On the Graded Injectivity of the Conze Embedding

Anthony Joseph (Weizmann Institute)

Let g be a complex semisimple Lie algebra with U(g) its enveloping algebra. Let p be a parabolic subalgebra of g and m the image of the nilradical of p under a Chevalley antiautomorphism. Let M be a generalized Verma module, that is to say a U(g) module obtained by induction from a one dimensional p module. Let F(M) (resp. A(M)) denote the subspace of locally finite elements of End M with respect to the diagonal action of g (resp. of m). The embedding of U(g)/Ann M into A(M) is called the Conze embedding. One may identify A(M) with the algebra of differential operators on m* and as such it inherits a g invariant filtration through the usual degree of a polynomial on m*.

The main result is that gr A(M) is an injective module in the appropriate version of the Bernstein-Gelfand-Gelfand (BGG) category. The proof is accomplished in three stages. First it is shown that A(M) is injective, whenever M is a simple module. Secondly using a Koszul resolution and spectral sequence argument (following B. Broer) together with some properties of the BGG category, the assertion is established for p being a Borel subalgebra. This involves some combinatorics which do not extend to the parabolic case. It implies that the isomorphism class of A(M) is independent of M and this is combined with the first result to establish the general case. Finally the multiplicity of each injective in A(M) is computed.

The main application of graded injectivity is to compute the multiplicity of a simple module in each filtration degree of F(M). Furthermore if the parameters which define M are taken to be independent variables (which then generate the centre Z(M) of F(M)) graded injectivity implies that F(M) is free over Z(M). This allows one to define and compute the so-called KPRV determinants.

Generically the map of U(g) into F(M), defined by its action on M, is surjective and moreover this holds for all parameters if the appropriate moment map is bijective with normal closure (through a result of Borho-Brylinski or directly). A description of the precise failure of surjectivity in general is an open problem.