kappa\left(P''_\ell\right) \otimes_R \left(R'' \otimes_R R^\dagger\right)_{M''} \tag{50}$$

is a localization of $\kappa\left(P''_\ell\right) \otimes_R \left(\kappa\left(P'_\ell\right) \otimes_R \left(R' \otimes_R R^\dagger\right)_{M'}\right)$. Hence (1) and (2) of Definition 4.1, applied to R', imply that $\kappa\left(P''_\ell\right) \otimes_R \left(R'' \otimes_R R^\dagger\right)_{M''}$ is an integral domain, so (1) of Definition 4.1 holds for R''. Replacing R' by R'' clearly does not affect the hypotheses about the non-existence of the extension L, so (2) of Definition 4.1 also holds for R''. $\square$

Next, we prove a technical result on which much of the rest of the paper is based:

Proposition 4.2 *Assume that R itself is $(\ell+1)$-stable and let $R' \in \mathcal{T}(R)$. Then for any $\overline{\beta} \in \frac{\Delta_\ell}{\Delta_{\ell+1}}$ we have*

$$\mathcal{P}'_{\overline{\beta}}R'^{\dagger} \cap R^{\dagger} = \mathcal{P}_{\overline{\beta}}R^{\dagger}. \tag{51}$$

Proof of the Proposition: As explained at the end of the previous section, since $R'^{\dagger}$ is faithfully flat over the ring $\left(R^{\dagger} \otimes_R R'\right)_{M'}$, we may replace $R'^{\dagger}$ by $\left(R^{\dagger} \otimes_R R'\right)_{M'}$ in (51). In other words, it is sufficient to prove that

$$\mathcal{P}'_{\overline{\beta}}\left(R^{\dagger} \otimes_R R'\right)_{M'} \bigcap R^{\dagger} = \mathcal{P}_{\overline{\beta}}R^{\dagger}. \tag{52}$$

One inclusion in (52) is trivial; we must show that

$$\mathcal{P}'_{\overline{\beta}}\left(R^{\dagger} \otimes_R R'\right)_{M'} \bigcap R^{\dagger} \subset \mathcal{P}_{\overline{\beta}}R^{\dagger}. \tag{53}$$

Lemma 4.1 *Let T' denote the image of the multiplicative set $\left(R' \otimes_R R^{\dagger}\right) \setminus M'$ under the natural map of R-algebras $R' \otimes_R R^{\dagger} \to \frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_{\ell+1}}} \otimes_R R^{\dagger}$. Then the map of R-algebras*

$$\bar{\pi} : \frac{R_{P_{\ell+1}}}{\mathcal{P}_{\overline{\beta}}R_{P_{\ell+1}}} \otimes_R R^{\dagger} \to (T')^{-1}\left(\frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_{\ell+1}}} \otimes_R R^{\dagger}\right) \tag{54}$$

induced by $\pi : R \to R'$ is injective.

Proof of Lemma 4.1: We start with the field extension

$$\kappa(P_{\ell+1}) \hookrightarrow \kappa(P'_{\ell+1})$$

induced by π. Since $R^{\dagger}$ is flat over R, the induced map $\pi_1 : \kappa(P_{\ell+1}) \otimes_R R^{\dagger} \to \kappa(P'_{\ell+1}) \otimes_R R^{\dagger}$ is also injective. By (1) of Definition 4.1, $\kappa(P'_{\ell+1}) \otimes_R R^{\dagger}$ is a domain. In particular,

$$\kappa\left(P'_{\ell+1}\right) \otimes_R R^{\dagger} = \left(\frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_{\ell+1}}} \otimes_R R^{\dagger}\right)_{red}. \tag{55}$$

The local ring $\frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_{\ell+1}}}$ is artinian because it can be seen as the quotient of $\frac{R'_{P'_{\ell+1}}}{P'_{\ell}R'_{P'_{\ell+1}}}$ by a valuation ideal corresponding to a rank one valuation. Since the ring is noetherian the valuation of the maximal ideal is positive, and since the group is archimedian, a power of the maximal ideal is contained in the valuation ideal.

Therefore, its only associated prime is its nilradical, the ideal $\frac{P'_{\ell+1}R'_{P'_{\ell+1}}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_{\ell+1}}}$; in particular, the (0) ideal in this ring has no embedded components. Since $R^{\dagger}$ is flat over R, $\frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_{\ell+1}}} \otimes_R R^{\dagger}$ is flat over $\frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_{\ell+1}}}$ by base change. Hence the (0) ideal of $\frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_{\ell+1}}} \otimes_R R^{\dagger}$ has no embedded components. In particular, since the multiplicative system T' is disjoint from the nilradical of $\frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_{\ell+1}}} \otimes_R R^{\dagger}$, the set T' contains no zero divisors, so localization by T' is injective.

By the definition of $\mathcal{P}_{\overline{\beta}}$, the map $\frac{R_{P_{\ell+1}}}{\mathcal{P}_{\overline{\beta}}R_{P_{\ell+1}}} \to \frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_{\ell+1}}}$ is injective, hence so is

$$\frac{R_{P_{\ell+1}}}{\mathcal{P}_{\overline{\beta}}R_{P_{\ell+1}}} \otimes_R R^\dagger \to \frac{R'_{P'_{\ell+1}}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_{\ell+1}}} \otimes_R R^\dagger$$

by the flatness of $R^\dagger$ over R. Combining this with the injectivity of the localization by T', we obtain that $\bar{\pi}$ is injective, as desired. Lemma 4.1 is proved. □

Again by the definition of $\mathcal{P}_{\overline{\beta}}$, the localization map $\frac{R}{\mathcal{P}_{\overline{\beta}}} \to \frac{R_{P_{\ell+1}}}{\mathcal{P}_{\overline{\beta}}R_{P_{\ell+1}}}$ is injective, hence so is the map

$$\frac{R}{\mathcal{P}_{\overline{\beta}}} \otimes_R R^\dagger \to \frac{R_{P_{\ell+1}}}{\mathcal{P}_{\overline{\beta}}R_{P_{\ell+1}}} \otimes_R R^\dagger \tag{56}$$

by the flatness of $R^\dagger$ over R. Combining this with Lemma 4.1, we see that the composition

$$\frac{R}{\mathcal{P}_{\overline{\beta}}} \otimes_R R^\dagger \to (T')^{-1}\left(\frac{R'_{P'_\ell}}{\mathcal{P}'_{\overline{\beta}}R'_{P'_\ell}} \otimes_R R^\dagger\right) \tag{57}$$

of (56) with $\bar{\pi}$ is also injective. Now, (57) factors through $\left(\frac{R'}{\mathcal{P}'_{\overline{\beta}}} \otimes_R R^\dagger\right)_{M'}$ (here we are guilty of a slight abuse of notation: we denote the natural image of M' in $\frac{R'}{\mathcal{P}'_{\overline{\beta}}} \otimes_R R^\dagger$ also by M'). Hence the map

$$\frac{R}{\mathcal{P}_{\overline{\beta}}} \otimes_R R^\dagger \to \left(\frac{R'}{\mathcal{P}'_{\overline{\beta}}} \otimes_R R^\dagger\right)_{M'} \tag{58}$$

is injective. Since $\frac{R}{\mathcal{P}_{\overline{\beta}}} \otimes_R R^\dagger \cong \frac{R^\dagger}{\mathcal{P}_{\overline{\beta}}R^\dagger}$ and $\left(\frac{R'}{\mathcal{P}'_{\overline{\beta}}} \otimes_R R^\dagger\right)_{M'} \cong \frac{(R'\otimes_R R^\dagger)_{M'}}{\mathcal{P}'_{\overline{\beta}}(R'\otimes_R R^\dagger)_{M'}}$, the injectivity of (58) is the same as (53). This completes the proof of the Proposition. □

Corollary 4.1 *Take an integer $\ell \in \{1,\dots,r+1\}$ and assume that R is $(\ell+1)$-stable. Then*

$$H_{2\ell+1} = \bigcap_{\beta\in\Delta_\ell} \mathcal{P}_\beta R^\dagger. \tag{59}$$

Proof: By Lemma 4 of Appendix 4 of [Z], the ideals $\mathcal{P}_{\overline{\beta}}$ are cofinal among the ideals $\mathcal{P}_\beta$ for $\beta \in \Delta_\ell$.

Now we are ready to state and prove the main Theorem of this section.

Theorem 4.1 *(1) Fix an integer $\ell \in \{1,\dots,r+1\}$. Assume that there exists $R' \in \mathcal{T}(R)$ which is $(\ell+1)$-stable. Then the ideal $H_{2\ell+1}$ is prime.*

(2) Let $i = 2\ell+2$. There exists an extension $\nu^\dagger_{i0}$ of $\nu_{\ell+1}$ to $\lim\limits_{\overrightarrow{R'}} \kappa(H'_{i-1})$, with value group

$$\Delta_{i-1,0} = \frac{\Delta_\ell}{\Delta_{\ell+1}}, \tag{60}$$

whose valuation ideals are described as follows. For an element $\overline{\beta} \in \frac{\Delta_\ell}{\Delta_{\ell+1}}$, the $\nu^\dagger_{i0}$-ideal of $\frac{R^\dagger}{H_{i-1}}$ of value $\overline{\beta}$, denoted by $\mathcal{P}^\dagger_{\overline{\beta}\ell}$, is given by the formula

$$\mathcal{P}^\dagger_{\overline{\beta},\ell+1} = \left(\lim_{\overrightarrow{R'}} \frac{\mathcal{P}'_{\overline{\beta}}R'^\dagger}{H'_{i-1}}\right) \cap \frac{R^\dagger}{H_{i-1}}. \tag{61}$$

Remark 4.3 *Once the even-numbered implicit prime ideals $H'_{2\ell}$ are defined below, we will show that $\nu^\dagger_{i0}$ is the unique extension of $\nu_{\ell+1}$ to $\lim\limits_{\overrightarrow{R'}}\kappa(H'_{i-1})$, centered in the local ring $\lim\limits_{\overrightarrow{R'}}\frac{R'^\dagger_{H'_{2\ell+2}}}{H'_{2\ell+1}R'^\dagger_{H'_{2\ell+2}}}$.*

Proof of Theorem 4.1: Let R' be a stable ring in $\mathcal{T}(R)$. Once Theorem 4.1 is proved for R', the same results for R will follow easily by intersecting all the ideals of $R'^\dagger$ in sight with $R^\dagger$. Therefore from now on we will replace R by R', that is, we will assume that R itself is stable.

Let Φ_ℓ denote the image of the semigroup $\nu(R\setminus\{0\})$ in $\frac{\Gamma}{\Delta_{\ell+1}}$. As we saw above, Φ_ℓ is well ordered. For an element $\overline{\beta}\in\Phi_\ell$, let $\overline{\beta}+$ denote the immediate successor of $\overline{\beta}$ in Φ_ℓ.

Take any element $x\in R^\dagger\setminus H_{i-1}$. By Corollary 4.1, there exists (a unique) $\overline{\beta}\in\Phi_\ell\cap\frac{\Delta_\ell}{\Delta_{\ell+1}}$ such that

$$x\in\mathcal{P}_{\overline{\beta}}R^\dagger\setminus\mathcal{P}_{\overline{\beta}+}R^\dagger \tag{62}$$

(where, of course, we allow $\overline{\beta}=0$). Let $\bar{x}$ denote the image of x in $\frac{R^\dagger}{H_{i-1}}$. We define

$$\nu^\dagger_{i0}(\bar{x})=\overline{\beta}.$$

Next, take another element $y\in R^\dagger\setminus H_{2\ell+1}$ and let $\gamma\in\Phi_\ell\cap\frac{\Delta_\ell}{\Delta_{\ell+1}}$ be such that

$$y\in\mathcal{P}_{\overline{\gamma}}R^\dagger\setminus\mathcal{P}_{\overline{\gamma}+}R^\dagger. \tag{63}$$

Let $(a_1,...,a_n)$ be a set of generators of $\mathcal{P}_{\overline{\beta}}$ and $(b_1,...,b_s)$ a set of generators of $\mathcal{P}_{\overline{\gamma}}$, with $\nu_{\ell+1}(a_1)=\overline{\beta}$ and $\nu_{\ell+1}(b_1)=\overline{\gamma}$. Let R' be a local blowing up along ν such that R' contains all the fractions $\frac{a_i}{a_1}$ and $\frac{b_j}{b_1}$. By Proposition 4.1 and Definition 4.1 (1), the ideal $P'_{\ell+1}R'^\dagger$ is prime. By construction, we have $a_1\mid x$ and $b_1\mid y$ in $R'^\dagger$. Write $x=za_1$ and $y=wb_1$ in $R'^\dagger$. The equality (51), combined with (62) and (63), implies that $z,w\notin P'_{\ell+1}R'^\dagger$, hence

$$zw\notin P'_{\ell+1}R'^\dagger \tag{64}$$

by the primality of $P'_{\ell+1}R'^\dagger$. We obtain

$$xy=a_1b_1zw. \tag{65}$$

Since ν is a valuation on R', we have $\left(\mathcal{P}'_{\overline{\beta}+\overline{\gamma}+}:(a_1b_1)R'\right)\subset P'_{\ell+1}$. By faithful flatness of $R'^\dagger$ over R' we obtain

$$\left(\mathcal{P}'_{\overline{\beta}+\overline{\gamma}+}R'^\dagger:(a_1b_1)R'^\dagger\right)\subset P'_{\ell+1}R'^\dagger. \tag{66}$$

Combining this with (64) and (65), we obtain

$$xy\notin\mathcal{P}_{\overline{\beta}+\overline{\gamma}+}R^\dagger, \tag{67}$$

in particular, $xy\notin H_{2\ell+1}$. We started with two arbitrary elements $x,y\in R^\dagger\setminus H_{2\ell+1}$ and showed that $xy\notin H_{2\ell+1}$. This proves (1) of the Theorem.

Furthermore, (67) shows that $\nu^\dagger_{i0}(\bar{x}\bar{y})=\overline{\beta}+\overline{\gamma}$, so $\nu^\dagger_{i0}$ induces a valuation of $\kappa(H_{i-1})$ and hence also of $\lim\limits_{\overrightarrow{R'}}\kappa(H'_{i-1})$. Equality (60) holds by definition and (61) by the assumed stability of R. $\square$

Next, we define the even-numbered implicit prime ideals $H'_{2\ell}$. The only information we need to use to define the prime ideals $H'_{2\ell}\subset H'_{2\ell+1}$ and to prove that $H'_{2\ell-1}\subset H'_{2\ell}$ are the facts that $H_{2\ell+1}$ is a prime lying over P_ℓ and that the ring homomorphism $R'\to R'^\dagger$ is regular.

Proposition 4.3 *There exists a unique minimal prime ideal $H_{2\ell}$ of $P_\ell R^\dagger$, contained in $H_{2\ell+1}$.*

Proof: Since $H_{2\ell+1}\cap R = P_\ell$, $H_{2\ell+1}$ belongs to the fiber of the map $Spec\ R^\dagger \to Spec\ R$ over P_ℓ. Since R was assumed to be excellent, $S := R^\dagger \otimes_R \kappa(P_\ell)$ is a regular ring (note that the excellence assumption is needed only in the case $R^\dagger = \hat{R}$; the ring homomorphism $R \to R^\dagger$ is automatically regular if $R^\dagger = \tilde{R}$ or $R^\dagger = R^e$). Hence its localization $\bar{S} := S_{H_{2\ell+1}S} \cong \frac{R^\dagger_{H_{2\ell+1}}}{P_\ell R^\dagger_{H_{2\ell+1}}}$ is a regular *local* ring. In particular, $\bar{S}$ is an integral domain, so (0) is its unique minimal prime ideal. The set of minimal prime ideals of $\bar{S}$ is in one-to-one correspondence with the set of minimal primes of P_ℓ, contained in $H_{2\ell+1}$, which shows that such a minimal prime $H_{2\ell}$ is unique, as desired. $\Box$

We have $P_\ell \subset H_{2\ell}\cap R \subset H_{2\ell+1}\cap R = P_\ell$, so $H_{2\ell}\cap R \subset P_\ell$.

Proposition 4.4 *We have $H_{2\ell-1} \subset H_{2\ell}$.*

Proof: Take an element $\beta \in \frac{\Delta_{\ell-1}}{\Delta_\ell}$ and a stable ring $R' \in \mathcal{T}$. Then $\mathcal{P}'_\beta \subset P'_\ell$, so

$$H'_{2\ell-1} \subset \mathcal{P}'_\beta {R'}^\dagger \subset P'_\ell {R'}^\dagger \subset H'_{2\ell}. \tag{68}$$

Intersecting (68) back with $R^\dagger$ we get the result. $\Box$

In §7 we will see that if $R^\dagger = \tilde{R}$ or $R^\dagger = R^e$ then $H_{2\ell} = H_{2\ell+1}$ for all ℓ.

Let the notation be the same as in Theorem 4.1.

Proposition 4.5 *The valuation $\nu^\dagger_{i0}$ is the unique extension of ν_ℓ to a valuation of $\lim\limits_{\overrightarrow{R'}} \kappa(H'_{i-1})$, centered in the local ring $\lim\limits_{\overrightarrow{R'}} \frac{{R'}^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}{R'}^\dagger_{H'_{2\ell}}}$.*

Proof: As usual, without loss of generality we may assume that R is stable. Take an element $x \in R^\dagger \setminus H_{2\ell-1}$. Let $\beta = \nu^\dagger_{i0}(\bar{x})$ and let R' be the blowing up of the ideal $\mathcal{P}_\beta = (a_1,\dots,a_n)$, as in the proof of Theorem 4.1. Write

$$x = za_1 \tag{69}$$

in R'. We have $z \in {R'}^\dagger \setminus P'_\ell {R'}^\dagger$, hence

$$\bar{z} \in \frac{{R'}^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}{R'}^\dagger_{H'_{2\ell}}} \setminus \frac{P'_\ell{R'}^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}{R'}^\dagger_{H'_{2\ell}}} = \frac{{R'}^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}{R'}^\dagger_{H'_{2\ell}}} \setminus \frac{H'_{2\ell}{R'}^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}{R'}^\dagger_{H'_{2\ell}}}. \tag{70}$$

If ν^* is any other extension of ν_ℓ to $\lim\limits_{\overrightarrow{R'}} \kappa(H'_{i-1})$, centered in $\lim\limits_{\overrightarrow{R'}} \frac{{R'}^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}{R'}^\dagger_{H'_{2\ell}}}$, then $\nu^*(\bar{a}_1) = \beta$, $\nu^*(z) = 0$ by (70), so $\nu^*(\bar{x}) = \beta = \nu^\dagger_{i0}(\bar{x})$. This completes the proof of the uniqueness of $\nu^\dagger_{i0}$. $\Box$

Remark 4.4 *If R' is stable, we have a natural isomorphism of graded algebras*

$$gr_{\nu^\dagger_{i0}} \frac{{R'}^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}{R'}^\dagger_{H'_{2\ell}}} \cong gr_{\nu_\ell} \frac{R'_{P'_\ell}}{P'_{\ell-1}R'_{P'_\ell}} \otimes_{R'} \kappa(H'_{2\ell}).$$

In particular, the residue field of $\nu^\dagger_{i0}$ is $k_{\nu^\dagger_{i0}} = \lim\limits_{\overrightarrow{R'}} \kappa(H'_{2\ell})$.

5 A classification of extensions of ν to rings of the form $\frac{R^\dagger}{P}$.

The purpose of this section is to give a systematic description of all the possible extensions $\nu^\dagger$ of ν to $R^\dagger$ as compositions of $2r$ valuations,

$$\nu^\dagger = \nu_1^\dagger \circ \cdots \circ \nu_{2r}^\dagger, \tag{71}$$

satisfying certain conditions. One is naturally led to consider the more general problem of extending ν not only to rings of the form $\frac{R^\dagger}{P}$ but also to the ring $\lim\limits_{\to} \frac{R'^\dagger}{P'}$, where P' is a tree of prime ideals of $R'^\dagger$, such that $P' \cap R' = (0)$. We deal in a uniform way with all the three cases $R^\dagger = \hat{R}$, $R^\dagger = \tilde{R}$ and $R^\dagger = R^e$, in order to be able to apply the results proved here to all three later in the paper. However, the reader should think of the case $R^\dagger = \hat{R}$ as the main case of interest and the cases $R^\dagger = \tilde{R}$ and $R^\dagger = R^e$ as auxiliary and slightly degenerate, since, as we shall see, in these cases the equality $H_{2\ell} = H_{2\ell+1}$ is satisfied for all ℓ and the extension $\nu^\dagger$ will later be shown to be unique.

We will associate to each extension $\nu^\dagger$ a chain

$$\tilde{H}'_0 \subset \tilde{H}'_1 \subset \cdots \subset \tilde{H}'_{2r} = m' R'^\dagger \tag{72}$$

of prime $\nu^\dagger$-ideals, corresponding to the decomposition (71) and prove some basic properties of this chain of ideals.

Now for the details. We wish to classify all the pairs $\left(\left\{\tilde{H}'_0\right\}, \nu^\dagger\right)$, where $\left\{\tilde{H}'_0\right\}$ is a tree of prime ideals of $R'^\dagger$, such that $\tilde{H}'_0 \cap R' = (0)$, and $\nu^\dagger$ is an extension of ν to the ring $\lim\limits_{\to} \frac{R'^\dagger}{\tilde{H}'_0}$.

Pick and fix one such pair $\left(\left\{\tilde{H}'_0\right\}, \nu^\dagger\right)$. We associate to it the following collection of data, which, as we will see, will in turn determine the pair $\left(\left\{\tilde{H}'_0\right\}, \nu^\dagger\right)$.

First, we associate to $\left(\left\{\tilde{H}'_0\right\}, \nu^\dagger\right)$ a chain (72) of $2r$ trees of prime $\nu^\dagger$-ideals. Let $\Gamma^\dagger$ denote the value group of $\nu^\dagger$. Defining (72) is equivalent to defining a chain

$$\Gamma^\dagger = \Delta_0^\dagger \supset \Delta_1^\dagger \supset \cdots \supset \Delta_{2r}^\dagger = \Delta_{2r+1}^\dagger = (0) \tag{73}$$

of $2r$ isolated subroups of $\Gamma^\dagger$ (the chain (73) will not, in general, be maximal, and $\Delta_{2\ell+1}^\dagger$ need not be distinct from $\Delta_{2\ell}^\dagger$).

We define the $\Delta_i^\dagger$ as follows. For $0 \le \ell \le r$, let $\Delta_{2\ell}^\dagger$ and $\Delta_{2\ell+1}^\dagger$ denote, respectively, the greatest and the smallest isolated subgroups of $\Gamma^\dagger$ such that

$$\Delta_{2\ell}^\dagger \cap \Gamma = \Delta_{2\ell+1}^\dagger \cap \Gamma = \Delta_\ell. \tag{74}$$

Lemma 5.1 *We have*

$$rk\ \frac{\Delta_{2\ell-1}^\dagger}{\Delta_{2\ell}^\dagger} = 1 \tag{75}$$

for $1 \le \ell \le r$.

Proof. Since by construction $\Delta_{2\ell}^\dagger \neq \Delta_{2\ell-1}^\dagger$, equality (75) is equivalent to saying that there is no isolated subgroup $\Delta^\dagger$ of $\Gamma^\dagger$ which is properly contained in $\Delta_{2\ell-1}^\dagger$ and properly contains $\Delta_{2\ell}^\dagger$. Suppose such an isolated subgroup $\Delta^\dagger$ existed. Then

$$\Delta_\ell = \Delta_{2\ell}^\dagger \cap \Gamma \subsetneqq \Delta^\dagger \cap \Gamma \subsetneqq \Delta_{2\ell-1}^\dagger \cap \Gamma = \Delta_{\ell-1}, \tag{76}$$

where the first inclusion is strict by the maximality of $\Delta^\dagger_{2\ell}$ and the second by the minimality of $\Delta^\dagger_{2\ell-1}$. Thus $\Delta^\dagger\cap\Gamma$ is an isolated subgroup of Γ, properly containing Δ_ℓ and properly contained in Δ_{l-1}, which is impossible since $rk\ \frac{\Delta_{\ell-1}}{\Delta_\ell}=1$. This is a contradiction, hence $rk\ \frac{\Delta^\dagger_{2\ell-1}}{\Delta^\dagger_{2\ell}}=1$, as desired. □

Definition 5.1 *Let $0\le i\le 2r$. The i-th prime ideal* ***determined*** *by $\nu^\dagger$ is the prime $\nu^\dagger$-ideal $\tilde H'_i$ of $R'^\dagger$, corresponding to the isolated subgroup $\Delta^\dagger_i$ (that is, the ideal $\tilde H'_i$ consisting of all the elements of $R'^\dagger$ whose values lie outside of $\Delta^\dagger_i$). The chain of trees (72) of prime ideals of $R'^\dagger$ formed by the $\tilde H'_i$ is referred to as the chain of trees* ***determined*** *by $\nu^\dagger$.*

The equality (74) says that

$$\tilde H'_{2\ell}\cap R'=\tilde H'_{2\ell+1}\cap R'=P'_\ell \tag{77}$$

By definitions, for $1\le i\le 2r$, $\nu^\dagger_i$ is a valuation of the field $k_{\nu^\dagger_{i-1}}$. In the sequel, we will find it useful to talk about the restriction of $\nu^\dagger_i$ to a smaller field, namely, the field of fractions of the ring $\lim\limits_{\overrightarrow{R'}}\frac{R'^\dagger_{\tilde H'_i}}{\tilde H'_{i-1}R'^\dagger_{\tilde H'_i}}$; we will denote this restriction by $\nu^\dagger_{i0}$. The field of fractions of $\frac{R'^\dagger_{\tilde H'_i}}{\tilde H'_{i-1}R'^\dagger_{\tilde H'_i}}$ is $\kappa(\tilde H'_{i-1})$, hence that of $\lim\limits_{\overrightarrow{R'}}\frac{R'^\dagger_{\tilde H'_i}}{\tilde H'_{i-1}R'^\dagger_{\tilde H'_i}}$ is $\lim\limits_{\overrightarrow{R'}}\kappa(\tilde H'_{i-1})$, which is a subfield of $k_{\nu^\dagger_{i-1}}$. The value group of $\nu^\dagger_{i0}$ will be denoted by $\Delta_{i-1,0}$; we have $\Delta_{i-1,0}\subset\frac{\Delta^\dagger_{i-1}}{\Delta^\dagger_i}$. If $i=2\ell$ is even then $\frac{R'_{P'_l}}{P'_{l-1}R'_{P'_l}}<\frac{R'^\dagger_{\tilde H'_i}}{\tilde H'_{i-1}R'^\dagger_{\tilde H'_i}}$, so $\lim\limits_{\overrightarrow{R'}}\frac{R'_{P'_l}}{P'_{l-1}R'_{P'_l}}<\lim\limits_{\overrightarrow{R'}}\frac{R'^\dagger_{\tilde H'_i}}{\tilde H'_{i-1}R'^\dagger_{\tilde H'_i}}$. In this case $rk\ \nu^\dagger_i=1$ and $\nu^\dagger_i$ an extension of the rank 1 valuation ν_ℓ from $\kappa(P_{\ell-1})$ to $k_{\nu^\dagger_{i-1}}$; we have $\frac{\Delta_{\ell-1}}{\Delta_\ell}\subset\Delta_{i-1,0}\subset\frac{\Delta^\dagger_{i-1}}{\Delta^\dagger_i}$.

Proposition 5.1 *Let $i=2\ell$. As usual, for an element $\overline\beta\in\left(\frac{\Delta_\ell}{\Delta_{\ell+1}}\right)_+$, let $\mathcal P_{\overline\beta}$ (resp. $\mathcal P'_{\overline\beta}$) denote the preimage in R (resp. in R') of the $\nu_{\ell+1}$-ideal of $\frac{R}{P_\ell}$ (resp. $\frac{R'}{P'_\ell}$) of value greater than or equal to $\overline\beta$. Then*

$$\bigcap_{\overline\beta\in\left(\frac{\Delta_\ell}{\Delta_{\ell+1}}\right)_+}\lim_{\overrightarrow{R'}}\left(\mathcal P'_{\overline\beta}R'^\dagger+\tilde H'_{i+1}\right)R'^\dagger_{\tilde H'_{i+2}}\cap R^\dagger\subset\tilde H_{i+1}. \tag{78}$$

The inclusion (78) should be understood as a condition on the tree of ideals. In other words, it is equally valid if we replace R by any other ring $R''\in\mathcal T$.

Proof of Proposition 5.1: Since $rk\frac{\Delta^\dagger_{i+1}}{\Delta^\dagger_{i+2}}=1$ by Lemma 5.1, $\frac{\Delta_\ell}{\Delta_{\ell+1}}$ is cofinal in $\frac{\Delta^\dagger_{i+1}}{\Delta^\dagger_{i+2}}$. Then for any $x\in\bigcap\limits_{\overline\beta\in\left(\frac{\Delta_\ell}{\Delta_{\ell+1}}\right)_+}\lim\limits_{\overrightarrow{R'}}\left(\mathcal P'_{\overline\beta}R'^\dagger+\tilde H'_{i+1}\right)R'^\dagger_{\tilde H'_{i+2}}\cap R^\dagger$ we have $\nu^\dagger(x)\notin\Delta^\dagger_i$, hence $x\in\tilde H_{i+1}$, as desired.
□

From now to the end of §6, we will assume that $\mathcal T$ contains a stable ring R', so that we can apply the results of the previous section, in particular, the primality of the ideals H'_i.

Proposition 5.2 *We have*

$$H'_i\subset\tilde H'_i\qquad \textit{for all } i\in\{0,\dots,2r+1\}. \tag{79}$$

Proof: For $\beta \in \Gamma^\dagger$ and $R' \in \mathcal{T}$, let $\mathcal{P}'^\dagger_\beta$ denote the $\nu^\dagger$-ideal of $R'^\dagger$ of value β. Fix an integer $\ell \in \{0,\dots,r\}$. For each $R' \in \mathcal{T}$, each $\beta \in \Delta_\ell$ and $x \in \mathcal{P}'_\beta$ we have $\nu^\dagger(x) = \nu(x) \geq \beta$, hence

$$\mathcal{P}'_\beta R'^\dagger \subset \mathcal{P}'^\dagger_\beta. \tag{80}$$

Taking the inductive limit over all $R' \in \mathcal{T}$ and the intersection over all $\beta \in \Delta_\ell$ in (80), and using the cofinality of Δ_ℓ in $\Delta^\dagger_{2\ell+1}$ and the fact that $\bigcap\limits_{\beta \in \Delta^\dagger_{2\ell}} \left(\lim\limits_{\overrightarrow{R'}} \mathcal{P}'^\dagger_\beta \right) = \lim\limits_{\overrightarrow{R'}} \tilde{H}'_{2\ell+1}$, we obtain the inclusion (79) for $i = 2\ell+1$. To prove (79) for $i = 2\ell$, note that $\tilde{H}'_{2\ell} \cap R' = \tilde{H}'_{2\ell+1} \cap R' = P_\ell$. By the same argument as in Proposition 4.3, excellence of R' implies that there is a unique minimal prime $H^*_{2\ell}$ of $P'_\ell R'^\dagger$, contained in $\tilde{H}'_{2\ell+1}$ and a unique minimal prime $H^{**}_{2\ell}$ of $P'_\ell R'^\dagger$, contained in $\tilde{H}'_{2\ell}$. Now, Proposition 4.3 and the facts that $H'_{2\ell+1} \subset \tilde{H}'_{2\ell+1}$ and $\tilde{H}'_{2\ell} \subset \tilde{H}'_{2\ell+1}$ imply that $H'_{2\ell} = H^*_{2\ell} = H^{**}_{2\ell}$, hence $H'_{2\ell} = H^{**}_{2\ell} \subset \tilde{H}'_{2\ell}$, as desired. □

Definition 5.2 *A chain of trees (72) of prime ideals of $R'^\dagger$ is said to be **admissible** if $H'_i \subset \tilde{H}'_i$ and (77) and (78) hold.*

Equalities (77), Proposition 5.1 and Proposition 5.2 say that a chain of trees (72) of prime ideals of $R'^\dagger$, determined by $\nu^\dagger$, is admissible.

Summarizing all of the above results, and keeping in mind the fact that specifying a composition of $2r$ valuation is equivalent to specifying all of its $2r$ components, we arrive at one of the main theorems of this paper:

Theorem 5.1 *Specifying the valuation $\nu^\dagger$ is equivalent to specifying the following data. The data will be described recursively in i, that is, the description of $\nu^\dagger_i$ assumes that $\nu^\dagger_{i-1}$ is already defined:*

(1) An admissible chain of trees (72) of prime ideals of $R'^\dagger$.

(2) For each i, $1 \leq i \leq 2r$, a valuation $\nu^\dagger_i$ of $k_{\nu^\dagger_{i-1}}$ (where $\nu^\dagger_0$ is taken to be the trivial valuation by convention), whose restriction to $\lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_{i-1})$ is centered at the local ring $\lim\limits_{\overrightarrow{R'}} \dfrac{R'^\dagger_{\tilde{H}'_i}}{\tilde{H}'_{i-1} R'^\dagger_{\tilde{H}'_i}}$

The data $\left\{\nu^\dagger_i\right\}_{1\leq i\leq 2r}$ is subject to the following additional condition: if $i = 2\ell$ is even then rk $\nu^\dagger_i = 1$, $\nu^\dagger_i$ is an extension of ν_ℓ to $k_{\nu^\dagger_{i-1}}$ (which is naturally an extension of $k_{\nu_{\ell-1}}$).

In particular, note that such extensions $\nu^\dagger$ always exist, and usually there are plenty of them. The question of uniqueness of $\nu^\dagger$ and the related question of uniqueness of $\nu^\dagger_i$, especially in the case when i is even, will be addressed in the next section.

6 Uniqueness properties of $\nu^\dagger$.

In this section we address the question of uniqueness of the extension $\nu^\dagger$. One result in this direction, which will be very useful here, was already proved in §4: Proposition 4.5. We give some necessary and some sufficient conditions both for the uniqueness of $\nu^\dagger$ once the chain (72) of prime ideals determined by $\nu^\dagger$ has been fixed, and also for the unconditional uniqueness of $\nu^\dagger$. In §7 we will use one of these uniqueness criteria to prove uniqueness of $\nu^\dagger$ in the cases $R^\dagger = \tilde{R}$ and $R^\dagger = R^e$. At the end of this section we generalize and give a new point of view of an old result of W. Heinzer and J. Sally (Proposition 6.5), which provides a sufficient condition for the uniqueness of $\nu^\dagger$; see also [17], Remarks 5.22.

For a ring $R' \in \mathcal{T}$ let K' denote the field of fractions of R'. For some results in this section we will need to impose an additional condition on the tree $\mathcal{T}$: we will assume that there exists $R_0 \in \mathcal{T}$ such that for all $R' \in \mathcal{T}(R_0)$ the field K' is algebraic over K_0. This assumption is needed in order to be able to control the height of all the ideals in sight. Without loss of generality, we may take $R_0 = R$.

Proposition 6.1 *Assume that for all $R' \in \mathcal{T}$ the field K' is algebraic over K. Consider a ring homomorphism $R' \to R''$ in $\mathcal{T}$. Take an $\ell \in \{0, \dots, r\}$. We have*

$$\text{ht } H''_{2\ell} \leq \text{ht } H'_{2\ell}. \tag{81}$$

If equality holds in (81) then

$$\text{ht } H''_{2\ell+1} \geq \text{ht } H'_{2\ell+1}. \tag{82}$$

Proof: We start by recalling a well known Lemma (for a proof see [20], Appendix 1, Propositions 2 and 3, p. 326):

Lemma 6.1 *Let $R \hookrightarrow R$ be an extension of integral domains, essentially of finite type. Let K and K' be the respective fields of fractions of R and R'. Consider prime ideals $P \subset R$ and $P' \subset R'$ such that $P = P' \cap R$. Then*

$$\text{ht } P' + tr.deg.(\kappa(P')/\kappa(P)) \leq \text{ ht } P + tr.deg.(K'/K). \tag{83}$$

Moreover, equality holds in (83) whenever R is universally catenarian.

Apply the Lemma to the rings R' and R'' and the prime ideals $P'_\ell \subset R'$ and $P''_\ell \subset R''$. In the case at hand we have $tr.deg.(K''/K') = 0$ by assumption. Hence

$$\text{ht } P''_\ell \leq \text{ ht } P'_\ell. \tag{84}$$

Since $H'_{2\ell}$ is a minimal prime of $P'_\ell R'^\dagger$ and $R'^\dagger$ is faithfully flat over R', we have $ht\ P'_\ell = ht\ H'_{2\ell}$. Similarly, ht P''_ℓ = ht $H''_{2\ell}$, and (81) follows. Furthermore, equality in (81) is equivalent to equality in (84).

To prove (82), let $\bar{R} = (R'' \otimes_{R'} R'^\dagger)_{M''}$, where $M'' = (m'' \otimes 1 + 1 \otimes m' R'^\dagger)$ and let $\bar{m}$ denote the maximal ideal of $\bar{R}$. We have the natural maps $R'^\dagger \overset{\iota}{\to} \bar{R} \overset{\sigma}{\to} R''^\dagger$. The homomorphism σ is nothing but the formal completion of the local ring $\bar{R}$; in particular, it is faithfully flat. Let

$$\bar{H} = H''_{2\ell+1} \cap \bar{R}, \tag{85}$$

$\bar{H}_0 = H''_0 \cap \bar{R}$. Since H''_0 is a minimal prime of $R''^\dagger$ and σ is faithfully flat, $\bar{H}_0$ is a minimal prime of $\bar{R}$.

Assume that equality holds in (81) (and hence also in (84)). Since equality holds in (84), by Lemma 6.1 (applied to the ring extension $R' \to R''$) the field $\kappa(P'')$ is algebraic over $\kappa(P')$.

Apply Lemma 6.1 to the ring extension $\frac{R'^\dagger}{H'_0} \hookrightarrow \frac{\bar{R}}{\bar{H}_0}$ and the prime ideals $\frac{H'_{2\ell+1}}{H'_0}$ and $\frac{\bar{H}}{\bar{H}_0}$. Since K'' is algebraic over K', $\kappa(\bar{H}_0)$ is algebraic over $\kappa(H'_0)$. Since $\kappa(P'')$ is algebraic over $\kappa(P')$, $\kappa(\bar{H})$ is algebraic over $\kappa(H'_{2\ell+1})$. Finally, $\hat{R}'$ is universally catenarian because it is a complete local ring. Now in the case $^\dagger = \hat{}$ Lemma 6.1 says that ht $\frac{H'_{2\ell+1}}{H'_0}$ = ht $\frac{\bar{H}}{\bar{H}_0}$. Since both $\hat{R}'$ and $\bar{R}$ are catenarian, this implies that

$$\text{ht } H'_{2\ell+1} = \text{ht } \bar{H}. \tag{86}$$

In the case where † stands for henselization or a finite étale extension, (86) is an immediate consequence of (85). Thus (86) is true in all the cases. Since σ is faithfully flat and in view of (85), ht$\bar{H} \leq$ ht $H''_{2\ell+1}$. Combined with (86), this completes the proof. □

Corollary 6.1 *For each i, $0 \le i \le 2r$, the quantity* ht H'_i *stabilizes for R' sufficiently far out in $\mathcal{T}$.*

The next Proposition is an immediate consequence of Theorem 5.1.

Proposition 6.2 *Suppose given an admissible chain of trees (72) of prime ideals of ${R'}^\dagger$. For each $\ell \in \{0, \dots, r-1\}$, consider the set of all $R' \in \mathcal{T}$ such that*

$$\text{ht } \tilde{H}'_{2\ell+1} - \text{ht } \tilde{H}'_{2_\ell} \le 1 \qquad \textit{for all even } i \tag{87}$$

and, in case of equality, the 1-dimensional local ring $\lim\limits_{\overrightarrow{R'}} \frac{{R'}^\dagger_{\tilde{H}'_{2\ell+1}}}{\tilde{H}'_{2\ell}{R'}^\dagger_{\tilde{H}'_{2\ell+1}}}$ is unibranch (that is, analytically irreducible). Assume that for each ℓ the set of such R' is cofinal in $\mathcal{T}$.

Assume that for each even $i = 2\ell$, ν_ℓ admits a unique extension $\nu^\dagger_{i0}$ to a valuation of $\lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_{i-1})$, centered in $\lim\limits_{\overrightarrow{R'}} \frac{{R'}^\dagger_{\tilde{H}'_i}}{\tilde{H}'_{i-1}{R'}^\dagger_{\tilde{H}'_i}}$. Then specifying the valuation $\nu^\dagger$ is equivalent to specifying for each i, $2 \le i \le 2r$, and extension $\nu^\dagger_i$ of the valuation $\nu^\dagger_{i0}$ of $\lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_{i-1})$ to its extension field $k_{\nu^\dagger_{i-1}}$ (in particular, such extensions $\nu^\dagger$ always exist). If for each i, $2 \le i \le 2r$, the field extension $\lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_{i-1}) \to k_{\nu^\dagger_{i-1}}$ is algebraic and the extension $\nu^\dagger_i$ of $\nu^\dagger_{i0}$ to $k_{\nu^\dagger_{i-1}}$ is unique then there is a unique extension $\nu^\dagger$ of ν such that the $\tilde{H}'_i$ are the prime ideals, determined by $\nu^\dagger$.

Conversely, assume that K' is algebraic over K and that there exists a unique extension $\nu^\dagger$ of ν such that the $\tilde{H}'_i$ are the prime $\nu^\dagger$-ideals, determined by $\nu^\dagger$. Then for each $\ell \in \{0, \dots, r-1\}$ and for all R' sufficiently far out in $\mathcal{T}$ (87) holds. For each even $i = 2\ell$, ν_ℓ admits a unique extension $\nu^\dagger_{i0}$ to a valuation of $\lim\limits_{\overrightarrow{R'}} \kappa\left(\tilde{H}'_{i-1}\right)$, centered in $\lim\limits_{\overrightarrow{R'}} \frac{{R'}^\dagger_{\tilde{H}'_i}}{\tilde{H}'_{i-1}{R'}^\dagger_{\tilde{H}'_i}}$; we have rk $\nu^\dagger_{i0} = 1$. For each odd i, the ring $\lim\limits_{\overrightarrow{R'}} \frac{{R'}^\dagger_{\tilde{H}'_i}}{\tilde{H}'_{i-1}{R'}^\dagger_{\tilde{H}'_i}}$ is a valuation ring of a (not necessarily discrete) rank 1 valuation. For each i, $1 \le i \le 2r$, the field extension $\lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_{i-1}) \to k_{\nu^\dagger_{i-1}}$ is algebraic and the extension $\nu^\dagger_i$ of $\nu^\dagger_{i0}$ to $k_{\nu^\dagger_{i-1}}$ is unique.

Remark 6.1 *We do not know of a simple criterion to decide when, given an algebraic field extension $K \hookrightarrow L$ and a valuation ν of K, is the extension of ν to L unique. See [6], [18] for more information about this question and an algorithm for arriving at the answer using MacLane's key polynomials.*

Next we describe three classes of extensions of ν to $\lim\limits_{\overrightarrow{R'}} {R'}^\dagger$, which are of particular interest for applications, and which we call **minimal**, **evenly minimal** and **tight** extensions.

Definition 6.1 *Let $\nu^\dagger$ be an extension of ν to $\lim\limits_{\overrightarrow{R'}} {R'}^\dagger$ and let the notation be as above. We say that $\nu^\dagger$ is **evenly minimal** if whenever $i = 2\ell$ is even, the following two conditions hold:*

(1)

$$\Delta_{i-1,0} = \frac{\Delta_{\ell-1}}{\Delta_\ell}. \tag{88}$$

(2) For an element $\overline{\beta}\in\frac{\Delta_{\ell-1}}{\Delta_\ell}$, the $\nu^\dagger_{i0}$-ideal of $\frac{R^\dagger_{\tilde H_i}}{\tilde H_{i-1}R^\dagger_{\tilde H_i}}$ of value $\overline{\beta}$, denoted by $\mathcal{P}^\dagger_{\overline{\beta},\ell}$, is given by the formula

$$\mathcal{P}^\dagger_{\overline{\beta},\ell}=\left(\lim_{\overrightarrow{R'}}\frac{\mathcal{P}'_{\overline{\beta}}R'^\dagger_{\tilde H'_i}}{\tilde H'_{i-1}R'^\dagger_{\tilde H'_i}}\right)\cap\frac{R^\dagger_{\tilde H_i}}{\tilde H_{i-1}R^\dagger_{\tilde H_i}}. \tag{89}$$

*We say that $\nu^\dagger$ is **minimal** if $\tilde H'_i=H'_i$ for each R' and each $i\in\{0,\dots,2r+1\}$. We say that $\nu^\dagger$ is **tight** if it is evenly minimal and*

$$\tilde H'_i=\tilde H'_{i+1}\qquad\text{for all even } i. \tag{90}$$

Remark 6.2 *The valuation $\nu^\dagger_{i0}$ is uniquely determined by conditions (88) and (89). Recall also that if $i=2\ell$ is even and we have:*

$$\tilde H'_i = H'_i \qquad \text{and} \tag{91}$$
$$\tilde H'_{i-1} = H'_{i-1} \tag{92}$$

then $\nu^\dagger_{i0}$ is uniquely determined by ν_ℓ by Proposition 4.5. In particular, if $\nu^\dagger$ is minimal (that is, if (91)–(92) hold for all i) then $\nu^\dagger$ is evenly minimal.

Examples. The extension $\hat\nu$ of Example 3.1 is minimal, but not tight. The valuation ν admits a unique tight extension $\hat\nu_2\circ\hat\nu_3$ to $\lim\limits_{\overrightarrow{R'}}\frac{R'}{H'_1}$; the valuation $\hat\nu$ is the composition of the discrete rank 1 valuation $\hat\nu_1$, centered in $\lim\limits_{\overrightarrow{R'}}\hat R'_{H'_1}$ with $\hat\nu_2\circ\hat\nu_3$.

The extension $\hat\nu^{(1)}$ of Example 3.2 is minimal. The extension $\hat\nu^{(2)}$ is evenly minimal but not minimal. Neither $\hat\nu^{(1)}$ nor $\hat\nu^{(2)}$ is tight. The valuation ν admits a unique tight extension $\hat\nu_2\circ\hat\nu_3$ to $\lim\limits_{\overrightarrow{R'}}\frac{R'}{\tilde H'_1}$, where $\tilde H'_1=\left(y-\sum\limits_{j=1}^{\infty}c_jx^j\right)$; the valuation $\hat\nu^{(2)}$ is the composition of the discrete rank 1 valuation $\hat\nu_1$, centered in $\lim\limits_{\overrightarrow{R'}}\hat R'_{\tilde H'_1}$ with $\hat\nu_2\circ\hat\nu_3$.

Remark 6.3 *As of this moment, we do not know of any examples of extensions $\hat\nu$ which are not evenly minimal. Thus, formally, the question of whether every extension $\hat\nu$ is evenly minimal is open, though we strongly suspect that such counterexamples do exist.*

Proposition 6.3 *Let $i=2\ell$ be even and let $\nu^\dagger_{i0}$ be the extension of ν_ℓ to $\lim\limits_{\overrightarrow{R'}}\kappa(\tilde H'_{i-1})$, centered at the local ring $\lim\limits_{\overrightarrow{R'}}\frac{R'^\dagger_{\tilde H'_i}}{\tilde H'_{i-1}R'^\dagger_{\tilde H'_i}}$, defined by (89). Then*

$$k_{\nu^\dagger_{i0}}=\lim_{\overrightarrow{R'}}\kappa(\tilde H'_i). \tag{93}$$

Proof: Take two elements $x,y\in\lim\limits_{\overrightarrow{R'}}\frac{R'^\dagger_{\tilde H'_i}}{\tilde H'_{i-1}R'^\dagger_{\tilde H'_i}}$, such that $\nu^\dagger_{i0}(x)=\nu^\dagger_{i0}(y)$. We must show that the image of $\frac{x}{y}$ in $k_{\nu^\dagger_{i0}}$ belongs to $\lim\limits_{\overrightarrow{R'}}\kappa(\tilde H'_i)$. Without loss of generality, we may assume that $x,y\in\frac{R^\dagger_{\tilde H_i}}{\tilde H_{i-1}R^\dagger_{\tilde H_i}}$. Let $\beta=\nu^\dagger_{i0}(x)=\nu^\dagger_{i0}(x)$. Choose $R'\in\mathcal{T}$ sufficiently far out in the direct system so that $x,y\in\frac{\mathcal{P}'_\beta R'^{\dagger}_{\tilde H'_i}}{\tilde H'_{i-1}R'^{\dagger}_{\tilde H'_i}}$. Let $R'\to R''$ be the blowing up of the ideal $\mathcal{P}'_\beta R'$. Then in $\frac{\mathcal{P}''_\beta R''^{\dagger}_{\tilde H''_i}}{\tilde H''_{i-1}R''^{\dagger}_{\tilde H''_i}}$ we can write

$$
\begin{aligned}
x &= az \qquad \text{and} &(94)\\
y &= aw, &(95)
\end{aligned}
$$

where $\nu^\dagger_{i0}(a)=\beta$ and $\nu^\dagger_{i0}(z)=\nu^\dagger_{i0}(w)=0$. Let $\bar z$ be the image of z in $\kappa(\tilde H''_i)$ and similarly for $\bar w$. Then the image of $\frac{x}{y}$ in $k_{\nu^\dagger_{i0}}$ equals $\frac{\bar z}{\bar w}\in\kappa(\tilde H''_i)$, and the result is proved. □

Remark 6.4 *Theorem 5.1 and the existence of the extension $\nu^\dagger_{2\ell,0}$ of ν_ℓ in the case when $\tilde H'_{2\ell}=H'_{2\ell}$ and $\tilde H'_{2\ell-1}=H'_{2\ell-1}$ guaranteed by Theorem 4.1 (2) allow us to give a fairly explicit description of the totality of minimal extensions as compositions of $2r$ valuations and, in particular, to show that they always exist. Indeed, minimal extensions $\nu^\dagger$ can be constructed at will, recursively in i, as follows. Assume that the valuations $\nu^\dagger_1,\dots,\nu^\dagger_{i-1}$ are already constructed. If i is odd, let $\nu^\dagger_i$ be an arbitrary valuation of the residue field $k_{\nu^\dagger_{i-1}}$ of the valuation ring $R_{\nu^\dagger_{i-1}}$. If $i=2\ell$ is even, let $\nu^\dagger_{i0}$ be the unique extension of ν_ℓ to $\lim\limits_{\overrightarrow{R'}}\kappa(H'_{i-1})$, centered at the local ring $\lim\limits_{\overrightarrow{R'}}\frac{R'^{\dagger}_{H'_i}}{H'_{i-1}R'^{\dagger}_{H'_i}}$, whose existence and uniqueness are guaranteed by Theorem 4.1 (2) and Proposition 4.5, respectively. Let $\nu^\dagger_i$ be an arbitrary extension of $\nu^\dagger_{i0}$ to the field $k_{\nu^\dagger_{i-1}}$. It is clear that all the minimal extensions $\nu^\dagger$ of ν are obtained in this way.*

In the next section we will use this remark to show that if $R^\dagger=\tilde R$ or $R^\dagger=R^e$ then ν admits a unique extension to $\frac{R^\dagger}{H_0}$, which is necessarily minimal.

We end this section by giving some sufficient conditions for the uniqueness of $\nu^\dagger$.

Proposition 6.4 *Suppose given an admissible chain of trees (72) of prime ideals of $R'^\dagger$. For each $\ell\in\{0,\dots,r-1\}$, consider the set of all $R'\in\mathcal{T}$ such that*

$$
ht\ \tilde H'_{2\ell+1}-ht\ \tilde H'_{2\ell}\le 1 \qquad \textit{for all even } i \tag{96}
$$

and, in case of equality, the 1-dimensional local ring $\lim\limits_{\overrightarrow{R'}}\frac{R'^{\dagger}_{\tilde H'_{2\ell+1}}}{\tilde H'_{2\ell}R'^{\dagger}_{\tilde H'_{2\ell+1}}}$ is unibranch (that is, analytically irreducible). Assume that for each ℓ the set of such R' is cofinal in $\mathcal{T}$.

Let $\nu^\dagger$ be an extension of ν such that the $\tilde H'_i$ are prime $\nu^\dagger$-ideals. Assume that $\nu^\dagger$ is evenly minimal. Then there is at most one such extension $\nu^\dagger$ and exactly one such $\nu^\dagger$ if

$$
\tilde H'_i=H'_i \qquad \textit{for all } i. \tag{97}
$$

(in the latter case $\nu^\dagger$ is minimal by definition).

Assume that $\nu^\dagger$ is tight. Then for each R' in our direct system the natural graded algebra extension $gr_\nu R'\to gr_{\nu^\dagger}R'^\dagger$ is scalewise birational.

Remark 6.5 *Proposition 6.4 allows us to rephrase Theorem 1.1 as follows: the valuation ν admits at least one tight extension $\nu^\dagger$.*

Proof of Proposition 6.4: By Theorem 4.1 (2) and Proposition 4.5, if (97) holds then $\nu^\dagger$ is minimal and for each even i the extension $\nu^\dagger_{i0}$ exists and is unique. Therefore we may assume that in all the cases $\nu^\dagger$ is evenly minimal and that $\nu^\dagger_{i0}$ exists whenever (97) holds.

The valuation $\nu^\dagger$, if it exists, is a composition of $2r$ rank valuations: $\nu^\dagger = \nu^\dagger_1 \circ \nu^\dagger_2 \circ \cdots \circ \nu^\dagger_{2r}$, subject to the conditions of Theorem 5.1. We prove the uniqueness of $\nu^\dagger$ by induction on r. Assume the result is true for $r-1$. This means that there is at most one evenly minimal extension $\nu^\dagger_3 \circ \nu^\dagger_4 \circ \cdots \circ \nu^\dagger_{2r}$ of $\nu_2 \circ \nu_3 \circ \cdots \circ \nu_r$ to $\lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_2)$, and exactly one in the case when (97) holds. To complete the proof of uniqueness of $\nu^\dagger$, it is sufficient to show that both $\nu^\dagger_1$ and $\nu^\dagger_2$ are unique and that the residue field of $\nu^\dagger_2$ equals $\lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_2)$.

We start with the uniqueness of $\nu^\dagger_1$. If (90) holds then $\nu^\dagger_1$ is the trivial valuation. Suppose, on the other hand, that equality holds in (96). Then the restriction of $\nu^\dagger_1$ to each $R' \in \mathcal{T}$ such that the local ring $\lim\limits_{\overrightarrow{R'}} \frac{\hat{R}'_{\tilde{H}'_1}}{\tilde{H}'_0 \hat{R}'_{\tilde{H}'_1}}$ is one-dimensional and unibranch is the unique divisorial valuation centered in that ring (in particular, its residue field is $\kappa(\tilde{H}'_1)$. By the assumed cofinality of such R', the valuation $\nu^\dagger_1$ is unique and its residue field equals $\lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_1)$. Thus, regardless of whether or not the inequality in (96) is strict, $\nu^\dagger_1$ is unique and we have the equality of residue fields

$$k_{\nu^\dagger_1} = \lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_1) \tag{98}$$

This equality implies that $\nu^\dagger_2 = \nu^\dagger_{20}$. Now, the valuation $\nu^\dagger_2 = \nu^\dagger_{20}$ is uniquely determined by the conditions (88) and (89), and its residue field is

$$k_{\nu^\dagger_2} = k_{\nu^\dagger_{20}} = \lim\limits_{\overrightarrow{R'}} \kappa(\tilde{H}'_2). \tag{99}$$

by Proposition 6.3. Furthermore, by Theorem 4.1 exactly one such $\nu^\dagger_2$ exists whenever (97) holds. This proves that there is at most one possibility for $\nu^\dagger$: the composition of $\nu^\dagger_1 \circ \nu^\dagger_2$ with $\nu^\dagger_3 \circ \nu^\dagger_4 \circ \cdots \circ \nu^\dagger_{2r}$, and exactly one if (97) holds.

Finally, assume that $\nu^\dagger$ is tight (that is, (90) holds) and take $R' \in \mathcal{T}$. Then the valuation $\nu_{2\ell+1}$ is trivial for all ℓ, so $\nu^\dagger = \nu^\dagger_2 \circ \nu^\dagger_4 \circ \cdots \circ \nu^\dagger_{2r}$. We must show that the graded algebra extension $\mathrm{gr}_\nu R' \to \mathrm{gr}_{\nu^\dagger} R'^\dagger$ is scalewise birational. Again, we use induction on r. Take an element $x \in R'^\dagger$. If $\nu^\dagger(x) \in \Delta_1$ then $\mathrm{in}_{\nu^\dagger} x \in \mathrm{gr}_{\nu^\dagger_4 \circ \cdots \circ \nu^\dagger_{2r}} \frac{R'^\dagger}{\tilde{H}'_2}$, hence by the induction assumption there exists $y \in R'$ with $\nu^\dagger(x) \in \Delta_1$ and $\mathrm{in}_{\nu^\dagger}(xy) \in \mathrm{gr}_\nu \frac{R'}{P_1}$. In this case, there is nothing more to prove. Thus we may assume that $\nu^\dagger(x) \notin \Delta_1$. It remains to show that there exists $y \in R'$ such that $\mathrm{in}_{\nu^\dagger}(xy) \in \mathrm{gr}_\nu R'$. Since local blowings up induce birational transformations of graded algebras, it is enough to find a local blowing up $R'' \in \mathcal{T}(R')$ and $y \in R''$ such that $\mathrm{in}_{\nu^\dagger}(xy) \in \mathrm{gr}_\nu R''$.

Now, Proposition 6.3 shows that there exists a local blowing up $R' \to R''$ such that $x = az$ (94), with $z \in R''$ and $\nu^\dagger_2(a) = \nu^\dagger_{2,0}(a) = 0$. The last equality means that $\nu^\dagger(a) \in \Delta_1$, and the result follows from the induction assumption, applied to a. □

The argument above also shows the following. Let $\Phi'^\dagger = \nu^\dagger\left(R'^\dagger \setminus \{0\}\right)$, take an element $\beta \in \Phi'^\dagger$ and let $\mathcal{P}'^\dagger_\beta$ denote the $\nu^\dagger$-ideal of $R'^\dagger$ of value β.

Corollary 6.2 *Take an element $x \in \mathcal{P}'^\dagger_\beta$. There exists a local blowing up $R' \to R''$ such that $\beta \in \nu(R'') \setminus \{0\}$ and $x \in \mathcal{P}''_\beta R''^\dagger$.*

The next Proposition gives a sufficient condition for the uniqueness of $\nu^\dagger$ (this result is due to Heinzer and Sally [4]).

Proposition 6.5 *Assume that K' is algebraic over K for all $R' \in \mathcal{T}$ and that the following conditions hold:*

(1) $ht\ H'_1 \leq 1$

(2) $ht\ H'_1 + rat.rk\ \nu = \dim\ R'$, where R' is taken to be sufficiently far out in the direct system.

Let $\nu^\dagger$ be an extension of ν to a ring of the form $\lim\limits_{\overrightarrow{R'}} \frac{R'^\dagger}{\tilde{H}'_0}$. Then either

$$\begin{aligned} \tilde{H}'_0 &= H'_0 \qquad or && (100)\\ \tilde{H}'_0 &= H'_1. && (101)\end{aligned}$$

The valuation ν admits a unique extension to $\lim\limits_{\overrightarrow{R'}} \frac{R'^\dagger}{H'_0}$ and a unique extension to $\lim\limits_{\overrightarrow{R'}} \frac{R'^\dagger}{H'_1}$. The first extension is minimal and the second is tight.

Proof: For $1 \leq \ell \leq r$, let r_i denote the rational rank of ν_ℓ. Let $\nu^\dagger$ be an extension of ν to a ring of the form $\lim\limits_{\overrightarrow{R'}} \frac{R'^\dagger}{\tilde{H}'_0}$, where $\tilde{H}'_0$ is a tree of prime ideals of $R'^\dagger$ such that $\tilde{H}'_0 \cap R' = (0)$. whenever $1 \leq i \leq 2r$.

By Corollary 6.1 $ht\ H'_i$ stabilizes for $1 \leq i \leq 2r$ and R' sufficiently far out in the direct system. From now on, we will assume that R' is chosen sufficiently far so that the stable value of $ht\ H'_i$ is attained. Now, let $i = 2\ell$. The valuation $\nu^\dagger_{i0}$ is centered in the local noetherian ring $\frac{R'^\dagger_{\tilde{H}'_i}}{\tilde{H}'_{i-1}R'^\dagger_{\tilde{H}'_i}}$, hence by Abhyankar's inequality

$$rat.rk\ \nu^\dagger_{i0} \leq \dim \frac{R'^\dagger_{\tilde{H}'_i}}{\tilde{H}'_{i-1}R'^\dagger_{\tilde{H}'_i}} \leq ht\ \tilde{H}'_i - ht\ \tilde{H}'_{i-1}. \qquad (102)$$

Since this inequality is true for all even i, summing over all i we obtain:

$$\begin{aligned} \dim R' &= \dim\ R'^\dagger = \sum_{i=1}^{2r}(ht\ \tilde{H}'_i - ht\ \tilde{H}'_{i-1}) \geq ht\ \tilde{H}'_1 + \sum_{\ell=1}^{r}(ht\ \tilde{H}'_{2\ell} - ht\ \tilde{H}'_{2\ell-1}) \geq \\ &\geq ht\ H'_1 + \sum_{\ell=1}^{r} rat.rk\ \nu^\dagger_{2\ell,0} \geq ht\ H'_1 + \sum_{\ell=1}^{r} r_\ell = ht\ H'_1 + rat.rk\ \nu = \dim R'. \end{aligned} \qquad (103)$$

Hence all the inequalities in (102) and (103) are equalities. In particular, we have

$$ht\ \tilde{H}'_1 = ht\ H'_1;$$

combined with (5.2) this shows that

$$\tilde{H}'_1 = H'_1. \qquad (104)$$

Together with the hypothesis (1) of the Proposition, this already proves the dichotomy (100)–(101). Furthermore, equalities in (102) and (103) prove that

$$ht\ \tilde{H}'_i = ht\ \tilde{H}'_{i-1}$$

for all odd $i > 1$, so that

$$\tilde{H}'_i = \tilde{H}'_{i-1} \qquad \text{whenever } i > 1 \text{ is odd} \tag{105}$$

and that

$$r_i = ht\ \tilde{H}'_i - ht\ \tilde{H}'_{i-1} \tag{106}$$

whenever i is even.

Now, consider the special case when $\tilde{H}'_i = H'_i$ for $i \geq 1$ and $\tilde{H}'_0$ is as in (100)–(101). According to Proposition 4.5 for each even $i = 2\ell$ there exists a unique extension $\nu^\dagger_{i0}$ of ν_l to a valuation of $\lim\limits_{\overrightarrow{R'}} \kappa(H'_{i-1})$, centered in the local ring $\lim\limits_{\overrightarrow{R'}} \frac{{R'}^\dagger_{H'_{2\ell}}}{H'_{2\ell-1}}$. Moreover, we have

$$k_{\nu^\dagger_{2\ell,0}} = \lim_{\overrightarrow{R'}} \kappa(H'_{2\ell}) \tag{107}$$

by Remark 4.4. By Theorem 5.1, there exists an extension $\nu^\dagger$ of ν to $\lim\limits_{\overrightarrow{R'}} \frac{{R'}^\dagger}{\tilde{H}'_0}$ such that the $\{\tilde{H}'_i\}$ as above is the chain of trees of prime ideals, determined by $\nu^\dagger$. In particular, (105) and (106) hold with $\tilde{H}'_i$ replaced by H'_i.

Now (106) and Proposition 5.2 imply that for *any* extension $\nu^\dagger$ we have $\tilde{H}'_i = H'_i$ for $i > 0$, so that the special case above is, in fact, the only case possible. Furthermore, by (105) we have $H'_{2\ell+1} = H'_{2\ell}$ for all $\ell \in \{1, \dots, r\}$. This implies that for all such ℓ $\nu^\dagger_{2\ell+1,0}$ is the trivial valuation of $\lim\limits_{\overrightarrow{R'}} \kappa(H'_{2\ell})$; in particular,

$$k_{\nu^\dagger_{2\ell+1,0}} = \lim_{\overrightarrow{R'}} \kappa(H'_{2\ell}) \tag{108}$$

for all $\ell \in \{1, \dots, r-1\}$.

If $\tilde{H}'_0 = H'_1 = \tilde{H}'_1$ then the only possibility for $\nu^\dagger_{10} = \nu^\dagger_1$ is the trivial valuation of $\lim\limits_{\overrightarrow{R'}} \kappa(H'_1)$; we have

$$k_{\nu^\dagger_1} = k_{\nu^\dagger_{10}} = \lim_{\overrightarrow{R'}} \kappa(H'_1). \tag{109}$$

If $\tilde{H}'_0 = H'_0$ then by the hypothesis (1) of the Proposition and the excellence of R the ring $\frac{{R'}^\dagger_{H'_1}}{H'_0}$ is a regular one-dimensional local ring (in particular, unibranch), hence the valuation $\nu^\dagger_1 = \nu^\dagger_{10}$ centered at $\lim\limits_{\overrightarrow{R'}} \frac{{R'}^\dagger_{H'_1}}{H'_0 {R'}^\dagger_{H'_1}}$ is unique and (109) holds also in this case.

By induction on i, it follows from (107), (108), the uniqueness of $\nu^\dagger_{2\ell,0}$ and the triviality of $\nu^\dagger_{2\ell+1,0}$ for $\ell \geq 1$ that $\nu^\dagger_i$ is uniquely determined for all i and $k_{\nu^\dagger_i} = \lim\limits_{\overrightarrow{R'}} \kappa(H'_i)$. This proves that in both cases (100) and (101) the valuation $\nu^\dagger = \nu^\dagger_1 \circ \dots \circ \nu^\dagger_{2r}$ is unique. The last statement of the Proposition is immediate from definitions. □

A related necessary condition for the uniqueness of $\nu^\dagger$ will be proved in §9.

7 Extending a valuation centered in an excellent local domain to its henselization.

Let $\tilde{R}$ denote the henselization of R, as above. The completion homomorphism $R \to \hat{R}$ factors through the henselization: $R \to \tilde{R} \to \hat{R}$. In this section, we will show that H_1 is a minimal prime of $\tilde{R}$, that ν extends uniquely to a valuation $\tilde{\nu}$ of rank r centered at $\frac{\tilde{R}}{H_1}$, and that H_1 is the unique prime ideal P of $\tilde{R}$ such that ν extends to a valuation of $\frac{\tilde{R}}{P}$. Furthermore, we will prove that $H_{2\ell+1}$ is a minimal prime of $P_\ell \tilde{R}$ for all ℓ and that these are precisely the prime $\tilde{\nu}$-ideals of $\tilde{R}$.

Studying the implicit prime ideals of $\tilde{R}$ and the extension of ν to $\tilde{R}$ is a logical intermediate step before attacking the formal completion, for the following reason. As we will show in the next section, if R is already henselian in (45) then $\mathcal{P}'_\beta \hat{R}'_{H'_{2\ell+1}} \cap \hat{R} = \mathcal{P}_\beta \hat{R}$ for all β and R' and thus we have $H_{2\ell+1} = \bigcap\limits_{\beta \in \Delta_\ell} \left(\mathcal{P}_\beta \hat{R}\right)$.

We state the main result of this section. In the case when R^e is an étale extension of R, contained in $\tilde{R}$, we use (47) with $R^\dagger = R^e$ as our definition of the implicit prime ideals.

Theorem 7.1 *Let R^e be a local étale extension of R, contained in $\tilde{R}$. Then:*

(1) The ideal $H_{2\ell+1}$ is prime for $0 \le l \le r$; it is a minimal prime of $P_\ell R^e$. In particular, H_1 is a minimal prime of R^e. We have $H_{2\ell} = H_{2\ell+1}$ for $0 \le l \le r$.

(2) The ideal H_1 is the unique prime P of R^e such that there exists an extension ν^e of ν to $\frac{R^e}{H_1}$; the extension ν^e is unique. The graded algebra $gr_{\nu^e} \frac{R^e}{H_1}$ is scalewise birational to $gr_\nu R$; in particular, rk $\nu^e = r$.

(3) The ideals $H_{2\ell+1}$ are precisely the prime ν^e-ideals of R^e.

Proof: By assumption, the ring R^e is a direct limit of local, strict étale extensions of R which are essentially of finite type. All the assertions (1)–(3) behave well under taking direct limits, so it is sufficient to prove the Theorem in the case when R^e is essentially of finite type over R. From now on, we will restrict attention to this case.

The next step is to describe explicitly those local blowings up $R \to R'$ for which R' is ℓ-stable. Their interest to us is that, according to Proposition 4.2, if R' is ℓ-stable then for all $R'' \in \mathcal{T}(R')$ and all $\beta \in \frac{\Delta_\ell}{\Delta_{\ell+1}}$, we have the equality

$$\mathcal{P}''_\beta (R'' \otimes_R R^e) \cap R^e = \mathcal{P}_\beta R^e; \tag{110}$$

in particular, the limit in (47) is attained, that is, we have the equality

$$H_{2\ell+1} = \bigcap_{\beta \in \Delta_\ell} \left(\left(\mathcal{P}'_\beta \left(R^e \otimes_R R'\right)_{M'}\right) \bigcap R^e \right). \tag{111}$$

Lemma 7.1 *Let $\nu_{\ell+1}$ be the valuation induced by ν on the quotient $\frac{R}{P_\ell}$. Let $\frac{R}{P_\ell} \to T$ be a finitely generated extension of $\frac{R}{P_\ell}$ contained in $\frac{R_\nu}{\mathbf{m}_\ell}$. There exist a ν-extension $R \to R'$ of R and a prime ideal $\mathbf{q}$ of T such that $\frac{R'}{P'_\ell} = T_{\mathbf{q}}$.*

Write $T = \frac{R}{P_\ell}[\overline{a}_1, \dots, \overline{a}_k]$, with $\overline{a}_i \in \frac{R_\nu}{\mathbf{m}_\ell}$, that is, $\nu_{\ell+1}(\overline{a}_i) \ge 0,\ 1 \le i \le k$. We can lift the $\overline{a}_i$ to elements a_i in R_ν such that $\nu(a_i) \ge 0$. Let us consider the ring $R'' = R[a_1, \dots, a_k] \subset R_\nu$ and its localization $R' = R''_{\mathbf{m}_\nu \cap R''}$. The ideal P'_ℓ is the kernel of the natural map $R' \to \frac{R_\nu}{\mathbf{m}_\ell}$. Thus both $\frac{R'}{P'_\ell}$ and $T_{\mathbf{q}}$ are equal to the $\frac{R}{P_l}$-subalgebra of $\frac{R_\nu}{\mathbf{m}_\ell}$, obtained by adjoining $\overline{a}_1, \dots, \overline{a}_k$ to $\frac{R}{P_l}$ inside $\frac{R_\nu}{\mathbf{m}_\ell}$ and then localizing at the preimage of the ideal $\frac{\mathbf{m}_\nu}{\mathbf{m}_\ell}$. This proves the Lemma. □

Let us now go back to our étale extension $R \to R^e$.

Lemma 7.2 *Fix an integer $l \in \{0, \dots, r\}$. There exists a local blowing up $R \to R'$ along ν having the following property: let P'_ℓ denote the ℓ-th prime ν-ideal of R'. Then the ring $\frac{R'}{P'_\ell}$ is analytically irreducible; in particular, $\frac{R'}{P'_\ell} \otimes_R R^e$ is an integral domain.*

Remark 7.1 *We are not claiming that there exists $R' \in \mathcal{T}$ such that $\frac{R'}{P'_\ell}$ is analytically irreducible for all ℓ (and we do not know how to prove such a claim), only that for each ℓ there exists an R', which may depend on ℓ, such that $\frac{R'}{P'_\ell}$ is analytically irreducible. On the other hand, below we will prove that there exists an ℓ-stable $R' \in \mathcal{T}$. According to Definition 4.1 (2) and Proposition 4.1, such a stable R' has the property that $\kappa\left(P''_\ell\right) \otimes_R (R'' \otimes_R R^e)_{M''}$ is a domain for all $R'' \in \mathcal{T}(R')$.*

Proof of Lemma 7.2: Since R is an excellent local ring, every homomorphic image of R is Nagata [13] (Theorems 72 (31.H), 76 (33.D) and 78 (33.H)). Let $\pi : \frac{R}{P_\ell} \to S$ be the normalization of $\frac{R}{P_\ell}$. Then S is a finitely generated $\frac{R}{P_\ell}$-algebra contained in $\frac{R_\nu}{\mathbf{m}_\ell}$, to which we can apply Lemma 9.1. We obtain a ν-extension $R \to R'$ such that the ring $\frac{R'}{P'_\ell} \cong \frac{R'}{P_\ell R'}$ is a localization of S at a prime ideal, hence it is an excellent normal local ring. In particular, it is analytically irreducible ([14], Proposition 1, chapter IX), as desired. □

Next, we fix $\ell \in \{0, \dots, r\}$ and study the ring $(T')^{-1}(\kappa(P'_\ell) \otimes_R R^e)$, in particular, the structure of the set of its zero divisors, as R' runs over $\mathcal{T}(R)$ (here T' is as in Remark 4.1). Since R^e is separable algebraic, essentially of finite type over R, the ring $(T')^{-1}(\kappa(P'_\ell) \otimes_R R^e)$ is finite over $\kappa(P'_\ell)$; this ring is reduced, but it may contain zero divisors. In fact, it is a direct product of fields which are finite separable extensions of $\kappa(P'_\ell)$ because R^e is separable and essentially of finite type over R.

Consider a chain $R \to R' \to R''$ of ν-extensions in $\mathcal{T}$. Let

$$\kappa(P_\ell) \otimes_R R^e = \prod_{j=1}^{n} K_j \tag{112}$$

$$(T')^{-1}\left(\kappa\left(P'_\ell\right) \otimes_R R^e\right) = \prod_{j=1}^{n'} K'_j \tag{113}$$

$$(T'')^{-1}\left(\kappa\left(P''_\ell\right) \otimes_R R^e\right) = \prod_{j=1}^{n''} K''_j \tag{114}$$

be the corresponding decompositions as products of finite field extensions of $\kappa(P_\ell)$ (resp. $\kappa(P'_\ell)$, resp. $\kappa(P''_\ell)$). We want to compare $(T')^{-1}\left(\kappa\left(P'_\ell\right) \otimes_R R^e\right)$ with $(T'')^{-1}\left(\kappa\left(P''_\ell\right) \otimes_R R^e\right)$.

Remark 7.2 *The ring $\kappa\left(P'_\ell\right) \otimes_R R^e$ is itself a direct product of finite extensions of $\kappa\left(P'_\ell\right)$; say $\kappa\left(P'_\ell\right) = \prod\limits_{j \in S'} K'_j$ for a certain set S'. In this situation, localization is the same thing as the natural projection to the product of the K'_j over a certain subset $\{1, \dots, n'\}$ of S'. Thus the passage from $(T')^{-1}\left(\kappa\left(P'_\ell\right) \otimes_R R^e\right)$ to $(T'')^{-1}\left(\kappa\left(P''_\ell\right) \otimes_R R^e\right)$ can be viewed as follows: first, tensor each K'_j with $\kappa\left(P''_\ell\right)$ over $\kappa\left(P'_\ell\right)$; then, in the resulting direct product of fields, remove a certain number of factors.*

Let $\bar{K}'_1, \dots, \bar{K}'_{\bar{n}'}$ be the distinct isomorphism classes of finite extensions of $\kappa(P'_\ell)$ appearing among $K'_1, \dots, K'_{n'}$, arranged in such a way that $\left[\bar{K}'_j : \kappa(P'_\ell)\right]$ is non-increasing with j, and similarly for $\bar{K}''_1, \dots, \bar{K}''_{\bar{n}''}$.

Lemma 7.3 *We have the inequality*

$$\left(\left[\bar{K}''_1 : \kappa\left(P''_\ell\right)\right], \dots, \left[\bar{K}''_{\bar{n}''} : \kappa\left(P''_\ell\right)\right], n''\right) \le \left(\left[\bar{K}'_1 : \kappa\left(P'_\ell\right)\right], \dots, \left[\bar{K}'_{\bar{n}'} : \kappa\left(P'_\ell\right)\right], n'\right) \qquad (115)$$

in the lexicographical ordering. Furthermore, either R' is ℓ-stable or there exists $R'' \in \mathcal{T}$ such that strict inequality holds in (115).

Proof: Fix a $q \in \{1, \dots, \bar{n}'\}$ and consider the tensor product $\bar{K}'_q \otimes_R \kappa(P''_\ell)$. Since $\bar{K}'_q$ is separable over $\kappa(P'_\ell)$, the ring $\bar{K}'_q \otimes_R \kappa(P''_\ell) = \prod\limits_{j \in S''_q} K''_j$ is a product of fields. Moreover, two cases are possible:

(a) there exists a non-trivial extension L of $\kappa(P'_\ell)$ which embeds both into $\kappa(P''_\ell)$ and $\bar{K}'_q$. In this case

$$\left[K''_j : \kappa\left(P''_\ell\right)\right] < \left[\bar{K}'_q : \kappa\left(P'_\ell\right)\right] \quad \text{for all } j \in S''_q. \qquad (116)$$

(b) there is no field extension L as in (a). In this case $\bar{K}'_q \otimes_R \kappa(P''_\ell)$ is a field, so

$$\# S''_q = 1 \qquad (117)$$

and

$$\left[K''_j : \kappa\left(P''_\ell\right)\right] = \left[\bar{K}'_q : \kappa\left(P'_\ell\right)\right] \quad \text{for } j \in S''_q. \qquad (118)$$

Now, if there exists $q \in \{1, \dots, \bar{n}'\}$ for which (a) holds, take the smallest such q. Then (116)–(118) imply that strict inequality holds in (115). On the other hand, if (b) holds for all $q \in \{1, \dots, \bar{n}'\}$ then (117) and (118) imply that

$$\left(\left[\bar{K}''_1 : \kappa\left(P''_\ell\right)\right], \dots, \left[\bar{K}''_{\bar{n}''} : \kappa\left(P''_\ell\right)\right]\right) = \left(\left[\bar{K}'_1 : \kappa\left(P'_\ell\right)\right], \dots, \left[\bar{K}'_{\bar{n}'} : \kappa\left(P'_\ell\right)\right]\right) \qquad (119)$$

and $n'' \le n'$, so again (115) holds.

Finally, assume that R' is not ℓ-stable. If there exists $R'' \in \mathcal{T}$ and $q \in \{1, \dots, \bar{n}'\}$ for which (a) holds, then by the above we have strict inequality in (115) and there is nothing more to prove. Assume there are no such R'' and q. Then $(T')^{-1}(\kappa(P'_\ell) \otimes_R R^e)$ is not a domain, so $n' > 1$.

Take $R'' \in \mathcal{T}(R')$ such that $\left(\frac{R''}{P''_l} \otimes_R R^e\right)_{M''}$ is an integral domain; such an R'' exists by Lemma 7.2. Then $n'' = 1 < n'$, as desired. □

Corollary 7.1 *There exists a stable $R' \in \mathcal{T}$. The limit in (47) is attained for this R'.*

Proof: In view of Proposition 4.1, it is sufficient to prove that there exists $R' \in \mathcal{T}$ which which is ℓ-stable for all $\ell \in \{0, 1, \dots, r\}$. First, we fix $\ell \in \{0, 1, \dots, r\}$. Lemma 7.3 implies that there exists $R' \in \mathcal{T}(R)$ which is ℓ-stable.

By Proposition 4.1, repeating the procedure above for each ℓ we can successively enlarge R' in such a way that it becomes stable.

The last statement follows from Proposition 4.2. □

We are now in the position to prove Theorem 7.1.

By Theorem 4.1 (1), $H_{2\ell-1}$ is prime. By Proposition 3.1, $H_{2\ell+1}$ maps to P_ℓ under the map $\pi^e : \text{Spec } R^e \to \text{Spec } R$. Since this map is étale, its fibers are zero-dimensional, which shows that $H_{2\ell+1}$ is a minimal prime of P_ℓ. This proves (1) of Theorem 7.1.

By Proposition 5.2, for $0 \leq i \leq 2r$, $\tilde{H}_i$ is a prime ideal of R^e, containing H_i. Since the fibers of π^e are zero-dimensional, we must have $\tilde{H}_i = H_i$, so $\tilde{H}_{2\ell} = \tilde{H}_{2\ell+1} = H_{2\ell} = H_{2\ell+1}$ for $0 \leq \ell \leq r$. In particular, $\tilde{H}_0 = H_1$. This shows that the unique prime $\tilde{H}_0$ of R^e such that there exists an extension ν^e of ν to $\frac{R^e}{\tilde{H}_0}$ is $\tilde{H}_0 = H_1$. Now (2) of the Theorem is given by Proposition 6.4.

(3) of Theorem 7.1 is now immediate. This completes the proof of Theorem 7.1. □

We note the following corollary of the proof of (2) of Theorem 7.1 and Corollary 6.2. Let $\Phi^e = \nu^e(R^e \setminus \{0\})$, take an element $\beta \in \Phi^e$ and let $\mathcal{P}^e_\beta$ denote the ν^e-ideal of R^e of value β.

Corollary 7.2 *Take an element $x \in \mathcal{P}^e_\beta$. There exists a local blowing up $R \to R'$ such that $\beta \in \nu(R') \setminus \{0\}$ and $x \in \mathcal{P}'_\beta R'^e$.*

8 The first Main Theorem: the primality of the implicit ideals.

In this section we study the ideals H_j for $\hat{R}$ instead of $\tilde{R}$. The main result of this section is

Theorem 8.1 *The ideal $H_{2\ell-1}$ is prime.*

Proof: For the purposes of this proof, let $H_{2\ell-1}$ denote the implicit ideals of $\hat{R}$ and $\tilde{H}_{2\ell-1}$ the implicit prime ideals of the henselization $\tilde{R}$ of R.

Let S be a local domain. By [14] (Proposition 1, Chapter IX) there exists bijective maps between the set of minimal prime ideals of the henselization $\tilde{S}$, the set of minimal primes of $\hat{S}$ and the maximal ideals of the normalization S^n. If S is a henselian local domain, its only minimal prime is the (0) ideal, hence by the above the same is true of $\hat{S}$. Thus $\hat{S}$ is also a domain.

This shows that any henselian local domain is analytically irreducible, hence $\tilde{H}_{2\ell-1}\hat{R}$ is prime for all $\ell \in \{0, \dots, r+1\}$. Let $\tilde{\nu}$ denote the unique extension of ν to $\frac{\tilde{R}}{\tilde{H}_1}$, constructed in the previous section. Let $H^*_{2\ell-1} \subset \frac{\hat{R}}{\tilde{H}_1}$ denote the implicit ideals associated to the henselian ring $\frac{\tilde{R}}{\tilde{H}_1}$ and the valuation $\tilde{\nu}$.

Claim. We have $H^*_{2\ell-1} = \frac{H_{2\ell-1}}{\tilde{H}_1}$.

Proof of Claim: For $\beta \in \Gamma$, let $\tilde{P}_\beta$ denote the $\tilde{\nu}$-ideal of $\frac{\tilde{R}}{\tilde{H}_1}$ of value β. For all β, we have $\frac{P_\beta}{\tilde{H}_1} \subset \tilde{P}_\beta$, and the same inclusion holds for all the local blowings up of R, hence $\frac{H_{2\ell-1}}{\tilde{H}_1} \subset H^*_{2\ell-1}$. To prove the opposite inclusion, we may replace $\tilde{R}$ by a finitely generated strict étale extension R^e of R. Now let $\Phi^e = \nu^e(R^e \setminus \{0\})$ and take an element $\beta \in \Phi^e \cap \Delta_{\ell-1}$. By Corollary 7.2, there exists a local blowing up $R \to R'$ such that $x \in P'_\beta R'^e$. Letting β vary over $\Phi^e \cap \Delta_{\ell-1}$, we obtain that if $x \in H^*_{2\ell-1}$ then $x \in \frac{H_{2\ell-1}}{\tilde{H}_1}$, as desired. This completes the proof of Claim.

The Claim shows that replacing R by $\frac{\tilde{R}}{\tilde{H}_1}$ in Theorem 8.1 does not change the problem. In other words, we may assume that R is a henselian domain and, in particular, that $\hat{R}$ is also a domain. Similarly, the ring $\frac{R}{P_\ell} \otimes_R \hat{R} \cong \frac{\hat{R}}{P_\ell}$ is a domain, hence so is its localization $\kappa(P_\ell) \otimes_R \hat{R}$.

Since R is a henselian excellent ring, it is algebraically closed in $\hat{R}$ [14]; of course, the same holds for $\frac{R}{P_\ell}$ for all ℓ. Then $\kappa(P_\ell)$ is algebraically closed in $\kappa(P_\ell) \otimes_R \hat{R}$. This shows that the ring R is stable. Now the Theorem follows from Theorem 4.1. This completes the proof of Theorem 8.1. □

9 Proof of Theorem 1.1, assuming local uniformization in lower dimension

In this section, we use the notations of the previous sections and we assume that all the $R' \in \mathcal{T}$ are birational to each other, so that all the fraction fields $K' = K$ and the homomorphisms $R' \to R''$ are local blowings up with respect to ν.

Since we know from section 2 that the result is true for valuations of rank one, the proof can proceed by induction on the rank of ν.
Moreover, according to Theorem 7.1 we may assume that R is henselian, which we shall do for the rest of this section. Then as we saw in the preceding section, R is stable. According to Corollary 6.1, we may assume that $ht\ H'_i$ is constant for each i after replacing R by some other ring sufficiently far in $\mathcal{T}$. From now on, we will make this assumption without always stating it explicitly. Let R be a local excellent integral domain dominated by a valuation ring R_ν of its field of fractions. Let ν_1 be the valuation of rank one with which ν is composed. Let R_ν/m_{ν_1} be the valuation ring inducing the residual valuation $\overline{\nu}$ on the quotient ring $\overline{R} = R/P_1$, where $P_1 = m_{\nu_1} \cap R$ is the center of ν_1 in R. Consider the tree of all local ν-extensions $R \to R'$, and the tree of the $\overline{\nu}$-extensions of quotients $\overline{R}' = R'/P'_1$ contained in $\frac{R_\nu}{m_{\nu_1}}$.

Lemma 9.1 *Every $\overline{\nu}$-extension of $\overline{R}$ is of the form R'/P'_1 for some ν-extension R' of R.*

An $\overline{\nu}$-extension $\overline{R''}$ of R'/P'_1 as above is a localization of an R'/P'_1-algebra $\overline{S}$ generated by finitely many elements of $\frac{R_\nu}{m_{\nu_1}}$. If we lift these elements to R_ν and consider the R'-algebra S which they generate, it has a natural surjection to $\overline{S}$ and after a suitable localization, we obtain a surjection $S_{\mathbf{q}} \to R''$ which is induced by the natural map $R_\nu \to \frac{R_\nu}{m_{\nu_1}}$. The result follows.□

As a consequence of this lemma, there is a natural morphism from the first tree to the second.

Set
$$\hat{\mathcal{R}} = \lim_{\overrightarrow{R'}} \hat{R}' \qquad \text{and} \qquad \hat{\overline{\mathcal{R}}} = \lim_{\overrightarrow{R'}} \hat{\overline{R}}'.$$

The previous remark translates into the fact that there is a natural surjection $r\colon \hat{\mathcal{R}} \to \hat{\overline{\mathcal{R}}}$ whose kernel is the ideal
$$P_1^\infty = \lim_{\overrightarrow{R'}} P'_1\hat{R}'.$$

Note that there are natural injections $R_\nu \hookrightarrow \hat{\mathcal{R}}$ and $\frac{R_\nu}{m_{\nu_1}} \hookrightarrow \hat{\overline{\mathcal{R}}}$ and that the surjection r induces the quotient map $R_\nu \to \frac{R_\nu}{m_{\nu_1}}$.
Let us consider more generally, for $\gamma_1 \in \Gamma_1 = \Gamma/\Delta_1$, the ideals
$$\mathcal{P}^\infty_{\gamma_1} = \lim_{\overrightarrow{R'}} \mathcal{P}'_{\gamma_1}\hat{R}', \quad \mathcal{P}^{\infty+}_{\gamma_1} = \lim_{\overrightarrow{R'}} \mathcal{P}'^{+}_{\gamma_1}\hat{R}'$$

so that $P_1^\infty = \mathcal{P}_0^{\infty+}$.
Define also the ideals
$$\mathcal{H}_1 = \bigcap_{\gamma_1 \in \Gamma_1} \mathcal{P}^\infty_{\gamma_1} = \bigcap_{\gamma \in \Gamma} \mathcal{P}^\infty_\gamma,$$

remember that we assume R to be an henselian local domain so it is analytically irreducible and that the maps $\hat{R}' \to \hat{\mathcal{R}}$ are injective, and note that we have in each completion $\hat{R}'$ the equality $H'_1 = \mathcal{H}_1 \cap \hat{R}'$.

Now let us consider the graded $\hat{\mathcal{R}}/P_1^\infty$-algebra

$$\mathrm{gr}_{\hat{\nu}_1}(\hat{\mathcal{R}}/\mathcal{H}_1) = \bigoplus_{\gamma_1 \in \Gamma_{1+}} \frac{\mathcal{P}^\infty_{\gamma_1}}{\mathcal{P}^{\infty+}_{\gamma_1}}$$

Proposition 9.1 *Assume that R is an henselian excellent local ring; then:*
1) The ring $\hat{\mathcal{R}}$ is a local integral domain and the ideals $\mathcal{H}_1$ and P_1^∞ are prime.
2) The algebra $\mathrm{gr}_{\hat{\nu}_1}(\hat{\mathcal{R}}/\mathcal{H}_1)$ is an integral domain.
3) For each analytically irreducible R', the map $\hat{R}' \to \hat{\mathcal{R}}$ is injective and we have

$$\mathcal{P}^\infty_{\gamma_1} \cap \hat{R}' = \mathcal{P}'_{\gamma_1}\hat{R}'.$$

Proof. In view of Lemma 7.2, the rings $\hat{\mathcal{R}}$ and $\hat{\overline{\mathcal{R}}} = \hat{\mathcal{R}}/P_1^\infty$ are the unions of local integral domains. The ideal $\mathcal{H}_1$ is prime as a consequence of Theorem 8.1. In order to prove 2), remark that in view of Corollary 6.2 since our tree $\{R'\}$ contains the localizations at the point determined by the valuation ν of the blowing-ups of the ideals $\mathcal{P}'_{\gamma_1}\hat{R}'$, an element x of $\mathcal{P}^\infty_{\gamma_1}$ can be written $x = ce_1$ with $e_1 \in \mathcal{P}'_{\gamma_1}(R') \setminus \mathcal{P}'^{+}_{\gamma_1}(R')$ an element of minimal ν valuation among a system of generators of the ideal $\mathcal{P}'_{\gamma_1}$ and $c \in \hat{\mathcal{R}} \setminus P_1^\infty$.
Let $x, y \in \hat{\mathcal{R}} \setminus \mathcal{H}_1$; since the rings R' in our tree are noetherian, the set of elements $\gamma \in \Gamma_1$ such that $x \notin \mathcal{P}^\infty_{\gamma_1}$ has a smallest element and since the valuation ν_1 is of rank one, this smallest element has a well defined immediate predecessor. So we may assume that $x \in \mathcal{P}^\infty_{\gamma_1} \setminus \mathcal{P}^{\infty+}_{\gamma_1}$ and $y \in \mathcal{P}^\infty_{\delta_1} \setminus \mathcal{P}^{\infty+}_{\delta_1}$. After a ν-extension $R \to R'$, we may write $x = ce$, $y = df$, with $e \in \mathcal{P}'_{\gamma_1}$, $f \in \mathcal{P}'_{\delta_1}$ so that $xy = cdef$ as above. Now $cd \notin P_1^\infty$ since this last is a prime ideal by 1), and $ef \in \mathcal{P}_{\gamma_1+\delta_1} \setminus \mathcal{P}^+_{\gamma_1+\delta_1}$ since $\mathrm{gr}_{\nu_1} R'$ is a domain. By the faithful flatness of each $\hat{R}'$ over R' and Proposition 4.2 we have $\mathcal{P}^{\infty+}_{\gamma_1+\delta_1} \cap R' = \mathcal{P}^+_{\gamma_1+\delta_1}(R')$. Thus $xy \in \mathcal{P}^\infty_{\gamma_1+\delta_1} \setminus \mathcal{P}^{\infty+}_{\gamma_1+\delta_1}$, which proves 2).
It follows from 2) that the valuation ν_1 extends uniquely to a valuation $\hat{\nu}_1$ of the quotients $\hat{R}'/\mathcal{H}_1 \cap \hat{R}'$ with the same value group Γ_1.

Let us now proceed to the proof of Theorem 1.1 by induction on the rank as announced at the beginning of the section. We assume that Theorem 1.1 is true for the ring $\overline{R} = \frac{R}{P_1}$ and the valuation $\overline{\nu}$, which is of rank $r-1$. Then for each $R' \in \mathcal{T}(R)$ there exists a sequence of ideals of $\hat{\overline{R}}' = \frac{\hat{R}'}{P'_1\hat{R}'}$, which we denote by

$$\tilde{\overline{H}}'_2 = \tilde{\overline{H}}'_3 \subset \tilde{\overline{H}}'_4 = \cdots \tilde{\overline{H}}'_{2r-2} = \tilde{\overline{H}}'_{2r-1} \subset \tilde{\overline{H}}'_{2r} = m\hat{\overline{R}}',$$

which determines a tight extension of $\overline{\nu}$ from $\overline{R}'$ to $\hat{\overline{R}}'/\tilde{\overline{H}}'_2$, which induces a scalewise birational map of the associated graded rings.
We can translate this in the language of trees as the existence of a sequence of prime ideals

$$\tilde{\overline{\mathcal{H}}}'_2 = \tilde{\overline{\mathcal{H}}}'_3 \subset \tilde{\overline{\mathcal{H}}}'_4 = \cdots \subset \tilde{\overline{\mathcal{H}}}'_{2r-2} = \tilde{\overline{\mathcal{H}}}'_{2r-1} \subset \tilde{\overline{\mathcal{H}}}'_{2r} = m_{\overline{\nu}}\hat{\overline{\mathcal{R}}}$$

in $\hat{\overline{\mathcal{R}}}$.
We have the following:

Lemma 9.2 • *Let M be a finitely generated module over a noetherian local domain S. For any prime ideal $\mathbf{q}$ of the completion $\hat{S}$ such that $\mathbf{q} \bigcap S = (0)$, the $\hat{S}_{\mathbf{q}}$-module $M \otimes_S \hat{S}_{\mathbf{q}}$ is free. In particular, if M is a finitely generated torsion S-module, then $M \otimes_S \hat{S}_{\mathbf{q}} = (0)$.*

- *A short exact sequence $0 \to M" \to M \to M' \to 0$ of finitely generated S-modules becomes after tensorization with $\hat{S}_{\mathbf{q}}$ a short exact sequence of finitely generated free $\hat{S}_{\mathbf{q}}$-modules.*

Proof: The Fitting ideals of the $\hat{S}_{\mathbf{q}}$-module $M \otimes_S \hat{S}_{\mathbf{q}}$ are the $F_k(M) \otimes_S \hat{S}_{\mathbf{q}}$ and their non zero generators are invertible in $\hat{S}_{\mathbf{q}}$ since they cannot belong to $\mathbf{q}$. The nonzero Fitting ideals of $M \otimes_S \hat{S}_{\mathbf{q}}$ are therefore invertible. The first claim now follows from ([3], 5.4.1-5.4.3). As a consequence, if $T \subset M$ is the submodule of torsion elements of M, the map $M \otimes_S \hat{S}_{\mathbf{q}} \to (M/T) \otimes_S \hat{S}_{\mathbf{q}}$ is an isomorphism of free $\hat{S}_{\mathbf{q}}$-modules, which proves the second claim. $\square$

Proposition 9.2 *Keeping the standing notations, let P_1 be the center of the valuation of rank one with which ν is composed. Let $\overline{H}'$ be a tree of prime ideals of the rings $\hat{\overline{R}}'$, where $\overline{R}' = R'/P_1'$, such that $\overline{H}' \bigcap \overline{R}' = (0)$. Then:*

- *The natural maps of graded algebras*

$$\mathrm{gr}_{\nu_1} R' \otimes_{\overline{R}'} \hat{\overline{R}}'_{\overline{H}'} \to \mathrm{gr}_{\nu_1} R'' \otimes_{\overline{R}''} \hat{\overline{R}}''_{\overline{H}''}$$

  associated to the maps $R' \to R''$ in $\mathcal{T}(R)$ which the maps are injective and make these graded algebras into a tree.

- *The $\mathrm{gr}_{\nu_1} R' \otimes_{\overline{R}'} \overline{H}'\hat{\overline{R}}'_{\overline{H}'}$ form a tree of prime ideals in the tree of graded algebras $\mathrm{gr}_{\nu_1} R' \otimes_{\overline{R}'} \hat{\overline{R}}'_{\overline{H}'}$.*

- *The inductive limit $\varinjlim_{R'} \mathrm{gr}_{\nu_1} R' \otimes_{\overline{R}'} \overline{H}'\hat{\overline{R}}'_{\overline{H}'}$ is a prime ideal $\mathrm{gr}_{\nu_1} R_\nu \otimes_{\frac{R_\nu}{m_{\nu_1}}} \overline{\mathcal{H}}\, \hat{\overline{\mathcal{R}}}_{\overline{\mathcal{H}}}$ in*

$$\varinjlim_{R'} \mathrm{gr}_{\nu_1} R' \otimes_{\overline{R}'} \hat{\overline{R}}'_{\overline{H}'} = \mathrm{gr}_{\nu_1} R_\nu \otimes_{\frac{R_\nu}{m_{\nu_1}}} \hat{\overline{\mathcal{R}}}_{\overline{\mathcal{H}}},$$

  where $\overline{\mathcal{H}} = \varinjlim_{R'} \overline{H}'$.

Proof: Let $R' \to R''$ be a map in $\mathcal{T}(R)$, and consider the injection $\mathrm{gr}_{\nu_1} R' \hookrightarrow \mathrm{gr}_{\nu_1} R''$ of $\overline{R'}$-algebras. Applying Theorem 4.1 to the valuation ν_1 and remarking that $\mathrm{gr}_{\nu_1} R \otimes_{\overline{R}} \hat{\overline{R}} = \mathrm{gr}_{\hat{\nu}_1} \frac{\hat{R}}{H_1}$, we see that the localization maps $\hat{\overline{R}}' \to \hat{\overline{R}}'_{\overline{H}'}$ are injections of integral domains and that furthermore the natural map of graded algebras:

$$\mathrm{gr}_{\nu_1} R' \otimes_{\overline{R}'} \hat{\overline{R}}' \hookrightarrow \mathrm{gr}_{\nu_1} R'' \otimes_{\overline{R}''} \hat{\overline{R}}''. \tag{120}$$

is injective.
Since the $\frac{\mathcal{P}'_{\gamma_1}}{\mathcal{P}'_{\gamma_1 +}}$ are torsion free $\frac{R'}{P'_1}$-modules, this inclusion extends to an inclusion

$$\mathrm{gr}_{\nu_1} R' \otimes_{\overline{R'}} \hat{\overline{R'}}_{\overline{H}'} \hookrightarrow \mathrm{gr}_{\nu_1} R'' \otimes_{\overline{R''}} \hat{\overline{R''}}_{\overline{H}''}. \tag{121}$$

Now we must show that an element of the first algebra whose image in the second is in $\overline{H}"\hat{\overline{R}}"_{\overline{H}"}$ is actually in $\mathrm{gr}_{\nu_1} R' \otimes_{\overline{R}'} \overline{H}'\hat{\overline{R}}'_{\overline{H}'}$. We remark that the assertion can be checked degree by degree and the inclusion map is induced for each γ_1 by the injection of $\overline{R}'$-modules

$$\frac{\mathcal{P}'_{\gamma_1}}{\mathcal{P}'^{+}_{\gamma_1}} \hookrightarrow \frac{\mathcal{P}"_{\gamma_1}}{\mathcal{P}"^{+}_{\gamma_1}}. \tag{122}$$

Let us first verify the result in a special case:
Let us fix a degree γ_1 and assume that the map $\overline{R'} \to \overline{R"}$ is obtained by localization at the center of $\overline{\nu}$ of the blowing-up of an ideal $\overline{J}'$. Then if we lift generators of $\overline{J}'$ to $f_1, \ldots, f_k \in R'$, the localization et the center of ν of the blowing up of the ideal J' generated by $f_1, \ldots, f_k \in R'$ will produce a birational map $R' \to R"$ such that $\frac{R"}{\mathcal{P}"_1} = \overline{R"}$. Writing $R" = R'[\frac{f_2}{f_1}, \ldots, \frac{f_k}{f_1}]_{m"}$ shows that if we take an element $\tilde{x} \in R"$ which represents $x \in \frac{\mathcal{P}"_{\gamma_1}}{\mathcal{P}"^{+}_{\gamma_1}}$, for sufficiently large t we have $f_1^t\tilde{x} \in \frac{\mathcal{P}'_{\gamma_1}}{\mathcal{P}'^{+}_{\gamma_1}}$, since $\overline{\nu}(f_1) = 0$. The element f_1 is not a zero divisor, and this shows that the cokernel of the inclusion (122) is a torsion module. A similar argument shows that the kernel is a torsion module. So we have proved:

Lemma. Assuming that the map $R' \to R"$ is obtained by localization at the center of ν of the blowing-up in R' of an ideal generated by elements whose ν_1 value is zero, the kernel and cokernel of the natural map $\frac{\mathcal{P}'_{\gamma_1}}{\mathcal{P}'^{+}_{\gamma_1}} \otimes_{\overline{R'}} \overline{R"} \to \frac{\mathcal{P}"_{\gamma_1}}{\mathcal{P}"^{+}_{\gamma_1}}$ are torsion $R"$-modules.□

Remark that this is an avatar of Lemma 5.13 in [17].

Now according to Lemma 9.2 the map

$$\frac{\mathcal{P}'_{\gamma_1}}{\mathcal{P}'^{+}_{\gamma_1}} \otimes_{\overline{R}'} \hat{\overline{R}}'_{\overline{H}'} \otimes_{\hat{\overline{R}}'_{\overline{H}'}} \hat{\overline{R}}"_{\overline{H}"} \to \frac{\mathcal{P}"_{\gamma_1}}{\mathcal{P}"^{+}_{\gamma_1}} \otimes_{\overline{R}"} \hat{\overline{R}}"_{\overline{H}"} \tag{123}$$

deduced from the map (122) and the natural inclusion $\hat{\overline{R}}'_{\overline{H}'} \hookrightarrow \hat{\overline{R}}"_{\overline{H}"}$ is an isomorphism of free $\hat{\overline{R}}"_{\overline{H}"}$-modules. It follows that if an element $\sum_i \overline{e_i} \otimes a_i \in \frac{\mathcal{P}'_{\gamma_1}}{\mathcal{P}'^{+}_{\gamma_1}} \otimes_{\overline{R}'} \hat{\overline{R}}'_{\overline{H}'}$, is in $\frac{\mathcal{P}"_{\gamma_1}}{\mathcal{P}"^{+}_{\gamma_1}} \otimes \overline{H}"\hat{\overline{R}}"_{\overline{H}"}$ the a_i must be in $\overline{H}" \bigcap \hat{\overline{R}}'_{\overline{H}'}$, which is $\overline{H}'\hat{\overline{R}}'_{\overline{H}'}$. This proves the result in this case.

If we assume that local uniformization holds for $\overline{R'}$ and the valuation $\overline{\nu}$ this special case shows that in order to prove the second statement of Proposition 9.2 we may assume that $\overline{R'}$ and $\frac{\overline{R'}}{\overline{H'}}$ are regular. Moreover, after blowing-up $\mathcal{P}'_{\gamma_1}$ in R' we may assume that $\mathcal{P}'_{\gamma_1}R"$ is principal, generated by e_1 say, where e_1 is a generator of minimal ν-value of $\mathcal{P}'_{\gamma_1}$. Finally, using local uniformization again, we may assume that $\overline{R"}$ and $\frac{\overline{R"}}{\overline{H"}}$ are also regular. This being the case, using the usual Cohen theorem, the morphism $\hat{\overline{R'}} \to \hat{\overline{R"}}$ can be put in the form

$$\frac{\hat{\overline{R'}}}{\overline{H}'}[[V_1, \ldots, V_s]] \to \frac{\hat{\overline{R"}}}{\overline{H}"}[[W_1, \ldots, W_t]] \tag{124}$$

with $\overline{H}' = (V_1, \ldots V_s)$ and $\overline{H}" = (W_1, \ldots W_t)$, so that the map is described by the canonical injection $\frac{\hat{\overline{R'}}}{\overline{H'}} \hookrightarrow \frac{\hat{\overline{R"}}}{\overline{H"}}$ and each V_i is mapped to a series without constant term in the W's.

To prove the result in this case, we may consider elements $(\overline{e}_j)_{j\in J} \in \frac{\mathcal{P}'_{\gamma_1}}{\mathcal{P}'^{+}_{\gamma_1}}$ such that the $\overline{e}_j \otimes 1$ form a basis of the free $\hat{\overline{R}}'_{\overline{H}'}$-module $\frac{\mathcal{P}'_{\gamma_1}}{\mathcal{P}'^{+}_{\gamma_1}} \otimes_{\overline{R}'} \hat{\overline{R}}'_{\overline{H}'}$. It suffices to show that if the image $\sum_j \overline{e}_j \otimes a_j$, with $a_j \in \hat{\overline{R'}}$ is in $\frac{\mathcal{P}"_{\gamma_1}}{\mathcal{P}"^{+}_{\gamma_1}} \otimes_{\overline{R"}} \overline{H}"\hat{\overline{R}}"_{\overline{H}"}$, then all the a_j are in $\overline{H}'$.

By our assumptions, the image an be written $\overline{e}_1 \otimes \sum_j a_j t_j$ where $t_j = \frac{\overline{e}_j}{\overline{e}_1} \in \overline{R"} \subset \frac{\hat{\overline{R"}}}{\overline{H"}} \subset \hat{\overline{R"}}$.

Since $\overline{H}"$ is a prime ideal, this image is in $\frac{\mathcal{P}"_{\gamma_1}}{\mathcal{P}"^{+}_{\gamma_1}} \otimes_{\overline{R"}} \overline{H}"\hat{\overline{R}}"_{\overline{H}"}$ if and only if $\sum_j a_j t_j$ is in $\overline{H}"$. If we read this condition inthe form (124), remembering that $t_j \in \overline{R"}$, and denoting by a_j^0

the constant term of $a_j \in \frac{\hat{\overline{R'}}}{\overline{H}'}[[V_1, \ldots, V_s]]$, we see that we must have

$$\sum_j a_j^0 t_j = 0.$$

This is possible only if the image of $\sum_j \overline{e}_j \otimes_{\hat{\overline{R'}}} a_j^0$ in $\frac{\mathcal{P}"_{\gamma_1}}{\mathcal{P}"^+_{\gamma_1}} \otimes_{\overline{R"}} \hat{\overline{R}}"_{\overline{H}"}$ is zero. But in view of the injectivity of the map (121) this implies that $\sum_j \overline{e}_j \otimes_{\hat{\overline{R'}}} a_j^0 = 0$. Since we consider the a_j^0 as elements of $\hat{\overline{R'}} \subset \hat{\overline{R'}}_{\overline{H}'}$ this gives a relation between the elements $\overline{e}_j$ which form a basis. This implies that all the a_j^0 must be zero, which means that the a_j are in $\overline{H}'$ and proves the second part of the Proposition.

The third part of the Proposition follows from the second. $\square$

From now on, using the induction assumption, we choose as $\overline{H}'$ the tree of prime ideals $\tilde{\overline{H}}'_2 = \tilde{\overline{H}}'_3 \subset \hat{\overline{R}}'$.

The ideals $(\mathcal{P}^{\infty}_{\gamma_1})_{\gamma_1 \in \Gamma_{1+}}$ define a filtration of $\hat{\mathcal{R}}$. Denoting by $\hat{\nu}_1$ the corresponding order function with values in Γ_1, we can define the following ideals:

$$\mathcal{S}_{\gamma_1} = \mathcal{P}^{\infty +}_{\gamma_1} + \langle x \in \hat{\mathcal{R}} | \hat{\nu}_1(x) = \gamma_1, \ \mathrm{in}_{\hat{\nu}_1}(x) \in \mathrm{gr}_{\nu_1} R_\nu \otimes_{\frac{R_\nu}{m_{\nu_1}}} \overline{\mathcal{H}} \ \hat{\overline{\mathcal{R}}}_{\overline{\mathcal{H}}} \rangle.$$

Now we can define the ideal

$$\tilde{\mathcal{H}} = \bigcap_{\gamma_1 \in \Gamma_{1+}} \mathcal{S}_{\gamma_1} \subset \hat{\mathcal{R}}.$$

Proposition 9.3 *In this situation, we have the following:*

- *The inclusions* $\hat{\mathcal{H}}_1 \subseteq \tilde{\mathcal{H}} \subset P_1^{\infty}$ *hold.*
- *The ideal* $\tilde{\mathcal{H}}$ *is prime and* $\tilde{\mathcal{H}} \bigcap R_\nu = (0)$.

Proof: The two inclusions follow directly from the definitions. To prove the second assertion, let $x, y \in \hat{\mathcal{R}}$ be such that $\mathrm{in}_{\hat{\nu}_1} xy \in \mathrm{gr}_{\nu_1} R_\nu \otimes_{\frac{R_\nu}{m_{\nu_1}}} \overline{\mathcal{H}} \ \hat{\overline{\mathcal{R}}}_{\overline{\mathcal{H}}}$. We may assume that $x, y \in \hat{R}'$ for some $R' \in \mathcal{T}(R)$. In view of the first inclusion of the preceding assertion and the fact that H'_1 is a prime ideal of $\hat{R}'$, we may assume that $xy \notin H'_1$.

Set $\gamma_1 = \hat{\nu}_1(x)$, $\delta_1 = \hat{\nu}_1(y)$. Since $\mathrm{in}_{\hat{\nu}_1} xy = \mathrm{in}_{\hat{\nu}_1} x . \mathrm{in}_{\hat{\nu}_1} y$, the product $\mathrm{in}_{\hat{\nu}_1} x \otimes \mathrm{in}_{\hat{\nu}_1} y$. The natural map

$$\Big(\frac{\mathcal{P}'_{\gamma_1}}{\mathcal{P}'^+_{\gamma_1}} \otimes \hat{\overline{R}}'_{\overline{H}'}\Big) \otimes \Big(\frac{\mathcal{P}'_{\delta_1}}{\mathcal{P}'^+_{\delta_1}} \otimes \hat{\overline{R}}'_{\overline{H}'}\Big) \to \frac{\mathcal{P}'_{\gamma_1+\delta_1}}{\mathcal{P}'^+_{\gamma_1+\delta_1}} \otimes \hat{\overline{R}}'_{\overline{H}'} \tag{125}$$

is injective since it is obtained by localization from the multiplication map in the integral domain (see Proposition 9.1) $\mathrm{gr}_{\hat{\nu}_1} \frac{\hat{R}'}{H'_1}$.

If we take bases $\mathrm{in}_{\nu_1} e_1 \otimes 1, \ldots, \mathrm{in}_{\nu_1} e_\ell \otimes 1$ and $\mathrm{in}_{\nu_1} f_1 \otimes 1, \ldots \mathrm{in}_{\nu_1} f_m \otimes 1$ for the free $\hat{\overline{R'}}_{\overline{H}'}$-modules appearing on the left of (125), with $e_i, f_j \in R$, the image $\mathcal{Q}$ of the map (125) is freely generated by the $\mathrm{in}_{\nu_1}(e_i f_j) \otimes 1$. Now we use Lemma **??** with

$$S = S' = \overline{R'}, \qquad M = \frac{\mathcal{P}'_{\gamma_1}}{\mathcal{P}'^+_{\gamma_1}} \otimes_{\overline{R'}} \frac{\mathcal{P}'_{\delta_1}}{\mathcal{P}'^+_{\delta_1}}, \qquad M' = \frac{\mathcal{P}'_{\gamma_1+\delta_1}}{\mathcal{P}'^+_{\gamma_1+\delta_1}}, \tag{126}$$

to prove that

$$\Big(\frac{\mathcal{P}'_{\gamma_1+\delta_1}}{\mathcal{P}'^{+}_{\gamma_1+\delta_1}}\otimes_{\overline{R'}}\overline{H}'\hat{\overline{R}}'_{\overline{H}'}\Big)\bigcap Q=\overline{H}'Q.$$

Thus we have to show that if $\mathrm{in}_{\hat{\nu}_1}xy\in\overline{H}'Q$ then either $\mathrm{in}_{\hat{\nu}_1}x\in\frac{\mathcal{P}'_{\gamma_1}}{\mathcal{P}'^{+}_{\gamma_1}}\otimes\overline{H}'\hat{\overline{R}}'_{\overline{H}'}$ or $\mathrm{in}_{\hat{\nu}_1}y\in\frac{\mathcal{P}'_{\delta_1}}{\mathcal{P}'^{+}_{\delta_1}}\otimes\overline{H}'\hat{\overline{R}}'_{\overline{H}'}$.
If we write $\mathrm{in}_{\hat{\nu}_1}x=\sum_i a_i\mathrm{in}_{\hat{\nu}_1}e_i$ and $\mathrm{in}_{\hat{\nu}_1}y=\sum_j b_j\mathrm{in}_{\hat{\nu}_1}f_j$, we have $\mathrm{in}_{\hat{\nu}_1}x.\mathrm{in}_{\hat{\nu}_1}y=\sum_{i,j}a_ib_j\mathrm{in}_{\hat{\nu}_1}e_if_j$. If all the a_ib_j are in $\overline{H}'$ and one a_i is not, then all the b_j must be in $\overline{H}'$ since $\overline{H}'$ is prime.
To prove the second assertion, it suffices to remark that since $\tilde{\overline{\mathcal{H}}}\bigcap\frac{R_\nu}{m_{\nu_1}}=(0)$, we have $\mathcal{S}_{\gamma_1}\bigcap R_\nu=\mathcal{P}^{+}_{\gamma_1}(R_\nu)$, inducing $\mathcal{P}'^{+}_{\gamma_1}$ on each $R'\in\mathcal{T}(R)$. We know that $\bigcap_{\gamma_1\in\Gamma_{1+}}\mathcal{P}^{+}_{\gamma_1}(R_\nu)=(0)$. □

Now we define the following sequence of ideals in $\hat{\mathcal{R}}$, using the canonical surjection $r\colon\hat{\mathcal{R}}\to\hat{\overline{\mathcal{R}}}$:

- For $i\geq 2$, define $\tilde{\mathcal{H}}_i=r^{-1}(\tilde{\overline{\mathcal{H}}}_i)$.
- For $i=0,1$, set $\tilde{\mathcal{H}}_0=\tilde{\mathcal{H}}_1=\tilde{\mathcal{H}}$.

We claim that if for each $R'\in\mathcal{T}(R)$ we set $\tilde{H}'_i=\tilde{\mathcal{H}}_i\bigcap\hat{R}'$, the sequence of prime ideals

$$\tilde{H}'_0=\tilde{H}'_1\subset\tilde{H}'_2=\tilde{H}'_3\subset\ldots\subset\tilde{H}'_{2r-2}=\tilde{H}'_{2r-1}$$

determines a valuation of $\hat{R}'$ such that the valuation induced on the quotient $\frac{\hat{R}'}{\tilde{H}'_0}$ has the property ot Theorem 1.1.
Proof: It suffices to check that the valuation ν of R' extends to $\frac{\hat{R}'}{\tilde{H}'_0}$ with the same value group. We apply Proposition 9.2 to the tree of prime ideals $\frac{\tilde{H}'_2}{P'_1\hat{R}'}\subset\frac{\hat{R}'}{P'_1\hat{R}'}=\hat{\overline{R}}'$. Given an element $x\in\hat{R}'\setminus\tilde{H}'_0$, it does not belong to H'_1. Let γ_1 be its ν_1-value. By the construction of the tree of ideals $\tilde{H}'_0$, the image of the ν_1-initial form of x in $\mathrm{gr}_{\nu_1}R''\otimes_{\overline{R}''}\hat{\overline{R}}''_{\tilde{\overline{H}}''_2}$ is not in $\mathrm{gr}_{\nu_1}R''\otimes_{\overline{R''}}\tilde{\overline{H}}''_2\hat{\overline{R}}''_{\tilde{\overline{H}}''_2}$.

For some $R''\in\mathcal{T}(R')$, for example after blowing-up the ideal $\mathcal{P}'_{\gamma_1}$ of R' and using the second part of Theorem 4.1, we may write $x=a''e''$ with $a''\in\hat{R}''\setminus P''_1\hat{R}''$ and $e''\in R''$ with $\nu_1(e'')=\gamma_1$. The image $\overline{a}''$ of a'' in $\hat{\overline{R''}}$ then does not belong to $\tilde{\overline{H}}''_2$ in view of Proposition 9.2, so by the induction hypothesis its image in $\frac{\hat{R}''}{\tilde{H}''_2}$ has a valuation $\hat{\overline{\nu}}(\overline{a}'')\in\Delta_1$ and we may define

$$\hat{\nu}(x)=\nu(e'')+\hat{\overline{\nu}}(\overline{a}'')\in\Gamma.$$

This value depends only upon the class of x modulo $\tilde{H}'_0$ and is equal to $\nu(x)$ if $x\in R'$ since $\hat{\overline{\nu}}$ extends the valuation $\overline{\nu}$.
The fact that it is a valuation follows from Proposition 9.1 □

The next Corollary gives a necessary condition for $\hat{\nu}$, including the data $\tilde{H}'_i$, to be uniquely determined by ν.

Corollary 9.1 *Suppose given a tree $\{\tilde{H}'_0\}$ is of minimal prime ideals of $\hat{R}'$ (in particular, $P'\cap\tilde{H}'_0=(0)$). The valuation ν admits a unique extension to a valuation $\hat{\nu}$ of $\varinjlim_{R'}\frac{\hat{R}'}{\tilde{H}'_0}$, if and only if the following conditions hold:*

(1) $\tilde{H}'_i=H'_i$, $0\leq i\leq 2r$
(2) ht $H'_1\leq 1$
(3) $H'_i=H'_{i-1}$ for all odd $i>1$.

Proof: "If" is given by Proposition 6.4. To prove "only if", □

10 An application to the Pierce-Birkhoff conjecture.

In this section we describe an application of Theorem 1.1 to the Pierce-Birkhoff conjecture, which was, in part, our motivation for developing a general theory of extending valuations to completions of local domains. We start by recalling the statement of the Pierce-Birkhoff conjecture. Let $u_1, \dots, u_n$ be independent variables and $A = \mathbf{R}[u_1, \dots, u_n]$.

Definition 10.1 *A function $f : \mathbf{R}^n \to \mathbf{R}$ is said to be* ***piecewise polynomial*** *if $\mathbf{R}^n$ can be covered by a finite collection of closed semi-algebraic sets S_i such that for all i there exists a polynomial $f_i \in A$ satisfying $f|_{S_i} = f_i|_{S_i}$.*

Clearly, such a function is continuous. Piecewise polynomial functions form a ring, containing A, which is denoted by $PW(A)$.

On the other hand, one can consider the (lattice-ordered) ring of all the functions obtained from A by iterating the operations of sup and inf. Since applying the operations of sup and inf to polynomials produces functions which are piecewise polynomial, this ring is contained in $PW(A)$ (this latter ring is closed under sup and inf). It is natural to ask whether the two rings coincide. The precise statement of the conjecture, as it appeared in the 1962 paper [5] by M. Henriksen and H. Isbell, is:

Conjecture 10.1 (Pierce-Birkhoff) *If $f : \mathbf{R}^n \to \mathbf{R}$ is in $PW(A)$, then there exists a finite family of polynomials $g_{ij} \in A$ such that $f = \sup_i \inf_j (g_{ij})$ (in other words, for all $x \in \mathbf{R}^n$, $f(x) = \sup_i \inf_j (g_{ij}(x))$).*

In 1989 J.J. Madden [12] reformulated this conjecture in terms of the real spectrum of A and separating ideals. We will now recall Madden's formulation together with the relevant definitions.

Let B be a ring. A point α in the real spectrum of B is, by definition, the data of a prime ideal $\mathfrak{p}$ of B, and a total ordering $\leq$ of the quotient ring $B/\mathfrak{p}$, or, equivalently, of the field of fractions of $B/\mathfrak{p}$. Another way of defining the point α is as a homomorphism from B to a real closed field, where two homomorphisms are identified if they have the same kernel $\mathfrak{p}$ and induce the same total ordering on $B/\mathfrak{p}$.

The ideal $\mathfrak{p}$ is called the support of α and denoted by $\mathfrak{p}_\alpha$, the quotient ring $B/\mathfrak{p}_\alpha$ by $B[\alpha]$, its field of fractions by $B(\alpha)$ and the real closure of $B(\alpha)$ by $k(\alpha)$. The total ordering of $B(\alpha)$ is denoted by $\leq_\alpha$. Sometimes we write $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$.

Definition 10.2 *The real spectrum of B, denoted by $Sper\ B$, is the collection of all pairs $\alpha = (\mathfrak{p}_\alpha, \leq_\alpha)$, where $\mathfrak{p}_\alpha$ is a prime ideal of B and $\leq_\alpha$ is a total ordering of $B/\mathfrak{p}_\alpha$.*

Next, we recall the notion of **separating ideal**, introduced by Madden in [12].

Definition 10.3 *Let B be a ring. For $\gamma, \delta \in Sper\ B$, the* ***separating ideal*** *of γ and δ, denoted by $< \gamma, \delta >$, is the ideal of B generated by all the elements $f \in B$ which change sign between γ and δ, that is, all the f such that $f(\gamma) \geq 0$ and $f(\delta) \leq 0$.*

Here and below, we commit the following standard abuse of notation: for an element $f \in B$, $f(\alpha)$ stands for the natural image of f in $B[\alpha]$ and the inequality $f(\alpha) > 0$ really means $f(\alpha) >_\alpha 0$.

Let f be a piecewise polynomial function on $\mathbf{R}^n$ and $\alpha \in \text{Sper } A$. Let the notation be as in Definition 10.1. The covering $\mathbf{R}^n = \bigcup_i S_i$ induces a corresponding covering $\text{Sper } A = \bigcup_i \tilde{S}_i$ of the real spectrum. Pick and fix an i such that $\alpha \in \tilde{S}_i$. We set $f_\alpha := f_i$. We refer to f_α as **a local polynomial representative of f at α**. In general, the choice of i is not uniquely determined by α. Implicit in the notation f_α is the fact that one such choice has been made.

In [12], Madden reduced the Pierce–Birkhoff conjecture to a purely local statement about separating ideals and the real spectrum. Namely, he showed that the Pierce-Birkhoff conjecture is equivalent to

Conjecture 10.2 (Pierce-Birkhoff conjecture, the abstract version) *Let f be a piecewise polynomial function and α, γ points in Sper A. Let $f_\alpha \in A$ be a local representative of f at α and $f_\gamma \in A$ a local representative of f at γ. Then $f_\alpha - f_\gamma \in <\alpha, \gamma>$.*

One advantage of this formulation is that it makes sense for an arbitrary commutative ring A rather than just the polynomial ring over $\mathbf{R}$; rings for which the Pierce-Birkhoff conjecture holds are called **Pierce–Birkhoff rings**.

Let B be a ring and α a point in Sper B. One can associate to α a valuation ν_α of $B(\alpha)$, as follows. First, we define the valuation ring R_α by

$$R_\alpha = \{x \in B(\alpha) \mid \exists z \in B[\alpha], |x| \leq_\alpha z\}.$$

That R_α is, in fact, a valuation ring, follows because for any $x \in B(\alpha)$, either $x \in R_\alpha$ or $\frac{1}{x} \in R_\alpha$. The maximal ideal of R_α is $M_\alpha = \left\{x \;\middle|\; |x| < \frac{1}{z}\ \forall z \in B[\alpha]\right\}$; its residue field k_{ν_α} comes equipped with a total ordering, induced by $\leq_\alpha$. By definition, we have a natural ring homomorphism

$$B \to R_\alpha \tag{127}$$

whose kernel is $\mathfrak{p}_\alpha$.

Remark: Conversely, the point α can be reconstructed from the ring R_α by specifying a certain number of sign conditions (finitely many conditions when B is noetherian), as we now explain. Take a prime ideal $\mathfrak{p} \subset B$ and a valuation ν of $\kappa(\mathfrak{p}) := \frac{B_\mathfrak{p}}{\mathfrak{p}B_\mathfrak{p}}$, with value group Γ. Let

$$r = \dim_{\mathbf{F}_2}(\Gamma/2\Gamma)$$

(if B is not noetherian, it may happen that $r = \infty$). Let $x_1, \ldots, x_r$ be elements of $\kappa(\mathfrak{p})$ such that $\nu(x_1), \ldots, \nu(x_r)$ induce a basis of the $\mathbf{F}_2$-vector space $\Gamma/2\Gamma$. Then for every $x \in B(\alpha)$, there exists $f \in B(\alpha)$, and a unit u of R_α such that $x = ux_1^{\epsilon_1} \cdots x_r^{\epsilon_r} f^2$ with $\epsilon_i \in \{0, 1\}$ (to see this, note that for a suitable choice of f and ϵ_j the value of the quotient u of x by the product $x_1^{\epsilon_1} \cdots x_r^{\epsilon_r} f^2$ is 0, hence u is invertible in R_α). Now, specifying a point $\alpha \in \text{Sper } B$ supported at $\mathfrak{p}$ amounts to specifying a valuation ν of $\frac{B}{\mathfrak{p}}$, whose residue field k_ν comes equipped with a total ordering, and the sign data sgn $x_1, \ldots,$ sgn x_r. For $x \notin \mathfrak{p}$, the sign of x is given by the product $\text{sgn}(x_1)^{\epsilon_1} \cdots \text{sgn}(x_r)^{\epsilon_r} \text{sgn}(u)$, where $\text{sgn}(u)$ is determined by the ordering of k_ν.

Now let R be a *henselian* local domain and consider a point $\alpha \in \text{Sper } R$.

Proposition 10.1 *There exists a prime ideal $\tilde{H}_1 \subset \hat{R}$ with $\tilde{H}_1 \cap R = (0)$ and a point $\hat{\alpha} \in$ Sper $\hat{R}$, having the following properties:*

(1) The valuation $\nu_{\hat{\alpha}}$ can be written as a composition $\nu_{\hat{\alpha}} = \hat{\nu}_1 \circ \hat{\nu}_+$, where $\hat{\nu}_1$ is centered in $\hat{R}_{\tilde{H}_1}$ and $\hat{\nu}_+$ in $\frac{\hat{R}}{\tilde{H}_1}$

(2) The natural inclusion $gr_\nu R \hookrightarrow gr_{\hat{\nu}_+} \frac{\hat{R}}{\tilde{H}_1}$ is scalewise birational

(3) $\tilde{H}_1 \subset \mathfrak{p}_{\hat{\alpha}}$, $\mathfrak{p}_{\hat{\alpha}} \cap R = \mathfrak{p}_\alpha$ and the total order $\leq_\alpha$ on $R[\alpha]$ is the restriction of the total order $\leq_{\hat{\alpha}}$ on $\hat{R}[\hat{\alpha}]$.

Proof: Use Theorem 1.1.

Corollary 10.1 *Let R be a henselian local ring and consider two points $\alpha, \gamma \in$ Sper R. There exist points $\hat{\alpha}, \hat{\gamma} \in$ Sper $\hat{R}$ such that we have the equality of separating ideals*

$$< \alpha, \gamma > = < \hat{\alpha}, \hat{\gamma} > \cap R.$$

Corollary 10.2 *Let R be a local, analytically irreducible noetherian domain. If the completion $\hat{R}$ of R is Pierce–Birkhoff then so is its henselization $\tilde{R}$.*

11 Appendix: Regular morphisms and G-rings.

In this Appendix we recall the definitions of regular homomorphism, G-rings and excellent and quasi-excellent rings.

Definition 11.1 *([13], Chapter 13, (33.A), p. 249) Let $\sigma : A \to B$ be a homomorphism of noetherian rings. We say that σ is* **regular** *if it is flat, and for every prime ideal $P \subset A$, the ring $B \otimes_A \kappa(P)$ is geometrically regular over $\kappa(P)$ (this means that for any finite field extension $\kappa(P) \to k'$, the ring $B \otimes_A k'$ is regular).*

Remark 11.1 *If $\kappa(P)$ is perfect, the ring $B \otimes_A \kappa(P)$ is geometrically regular over $\kappa(P)$ if and only if it is regular.*

Remark 11.2 *It is known that a morphism of finite type is regular in the sense above if and only if it is smooth (that is, formally smooth in the sense of Grothendieck with respect to the discrete topology), though we do not use this fact in this paper.*

Regular morphisms come up in a natural way when one wishes to pass to the formal completion of a local ring:

Definition 11.2 *([13], (33.A) and (34.A)) Let R be a noetherian ring. For a maximal ideal m of R, let $\hat{R}_m$ denote the m-adic completion of R. We say that R is a* **G-ring** *if for every maximal ideal m of R, the natural map $R \to \hat{R}_m$ is a regular homomorphism.*

The property of being a G-ring is preserved by localization and passing to rings essentially of finite type over R.

Definition 11.3 *([13], Definition 2.5, (34.A), p. 259) Let R be a noetherian ring. We say that R is* **quasi-excellent** *if the following two conditions hold:*

(1) R is J-2, that is, for any scheme X, which is reduced and of finite type over Spec R, $Reg(X)$ is open in the Zariski topology.

(2) For every maximal ideal $m \subset R$, R_m is a G-ring.

It is known [13] that a *local* G-ring is automatically J-2, hence automatically quasi-excellent. Thus for local rings "G-ring" and "quasi-excellent" are one and the same thing. A scheme is said to be **excellent** if it is quasi-excellent and universally catenary, but we do not need the catenary condition in this paper.

Both excellence and quasi-excellence are preserved by localization and passing to rings of finite type over R ([13], Chapter 13, (33.G), Theorem 77, p. 254). In particular, any scheme essentially of finite type over a field, $\mathbf{Z}$, $\mathbf{Z}_{(p)}$, $\mathbf{Z}_p$, the Witt vectors or any other excellent Dedekind domain is excellent. See [14] (Appendix A.1, p. 203) for some examples of non-excellent rings.

Rings which arise from natural constructions in algebra and geometry are excellent. Complete and complex-analytic local rings are excellent (see [13], Theorem 30.D) for a proof that any complete local ring is excellent).

Finally, we remark that the category of quasi-excellent rings is a natural one for doing algebraic geometry, since it is the largest reasonable class of rings for which resolution of singularities holds. Namely, let R be a noetherian ring. Grothendieck ([2], IV.7.9) proves that if all of the irreducible closed subschemes of Spec R and all of their finite purely inseparable covers admit resolution of singularities, then R must be quasi-excellent. Grothendieck's result means that the largest *class* of noetherian rings, closed under homomorphic images and finite purely inseparable extensions, for which resolution of singularities could possibly exist, is *quasi-excellent* rings.

References

[1] [A] S. Abhyankar, *Local uniformization on algebraic surfaces over ground fields of characteristic $p \neq 0$*, Ann. of Math., 63 (1956) 491–526.

[2] [EGA] A. Grothendieck, J. Dieudonné, Elément de Géométie algébrique, Chap. IV, Pub. Math. IHES, No. 24, 1965.

[3] [G-R] L. Gruson and M. Raynaud, *Critères de platitude et de projectivité* Inventiones math., 13 (1971), 1-89.pp. 175–186, (1991).

[4] [HeSa] W. Heinzer and J. Sally, *Extensions of valuations to the completion of a local domain.* J. Pure Appl. Algebra, Vol. 71, no 2–3, pp. 175–186, (1991).

[5] [HI] M. Henriksen and J. Isbell, *Lattice-ordered rings and function rings.* Pacific J. Math. 11, 533–566 (1962).

[6] [HOS] J. Herrera, M.A. Olalla, M. Spivakovsky, *Valuations in algebraic field extensions*, J. Algebra 312 (2007), no. 2, 1033–1074.

[7] [Kap1] I. Kaplansky, *Maximal fields with valuations I.* Duke Math. J., 9:303–321 (1942).

[8] [Kap2] I. Kaplansky, *Maximal fields with valuations II.* Duke Math. J., 12:243–248 (1945).

[9] [L] J. Lipman *Desingularization of two-dimensional schemes*, Ann. Math. 107 (1978) 151–207.

[10] [LMSS1] F. Lucas, J.J. Madden, D. Schaub, M. Spivakovsky, *On connectedness of sets in the real spectra of polynomial rings*, arXiv:math/0601671v2 [math.AG], submitted for publication.

[11] [LMSS2] F. Lucas, J.J. Madden, D. Schaub, M. Spivakovsky, *On the Pierce–Birkhoff and the separation conjecture in dimension 3,* in preparation.

[12] [Mad] J. J. Madden, *Pierce–Birkhoff rings.* Arch. Math. 53, 565–570 (1989).

[13] [Mat] H. Matsumura, *Commutative Algebra.* Benjamin/Cummings Publishing Co., Reading, Mass., 1970.

[14] [Nag] M. Nagata, *Local Rings.*, Interscience Publishers, 1960.

[15] [Spi1] M. Spivakovsky, *Valuations in function fields of surfaces.* Amer. J. Math 112, 1, 107–156 (1990).

[16] [Spi2] M. Spivakovsky, *Resolution of Singularities I: Local Uniformization.* In preparation.

[17] [Te] B. Teissier, *Valuations, deformations, and toric geometry*, Proceedings of the Saskatoon Conference and Workshop on valuation theory (second volume), F-V. Kuhlmann, S. Kuhlmann, M. Marshall, editors, Fields Institute Communications, **33**, 2003, 361-459.

[18] [V] M. Vaquié, *Famille admissible de valuations et défaut d'une extension*, J. Algebra 311 (2007), no. 2, 859–876.

[19] [Z] O. Zariski, *Local uniformization theorem on algebraic varieties*, Ann. of Math., 41 (1940), 852–896.

[20] [ZS] O. Zariski, P. Samuel *Commutative Algebra, Vol. II*, Springer-Verlag (1960).

Comments and questions

I wonder if it is possible that $\hat{\mathcal{R}}$ integral domain implies at least that R has geometrically reduced formal fibers (see Prop. 9.1). If the normalization is not finite, how do we prove integrality of $\hat{\mathcal{R}}$?
Question: Do we put this question in the text?
Comment: I am unhappy with the notation $\hat{\nu}$. In the functorial notation, it should denote an extension of ν to $\hat{R}$. However, in the text it denotes an extension of ν to $\frac{\hat{R}}{\tilde{H}_0}$. Any suggestions? In fact you have to use some notation like that in the paragraph on Pierce-Birhkoff. How about $\hat{\nu}$ on $\hat{R}$ and $\hat{\nu}_+$ or $\hat{\nu}_-$ on $\hat{R}/\tilde{H}_0$?
Comment: The proof I propose in parag. 9 does again uses local uniformization in lower dimension. The new ideas are the (trivial) lemma 9.2 , the first statement of prop. 9.2 and the definition of the ideals $\mathcal{S}_{\gamma_1}$. Question: It seems possible that in the situation of parag. 9, at least when $\hat{R}$ and $\hat{\overline{R}}$ are regular, and the same for R', then the map $\hat{R}\to\hat{R}'$ between them can be put in the form

$$\frac{\hat{R}}{H}[[V_1,\dots,V_r]]\to\frac{\hat{R}'}{H'}[[V_1,\dots,V_r]]$$

with $V_i\mapsto V_i$. It would imply that the H'-adic valuation in $\hat{R}'_{H'}$ induces the $H-adic$ in $\hat{R}_H$, which seems necessary for a good theory. What do you think?
Comment: As it is written it seems that there is a canonical extension of ν giving a scalewise birational map of the associated graded rings. OK if the answer to the previous question is positive.
Comment: In Prop 6.4, should we not state and prove that if the extension $\nu^\dagger$ has the scalewise birationality property for the graded rings, it is tight? More generally, I find that it would be better to make more explicit the three definitions minimal, etc. in terms of the associated graded rings of the $2r$ valuations of $\hat{R}$.