Introduction to q-characters on quantum affine algebras

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Abstract

This is only a preliminary version.

Contents

1 Finite dimensional representations of $U_q(\widehat{\mathfrak{sl}_2})$

Assume that we are working on the field $\mathbb{C}$, $q \in \mathbb{C}^*$ is not a root of unity.

1.1 Definition of $U_q(\widehat{\mathfrak{sl}_2})$

Jimbo presentation

We call the presentation of quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$ by Chevalley generators a Jimbo presentation and denote it by $(U_q(\widehat{\mathfrak{sl}_2}), R_1)$.

As an algebra, it is generated by generators $e_0^{\pm}, e_1^{\pm}, K_0^{\pm}, K_1^{\pm}$ and relations:

$$K_0K_1 = K_1K_0, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_ie_i^{\pm} = q^{\pm 2}e_i^{\pm}K_i, \ (i = 0, 1)$$
$$K_ie_j^{\pm} = q^{\mp 2}e_j^{\pm}K_i, \ (i \neq j)$$
$$e_0^{\pm}e_1^{\mp} = e_1^{\mp}e_0^{\pm},$$
$$e_i^+e_i^- - e_i^-e_i^+ = \frac{K_i - K_i^{-1}}{q - q^{-1}}, \ (i = 0, 1),$$

and quantized Serre relations:

$$(e_i^{\pm})^3e_j \pm -[3](e_i^{\pm})^2e_j^{\pm}e_i^{\pm} + [3]e_i^{\pm}e_j^{\pm}(e_i^{\pm})^2 - e_j^{\pm}(e_i^{\pm})^3 = 0, \ (i \neq j).$$

The Hopf algebra structure over $U_q(\widehat{\mathfrak{sl}_2})$ is given by:

$$\Delta(e_i^+) = e_i^+ \otimes K_i + 1 \otimes e_i^+, \quad \varepsilon(e_i^+) = 0, \quad S(e_i^+) = -e_i^+K_i^{-1},$$
$$\Delta(e_i^+) = e_i^- \otimes 1 + K_i^{-1} \otimes e_i^-, \quad \varepsilon(e_i^-) = 0, \quad S(e_i^+) = -K_ie_i^-,$$
$$\Delta(K_i^{\pm}) = K_i^{\pm} \otimes K_i^{\pm}, \quad \varepsilon(K_i^{\pm}) = 1, \quad S(K_i^{\pm}) = K_i^{\mp}.$$

Drinfel'd presentation

Drinfel'd gave another realization of quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$ by generators and relations. This is essential in the study of representation theory of such algebras because it gives a finer structure of the "torus part", which will offer us more information about weight spaces of a representation.

The Drinfel'd realization $(U_q(\widehat{\mathfrak{sl}_2}), R_2)$ is defined by generators $x^{\pm}_m, h_r, K^{\pm 1}, c^{\pm 1}$ where $m \in \mathbb{Z}$, $r \in \mathbb{Z} - \{0\}$, and relations:

$$c^{\pm 1} \text{ are in the center,}$$
$$KK^{-1} = K^{-1}K = 1, \quad cc^{-1} = c^{-1}c = 1, \quad [K, h_r] = 0,$$
$$Kx^{\pm}_k = q^{\pm 2} x^{\pm}_k K,$$
$$[h_k, x^{\pm}_l] = \pm \frac{1}{k}[2k] c^{\mp |k|} x^{\pm}_{k+l},$$
$$x^{\pm}_{k+1} x^{\pm}_l - q^{\pm 2} x^{\pm}_l x^{\pm}_{k+1} = q^{\pm 2} x^{\pm}_k x^{\pm}_{l+1} - x^{\pm}_{l+1} x^{\pm}_k,$$
$$[h_k, h_l] = \delta_{k,-l} \frac{1}{k}[2k] \frac{c^k - c^{-k}}{q - q^{-1}},$$
$$[x^+_k, x^-_l] = \frac{1}{q - q^{-1}}[c^{k-l}\psi_{k+l} - \phi_{k+l}].$$

where $\psi_k, \ \phi_k$ are given by:

$$\sum_{m=0}^{\infty} \psi_m z^m = K exp\left((q - q^{-1}) \sum_{s=1}^{\infty} h_s z^s \right),$$
$$\sum_{m=0}^{\infty} \phi_{-m} z^{-m} = K^{-1} exp\left(-(q - q^{-1}) \sum_{s=1}^{\infty} h_{-s} z^{-s} \right),$$

and $\psi_k = 0, \ \phi_{-k} = 0$, for $k < 0$.

Example 1. *It is easy to see that $\psi_0 = K$, $\phi_0 = K^{-1}$, and in general,*

$$\psi_s = \frac{(q - q^{-1})^s}{s!} K \sum_{l=1}^{s} \sum_{i_1 + \cdots + i_l = s} h_{i_1} \cdots h_{i_l}.$$

This formula tells that it is possible to write ψ_s as a polynomial in h_k, $(k > 0)$ and K.

For ϕ_s, we may obtain a similar formula, so ϕ_s can be written as a polynomial in h_{-k} $(k > 0)$, and K^{-1}.

Relation between R_1 and R_2

1.2 Evaluation representations

Evaluation representation is a particular phenomenon for quantum affine algebras of type A and is a substantial tool for understanding the finite dimensional irreducible representations. The main idea is to define a family of surjective algebra morphisms from $U_q(\widehat{\mathfrak{sl}_2})$ to the quantum group $U_q(\mathfrak{sl}_2)$ and then pull back all irreducible finite dimensional representations of the latter, which is completely clear.

At first, we define two $U_q(\mathfrak{sl}_2)$ structures in $U_q(\widehat{\mathfrak{sl}_2})$.

Definition 1. *We define two algebra morphisms $I_0, I_1 : U_q(\mathfrak{sl}_2) \to (U_q(\widehat{\mathfrak{sl}_2}), R_1)$ by:*

$$I_0(e^{\pm}) = e_1^{\pm}, \quad I_0(K^{\pm 1}) = K_1^{\pm 1},$$

$$I_1(e^{\pm}) = e_0^{\mp}, \quad I_1(K^{\pm 1}) = K_0^{\mp 1}.$$

I_0 and I_1 are both injections, we thus found two copies of $U_q(\mathfrak{sl}_2)$ in $(U_q(\widehat{\mathfrak{sl}_2}), R_1)$.

Now we define the evaluation representation.

Definition 2. *For any $a \in \mathbb{C}^*$, define an algebra morphism $ev_a : (U_q(\widehat{\mathfrak{sl}_2}), R_1) \to U_q(\mathfrak{sl}_2)$ as follows:*

$$ev_a(e_0^{\pm}) = q^{\mp 1} a^{\pm 1} e^{\mp}, \quad ev_a(e_1^{\pm}) = e^{\pm},$$
$$ev_a(K_0) = K^{-1}, \quad ev_a(K_1) = K.$$

Remark 1. *$ev_a \circ I_0 = id$, $ev_q \circ I_1 = id$. This means that for any $a \in \mathbb{C}^*$, ev_a is surjective.*

Let ${}_{U_q}mod$ and ${}_{\widehat{U_q}}mod$ be categories of finite dimensional representations of $U_q(\mathfrak{sl}_2)$ and $U_q(\widehat{\mathfrak{sl}_2})$ respectively, then for any $a \in \mathbb{C}^*$, ev_a induces a functor

$$\widetilde{ev_a} : {}_{U_q}mod \to {}_{\widehat{U_q}}mod.$$

Thus we got a family of functors $\widetilde{ev_a}$.

Let V_n be the irreducible representation of $U_q(\mathfrak{sl}_2)$ of dimension $n+1$, we denote the image of V_n under the functor $\widetilde{ev_a}$ by $V_n(a)$, it is also irreducible.

Definition 3. *For $a \in \mathbb{C}^*$ and $n \in \mathbb{N}$, we call these $V_n(a)$ evaluation representations of quantum affine algebra $U_q(\widehat{\mathfrak{sl}_2})$.*

Now we want to lift the evaluation morphism to $(U_q(\widehat{\mathfrak{sl}_2}), R_2)$.

Recall that there is an algebra isomorphism

$$\mathcal{F} : (U_q(\widehat{\mathfrak{sl}_2}), R_1) \to (U_q(\widehat{\mathfrak{sl}_2}), R_2),$$

then $ev_a \circ \mathcal{F}^{-1}$ gives out the evaluation morphism of $(U_q(\widehat{\mathfrak{sl}_2}), R_2)$.

Proposition 1. *$ev_a : (U_q(\widehat{\mathfrak{sl}_2}), R_2) \to U_q(\mathfrak{sl}_2)$ is given by:*

$$ev_a(c^{\pm 1}) = 1, \quad ev_a(K) = K,$$

$$ev_a(x_k^+) = q^{-k} a^k K^k e^+, \quad ev_a(x_k^-) = q^{-k} a^k e^- K^k.$$

Recall that for the representation V_n of $U_q(\mathfrak{sl}_2)$ of dimension $n+1$, there exists a basis $v_0, \cdots, v_n$ of V_n such that

$$K.v_i = q^{n-2i} v_i, \quad e^+.v_i = [n-i+1] v_{i-1}, \quad e^-.v_i = [i+1] v_{i+1}.$$

As a vector space, $V_n(a)$ is nothing but V_n, and the action of $U_q(\widehat{\mathfrak{sl}_2})$ is given by:

Proposition 2.

$$x_k^+.v_i = a^k q^{k(n-2i+1)} [n-i+1] v_{i-1},$$
$$x_k^-.v_i = a^k q^{k(n-2i-1)} [i+1] v_{i+1}.$$

1.3 Highest weight representations

In the following sections, $U_q(\widehat{\mathfrak{sl}_2})$ will stand for the Drinfel'd realization if not specified.

Highest weight modules

Denote H the subalgebra of $U_q(\widehat{\mathfrak{sl}_2})$ generated by $c^{\pm 1}, K^{\pm 1}, h_k, \psi_k$ $(k > 0), \phi_k$ $(k < 0)$. It is easy to see from relations between ψ_k, ϕ_s and h_i that it is also generated by $c^{\pm 1}, \phi_k$ $(k \leq 0)$ and ψ_k $(k \geq 0)$.

Definition 4. *Let V be a representation of $U_q(\widehat{\mathfrak{sl}_2})$, $\Omega \in V$ is called a highest weight vector if it satisfies:*
(1). Ω is annihilated by all x_k^+, $k \in \mathbb{Z}$;
(2). Ω is a common eigenvector of H, it is to say, $c^{\pm 1}.\Omega = \varepsilon \Omega$, for some $\varepsilon \in \{\pm 1\}$;

$\psi_k.\Omega = d_k^+\Omega$, for $k \geq 0$ and some $d_k^+ \in \mathbb{C}$; $\phi_k.\Omega = d_k^-\Omega$, for $k \leq 0$ and some $d_k^- \in \mathbb{C}$. If V is generated by a highest weight vector, we call it a highest weight module. A highest weight module is called of type 1 if the action of c on the highest weight vector is give by identity and when restricted to $U_q(\mathfrak{sl}_2)$, it is a representation of type 1.

Remark 2. *In general, contract to the $U_q(\mathfrak{sl}_2)$ case, H is not commutative because there is the following relation:*

$$[h_k, h_{-k}] = \frac{1}{k}[2k]\frac{c^k - c^{-k}}{q - q^{-1}}.$$

But if we consider a highest weight module, the action of c gives ± 1, so we always have $c^k = c^{-k}$, then the action of H on V is commutative, it is to say, the image of H in $End(V)$ is a commutative subalgebra.

Example 2. *We will show that evaluation representations $V_n(a)$ for $a \in \mathbb{C}^*$ are highest weight representations.*

At first, it is easy to see from Proposition 2 that v_0 is annihilated by all x_k^+.

The action of ϕ_k and ψ_k can be easily computed:

$$\psi_0.v_0 = q^n v_0, \quad \phi_0.v_0 = q^{-n}v_0,$$

$$d_k^+ = (q - q^{-1})a^k q^{k(n-1)}[n], \quad (k > 0),$$
$$d_{-k}^- = -(q - q^{-1})a^{-k} q^{-k(n-1)}[n], \quad (k > 0).$$

In general, we have:

Proposition 3. *All finite dimensional irreducible representations of $U_q(\widehat{\mathfrak{sl}_2})$ are of highest weight.*

Drinfel'd polynomials

From this subsection until the end, suppose that all highest weight modules are of type 1.

For any type 1 highest weight representation, we know that the weight of the highest weight vector is determined by a series of complex numbers $(d_k^+, d_{-k}^-)_{k\in\mathbb{N}}$ such that $d_0^+ d_0^- = 1$. But we really do not want to write down an infinite series every time because there exists relations between these numbers. For example, as has calculated in Example 2, for $k > 0$, the series d_k^+ is geometric. So forming the generating function will definitely leads us to a more simple "moduli". This is the following theorem.

Theorem 1 (Chari-Pressley)**.** *(1). Let V be a finite dimensional highest weight representation of $U_q(\widehat{\mathfrak{sl}_2})$, then there exists a polynomial $P_V(z) \in \mathbb{C}[z]$ such that $P_V(0) \neq 0$ and*

$$\sum_{k=0}^{\infty} d_k^+ z^k = \sum_{k=0}^{\infty} d_{-k}^- z^{-k} = q^{degP_V} \frac{P_V(q^{-2}z)}{P_V(z)}.$$

(2). For any series of complex numbers $\underline{d} = (d_k^+, d_{-k}^-)_{k\in\mathbb{N}}$ such that $d_0^+ d_0^- = 1$, there exists a finite dimensional irreducible highest weight module $V(\underline{d})$.

The polynomial $P(z)$ in the theorem is called Drinfel'd polynomial.

Remark 3. *After a normalization, we may suppose that $P_V(0) = 1$, which gives a bijection between the isomorphism class of finite dimensional irreducible representations $V = V(\underline{d})$ and the polynomial $P(z) \in \mathbb{C}[z]$ such that $P(0) = 1$.*

Example 3. *(Drinfel'd polynomial for evaluation representation $V_n(a)$) We have already seen that $d_k^+ = (q - q^{-1})a^k q^{k(n-1)}[n]$, then*

$$\begin{aligned}
\sum_{k=0}^{\infty} d_k^+ z^k &= q^n + \sum_{k=1}^{\infty}(q-q^{-1})a^k q^{k(n-1)}[n]z^k \\
&= q^n + \sum_{k=1}^{\infty}(q-q^{-1})[n](aq^{n-1}z)^k \\
&= q^n \frac{1 - aq^{-n-1}z}{1 - aq^{n-1}z} \\
&= q^n \frac{1 - aq^{-n-1}z}{1 - aq^{-n+1}z} \frac{1 - aq^{-n+1}z}{1 - aq^{-n+3}z} \cdots \frac{1 - aq^{n-3}z}{1 - aq^{n-1}z}
\end{aligned}$$

Taking

$$P(z) = \prod_{k=1}^{n}(1 - aq^{n-2k+1}z),$$

gives?

$$\sum_{k=0}^{\infty} d_k^+ z^k = q^n \frac{P(q^{-2}z)}{P(z)}.$$

In the same method, it is easy to see that

$$\sum_{k=0}^{\infty} d_{-k}^- z^{-k} = q^n \frac{P(q^{-2}z)}{P(z)}.$$

Definition 5. *Let V be a highest weight representation, $P_V(z)$ its Drinfel'd polynomial,*
(1). if $P_V(z) = 1 - az$ for some $a \in \mathbb{C}^*$, we call V a fundamental representation;
(2). if $P_V(z) = (1 - aq^{-n+1}z)(1 - aq^{-n+3}z) \cdots (1 - aq^{n-1}z)$ for some $a \in \mathbb{C}^*$, we call V a Kirillov-Reshetikhin module.

Remark 4. *Kirillov-Reshetikhin modules are a generalization of evaluation representations. As has been remarked, an irreducible representation is uniquely determined by its Drinfel'd polynomial, but maybe there are many highest weight representations sharing the same one. So from our computation above, all evaluation representations are Kirillov-Reshetikhin modules, but not vice-versa.*

1.4 q-strings

As we have already seen, an evaluation representation could be uniquely determined by its Drinfel'd polynomial and then by the inverse of its roots. These roots form a set of type $\{aq^{n-1}, aq^{n-3}, \cdots, aq^{-n+1}\}$.

Definition 6. *Let $S \subset \mathbb{C}^*$ be a non-empty finite set, we call S a q-string if*

$$S = \{\zeta, q^{-2}\zeta, q^{-4}\zeta, \cdots, q^{-2r}\zeta\}$$

for some $\zeta \in \mathbb{C}^*$ and $r \in \mathbb{Z}_+$.

Example 4. *As we have seen, the set of inverse roots of Drinfel'd polynomial for the evaluation representation is a q-string if we choose $\zeta = aq^{n-1}$ and $r = n-1$. We denote this q-string by $\Sigma_{a,n}$.*

Definition 7. *Let S_1, S_2 be two q-strings, we call them in general position if one of the following conditions holds:*
(1). $S_1 \cup S_2$ is not a q-string;
(2). $S_1 \subset S_2$ or $S_2 \subset S_1$.
Two q-strings which are not in general position are called in special position.

Example 5. *It is easy to see that: q-strings $\Sigma_{a,n}$ and $\Sigma_{b,m}$ are in special position if and only if*

$$\frac{b}{a} = q^{n+m-2k}, \quad 0 \le k < min(m,n).$$

Being in general position is a property related to the irreducibility of tensor product representations.

Theorem 2. *Tensor products of evaluation representations* $V_{n_1}(a_1) \otimes \cdots V_{n_r}(a_r)$ *is irreducible if and only if any two q-strings among* $\Sigma_{a_1,n_1}, \cdots, \Sigma_{a_r,n_r}$ *are in general position.*

This theorem will simply lead to the classification of finite dimensional irreducible representations of $U_q(\widehat{\mathfrak{sl}_2})$ of type 1. We have already seen that such representations have a one-to-one correspondence with Drinfel'd polynomials.

Proposition 4. *Any set of complex numbers with multiplicities can be uniquely written as a union of q-strings such that any two of them are in general position.*

So we can write the inverse roots of Drinfel'd polynomial as a union of q-strings as in the proposition above, which signifies the following classification theorem:

Theorem 3 (Chari-Pressley)**.** *Any finite dimensional irreducible representation of* $U_q(\widehat{\mathfrak{sl}_2})$ *of type* 1 *is isomorphic to a tensor product of evaluation representations. Two such tensor products are isomorphic if and only if one can be obtained by the other by permuting the tensor product factors.*

2 q-characters on $U_q(\widehat{\mathfrak{sl}_2})$

In this section, we will introduce q-characters on quantum affine algebras. All modules are left if not specified. Recall that once a highest weight representation is mentioned, it is always of type 1.

2.1 Introduction

As in the case of finite groups and compact Lie groups, the aim of character theory is to recogonize irreducible representations. It is to say, given two irreducible representations, whether they are isomorphic?

Essentially, the character theory is a kind of invariant theory. We want to construct a functor from the module category to the category of some algebraic structure (for example, groups, rings, algebras, etc.).

Recall that ${}_{\widehat{U_q}}mod$ is the category of finite dimensional representations of $U_q(\widehat{\mathfrak{sl}_2})$. Because $U_q(\widehat{\mathfrak{sl}_2})$ is a Hopf algebra (of course, a bialgebra), ${}_{\widehat{U_q}}mod$ is a monoidal category, denote $Rep(U_q(\widehat{\mathfrak{sl}_2}))$ its Grothendieck ring. Because we are working under the finite dimensional assumption, the Jordan-Hölder theorem holds in this category, so the Grothendieck ring, as an abelian group, is a free $\mathbb{Z}$-module with a basis formed by the isomorphism class of irreducible representations.

A character should be a morphism from $Rep(U_q(\widehat{\mathfrak{sl}_2}))$ to a ring. As a character, it should recognize irreducible representations, so could tell off different evaluation representations. At first, we will try a naïve idea.

Naïve idea

Whether can we use the character theory of $U_q(\mathfrak{sl}_2)$ directly?

Be more concrete, there is an injection $U_q(\mathfrak{sl}_2) \to U_q(\widehat{\mathfrak{sl}_2})$, so it is possible to restrict the representation of $U_q(\widehat{\mathfrak{sl}_2})$ to $U_q(\mathfrak{sl}_2)$ and then use the character theory on the latter. But as we will see in the following example, this is indeed naïve.

Example 6. *Consider the evaluation representation $V_1(a)$. As a $U_q(\mathfrak{sl}_2)$-module, it is just $V_1(a) = \mathbb{C}v_0 \oplus \mathbb{C}v_1$, v_0 is the highest weight vector of weight q, v_1 of weight q^{-1}, so*

$$\chi(V_1(a)) = e(1) + e(-1),$$

where $e(i)$ is the formal element. The character is independent with the invertible complex number a! So even if $a' \neq a$, the characters are all the same. But these two representations are not isomorphic because they have different highest weight!

General idea

To define the character theory on quantum affine algebras, a refined structure should be explored.

Note that in the above example, we only used the action of elements K and K^{-1} in H, which is only a tiny part (they are ψ_0 and ϕ_0).

A more natural idea is to consider the action of all ϕ_k and ψ_k. Recall that when considering a highest weight representation, the action of ψ_k (resp. ϕ_k) on the highest weight vector v will give out a series (d_k^+) (resp. d_{-k}^-). From the Chari-Pressley theorem, an irreducible representation could be characterized by its Drinfel'd polynomial, which comes from the series d_k^+ (and so d_{-k}^-). So the information contained in series d_k^+ is sufficient to recognize irreducible representations. This is the motivation of the following definition of q-characters.

2.2 q-characters

Definition and examples

To define q-characters,a result of Jordan decomposition is needed.

Proposition 5. *Let V be a finite dimensional representation of $U_q(\widehat{\mathfrak{sl}_2})$, $\underline{d} = (d_k^+)_{k\in\mathbb{N}}$ where $d_k^+ \in \mathbb{C}$,*

$$V_{\underline{d}} = \{v \in V|\ \exists p, \forall k \geq 0, (\psi_k - d_k^+)^p.v = 0\},$$

then

$$V = \bigoplus_{\underline{d}} V_{\underline{d}}$$

is a decomposition of the common characteristic spaces of H.

Then we can define q-character as the ordinary character associated to the Jordan decomposition above.

Definition 8. *Let V be a finite dimensional representation of $U_q(\widehat{\mathfrak{sl}_2})$, define*

$$\chi_q(V) = \sum_{\underline{d}} dim(V_{\underline{d}})e(\underline{d}),$$

where $e(\underline{d})$ are formal elements. We call χ_q the q-character of representation V.

Example 7. *(Computation of the evaluation representation $V_1(a)$) As a vector space, it is just $\mathbb{C}v_0 \oplus \mathbb{C}v_1$. We have already seen that*

$$\psi_0.v_0 = qv_0, \quad \psi_0.v_1 = q^{-1}v_1,$$

$$\psi_k.v_0 = (q - q^{-1})a^k v_0, \quad \psi_k.v_1 = -(q - q^{-1})a^k v_1, \cdots$$

it is easy to calculate in general, if we denote

$$\underline{d}_0 = (q, (q - q^{-1})a, (q - q^{-1})a^2, \cdots), \quad \underline{d}_1 = (q^{-1}, -(q - q^{-1})a, -(q - q^{-1})a^2, \cdots),$$

and $V = V_1(a)$, then the decomposition in the Frenkel-Reshetikhin theorem is given by:

$$V_{\underline{d}_0} = \mathbb{C}v_0, \quad V_{\underline{d}_1} = \mathbb{C}v_1, V = V_{\underline{d}_0} \oplus V_{\underline{d}_1}.$$

So $\chi_q(V) = e(\underline{d}_0) + e(\underline{d}_1)$.

The same reason as in the remark before Theorem 1, we may rewrite the formal element in a more compact form. Encore une fois, we calculate the generating function.

Example 8.

$$\begin{aligned} d_0(z) &= \sum_{i=0}^{\infty} (\underline{d}_0)_i z^i \\ &= q + \sum_{i=1}^{\infty} (q - q^{-1}) a^i z^i \\ &= q + (q - q^{-1}) \frac{az}{1 - az} = q \frac{1 - q^{-2}az}{1 - az}. \end{aligned}$$

Let $R(z) = 1 - aq^{-1}z$*, then*

$$d_0(z) = q \frac{R(q^{-1}z)}{R(qz)}.$$

The same calculation tells us that

$$d_1(z) = \sum_{i=0}^{\infty} (\underline{d}_1)_i z^i = q^{-1} \frac{1 - aq^2 z}{1 - az},$$

let $Q(z) = 1 - aqz$*, then*

$$d_1(z) = q^{-1} \frac{Q(qz)}{Q(q^{-1}z)}.$$

In general, we have the following theorem, it is a generalization of Chari-Pressley theorem, which concerns only highest weight vector.

Theorem 4 (Frenkel-Reshetikhin)**.** *Let* V *be a finite dimensional representation of* $U_q(\widehat{\mathfrak{sl}_2})$, $\underline{d} = (d_0, d_1, \cdots)$ *such that* $V_{\underline{d}} \neq \{0\}$*, then there exists polynomials* $R(z), Q(z) \in \mathbb{C}[z]$*,* $R(0) = Q(0) = 1$*, such that*

$$\sum_{i=0}^{\infty} d_i z^i = q^{deg(R) - deg(Q)} \frac{R(q^{-1}z)}{R(qz)} \frac{Q(qz)}{Q(q^{-1}z)}.$$

The proof will be given later, after been examined in some examples.

Remark 5. *In the theorem, if* V *is a highest weight module with highest weight vector* v_0 *and weight* $(d_k^+, d_{-k}^-)_{k \geq 0}$*, then Theorem 1 tells that once the Drinfel'd polynomial* $P(z)$ *is normalized by* $P(0) = 1$*, polynomials in the theorem will say* $R(z) = P(qz)$*,* $Q(z) = 1$*.*

Example 9. *In the previous example, we have already computed that for* $\underline{d}_0$*,* $R(z) = 1 - aq^{-1}z$*,* $Q(z) = 1$ *and for* $\underline{d}_1$*,* $R(z) = 1$*,* $Q(z) = 1 - aqz$*.*

The generating function of the infinite series $\underline{d}$ can be determined by two polynomials $R(z)$ and $Q(z)$, so by its roots and multiplicities. An observation to the formula in the theorem above tells us that only the difference of the multiplicity contributes. Then we could write the q-character in a more compact form as follows.

Let $\underline{d}$ be an infinite sequence such that $V_{\underline{d}} \neq \{0\}$ as in the theorem, then we obtain complex polynomials $R(z)$ and $Q(z)$. These polynomials can be factorized as:

$$R(z) = \prod_{a\in\mathbb{C}^*}(1 - az)^{\lambda_a}, \quad \lambda_a \geq 0,$$

$$Q(z) = \prod_{a\in\mathbb{C}^*}(1 - az)^{\mu_a}, \quad \mu_a \geq 0.$$

Definition 9. *Let* $\mathcal{Y} = \mathbb{Z}[Y_a^{\pm 1}]_{a\in\mathbb{C}^*}$*. For the series* $\underline{d}$ *as above, define*

$$m_{\underline{d}} = \prod_{a\in\mathbb{C}^*} Y_a^{\lambda_a - \mu_a} \in \mathcal{Y}.$$

Then for a finite dimensional representation V *of* $U_q(\widehat{\mathfrak{sl}_2})$*, its q-character is defined by*

$$\chi_q(V) = \sum_{\underline{d}} dim(V_{\underline{d}}) m_{\underline{d}} \in \mathcal{Y}.$$

Thus we transformed the moduli space generated by formal elements into a polynomial algebra, which is much easier to manipulate.

Example 10. *Again we deal with the case* $V_1(a)$*. It is calculated in the last example that for* $\underline{d}_0$*,* $R(z) = 1 - aq^{-1}z$*,* $Q(z) = 1$ *and for* $\underline{d}_1$*,* $R(z) = 1$*,* $Q(z) = 1 - aqz$*. So* $m_{\underline{d}_0} = Y_{aq^{-1}}$*,* $m_{\underline{d}_1} = Y_{aq}^{-1}$ *and*

$$\chi_q(V_1(a)) = Y_{aq^{-1}} + Y_{aq}^{-1}.$$

So if $a \neq a' \in \mathbb{C}^*$*, it is quite easy to see that* $\chi_q(V_1(a)) \neq \chi_q(V_1(a'))$*.*

Example 11. *Now we compute the q-character for all evaluation representations. For the reason explained in Remark 5, denote* $W_n(a) = V_n(qa)$*, this will essentially change nothing but give a direct relation between Theorem 1 and Theorem 5.*

$W_n(a)$ *admits a linear space decomposition as* $U_q(\mathfrak{sl}_2)$*-module:*

$$W_n(a) = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \oplus \cdots \mathbb{C}v_n.$$

From Proposition 2, the action of elements in $U_q(\widehat{\mathfrak{sl}_2})$ is given by:

$$x_k^+.v_i = (aq)^k q^{k(n-2i+1)}[n-i+1]v_{i-1},$$

$$x_k^-.v_i = (aq)^k q^{k(n-2i-1)}[i+1]v_{i+1}.$$

So a direct computation shows that

$$\psi_k.v_i = (q-q^{-1})a^k q^{k(n-2i)}([i+1][n-i] - q^{2k}[n-i+1][i])v_i.$$

Then

$$\begin{aligned}\sum_{k=0}^{\infty} d_{k,i}^+ z^k &= q^{n-2i} + \sum_{k=1}^{\infty}(q-q^{-1})a^k q^{k(n-2i)}([i+1][n-i] - q^{2k}[n-i+1][i]) \\ &= q^{n-2i} + (q-q^{-1})\left([i+1][n-i]\frac{aq^{n-2i}z}{1-aq^{n-2i}z} - [n-i+1][i]\frac{aq^{n-2i+2}z}{1-aq^{n-2i+2}z}\right) \\ &= q^{n-2i}\frac{(1-aq^{n+2}z)(1-aq^{-n}z)}{(1-aq^{n-2i}z)(1-aq^{n-2i+2}z)}.\end{aligned}$$

Denote

$$R_i(z) = (1-aq^{n-1}z)(1-aq^{n-3}z)\cdots(1-aq^{-n+1}z) = \prod_{k=1}^{n}(1-aq^{n-2k+1}z),$$

$$Q_i(z) = \prod_{k=1}^{i}(1-aq^{n-2k+3}z)\prod_{k=1}^{i}(1-aq^{n-2k+1}z),$$

then

$$\frac{R_i(q^{-1}z)}{R_i(qz)} = \frac{1-aq^{-n}z}{1-aq^{n}z}, \quad \frac{Q_i(qz)}{Q_i(q^{-1}z)} = \frac{(1-aq^{n+2}z)(1-aq^{n}z)}{(1-aq^{n-2i}z)(1-aq^{n-2i+2}z)},$$

so

$$\sum_{k=0}^{\infty} d_{k,i}^+ z^k = q^{deg(R_i)-deg(Q_i)}\frac{R_i(q^{-1}z)}{R_i(qz)}\frac{Q_i(qz)}{Q_i(q^{-1}z)}.$$

From the definition of q-character, we obtain:

$$\begin{aligned}\chi_q(W_n(a)) &= \sum_{i=0}^{n}\prod_{k=1}^{n} Y_{aq^{n-2k+1}}\prod_{k=1}^{i} Y^{-1}_{aq^{n-2k+3}}Y^{-1}_{aq^{n-2k+1}} \\ &= \prod_{k=1}^{n} Y_{aq^{n-2k+1}}\left(1 + \sum_{i=1}^{n}\prod_{k=1}^{i} Y^{-1}_{aq^{n-2k+3}}Y^{-1}_{aq^{n-2k+1}}\right).\end{aligned}$$

Once denoting $A_a = Y_{aq^{-1}}Y_{aq}$ *and* $A_a^{-1} = Y_{aq^{-1}}^{-1}Y_{aq}^{-1}$*, the formula above could be rewritten as:*

$$\prod_{k=1}^{n} Y_{aq^{n-2k+1}} \left(1 + \sum_{i=1}^{n} \prod_{k=1}^{i} A^{-1}_{aq^{n-2k+2}}\right).$$

Example 12. *We write down explicitely in the case* $n = 2$ *and* $n = 3$*.*

For $n = 2$*, as we computed in the last example,*

$$\chi_q(W_2(a)) = Y_{aq^{-1}}Y_{aq}(1 + A^{-1}_{aq^2} + A^{-1}_a A^{-1}_{aq^2}),$$

once been expanded,

$$\chi_q(W_2(a)) = Y_{aq^{-1}}Y_{aq} + Y_{aq^{-1}}Y^{-1}_{aq^3} + Y^{-1}_{aq}Y^{-1}_{aq^3}.$$

For $n = 3$*,*

$$\chi_q(W_3(a)) = Y_{aq^2}Y_aY_{aq^{-2}}(1 + A^{-1}_{aq^3} + A^{-1}_{aq^3}A^{-1}_{aq} + A^{-1}_{aq^3}A^{-1}_{aq}A^{-1}_{aq^{-1}}),$$

once been expanded,

$$\chi_q(W_3(a)) = Y_{aq^{-2}}Y_aY_{aq^2} + Y_{aq^{-2}}Y_aY^{-1}_{aq^4} + Y_{aq^{-2}}Y^{-1}_{aq^2}Y^{-1}_{aq^4} + Y^{-1}_a Y^{-1}_{aq^2}Y^{-1}_{aq^4}.$$

Proof of Frenkel-Reshetikhin theorem

(1). At first, note that is has been proved for evaluation representations.
(2). Now we want toattack finite dimensional irreducible representations.

Recall that from Chari-Pressley theorem, every finite dimensional irreducible representation of $U_q(\widehat{\mathfrak{sl}_2})$ is isomorphic to a tensor product of evaluation representations. So what we need is the following technical lemma:

Lemma 1. *Let* V *and* W *be two finite dimensional representations of* $U_q(\widehat{\mathfrak{sl}_2})$*, we identify* $\underline{d} = (d_0, d_1, \cdots)$ *with* $d(z) = \sum_{i=0}^{\infty} d_i z^i$*, then*

$$V_{d(z)} \otimes W_{d'(z)} \subset (V \otimes W)_{d(z)d'(z)}.$$

We will not prove this lemma here but remark that it depends on the coproduct formula of ψ_k.

Now, suppose that V is a finite dimensional irreducible representation of $U_q(\widehat{sl}_2)$, it can be written as $V = V_{n_1}(a_1) \otimes \cdots \otimes V_{n_r}(a_r)$. If $\underline{d} = (d_0, d_1, \cdots)$ such that $V_{\underline{d}} \neq 0$ and $v \in V_{\underline{d}}$, then there exists $v_i \in (V_{n_i}(a_i))_{d_i(z)}$ for some $d_i(z)$ such that $v = v_1 \otimes v_2 \otimes \cdots \otimes v_r$ and then $d(z) = d_1(z) \cdots d_r(z)$.

From the case of evaluation representation, we know that for each $d_i(z)$, there exists $R_i(z)$ and $Q_i(z)$ such that

$$d_i(z) = q^{degR_i - degQ_i} \frac{R_i(q^{-1}z)Q_i(qz)}{R_i(qz)Q_i(q^{-1}z)},$$

so for $d(z)$, it suffices to consider $R(z) = R_1(z) \cdots R_r(z)$ and $Q(z) = Q_1(z) \cdots Q_r(z)$.
(3). For general V, choose a filtration

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_m = V,$$

such that every successive quotient are simple. (Note that we have Jordan-Hölder.) If $V_{d(z)} \neq 0$ for some $d(z)$, there exists some i, such that $(V_i/V_{i-1})_{d(z)} \neq 0$, then use the result for simple modules.

2.3 Properties of q-characters

Fundamental property

At first, it is easy to see that once given a short exact sequence $0 \to L \to M \to N \to 0$, we have $\chi_q(M) = \chi_q(L) + \chi_q(N)$, so χ_q induces a map from the Grothendieck ring as:

$$\chi_q : Rep(U_q(\widehat{\mathfrak{sl}_2})) \to \mathcal{Y} = \mathbb{Z}[Y_a^{\pm 1}]_{a \in \mathbb{C}^*}.$$

This section will be devoted to giving some algebraic properties of these characters.

Proposition 6. *χ_q is an injective ring morphism.*

Proof. **Additive.** This comes from the simple fact that for $d(z) = \sum_{i=0}^{\infty} d_i z^i$,

$$(V \oplus W)_{d(z)} = V_{d(z)} \oplus W_{d(z)}.$$

Multiplicative. Let V, W be two elements in the Grothendieck ring, $d(z)$ and $f(z)$ are two generating functions of corresponding infinite sequences. Then from Lemma 1,

$$V_{d(z)} \otimes W_{f(z)} \subset (V \otimes W)_{d(z)f(z)}.$$

From the definition of $m_{d(z)}$, this gives

$$m_{d(z)f(z)} = m_{d(z)} m_{f(z)},$$

so from the definition of χ_q,

$$\chi_q(V \otimes W) = \chi_q(V)\chi_q(W).$$

Injectivity. Because of the existence of the Jordan-Hölder sequence, it suffices to prove that the q-characters for simple modules are linearly independent.

In the polynomial ring $\mathbb{Z}[Y_a^{\pm 1}]_{a\in\mathbb{C}^*}$, let $deg(Y_a) = 1$ and $deg(Y_a^{-1}) = -1$. For a simple representation $V = V(P)$, where P is Drinfel'd polynomial, it is easy to know that the monomial m_P corresponds to the highest weight vector is determined by the Drinfel'd polynomial, it is to say, if $P \neq P'$, then m_P and $m_{P'}$ are linearly independent. From Chari-Pressley theorem and the formula of q-characters for evaluation representations, the other monomials in $\chi_q(V(P))$ have degree strictly less than m_P.

Now, if there exists a linear relation between the q-character of irreducible representations, a glimpse of the monomial with highest degree in each q-character will give the result. □

Examples

Example 13. *We calculate the q-character for $W_1(a) \otimes W_1(aq^2)$:*

$$\begin{aligned}
\chi_q(W_1(a) \otimes W_1(aq^2)) &= \chi_q(W_1(a))\chi_q(W_1(aq^2)) \\
&= (Y_{aq^{-1}} + Y_{aq}^{-1})(Y_{aq} + Y_{aq^3}^{-1}) \\
&= Y_{aq^{-1}}Y_{aq} + Y_{aq^{-1}}Y_{aq^3}^{-1} + Y_{aq}^{-1}Y_{aq^3}^{-1} + 1 \\
&= \chi_q(W_2(a)) + \chi_q(W_0(a)).
\end{aligned}$$

So

$$\chi_q(W_1(a) \otimes W_1(aq^2)) = \chi_q(W_2(a) \oplus W_0(a)).$$

But it should be pointed out that $W_1(a) \otimes W_1(aq^2)$ is not isomorphic to $W_2(a) \oplus W_0(a)$!
Exercise. *Prove the last statement. (Hint: maybe Chevalley generators will help.)*

This gives an example for the non-semisimplicity of finite dimensional $U_q(\widehat{\mathfrak{sl}_2})$-modules. But the injectivity of χ_q says that they are equal in the Grothendieck ring, which means that they have the same simple blocks, but there exists extension between them.

Note that we always have an exact sequence:

$$0 \to W_0(a) \to W_1(a) \otimes W_1(aq^2) \to W_2(a) \to 0$$

but it is not split.

2.4 Monomials

In the calculation of the q-characters of evaluation representations, we have seen that:

$$\chi_q(W_n(a)) = \prod_{k=1}^{n} Y_{aq^{n-2k+1}} \left(1 + \sum_{i=1}^{n} \prod_{k=1}^{i} A^{-1}_{aq^{n-2k+2}} \right).$$

Observation. (1). Among monomials appear in $\chi_q(W_n(a))$, there is only one term which does not contain negative powers of Y_a.
(2). This term uniquely corresponds to a Drinfel'd polynomial, thus determines the original evaluation representation.

In this section, we will consider the relation between some kinds of monomials in q-characters and irreducible representations themselves.

Definition 10. *(1). Let $\mathcal{M}$ be the set of all Laurent monomials generated by Y_a, $a \in \mathbb{C}^*$ and for $m \in \mathcal{M}$, we call m dominant if it contains no negative powers. All dominant monomials forms a subset $\mathcal{M}_+ \subset \mathcal{M}$.*
(2). Let S be a finite dimensional irreducible representation, denote $\mathcal{M}(S)$ the set of all monomials appear in $\chi_q(S)$.

Let S be a finite dimensional irreducible representation, its Drinfel'd polynomial can be written as

$$P_S(z) = \prod_{i=1}^{n} (1 - \alpha_i z).$$

From Theorem 3, S is a tensor product of evaluation representations which are 2 by 2 in general position. Recall the form of q-character for evaluation representations, it is easy to be convinced that $\chi_q(S)$ contains a monomial

$$m_S = \prod_{i=1}^{n} Y_{\alpha_i}.$$

We want to characterize this element by defining a partial order on $\mathcal{M}(S)$.

Definition 11. *Define a partial order $\leq$ on $\mathcal{M}$ by: for any $m, m' \in \mathcal{M}$, $m \leq m'$ if and only if $\frac{m'}{m}$ is a monomial in A_a, $(a \in \mathbb{C}^*)$.*

Then we have

Theorem 5 (Frenkel-Mukhin)**.** *Let S be a finite dimensional irreducible representation, then:*

(1). m_S is the maximal element under the partial order $\leq$ in $\mathcal{M}(S)$;
(2).

$$\chi_q(S) = m_S(1 + \sum_p M_p),$$

where M_p are monomials in A_a^{-1}.

Proof. Because irreducible representations are tensor products of evaluation ones, the theorem can be obtained by q-characters for the latter. □

Remark 6. *This theorem is merely a special case of the general theory. The easiness of the proof depends heavily on the Theorem 3, which is not true in general. For general case, the correctness of this theorem is provided by the fact that the irreducible representation could be obtained from a subrepresentation in the tensor products of fundamental ones, and then take quotient. We will meet this later.*

So in the q-character of irreducible representations, there exists a unique dominant monomial. But this does not hold for arbitrary ones.

Example 14. *We calculate the q-character for $W_2(a) \otimes W_2(aq^{-2})$.*

$$\begin{aligned} & \chi_q(W_2(a) \otimes W_2(aq^{-2})) \\ = \ & (Y_{aq^{-1}}Y_{aq} + Y_{aq^{-1}}Y_{aq^3}^{-1} + Y_{aq}^{-1}Y_{aq^3}^{-1})(Y_{aq^{-3}}Y_{aq^{-1}} + Y_{aq^{-3}}Y_{aq}^{-1} + Y_{aq^{-1}}^{-1}Y_{aq}^{-1}) \\ = \ & Y_{aq^{-3}}Y_{aq^{-1}}^2 Y_{aq} + Y_{aq^{-3}}Y_{aq^{-1}} + 1 + \cdots . \end{aligned}$$

For a resume, we have a bijection between isomorphism classes of simple modules in ${}_{\widehat{U_q}}mod$ and the set of dominant monomials $\mathcal{M}_+$.

At the end of this section, we describe an easy but useful algorithm, which will frequently occur in Frankel-Mukhin algorithm later.

Algorithm. Suppose that a dominant monomial is given, denote Λ the set with multiplicities of roots appear in it:
(1). start from a root in Λ, find the longest q-string pass through it and then take this q-string away from Λ;
(2). repeat (1) until all q-strings are obtained, it is to say, Λ is empty (so they are in general position);
(3). correspond these q-strings with evaluation representations;
(4). once multiply q-characters of these evaluation representations, we obtain the q-character of the irreducible representation associated to this dominant monomial.

3 Quantum affine algebra $U_q(\widehat{\mathfrak{g}})$

3.1 Definitions

Introduction

Let $\mathfrak{g}$ be a finite dimensional complex semi-simple Lie algebra. We want to construct the quantum affine algebra associated to such Lie algebras. This subsection is devoted to understand the following diagram:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\text{Affinization}} & \hat{\mathfrak{g}} \\ \Big\downarrow{\scriptstyle\text{Quantization}} & & \Big\downarrow{\scriptstyle\text{Quantization}} \\ \mathcal{U}_q(\mathfrak{g}) & \xrightarrow{\text{Affinization}} & \mathcal{U}_q(\hat{\mathfrak{g}}) \end{array}$$

One route means that we do affinization at first then quantization, and the other one reverse the order.

We should remark that if we adopt the first way, we will get Drinfel'd realization at last, and the second way leads to Jimbo realization, as have already been seen in the $\mathfrak{sl}_2$ case.

Les donneés

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra and $n = dim\mathfrak{h} = rank\mathfrak{g}$ its rank. From the simplicity, there is a nondegenerate invariant bilinear form given by the Killing form $(\cdot,\cdot)$ on $\mathfrak{h}$, which identifies $\mathfrak{h}$ and its linear dual $\mathfrak{h}^*$,let $\mathcal{R}$ be the corresponding root system, $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ be the set of simple roots, $\mathcal{P}$ the weight lattice, $\mathcal{P}_{++}$ the set of dominant weights.

For two roots α, β, we define $< \alpha, \beta >= 2(\alpha,\beta)/(\beta,\beta)$, let $\{\varpi_1, \cdots, \varpi_n\}$ be the set of fundamental weights such that $< \alpha_i, \varpi_j >= \delta_{ij}$, where δ_{ij} is the Kronecker notation. Let $C = (C_{ij}) = (< \alpha_j, \alpha_i >)$ be a $n \times n$ matrix, it is the Cartan matrix associated to the root system. Let $D = diag(d_1, \cdots, d_n)$ be a diagonal matrix such that $B = DC$ is symmetric.

For a simple Lie algebra, we have Chevalley generators $\{e_i, f_i, h_i\}_{i=1,\cdots,n}$.

Jimbo realization

As we have seen in the beginning of this section, Jimbo realization corresponds to the procedure "quantization→affinization".

The Jimbo realization for $U_q(\widehat{\mathfrak{g}})$ is easy to understand because what we need to do is just changing the finite type Cartan matrices into affine ones.

Definition 12. *An integer matrix $C \in M_{n+1}(\mathbb{Z})$ is called a Cartan matrix of affine type if it satisfies:*
(1). For any $i = 0, 1, \cdots, n$, $C_{ii} = 2$;
(2). For $i \neq j$, $C_{ij} \leq 0$ and if $C_{ij} = 0$, then $C_{ji} = 0$;
(3). $det(C) = 0$;
(4). $det((C_{ij})_{1 \leq i,j \leq n'}) > 0$ for any $n' \leq n$.

Remark 7. *Once the 0-th row and column are deleted, we will get a finite type Cartan matrix.*

Let C be an affine type Cartan matrix and $C' = (C_{ij})_{1 \leq i,j \leq n}$ be the finite type Cartan martix associated to it.

Start from the Cartan matrix C', we can construct a semi-simple Lie algebra $\mathfrak{g}(C')$. In the construction of quantum groups after Drinfel'd-Jimbo, for any semi-simple Lie algebra $\mathfrak{g}$, there exists a quantized enveloping algebra $U_q(\mathfrak{g})$ associated to $\mathfrak{g}$ defined by generators E_i, F_i and $K_i^{\pm}$ for $i = 1, \cdots, n$ and relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$K_i E_j K_i^{-1} = q_i^{C_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-C_{ij}} F_j,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

and quantized Serre relations

$$\sum_{s=0}^{1-C_{ij}} (-1)^s E_i^{(1-C_{ij}-s)} E_j E_i^{(s)} = 0, \quad for \quad i \neq j,$$

$$\sum_{s=0}^{1-C_{ij}} (-1)^s F_i^{(1-C_{ij}-s)} F_j F_i^{(s)} = 0, \quad for \quad i \neq j,$$

where $E_i^{(s)} = \dfrac{E_i^s}{[s]_{q_i}!}$, $q_i = q^{d_i}$.

In this construction, we used nothing but a Cartan matrix of finite type, once changing it into an affine one, the same procedure will offer a quantum affine algebra.

Definition 13. *Let C be a Cartan matrix of affine type, C' be the Cartan matrix of finite type as we denoted above, $\mathfrak{g} = \mathfrak{g}(C')$ be a semi-simple Lie algebra associated to C'. The quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ is defined by the procedure above with C the Cartan matrix.*

Remark 8. *From a semi-simple Lie algebra to a quantum group, in some extent, we change the Chevalley generators e_i, f_i, h_i by $E_i, F_i, K_i = e^{\frac{h}{2}h_i} = q^{h_i}$.*

Drinfel'd realization

The Drinfel'd realization comes from the procedure "affinization→quantization". We should remark that this realization only fits for the "non-tordue" case.

We define $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ be the loop space of $\mathfrak{g}$, then there exists a Lie algebra structure over $\mathcal{L}\mathfrak{g}$ given by:

$$[x \otimes P(t), y \otimes Q(t)] = [x, y] \otimes P(t)Q(t), \quad for\ x, y \in \mathfrak{g},\ P(t), Q(t) \in \mathbb{C}[t, t^{-1}].$$

Remark 9. *$\mathcal{L}\mathfrak{g}$ is a infinite dimensional Lie algebra.*

Define $\widehat{\mathfrak{g}}$ as an central extension of $\mathcal{L}\mathfrak{g}$ by an central element c, it is to say, there is a short exact sequence

$$0 \to \mathbb{C}c \to \widehat{\mathfrak{g}} \to \mathcal{L}\mathfrak{g} \to 0.$$

Remark 10. *There exists a Lie algebra structure on $\widehat{\mathfrak{g}}$ which comes from the Killing form on $\mathfrak{g}$ and the residue map.*

The Chevalley generators of $\mathfrak{g}$ gives those for $\widehat{\mathfrak{g}}$. If we denote the former by $\{e_i, f_i, h_i\}_{i=1,\cdots,n}$, the latter is given by $\{e_i \otimes t^j, f_i \otimes t^j, h_i \otimes t^j\}_{i=1,\cdots,n; j\in\mathbb{Z}}$.

Denote $I = \{1, \cdots, n\}$. The Drinfel'd realization of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ is given by generators $x^{\pm}_{i,m}$, $h_{i,s}$, $K_i^{\pm 1}$, $c^{\pm 1}$ for $m \in \mathbb{Z}$, $s \in \mathbb{Z}^*$, $i \in I$, and relations:

$$c^{\pm 1} \text{ are in the center,}$$

$$K_i K_j = K_j K_i, \quad K_i h_{j,n} = h_{j,n} K_i, \quad K_i x^{\pm}_{j,n} K_i^{-1} = q^{\pm C_{ij}} x^{\pm}_{j,n},$$

$$[h_{i,n}, x^{\pm}_{j,m}] = \pm [nC_{ij}] c^{\mp |n|} x^{\pm}_{j,n+m},$$

$$x^{\pm}_{i,n+1} x^{\pm}_{j,m} - q^{\pm C_{ij}} x^{\pm}_{j,m} x^{\pm}_{i,n+1} = q^{\pm C_{ij}} x^{\pm}_{i,n} x^{\pm}_{j,m+1} - x^{\pm}_{j,m+1} x^{\pm}_{i,n},$$

$$[h_{i,n}, h_{j,m}] = \delta_{n,-m} \frac{1}{n} [nC_{ij}] \frac{c^n - c^{-n}}{q - q^{-1}},$$

$$[x^+_{i,n}, x^-_{j,m}] = \delta_{ij}\frac{c^{n-m}\psi_{i,n+m} - \phi_{i,n+m}}{q_i - q_i^{-1}},$$

$$\sum_{\pi\in\mathfrak{S}_s}\sum_{k=0}^{s}(-1)^k \begin{bmatrix} s \\ k \end{bmatrix}_{q_i} x^{\pm}_{i,n_{\pi(1)}}\cdots x^{\pm}_{i,n_{\pi(k)}} x_{j,m} x^{\pm}_{i,n_{\pi(k+1)}}\cdots x^{\pm}_{i,n_{\pi(s)}} = 0, \quad (i \neq j),$$

for $s = 1 - C_{ij}$ and all sequences of integers $n_1, \cdots, n_s$. The elements $\psi_{i,n}$ and $\phi_{i,n}$ are defined by:

$$\sum_{n=0}^{\infty}\psi_{i,n}z^n = K_i exp\left((q-q^{-1})\sum_{m=1}^{\infty}h_{i,m}z^m\right),$$

$$\sum_{n=0}^{\infty}\psi_{i,-n}z^{-n} = K_i^{-1} exp\left(-(q-q^{-1})\sum_{m=1}^{\infty}h_{i,-m}z^{-m}\right),$$

and for $n < 0$, $\psi_{i,n} = \phi_{i,-n} = 0$ for any $i \in I$.

Just like in the quantum group case, we have the following correspondence

$$e_i \otimes t^j \mapsto x^+_{i,j}, \quad f_i \otimes t^j \mapsto x^-_{i,j},$$

$$h_i \otimes t^j \mapsto e^{\frac{h}{2}h_i\otimes t^j} = q^{h_i\otimes t^j} = h_{i,j}, \quad (j \neq 0),$$

$$h_i \otimes 1 \mapsto e^{\frac{h}{2}h_i\otimes 1} = q^{h_i\otimes 1} = K_i,$$

$$c \mapsto e^{\frac{h}{2}c} = q^c = c.$$

This explains Drinfel'd realization in a more natural way.

3.2 Highest weight representations

As in the $\mathfrak{sl}_2$ case, define $H \subset U_q(\widehat{\mathfrak{g}})$ be the subalgebra of $U_q(\widehat{\mathfrak{g}})$ generated by $h_{i,k}$, $\psi_{i,s}$, $\phi_{i,s}$, $K_i^{\pm 1}$ and $c^{\pm 1}$ for $k \in \mathbb{Z}^*$, $s \in \mathbb{Z}$ and $i \in I$. It can also be generated by $\psi_{i,s}$, $\phi_{i,s}$ and $c^{\pm 1}$ as we have already seen in $U_q(\widehat{\mathfrak{sl}_2})$.

Definition 14. *Let V be a finite dimensional representation of $U_q(\widehat{\mathfrak{g}})$,*
(1). we call V of type 1 if the action of central element c is identity and once been restricted to a representation of $U_q(\mathfrak{g})$, it is also of type 1;
In the following discussion, we only consider type 1 representations.
(2). $v \in V$ is called a highest weight vector if for any $i = 1, \cdots, n$, $s \in \mathbb{Z}$,

$$x^+_{i,s}.v = 0, \quad \psi_{i,s}.v = d^+_{i,s}.v, \quad \phi_{i,s}.v = d^-_{i,s}.v, \quad c.v = v,$$

where $d^{\pm}_{i,s} \in \mathbb{C}$ are complex numbers;
(3). V is called a highest weight representation if there exists a highest weight vector v such that $V = U_q(\widehat{\mathfrak{g}}).v$, we call $(d^{\pm}_{i,r})$ the highest weight of v.

As in $\mathfrak{sl}_2$ case, for finite dimensional irreducible representations, there exists corresponding Drinfel'd polynomials. But here, the set of Drinfel'd polynomials is a little different from the $\mathfrak{sl}_2$ case because we need to consider a family of polynomials in the general case.

Definition 15. *Define $\mathcal{P}$ be the set of I-tuples $(P_i)_{i\in I}$ with $P_i(z) \in \mathbb{C}[z]$, $P_i(0) = 1$ for any $i \in I$.*

Notations.

$$\psi_i(z) = \sum_{n=0}^{\infty} \psi_{i,n} z^n, \quad \phi_i(z) = \sum_{n=0}^{\infty} \phi_{i,-n} z^{-n},$$

$$\Psi_i(z) = \sum_{n=0}^{\infty} d^{+}_{i,n} z^n, \quad \Phi_i(z) = \sum_{n=0}^{\infty} d^{-}_{i,-n} z^{-n}.$$

Theorem 6 (Chari-Pressley)**.** *(1). Any finite dimensional irreducible representation of $U_q(\widehat{\mathfrak{g}})$ is highest weight.*
(2). Let V be a finite dimensional irreducible representation of $U_q(\widehat{\mathfrak{g}})$ with highest weight $(d^{\pm}_{i,r})$, then there exists $\mathbf{P} = (P_i)_{i\in I} \in \mathcal{P}$, such that

$$\Psi_i(z) = \Phi_i(z) = q_i^{degP_i} \frac{P_i(q_i^{-2} z)}{P_i(z)}.$$

This gives a bijection between $\mathcal{P}$ and the isomorphism classes of finite dimensional irreducible representations of $U_q(\widehat{\mathfrak{g}})$, we denote $V(\mathbf{P})$ the representation corresponds to $\mathbf{P} \in \mathcal{P}$.
(3). Let $\mathbf{P}, \mathbf{Q} \in \mathcal{P}$, $v_{\mathbf{P}}$ and $v_{\mathbf{Q}}$ are highest weight vectors in $V(\mathbf{P})$ and $V(\mathbf{Q})$ respectively, denote

$$\mathbf{P} \otimes \mathbf{Q} = (P_i Q_i)_{i\in I},$$

then $V(\mathbf{P} \otimes \mathbf{Q})$ is isomorphic to a quotient of the subrepresentation of $V(\mathbf{P}) \otimes V(\mathbf{Q})$ generated by $v_{\mathbf{P}} \otimes v_{\mathbf{Q}}$.

In general, we have no evaluation representations, so fundamental representations become much more important than $\mathfrak{sl}_2$ case.

Remark 11. *We know that in the* $\mathfrak{sl}_2$ *case, the Drinfel'd polynomial of a fundamental representation is given by* $1-az$ *for some* $a \in \mathbb{C}^*$*, so any Drinfel'd polynomial in that case can be written as a product of fundamental ones. From (3) of Theorem 6, any irreducible representation can be obtained from the tensor products of fundamental ones. Contract to the result for evaluation representations, the fundamental one also holds for any* $\mathfrak{g}$*, as we will see below.*

Definition 16. *Let* $k \in I$*,* $a \in \mathbb{C}^*$*, define* $\mathbf{P}_a^{(k)}$ *be the series of polynomial* $(P_i(z))_{i\in I}$*, where* $P_k(z) = 1 - az$*,* $P_j(z) = 1$ *for any* $j \neq k$*. This is a Drinfel'd polynomial and the correspondent irreducible representation* $V(\mathbf{P}_a^{(k)})$ *is denoted by* $V_{\varpi_k}(a)$*, called the k-th fundamental representation of* $U_q(\widehat{\mathfrak{g}})$*.*

Remark 12. $V_{\varpi_i}(a)$*, viewed as a* $U_q(\mathfrak{g})$*-module, maybe reducible. This induces many difficulties in the general case.*

Theorem 7 (Chari-Pressley)**.** *Let* V *be a finite dimensional irreducible representation of* $U_q(\widehat{\mathfrak{g}})$*, then* V *is isomorphic to a quotient of the submodule of* $V_{\varpi_1}(a_1) \otimes \cdots \otimes V_{\varpi_n}(a_n)$ *generated by the tensor product of highest weight vectors. In particular, the couples* $(\varpi_i, a_i)_{i=1,\cdots,n}$ *are uniquely determined by* V*, up to a permutation.*

Because $U_q(\widehat{\mathfrak{g}})$ is a Hopf algebra, the category of finite dimensional representations of $U_q(\widehat{\mathfrak{g}})$ is a monoidal category, and the Grothendieck group $Rep(U_q(\widehat{\mathfrak{g}}))$ of this category is a ring. Moreover, the Chari-Pressley theorem can be rewritten in the language of Grothendieck ring.

Corollary 1. *The Grothendieck ring* $Rep(U_q(\widehat{\mathfrak{g}}))$ *is the polynomial ring over* $\mathbb{Z}$ *generated by isomorphism classes of fundamental representations.*

As in the $\mathfrak{sl}_2$ case, there exists a notion of Kirillov-Reshetikhin modules.

Definition 17. *For* $k \in \mathbb{N}$*,* $a \in \mathbb{C}^*$ *and* $j \in I$*, define* $\mathbf{P} = (P_i(z))_{i\in I} \in \mathcal{P}$ *be:*

$$P_j(z) = \prod_{i=1}^{k}(1 - aq^{j-2i+1}z), \quad P_i(z) = 1, \quad \text{for } i \neq j.$$

The highest weight representation corresponds to this Drinfel'd polynomial $\mathbf{P}$ *is called a Kirillov-Reshetikhin module. If it is irreducible, denote it by* $V_n^{(i)}(a)$*.*

As before, denote $W_n^{(i)}(a) = V_n^{(i)}(qa)$.

Remark 13. *These modules are not as important as in the* $\mathfrak{sl}_2$*-case for the reason of absence of evaluation representations.*

4 q-characters for $U_q(\hat{\mathfrak{g}})$

We suppose in this section that all representations are of type 1.

4.1 Definition

In some aspects, the definition of q-characters in the general case is a direct generalization of the case $\mathfrak{sl}_2$. Essentially, there is no new idea.

At first, we do the Jordan decomposition related to the "torus part".

Proposition 7. *Let V be a finite dimensional representation of $U_q(\hat{\mathfrak{g}})$, $\underline{d} = (d^+_{i,k})_{i\in I,k\in\mathbb{N}}$, where $d^+_{i,k} \in \mathbb{C}$, denote*

$$V_{\underline{d}} = \{v \in V | \ \exists p, \forall k \geq 0, \forall i \in I, (\psi_{i,k} - d^+_{i,k})^p = 0\},$$

then $V = \bigoplus_{\underline{d}} V_{\underline{d}}$ is a decomposition of common characteristic spaces of H.

The second step is finding polynomials which characterize all weights.

Theorem 8 (Frenkel-Reshetikhin)**.** *Let V be a finite dimensional representation of $U_q(\hat{\mathfrak{g}})$, $\underline{d} = (d^+_{i,k})_{i\in I,k\in\mathbb{N}}$ such that $V_{\underline{d}} \neq \{0\}$, then there exists polynomials $R_i(z)$, $Q_i(z) \in \mathbb{C}[z]$, $i = 1, \cdots, n$, such that $R_i(0) = Q_i(0) = 1$ and for any $i \in I$,*

$$\sum_{k=0}^{\infty} d^+_{i,k} z^k = q_i^{degR_i - degQ_i} \frac{R_i(q_i^{-1}z)Q_i(q_i z)}{R_i(q_i z)Q_i(q_i^{-1}z)}.$$

Denote

$$R_i(z) = \prod_{j=1}^{k_i}(1 - a_j^{(i)} z), \quad Q_i(z) = \prod_{j=1}^{l_i}(1 - b_j^{(i)} z),$$

and $\mathcal{Y} = \mathbb{Z}[Y^{\pm 1}_{i,a}]_{i\in I,a\in\mathbb{C}^*}$, then we can define the q-character as in the case $\mathfrak{sl}_2$:

Definition 18. *Let $\underline{d}$ be as in above theorem such that $V_{\underline{d}} \neq \{0\}$, define*

$$m_{\underline{d}} = \prod_{i\in I}\prod_{r=1}^{k_i} Y_{i,a_r^{(i)}} \prod_{s=1}^{l_i} Y^{-1}_{i,b_s^{(i)}},$$

and

$$\chi_q(V) = \sum_{\underline{d}} dim(V_{\underline{d}}) m_{\underline{d}} \in \mathcal{Y}.$$

χ_q is called the q-character of V.

4.2 Properties of q-characters

With the same argument as in $\mathfrak{sl}_2$ case, once having a short exact sequence $0 \to L \to M \to N \to 0$, we have:

$$\chi_q(M) = \chi_q(L) + \chi_q(N)$$

and it induces a map

$$\chi_q : Rep(U_q(\widehat{\mathfrak{g}})) \to \mathcal{Y}.$$

Proposition 8. *χ_q is an injective ring morphism.*

Now we want to show that the q-character introduced here is really a generalization of the character theory over quantized enveloping algebra $U_q(\mathfrak{g})$.

Define a graduation on $\mathcal{Y}$ by requiring the variable $Y_{i,a}^{\pm 1}$ being of weight $\pm\varpi_i$, then $A_{i,a}^{\pm 1}$ is of weight $\pm\alpha_i$.

Recall the character on $U_q(\mathfrak{g})$: it is a ring morphism $\chi : Rep(U_q(\mathfrak{g})) \to \mathbb{Z}[y_i^{\pm 1}]_{i\in I}$. If $V = \bigoplus_{\mu\in\mathcal{P}} V_\mu$ is a decomposition of $V \in Rep(U_q(\mathfrak{g}))$ into its weight spaces, define

$$\chi(V) = \sum_{\mu\in\mathcal{P}} dim(V_\mu) y^\mu,$$

where $y^\mu = \prod_{i=1}^n y_i^{m_i}$ if $\mu = \sum_{i=1}^n m_i\varpi_i$.

Define a ring morphism

$$\beta : \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*} \to \mathbb{Z}[y_i^{\pm 1}]_{i\in I}, \quad Y_{i,a}^{\pm 1} \mapsto y_i^{\pm 1}.$$

As in the $\mathfrak{sl}_2$ case, we have the restriction functor

$$res : {}_{\widehat{U_q}}mod \to {}_{U_q}mod,$$

and it induces a ring morphism

$$res : Rep(U_q(\widehat{\mathfrak{g}})) \to Rep(U_q(\mathfrak{g})).$$

The following proposition makes the remark above more precise.

Proposition 9. *The following diagram is commutative:*

$$\begin{array}{ccc} Rep(U_q(\widehat{\mathfrak{g}})) & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*} \\ \downarrow{\scriptstyle res} & & \downarrow{\scriptstyle \beta} \\ Rep(U_q(\mathfrak{g})) & \xrightarrow{\chi} & \mathbb{Z}[y_i^{\pm 1}]_{i\in I} \end{array}.$$

4.3 Monomials

Similarly, we can define the set of all Laurent monomials $\mathcal{M}$ in $\mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^*}$ and the set of all dominant monomials $\mathcal{M}_+$. For an irreducible representation S, $\mathcal{M}(S)$ is the set of all monomials appear in $\chi_q(S)$.

Define

$$A_{i,a} = Y_{i,aq_i^{-1}} Y_{i,aq_i} \prod_{C_{ji}=-1} Y_{j,a}^{-1} \prod_{C_{ji}=-2} Y_{j,aq^{-1}}^{-1} Y_{j,aq}^{-1} \prod_{C_{ji}=-3} Y_{j,aq^{-2}}^{-1} Y_{j,a}^{-1} Y_{j,aq^2}^{-1}.$$

This is an analogue of the monomial A_a in the case $\mathfrak{sl}_2$.

Proposition 10 (Frenkel-Reshetikhin). *Let V be a finite dimensional irreducible representation, $V = V(\mathbf{P})$, $\mathbf{P} = (P_i)_{i\in I}$,*

$$P_i(z) = \prod_{k=1}^{n_i} (1 - a_k^{(i)} z),$$

then

$$\chi_q(V) = \prod_{i\in I} \prod_{k=1}^{n_i} Y_{i,a_k^{(i)}} (1 + \sum_p M_p),$$

where M_p are monomials in $A_{j,c}^{\pm 1}$. Moreover,

$$m_V = \prod_{i\in I} \prod_{k=1}^{n_i} Y_{i,a_k^{(i)}}$$

is the highest weight monomial.

Remark 14. *"Highest weight monomial" means the monomial corresponds to the highest weight vector in the irreducible representation V.*

In their article, E.Frenkel and N.Reshetikhin conjectured that there will be no positive powers of $A_{j,c}$ appear in M_p in the formula above. It is solved finally by E.Frenkel and E.Mukhin.

Theorem 9 (Frenkel-Mukhin). *Let V be an irreducible representation of $U_q(\widehat{\mathfrak{g}})$, those M_p in the last proposition are monomials in $A_{j,c}^{-1}$ for some $j \in I$ and $c \in \mathbb{C}^*$, moreover, m_V is the maximal monomial in the partial order defined on $\mathcal{M}(S)$.*

Proposition 11. *Let V, W be two finite dimensional representations of $U_q(\widehat{\mathfrak{g}})$, $\chi_q(V)$ and $\chi_q(W)$ have the same dominant monomials and multiplicities, then $\chi_q(V) = \chi_q(W)$.*

As a summary, we have already got a bijection between $\mathcal{M}_+$ and the set formed by isomorphic classes of irreducible representations of $U_q(\widehat{\mathfrak{g}})$. For a dominant monomial m, we associate it to $L(m)$.

4.4 Frenkel-Mukhin Algorithm

In Proposition 11, we have seen that for the property of having the same q-characters, it suffices to consider the dominant monomials with multiplicities. So it is natural to ask the following question:
Problem. Whether can we get the q-character by knowing its dominant monomials?

Frenkel-Mukhin algorithm is such a computational method to associate a polynomial $FM(m)$ to a single dominant monomial m. In some cases, this will give the q-character for the irreducible representation corresponds to m.

In the $\mathfrak{sl}_2$ case, we have seen an algorithm. The idea for the general case is to expand the given monomial in all possible $\mathfrak{sl}_2$ directions until touching the lowest monomial, it is an analogue of finding all weights with multiplicities in an irreducible representation of a simple Lie algebra. But here, we have no Weyl character formula.

At first, we recall that $\mathcal{M}$ is the set of all monomials, $\mathcal{M}_+$ the set of all dominant monomials.

Definition 19. *Let $m \in \mathcal{M}$ and $i \in I$, we call m is i-dominant if there is no negative powers of $Y_{i,a}$ in m. The set of all i-dominant monomials is denoted by $\mathcal{M}_{i,+}$.*

Now we describe the Frenkel-Mukhin algorithm.

The algorithm.

Fix a dominant monomial $m \in \mathcal{M}_+$.
(1). If m is i-dominant, we define another Laurent polynomial $\varphi_i(m)$ in this step.

(1.1). If $j \neq i$, replace all $Y_{j,a}$ appear in m by 1 and $Y_{i,a}$ by Y_a, thus we obtain a monomial in $\mathbb{Z}[Y_a]_{a\in\mathbb{C}^*}$, denote it by $\overline{m}$.

(1.2). The monomial $\overline{m}$ can be treated as a dominant one in the character theory of $U_{q_i}(\widehat{\mathfrak{sl}_2})$, the algorithm mentioned in $\mathfrak{sl}_2$ case gives an irreducible representation V

of $U_{q_i}(\widehat{\mathfrak{sl}_2})$ which has q-character

$$\chi_q(V) = \overline{m}(1 + \sum_p \overline{M_p}),$$

where each $\overline{M_p}$ is a monomial in A_a^{-1}.

(1.3). Replace every A_a^{-1} by $A_{i,a}^{-1}$ in all these $\overline{M_p}$, we get monomials M_p in $A_{i,a}^{-1}$.

(1.4). Define $\varphi_i(m) = m(1 + \sum_p M_p)$.

(2). Define a set of monomials $D_m \subset \mathcal{M}$ for $m \in \mathcal{M}_+$.

A monomial m' is in D_m if and only if there exists a finite length sequence $m = m_0, m_1, \cdots, m_t = m'$ such that for any $r = 1, \cdots, t$, there exists i such that $m_{r-1} \in \mathcal{M}_{i,+}$ and m_r appears as a monomial in $\varphi_i(m_{r-1})$.

Proposition 12. *D_m is countable, and for any $m' \in D_m$, $m' \leq m$.*

(3). Define inductively some coefficients $(s(m_r))_{r\geq 0}$ and $(s_i(m_r))_{r\geq 0}$.

The initial condition is given by $s(m_0) = 1$, $s_i(m_0) = 1$ and in general:

$$s_i(m_t) = \sum_{r<t, m_r \in \mathcal{M}_{i,+}} (s(m_r) - s_i(m_r))[\varphi_i(m_r) : m_t],$$

$$s(m_t) = max\{s_i(m_t) |\ i \in I\},$$

where $[\varphi_i(m_r) : m_t]$ is the coefficient of m_t in $\varphi_i(m_r)$.

(4). Define $FM(m) = \sum_{r\geq 0} s(m_r)m_r$ and call it Frenkel-Mukhin polynomial.

Example.

Let $\mathfrak{g} = \mathfrak{sl}_3$ be of type A_2, $m = Y_{1,q^2}Y_{2,q^{-1}}$ be a dominant monomial. This section is devoted to the calculation of its Frenkel-Mukhin polynomial $FM(m)$.

Recall that

$$A_{1,a} = Y_{1,aq^{-1}}Y_{1,aq}Y_{2,a}^{-1}, \quad A_{2,a} = Y_{1,a}^{-1}Y_{2,aq^{-1}}Y_{2,aq}.$$

The monomial m is both 1-dominant and 2-dominant, so we need to calculate $\varphi_1(m)$ and $\varphi_2(m)$.

(1). For $\varphi_1(m)$, we have the following procedure:

$$Y_{1,q^2}Y_{2,q^{-1}} \mapsto Y_{q^2} \mapsto Y_{q^2}(1 + A_{q^3}^{-1}) \mapsto Y_{1,q^2}Y_{2,q^{-1}}(1 + A_{1,q^3}^{-1}),$$

once expanded,

$$\varphi_1(m) = Y_{1,q^2}Y_{2,q^{-1}} + Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3} = m + m_1,$$

here $m_1 = Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3}$ is 2-dominant.

(2). For $\varphi_2(m)$, we have the following procedure:

$$Y_{1,q^2}Y_{2,q^{-1}} \mapsto Y_{q^{-1}} \mapsto Y_{q^{-1}}(1 + A_1^{-1}) \mapsto Y_{1,q^2}Y_{2,q^{-1}}(1 + A_{2,1}^{-1}),$$

once expanded,

$$\varphi_2(m) = Y_{1,q^2}Y_{2,q^{-1}} + Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1} = m + m_2,$$

here $m_2 = Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1}$ is 1-dominant.

(3). For $\varphi_2(m_1)$, we have the following procedure:

$$Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3} \mapsto Y_{q^{-1}}Y_{q^3} \mapsto Y_{q^{-1}}Y_{q^3}(1+A_1^{-1})(1+A_{q^4}^{-1}) \mapsto Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3}(1+A_{2,1}^{-1})(1+A_{2,q^4}^{-1}),$$

once expanded,

$$\varphi_2(m_1) = Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3}+Y_{1,1}Y_{1,q^4}^{-1}Y_{2,q^3}Y_{2,q}^{-1}+Y_{2,q^{-1}}Y_{2,q^5}^{-1}+Y_{1,1}Y_{2,q}^{-1}Y_{2,q^5}^{-1} = m_1+m_3+m_4+m_5,$$

here $m_5 = Y_{1,1}Y_{2,q}^{-1}Y_{2,q^5}^{-1}$ is 1-dominant.

(4). For $\varphi_1(m_2)$, we have the following procedure:

$$Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1} \mapsto Y_1Y_{q^2} \mapsto Y_1Y_{q^2}(1+A_{q^3}^{-1}+A_q^{-1}A_{q^3}^{-1}) \mapsto Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1}(1+A_{1,q^3}^{-1}+A_{1,q}^{-1}A_{1,q^3}^{-1}),$$

once expanded,

$$\varphi_1(m_2) = Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1} + Y_{1,1}Y_{1,q^4}^{-1}Y_{2,q^3}Y_{2,q}^{-1} + Y_{1,q}^{-1}Y_{1,q^4}^{-1}Y_{2,q^3} = m_2 + m_3 + m_6,$$

here $m_6 = Y_{1,q}^{-1}Y_{1,q^4}^{-1}Y_{2,q^3}$ is 2-dominant.

(5). With the same method,

$$\varphi_1(m_5) = Y_{1,1}Y_{2,q}^{-1}Y_{2,q^5}^{-1} + Y_{1,q^2}^{-1}Y_{2,q^5}^{-1} = m_5 + m_7,$$

$$\varphi_2(m_6) = Y_{1,q^2}^{-1}Y_{1,q^4}^{-1}Y_{2,q^3} + Y_{1,q^2}^{-1}Y_{2,q^5}^{-1} = m_6 + m_7.$$

As a summary,

$$m_0 = Y_{1,q^2}Y_{2,q^{-1}}, \quad m_1 = Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3}, \quad m_2 = Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1},$$

$$m_3 = Y_{1,1}Y_{1,q^4}^{-1}Y_{2,q^3}Y_{2,q}^{-1}, \quad m_4 = Y_{2,q^{-1}}Y_{2,q^5}^{-1}, \quad m_5 = Y_{1,1}Y_{2,q}^{-1}Y_{2,q^5}^{-1},$$

$$m_6 = Y_{1,q}^{-1}Y_{1,q^4}^{-1}Y_{2,q^3}, \quad m_7 = Y_{1,q^2}^{-1}Y_{2,q^5}^{-1}.$$

Then it is easy to compute that all $s(m_j) = 1$ for $j = 0, \cdots, 7$, and the Frenkel-Mukhin polynomial

$$\begin{aligned} FM(m) = Y_{1,q^2}Y_{2,q^{-1}} &+ Y_{1,q^4}^{-1}Y_{2,q^{-1}}Y_{2,q^3} + Y_{1,1}Y_{1,q^2}Y_{2,q}^{-1} + Y_{1,1}Y_{1,q^4}^{-1}Y_{2,q^3}Y_{2,q}^{-1} \\ &+ Y_{2,q^{-1}}Y_{2,q^5}^{-1} + Y_{1,1}Y_{2,q}^{-1}Y_{2,q^5}^{-1} + Y_{1,q}^{-1}Y_{1,q^4}^{-1}Y_{2,q^3} + Y_{1,q^2}^{-1}Y_{2,q^5}^{-1}. \end{aligned}$$

Relations with q-characters

In general, even been restricted to the simple modules, the Frenkel-Mukhin polynomial will not coincide with the q-character. But this is true for some cases we are interested in.

Definition 20. *Let $m \in \mathcal{M}_+$ and $L(m)$ be the simple module with the highest weight monomial m, we call $L(m)$ minuscule if m is the only dominant monomial in $\chi_q(L(m))$.*

Theorem 10 (Frenkel-Mukhin)**.** *For $m \in \mathcal{M}_+$ such that $L(m)$ is minuscule, then $\chi_q(L(m)) = FM(m)$. Moreover, all fundamental modules are minuscule.*

Theorem 11 (Nakajima)**.** *All Kirillov-Reshetikhin modules are minuscule.*

Example 15. *In the example of subsection 4.4, we calculated the Frenkel-Mukhin polynomial for $L = L(Y_{1,q^2}Y_{2,q^{-1}})$. As proved by D.Hernandez in the framework of minimal affinization, L is minuscule, so*

$$\chi_q(L) = FM(Y_{1,q^2}Y_{2,q^{-1}}).$$

5 Applications to monoidal categorification

This section is devoted to giving an application of q-characters and Frenkel-Mukhin algorithm in the monoidal categorification of cluster algebras.

For convenience, denote $V_{i,a}$ the fundamental representation $V_{\varpi_i}(a)$ defined in the last section.

For the accordance with notations in cluster algebras, denote $W_{n,a}^{(i)}$ be the Kirillov-Reshetikhin module given by $W_{n,a}^{(i)} = W_n^{(i)}(aq^{n-1})$. So the Drinfel'd polynomial for the Kirillov-Reshetikhin module $W_{n,a}$ in the case $\mathfrak{sl}_2$ is given by

$$P(z) = (1 - az)(1 - aq^2z) \cdots (1 - aq^{2n-2}z).$$

5.1 Categories with level

Definitions

Assume that $\mathfrak{g}$ is simply laced, it is to say, of type A, D, E. Then the Dynkin diagram of $\mathfrak{g}$ is a bipartite graph. A graph is called bipartite if it has an orientation such that each vertex is either sink or source.

Let $I = \{1, \cdots, n\}$ be the set of vertexes in Dynkin diagram, then I can be written into two disjoint subsets I_0 and I_1 such that each edge connects a vertex of I_0 with one in I_1. (For example, we take I_0 be the set of sources and I_1 sinks.)

For $i \in I_0$, define $\xi_i = 0$ and for $i \in I_1$, define $\xi_i = 1$. Denote $\varepsilon_i = (-1)^{\xi_i}$. Then $i \mapsto \xi_i$ is completely determined by a partition of I, we remark that there are only two possible partitions and the other one can be obtained by changing sinks and sources.

Definition 21. *(1). Define $\mathcal{C}$ be the category of finite dimensional representations of $U_q(\widehat{\mathfrak{g}})$.*
(2). Define $\mathcal{C}_{\mathbb{Z}}$ be the full subcategory of $\mathcal{C}$ containing objects V such that for any simple composition factor S of V and $i \in I$, roots of Drinfel'd polynomial for S belong to the set $\{q^{2k+\xi_i} |\ k \in \mathbb{Z}\}$.
(3). For $l \in \mathbb{N}$, define $\mathcal{C}_l$ be the full subcategory of $\mathcal{C}_{\mathbb{Z}}$ containing objects V such that for any simple composition factor S of V and $i \in I$, roots of Drinfel'd polynomial for S belong to the set $\{q^{-2k-\xi_i} |\ 0 \leq k \leq l\}$.

Remark 15. *It is easy to see that $\mathcal{C}_{\mathbb{Z}}$ is a monoidal subcategory of $\mathcal{C}$ and the Grothendieck ring $R_{\mathbb{Z}}$ of $\mathcal{C}_{\mathbb{Z}}$ is generated by isomorphism classes $[V_{i,q^{2k+\xi_i}}]$ ($i \in I$, $k \in \mathbb{Z}$) as a subring of R-the Grothendieck ring of $\mathcal{C}$.*

Proposition 13. *For $l \in \mathbb{N}$, $\mathcal{C}_l$ is a monoidal category, with Grothendieck ring the polynomial ring generated by isomorphism classes $[V_{i,q^{2k+\xi_i}}]$ for $0 \leq k \leq l$.*

5.2 Examples

In this section, we concentrate on the case $\mathfrak{sl}_2$, the Dynkin diagram of $\mathfrak{sl}_2$ has only one vertex and no edge.

Category $\mathcal{C}_0$

At first, we consider the category $\mathcal{C}_0$, from the definition, the only possible evaluation representations appear in it are $W_{0,1}$ and $W_{1,1}$. The q-character of them are:

$$\chi_q(W_{0,1}) = Y_1, \quad \chi_q(W_{1,1}) = Y_1 + Y_{q^2}^{-1}.$$

The Grothendieck ring R_0 is generated by $[W_{0,a}]$ and $[W_{1,1}]$. The representation $W_{0,1}$ is trivial so of no interest. For $W_{1,1}$, it is easy to calculate

$$\chi_q(W_{1,1} \otimes W_{1,1}) = Y_1^2 + 2Y_1 Y_{q^2}^{-1} + Y_{q^2}^{-2},$$

it means that $W_{1,1} \otimes W_{1,1}$ is irreducible.
Conclusion. In $\mathcal{C}_0$, tensor products of irreducible modules are also irreducible. This could be also seen from the fact that the admissible q-chains are too short to form a special position.

Category $\mathcal{C}_1$

From definition, $l = 1$ means that the inverse roots of the Drinfel'd polynomial must be 1 or q^2, so $\mathcal{C}_1$ contains three evaluation representations: $W_{0,1}$, $W_{1,1}$ and $W_{2,1}$. The q-character of $W_{2,1}$ is:

$$\chi_q(W_{2,1}) = Y_1 Y_{q^2} + Y_1 Y_{q^4}^{-1} + Y_{q^2}^{-1} Y_{q^4}^{-1}.$$

So the dominant monomials appear in q-characters of elements in $\mathcal{C}_1$ contain no variables with parameter except 1, q and q^2.

So if we define $\chi_q(V)_{\leq 2}$ by keeping monomials in $\chi_q(V)$ having only variables with parameters 1, q and q^2 and sending another ones to 0, this is a ring morphism from R_1 to $\mathbb{Z}[Y_1^{\pm 1}, Y_q^{\pm 1}, Y_{q^2}^{\pm 1}]$. Moreover, it is injective, as we explained above. This is why the truncated q-character works well for category $\mathcal{C}_1$ in the framework of Hernandez-Leclerc.

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