Non-commutative desingularizations and partial resolutions

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We start from the (partial) resolution of singularities

$$G - Hilb \longmapsto \mathbb{C}^n/G$$

of the quotient singularity by means of the Hilbert scheme of G-clusters and generalize this to the following setting. Let (Q,R) be a quiver Q with relations R and let α be a dimension vector of a simple representation of (Q,R). We assume that there is a stability structure θ such that the set $\mathtt{rep}_\alpha^\theta(Q,R)$ of all θ-semistable representations of (Q,R) is a smooth variety and consider the projective morphism

$$\mathtt{moduli}_\alpha^\theta(Q,R) \longmapsto \mathtt{iss}_\alpha(Q,R)$$

from the moduli space of θ-semistable representations to the quotient variety of semi-simple representations of (Q,R). In the above case, (Q,α) is the McKay quiver-setting, R the set of commuting matrix-relations and θ a suitable stability structure. We show that there is a sheaf of smooth non-commutative algebras $\mathcal{A}$ over $\mathtt{moduli}_\alpha^\theta(Q,R)$ such that its non-commutative variety

$$\mathtt{spec}\ \mathcal{A} \longmapsto \mathtt{moduli}_\alpha^\theta(Q,R) \longmapsto \mathtt{iss}_\alpha(Q,R)$$

can be viewed as a non-commutative desingularization of $\mathtt{iss}_\alpha(Q,R)$. However, the central scheme $\mathtt{moduli}_\alpha^\theta(Q,R)$ may still have singularities. We classify the types of singularities with can occur for a given dimension n of $\mathtt{iss}_\alpha(Q,R)$. If $n=2$ there are none, if $n=3$ there is just one type (the conifold), for $n=4$ there are three, for $n=5$ ten etc.