Lectures on the moduli stack of vector bundles on a curve

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Abstract. These are lecture notes of a short course on the moduli stack of vector bundles on an algebraic curve. The aim of the course was to use this example to introduce the notion of algebraic stacks and to illustrate how one can work with these objects. Applications given are the (non)-existence of universal families on coarse moduli spaces and the computation of the cohomology of the moduli stack.

Introduction

This text consists of my notes for a course on algebraic stacks given at the German-Spanish Workshop on vector bundles on algebraic curves in Essen and Madrid 2007, organized by L. Álvarez-Cónsul, O. García-Prada, and A. Schmitt. The aim of the course was to use the example of vector bundles on curves to introduce the basic notions of algebraic stacks and to illustrate how one can work with these objects.

The course consisted of 5 one-hour lectures and was meant to be introductory. We start with the definition of stacks. In order to get to some interesting applications we chose to give only those parts of the basic theory that are needed in our applications. Also we often chose to give ad hoc definitions, whenever these are easier to digest than the abstract ones. The time constraints had the side effect that I deliberately skipped some of the technical fine print in the first lectures. On the one hand I hope that this makes the subject more accessible, because the ideas are most of the time not so difficult to understand. On the other hand for written notes I feel that some of the fine print should at least be indicated. Since there are excellent references for these points available I will try to include some comments indicating where one can find more information. In order not to distract the reader who wants to get a first idea what the subject is about, I will put these comments in fine print.

Finally, I gave a set of exercises for the course, these are included in the text.

The structure of the lectures was as follows: The first lecture gives the definition of an algebraic stack. The second lecture explaines why geometric notions make sense for such stacks and introduce sheaves on stacks. As an example on how to work with these we begin Lecture 3 by proving the innocuous technical result of Laumon–Moret-Bailly that on a noetherian stack any quasi-coherent sheaf is the limit of its coherent subsheaves. We then consider the relation with coarse moduli spaces. We introduce the notion of a gerbe in order to give a simple proof of the classical result on the non existence of a universal family of vector bundles on coarse moduli spaces of bundles when rank and degree are not coprime. This uses the theorem proven in the beginning of the lecture. In the 4th lecture we introduce cohomology of constructible sheaves and, as an example we indicate how one can compute the cohomology ring of the moduli stack of vector bundles on a curve. In the last lecture we give some ideas how one can deduce results on the cohomology of the coarse moduli spaces.

Of course this material is not original. The basic results on algebraic stacks are explained in the book of Laumon and Moret-Bailly [20], the standard reference on the subject. The results on $\mathbb{G}_m$-gerbes can be found in Lieblich's thesis [22], the application to the non-existence of universal families has been explained in great generality by Biswas and Hoffmann [8]. The calculation of the cohomology ring of the moduli stack owes much to the classical work of Atiyah-Bott [2]. The reformulation in terms of stacks was the subject G. Harder suggested as subject of my Diploma-thesis. However we use Beauville's trick here in order to avoid the usage of the Lefschetz-trace formula. A more general result is proven in [19].

Also, by now there are several very good introductory notes on the basic definitions on stacks available (for example [11],[15]), each giving an introduction form a different point of view.

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1. Lecture: Algebraic stacks

1.1. Motivation and definition

The main motivation to introduce algebraic stacks is that we want to study moduli spaces. One of the simplest problems would be to look for a classifying space for vector bundles, say of rank n. So we would like to have a space $B\mathsf{GL}_n$ such that for any test scheme T

$$\mathrm{Mor}(T, B\mathsf{GL}_n) = \langle \text{vector bundles of rank } n \text{ on } T\rangle/\text{isomorphism}. \qquad (1.1)$$

This would be nice, because to construct functorial invariants, like Chern classes it would then be sufficient to construct cohomology classes of $B\mathsf{GL}_n$.

However, such a space cannot exist, because every vector bundle $\mathcal{E}$ on T is locally trivial, so the map $T \to B\mathsf{GL}_n$ corresponding to $\mathcal{E}$ would have to be locally constant so $\mathcal{E}$ would have to be trivial globally.

This is a bit disappointing, since topologists do know a classifying space $B\mathsf{GL}_n$, such that for any space T the homotopy classes of maps $f : T \to B\mathsf{GL}_n$ correspond to isomorphism classes of vector bundles on T.

Since we do not have a good algebraic replacement for homotopy classes of maps, we have (at least) two other ways to circumvent the problem:

1. (see Georg Hein's lectures) Restrict the functor and the expectations on the representing space, i.e. consider coarse moduli spaces.
2. Don't pass to isomorphism classes in (1.1)!

The second option is used for the definition of stacks. Before giving the definition, recall that the Yoneda-Lemma (valid in any category) tells us that any scheme X is determined by its functor of points, i.e. X is determined by the functor

$$\mathrm{Mor}(\_\_, X) : \mathrm{Schemes} \to \mathrm{Sets}$$

sending a scheme T to the set $\mathrm{Mor}(T, X)$. Furthermore this functor $\mathrm{Mor}(\_\_, X)$ is a sheaf, in the sense that a morphism $T \to X$ can be obtained from glueing morphisms on a covering of T.

The definition of a stack follows this idea, we first define a stack to be given by its functor of points. So in the case of $B\mathsf{GL}_n$ we just define $B\mathsf{GL}_n(T)$ to be the category of vector bundles of rank n on T. Vector bundles can also be obtained by glueing bundles on an open covering, so the general definition of a stack will be that it is a sheaf of categories (more precisely a sheaf of groupoids). In writing down the axioms that should be satisfied by such an assignment we keep the example $B\mathsf{GL}_n$ in mind.

In a second step we will then try to see how one can do geometry using such objects.

Definition 1.1. A *stack* is a sheaf of groupoids:

$$\mathcal{M} : \mathrm{Sch} \to \mathrm{Groupoids} \subset \mathrm{Categories}$$

i.e., an assignment

1. for any scheme T a category $\mathcal{M}(T)$ in which all morphisms are isomorphisms.
2. for any morphism $f : X \to Y$ a functor $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$.
3. for any pair of composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ a natural transformation $\phi_{f,g} : f^* \circ g^* \Rightarrow (g \circ f)^*$. These transformations have to be associative for composition, in particular we assume this transformation to be the identity if one of the morphisms is the identity.

satisfying the following gluing conditions:

1. (Objects glue) Given a covering[1] $U_i \twoheadrightarrow T$, objects $\mathcal{E}_i \in \mathcal{M}(U_i)$ and isomorphisms $\phi_{ij} : \mathcal{E}_i|_{U_i \cap U_j} \to \mathcal{E}_j|_{U_i \cap U_j}$ satisfying a cocycle condition on 3-fold intersections, there exists an object $\mathcal{E} \in \mathcal{M}(T)$, unique up to isomorphism together with isomorphisms $\psi_i : \mathcal{E}|_{U_i} \to \mathcal{E}_i$ such that $\phi_{ij} = \psi_j \circ \psi_i^{-1}$.
2. (Morphisms glue) Given a covering $U_i \twoheadrightarrow T$, objects $\mathcal{E}, \mathcal{F} \in \mathcal{M}(T)$ and morphisms $\phi_i : \mathcal{E}|_{U_i} \to \mathcal{F}|_{U_i}$ such that $\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}$ then there is a unique morphism $\phi : \mathcal{E} \to \mathcal{F}$ such that $\phi|_{U_i} = \phi_i$.

Remark 1.2. In the definition we used the word *covering* to mean one of the following choices: In complex geometry we use the analytic topology. Otherwise we mean a covering in either the étale topology or the *fppf* topology (i.e. a surjective map $U_i \to X$ which is étale or flat of finite presentation respectively). In this case the intersection $U_i \cap U_j$ has to be defined as $U_i \cap U_j := U_i \times_X U_j$.

Finally, the notation $\mathcal{E}|_{U_i}$ means the pull back of $\mathcal{E}$ to U_i given by the map $U_i \to X$.

Actually to make the above definition precise, we would need to spell out the canonical isomorphisms of $\mathcal{E}|_{U_i}|_{U_i \cap U_j} \to \mathcal{E}|_{U_j}|_{U_i \cap U_j}$ given by the last part of the bi-functor. This would make the definition more difficult to read.

Also it is sometimes not quite obvious how to define all pull back functors functorially. In fact one often has to make a choice here. This can be avoided using the language of fibred categories as in [20]. This means that similarly to the procedure of replacing a sheaf by its espace étalé one considers the large category $\coprod_{T \in \mathrm{Sch}} \mathcal{M}(T)$ instead of the different $\mathcal{M}(T)$. This has other advantages (see the last remark in [16] VI) but to me the above definition seems easier to digest at first.

Let us collect some examples

Example 1.3. Let C be a smooth projective curve (a general scheme which is flat of finite type over the base would do here). Then let $\mathrm{Bun}_{n,C}$ be the stack given by[2]

$$\mathrm{Bun}_{n,C}(T) := \langle \text{vector bundles of rank } n \text{ on } C \times T \rangle.$$

Here the morphisms in the category are isomorphisms of vector bundles and the functors f^* are given by the pull-back of bundles. The gluing conditions are satisfied, by descent for vector bundles ([16] Exposé VIII, Théorème 1.1 and Proposition 1.10).

Similarly Coh_C is the stack of coherent sheaves on C:

$$\mathrm{Coh}_C(T) := \langle \text{coherent sheaves on } C \times T \text{ flat over T} \rangle$$

Example 1.4. Let G be an affine algebraic group, then we denote by

$$BG(T) := \langle \text{principal } G - \text{bundles on } T \rangle$$

the classifying stack of G.

Example 1.5. Let X be a scheme. Then $\underline{X}(T) := \mathrm{Mor}(T, X)$ is a stack. Here we consider the set $\mathrm{Mor}(T, X)$ as a category in which the only morphisms are

[1] see the remark following the definition
[2] We denote categories by $\langle\rangle$ to distinguish them from sets.

identities, the pull-back functors f^* for $f : S \to T$ being given by composition with f. Such a stack is called a *representable stack*.

Example 1.6. (Quotient stacks) Let X be a scheme (say over some field k in order to avoid a flatness condition in what follows) and G be an algebraic group acting on X. Then we define the quotient stack $[X/G]$ by

$$[X/G](T) := \left\langle \begin{array}{c} P \overset{g}{\to} X \\ \downarrow p \\ T \end{array} \middle| \begin{array}{c} P \to T \text{ is a } G \text{ bundle} \\ P \to X \text{ is a } G\text{-equivariant map} \end{array} \right\rangle$$

Morphisms in this category are isomorphisms of G-bundles commuting with the map to X.

To check that this definition makes sense let us consider the case that there exist a quotient X/G of X by G such that the map $X \to X/G$ is a $G-$bundle. In this case any diagram $\begin{array}{ccc} P & \overset{g}{\to} & X \\ \downarrow p & & \downarrow \\ T & & X/G \end{array}$ defines a map $\overline{g} : T \to X/G$ and in this way P becomes canonically isomorphic to the pull-back of the G-bundle $\overline{g}^*X = X \times_{X/G} T$ over T. So in this case the category $[X/G](T)$ is canonically equivalent to the set $X/G(T)$, which we consider as a category in which the only morphisms are the identities of elements.

Remark 1.7. Stacks form a $2-$category. Morphisms of stacks are given by functors between the corresponding categories: A morphism $F : \mathcal{M} \to \mathcal{N}$ is given by a collection of functors $F_T : \mathcal{M}(T) \to \mathcal{N}(T)$ for all T together with for every $f : X \to Y$ a natural transformation $F_f : F_X \circ f^* \to f^* \circ F_Y$ satisfying an associativity constraint.

A 2-morphism is a morphism between such functors F, G. It is given by a natural transformation $\phi : F_T \to G_T$ for all T, compatible with the pull-back functors f^*.

Example 1.8. 1. The tensor product defines a morphism $\mathrm{Bun}_{n,C} \times \mathrm{Bun}_{m,C} \to \mathrm{Bun}_{nm,C}$ sending a pair of vector bundles $\mathcal{E}, \mathcal{F}$ to $\mathcal{E} \otimes \mathcal{F}$.

2. An example of a $2-$morphism is as follows. Consider the identity functor $id : \mathrm{Bun}_{n,C} \to \mathrm{Bun}_{n,C}$. Fix an invertible scalar α. A 2-morphism from $id \to id$ is then given by multiplication by α on all objects.

The first important observation is that the examples of stacks given above are indeed moduli-spaces:

Lemma 1.9 (Yoneda lemma for stacks). *Let $\mathcal{M}$ be a stack. Then for any scheme T there is a natural equivalence of categories:*

$$\mathrm{Mor}_{Stacks}(\underline{T}, \mathcal{M}) \cong \mathcal{M}(T).$$

Proof. First note that we can define a functor:

$$\mathrm{Mor}_{\mathrm{Stacks}}(\underline{T}, \mathcal{M}) \to \mathcal{M}(T)$$

by sending $F : \underline{T} \to \mathcal{M}$ to $F(id_T) \in \mathcal{M}(T)$.

Conversely, given an object $\mathcal{E} \in \mathcal{M}(T)$ we can define a morphism $F_{\mathcal{E}} : \underline{T} \to \mathcal{M}$ by sending $f \in T(S) = \mathrm{Mor}(S,T)$ to $f^*(\mathcal{E}) \in \mathcal{M}(S)$.

Note that the composition $\mathcal{E} \mapsto F_{\mathcal{E}} \mapsto F_{\mathcal{E}}(id_T) = id^*\mathcal{E} = \mathcal{E}$ is the identity.

Conversely, let us compute the composition $F \mapsto F(id_T) \mapsto F_{F(id_T)}$. We have $F_{F(id_T)}(f : S \to T) = f^*(F(id_T))$. But $F_f : F(f : S \to T) \to f^*(F(id_T))$ then gives a natural isomorphism. □

Because of this lemma we will often simply write T instead of $\underline{T}$.

1.2. How to make this geometric?

In order to make sense of geometric notions for stacks, we look for a notion of charts for an algebraic stack. To see why this could make sense let us begin by computing a fibre product in a simple example:

Take G a smooth group and consider our stack BG, classifying G-bundles. Let $pt = \mathrm{Spec}\, k$ be a point, and $\mathcal{E}$ be a G-bundle on some other scheme X. By the Yoneda lemma, $\mathcal{E}$ defines a morphism $F_{\mathcal{E}} : \underline{X} \to BG$ and the trivial bundle defines a morphism $triv : pt \to BG$:

$$\begin{array}{ccc} & & pt \\ & & \downarrow{\scriptstyle triv} \\ X & \xrightarrow{F_{\mathcal{E}}} & BG \end{array} .$$

We want to compute the fibre product of this diagram. For any scheme T this is given by:

$$\begin{aligned} X \times_{BG} pt(T) &= \left\langle \begin{array}{ccc} T & \xrightarrow{p} & pt \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle triv} \\ X & \xrightarrow{F_{\mathcal{E}}} & BG \end{array} ; \phi : triv \circ p \xrightarrow{\cong} F_{\mathcal{E}} \circ f \right\rangle \\ &= \langle (f,p,\phi) | \phi : f^*\mathcal{E} \cong p^*(triv) = T \times G \rangle \\ &= \{(f,s) | s : T \to f^*\mathcal{E} \text{ a section}\} \\ &= \mathcal{E}(T). \end{aligned}$$

Thus the T-valued points of the fibre product are a set and not only a category and the resulting stack is equivalent to the G-bundle $\mathcal{E}$. This means:

1. For every $F_{\mathcal{E}} : X \to BG$ the pull back of the morphism $pt \to BG$ is the G-bundle $\mathcal{E}$, so $pt \to BG$ is the universal G-bundle on BG!
2. The map $pt \to BG$ becomes a smooth surjection after every base-change.

The second point means that we should regard the map $pt \to BG$ as a smooth covering of BG so we could consider it as an atlas for BG. The existence of such a map will be the main part of the definition of algebraic stacks.

More generally, let $\mathcal{M}$ be any stack and $x : X \to \mathcal{M}, y : Y \to \mathcal{M}$ be two morphisms. Then for any scheme T:

$$X \times_{\mathcal{M}} Y(T) = \left\langle \begin{array}{c} \nearrow^{f} X \\ T \\ \searrow_{g} Y \end{array}, \phi : f^*x \cong g^*y \right\rangle =: \mathrm{Isom}(x, y)$$

is a sheaf and in all the examples we have seen so far it is even represented by a scheme.

Definition 1.10. A stack $\mathcal{M}$ is called *algebraic* if

1. For all $X \to \mathcal{M}, Y \to \mathcal{M}$ the fibre product $X \times_{\mathcal{M}} Y$ is representable.
2. There exists a scheme $u : U \to \mathcal{M}$ such that for all schemes $X \to \mathcal{M}$ the projection $X \times_{\mathcal{M}} U \to X$ is a smooth surjection.
3. The forgetful map $\mathrm{Isom}(u, u) = U \times_{\mathcal{M}} U \to U \times U$ is quasicompact and separated.

Remark 1.11. The last condition is a technical condition. It implies that $\mathrm{Isom}(x, y)$ is always separated. In particular we are not allowed to consider non-separated group schemes and it also rules out quotients like $[\mathbb{A}^1_{\mathbb{C}}/\mathbb{Q}^{discrete}]$.

We call a map $u : U \to \mathcal{M}$ as in 2. an atlas of $\mathcal{M}$.

Remark 1.12. There is a second technical problem. In order to get a definition in which algebraicity of a stack can be checked by deformation-theoretic conditions it is more natural to replace the requirement that $X \times_{\mathcal{M}} Y$ is a scheme by the weaker condition that it is an algebraic space. Once one is used to algebraic stacks this will not be a difficult concept, because the definition is exactly the same as the above, if one adds the condition that the stack $\mathcal{M}$ is actually a sheaf, i.e. that all $\mathcal{M}(T)$ are sets. In this context the last technical condition is then needed to make the condition on fibred products to be schemes to be reasonable.

Example 1.13. We have just seen that the stack BG is algebraic. Analogously quotient stacks $[X/G]$ are algebraic, the canonical map $X \to [X/G]$ given by the trivial G-bundle $G \times X \to X$ is an atlas.

The most important example in this course will be the following:

Example 1.14. Let C be a smooth projective curve. And denote by Bun_n the stack of vector bundles of rank n on C. This is an algebraic stack. We know that for two bundles $\mathcal{E}, \mathcal{F}$ on $C \times X$ and $C \times Y$ the sheaf $\mathrm{Isom}(\mathcal{E}, \mathcal{F}) \subset \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is an open subscheme. And $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \to X \times Y$ is affine.

An atlas of Bun_n is given as follows: Choose an ample bundle $\mathcal{O}(1)$ on C:

$$U := \coprod_{N \in \mathbb{N}} \left\langle (\mathcal{E}, s_i) \middle| \begin{array}{c} \mathcal{E} \text{ a bundle on C such that } \mathcal{E} \otimes \mathcal{O}_C(N) \\ \text{globally generated, } H^1(C, \mathcal{E} \otimes \mathcal{O}_C(N)) = 0 \\ s_i \text{ a basis of } H^0(C, \mathcal{E} \otimes \mathcal{O}_C(N)) \end{array} \right\rangle$$

This is a representable functor by the theory of Hilbert- (or Quot-)schemes.

The condition that fibre products are representable gets a name:

Definition 1.15. A morphism $F : \mathcal{M} \to \mathcal{N}$ of stacks is called *representable* if for all $X \to \mathcal{N}$ the fibre product $X \times_{\mathcal{N}} \mathcal{M}$ is representable.

As before, in this definition one should again use algebraic spaces to define representability. This will not make a difference in our examples.

Example 1.16. The standard example for a non-representable morphism is the projection $BG \to pt$. More generally it is not difficult to check that representable morphisms induce injections on the automorphism groups of objects. This condition is actually a sufficent condition for morphisms between algebraic stacks, if one takes the above fine print on the notion of representability into account, i.e. if one uses the larger category of algebraic spaces instead of schemes as representable stacks.

Exercise 1.17. Show that the property of a morphism to be representable is stable under pull-backs.

Exercise 1.18. Show that the fibre product $X \times_{\mathcal{M}} Y$ is representable for all X, Y if and only if the diagonal morphism $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is representable.

The main idea - which will be explained in the next lecture - is that this concept of algebraic stacks allows to translate every notion for schemes that can be checked on a smooth covering into a notion for stacks. For example smoothness, closed and open substacks. But also sheaves and cohomology, we will simply define sheaves to be sheaves on one atlas together with a descent datum to $\mathcal{M}$.

Remark 1.19. One might wonder why one does not replace "smooth" by "flat" in the definition of algebraic stacks. The reason for this is a theorem of Artin ([1] Thm. 6.1), which says that this would not give a more general notion!

Exercise 1.20 (2-Fibred Products). For any stack $\mathcal{M}$ one defines its inertia stack $I(\mathcal{M})$ as

$$I(\mathcal{M})(T) \cong \langle (t, \phi) | t \in \mathcal{M}(T), \phi \in Aut(t) \rangle.$$

Show that $I(\mathcal{M}) \cong \mathcal{M} \times_{\mathcal{M} \times \mathcal{M}} \mathcal{M}$ where the map $\mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is the diagonal.

Exercise 1.21. Given G an algebraic group and $H \subset G$ a closed subgroup we consider the canonical map $BH \to BG$ mapping any H-bundle to the induced G-bundle. Show that there is a (2-)cartesian diagram:

$$\begin{array}{ccc} G/H & \longrightarrow & BH \\ \downarrow & & \downarrow \\ pt & \longrightarrow & BG. \end{array}$$

If $H \subset G$ is normal, then there is a (2-)cartesian diagram

$$\begin{array}{ccc} BH & \longrightarrow & BG \\ \downarrow & & \downarrow \\ pt & \longrightarrow & B(G/H). \end{array}$$

(Of course these diagrams are well-known in topology as homotopy-fibrations.)

Exercise 1.22 (Points of a stack). Let $\mathcal{M}$ be an algebraic stack. We define its set of points $|\mathcal{M}|$ as the union

$$(\amalg_{k \subset K \text{ a field}} \text{Objects}(\mathcal{M}(K)))/\sim$$

where $\sim$ declares $x \in \mathcal{M}(K)$ equivalent to $y \in \mathcal{M}(K')$ if there is a field extension K'' containing K and K' such that $x|_{K''} \cong y|_{K''}$.

Let $X \to \mathcal{M}$ be an atlas of $\mathcal{M}$. Show that $|\mathcal{M}| = |X|/\sim_{\mathcal{M}}$ where $\sim_{\mathcal{M}}$ is the equivalence relation defined by $|X \times_{\mathcal{M}} X| \to |X| \times |X|$.

(You might also want to define a Zariski-topology on $|\mathcal{M}|$.)

2. Lecture: Geometric properties of algebraic stacks

2.1. Properties of stacks and morphisms

Recall that the essential point in the definition of algebraicity of a stack $\mathcal{M}$ is the existence of a smooth surjection $u : U \twoheadrightarrow \mathcal{M}$ from a scheme U to $\mathcal{M}$. Let us use this to define some first geometric properties of algebraic stacks:

Definition 2.1. An algebraic stack $\mathcal{M}$ is called *smooth (resp. normal/reduced/locally of finite presentation/locally noetherian/regular)* if there exists an atlas $u : U \twoheadrightarrow \mathcal{M}$ with U being smooth (resp. normal/reduced/locally of finite presentation/locally noetherian/regular).

Note that for schemes this definition gives nothing new, because all the above properties can be checked locally on a smooth covering of a scheme.

Similarly properties of morphisms which can be checked after a smooth base change extend to properties of representable morphisms of algebraic stacks:

Definition 2.2. Let P be a property of morphisms of schemes $f : X \to Y$ such that f has P if and only if for some smooth surjective $Y' \to Y$ the induced morphism $f' : X \times_Y Y' \to Y'$ has P (e.g. closed immersion, open immersion, affine, finite, proper).

We say that *a representable morphism* $F : \mathcal{M} \to \mathcal{N}$ of algebraic stacks *has property* P if for some (equivalently any) atlas $u : U \twoheadrightarrow \mathcal{N}$ the morphism $\mathcal{M} \times_{\mathcal{N}} U \to U$ has P.

Remark 2.3. In particular the above definition gives us a notion of closed and open substacks of an algebraic stack.

Let us give some examples:

1. If $\mathcal{M} = [X/G]$ is a quotient stack, then open/closed substacks are of the form $[Y/G]$ where $Y \subset X$ is an open/closed subscheme.
2. In our example Bun_n, the stack of vector bundles on a projective curve, the substack Bun_n^{ss} of semistable vector bundles[3] is an open substack, because for any family of bundles, instability is a closed condition.

Finally we can also define properties of arbitrary morphisms of stacks as long as we can check these properties locally in the source and the image of a morphism:

Definition 2.4. Let P be a property of morphisms of schemes $f : X \to Y$ such that f has P if and only if there exists some commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\tilde{f}} & Y' \\ \downarrow{\scriptstyle smooth} & & \downarrow{\scriptstyle smooth} \\ X & \xrightarrow{f} & Y \end{array}$$

such that $\tilde{f}$ has P. For example being smooth, flat, locally of finite presentation.

Then *a morphism* of algebraic stacks $F : \mathcal{M} \to \mathcal{N}$ *has* P if for some atlases $v : V \to \mathcal{M}$, $u : U \to \mathcal{N}$ there exists a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\tilde{f}} & U \\ \downarrow{\scriptstyle smooth} & & \downarrow{\scriptstyle smooth} \\ \mathcal{M} & \xrightarrow{F} & \mathcal{N} \end{array}$$

such that $\tilde{f}$ has P.

Example 2.5. In geometric invariant theory one often encounters the situation that one has an action of GL_n on a scheme X, such that the center of GL_n acts trivially, i.e. the action actually factors through an action of the group PGL_n on X. In this case the canonical morphism $[X/\mathsf{GL}_n] \to [X/\mathsf{PGL}_n]$ is smooth and surjective, but not representable.

Finally, we claim that our main example of an algebraic stack Bun_n is a smooth stack. One way to do this would be to directly apply the definition above to the atlas given in the last lecture and try to check that this atlas is smooth. However there is an intrinsic way to show smoothness, avoiding the choice of an atlas. This is as follows:

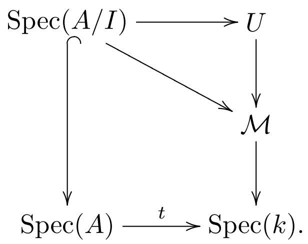
Recall the lifting criterion for smoothness: A morphism of schemes $f : X \to Y$ is smooth if and only if f is locally of finite presentation and for all (local) Artin algebras A with an ideal $I \subset A$ with $I^2 = (0)$ one can complete any diagram:

$$\begin{array}{ccc} \mathrm{Spec}(A/I) & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ \mathrm{Spec}(A) & \longrightarrow & Y. \end{array}$$

[3]Recall that a vector bundle $\mathcal{E}$ on a curve is called semistable if for all subbundles $\mathcal{F} \subset \mathcal{E}$ we have $\frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} \leq \frac{\deg(\mathcal{E})}{\mathrm{rk}(\mathcal{E})}$.

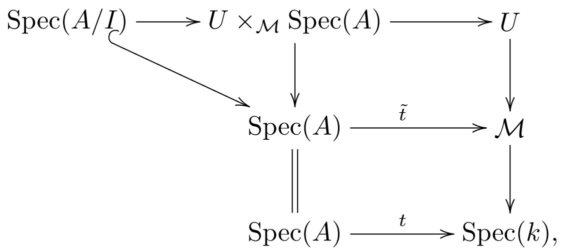
Lemma 2.6. *Let $\mathcal{M}$ be an algebraic stack, locally of finite presentation over* $\mathrm{Spec}(k)$ *such that the structure morphism* $\mathcal{M} \to \mathrm{Spec}(k)$ *satisfies the above lifting criterion for smoothness. Then $\mathcal{M}$ is smooth.*

Proof. Let $u : U \twoheadrightarrow \mathcal{M}$ be an atlas. We have to show that U is smooth, i.e., that the lifting criterion holds for $U \to \mathrm{Spec}(k)$. So assume that we are given:



Since $\mathcal{M} \to \mathrm{Spec}(k)$ satisfies the lifting criterion by assumption, we can lift t to $\tilde{t} : \mathrm{Spec}(A) \to \mathcal{M}$.

Knowing that u is smooth, implies that the projection $U \times_{\mathcal{M}} \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ is a smooth morphism of schemes. So we can also find a lifting in:



which proves our claim. □

Let us apply this to show that Bun_n is smooth: Giving a morphism $\mathrm{Spec}(A/I) \to \mathrm{Bun}_n$ is the same as giving a family of vector bundles $\overline{\mathcal{E}}$ on $C \times \mathrm{Spec}(A/I)$. We have to check that we can extend any such family to a vector bundle $\mathcal{E}$ on $C \times \mathrm{Spec}(A)$. Denote by $\mathfrak{m}$ the maximal ideal of A and $k = A/\mathfrak{m}$. By induction we can assume that $I = (\nu)$ is generated by one element and that $\nu \cdot \mathfrak{m} = 0$. Denote by $\mathcal{E}_0 := \mathcal{E} \otimes_{A/I} k$.

One way to see that an extension $\mathcal{E}$ exists is to describe $\overline{\mathcal{E}}$ by gluing cocycles and to lift these cocycles. (The lack of the cocycle condition for the lifted elements gives an element in $H^2(C, \mathcal{E}\mathrm{nd}(\mathcal{E}_0)) \otimes_k I$. This group is zero, because C is 1-dimensional.) Again one can avoid cocycles here. First, since we assumed that $I = (\nu)$ is generated by one element and that $\nu \cdot \mathfrak{m} = 0$ we have an exact sequence of A/I-modules:

$$0 \to k \xrightarrow{\nu \cdot} \mathfrak{m} \to A/I \to k = A/\mathfrak{m} \to 0 \tag{2.1}$$

Claim 2.7. $\overline{\mathcal{E}}$ extends to a bundle $\mathcal{E}$ on $C \times \mathrm{Spec}(A)$ if and only if the class

$$obs(\overline{\mathcal{E}}) := (2.1) \otimes_{A/I} \overline{\mathcal{E}} \in \mathrm{Ext}^2(\mathcal{E}_0, \mathcal{E}_0)$$

vanishes.

Of course this condition is automatic here, because $\mathrm{Ext}^2(\mathcal{E}_0, \mathcal{E}_0) = 0$, on a curve C.

Proof. Let us decompose the sequence 2.1 into two short exact sequences: Denote by $\mathfrak{m}_{A/I}$ the maximal ideal of A/I.

$$0 \to k \to \mathfrak{m} \to \mathfrak{m}_{A/I} \to 0$$

$$0 \to \mathfrak{m}_{A/I} \to A/I \to k \to 0$$

Tensoring the second sequence with $\overline{\mathcal{E}}$ we get a short exact sequence $0 \to \mathfrak{m}_{A/I} \otimes \overline{\mathcal{E}} \to \overline{\mathcal{E}} \to \mathcal{E}_0 \to 0$. This defines a long exact sequence:

$$\cdots \to \mathrm{Ext}^1(\overline{\mathcal{E}}, \mathcal{E}_0) \to \mathrm{Ext}^1(\mathfrak{m}_{A/I} \otimes \overline{\mathcal{E}}, \mathcal{E}_0) \xrightarrow{\partial} \mathrm{Ext}^2(\mathcal{E}_0, \mathcal{E}_0) \to \cdots .$$

The class $\mathrm{obs}(\overline{\mathcal{E}})$ is then (by definition) given by the image of $0 \to \mathcal{E}_0 \to \mathfrak{m} \otimes \overline{\mathcal{E}} \to \mathfrak{m}_{A/I} \otimes \overline{\mathcal{E}} \to 0$ under the boundary map ∂.

Assume first that $\mathcal{E}$ exists. Then we can tensor the diagram

$$\begin{array}{ccccc} k & \longrightarrow & \mathfrak{m} & \longrightarrow & \mathfrak{m}_{A/I} \\ \downarrow & & \downarrow & & \downarrow \\ k & \longrightarrow & A & \longrightarrow & A/I \end{array}$$

with $\mathcal{E}$:

$$\begin{array}{ccccc} \mathcal{E}_0 & \longrightarrow & \mathfrak{m} \otimes \overline{\mathcal{E}} & \longrightarrow & \mathfrak{m}_{A/I} \otimes \overline{\mathcal{E}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{E}_0 & \longrightarrow & \mathcal{E} & \longrightarrow & \overline{\mathcal{E}} \end{array} .$$

This shows that $\mathrm{obs}(\overline{\mathcal{E}})$ is zero, because of the long exact sequence above.

Conversely given such a diagram one can reconstruct the A-module structure on $\mathcal{E}$ by defining multiplication with an element in $\mathfrak{m}$ by the composition $\mathcal{E} \to \overline{\mathcal{E}} \to \mathfrak{m} \otimes \overline{\mathcal{E}} \to \mathcal{E}$. □

2.2. Sheaves on stacks

Recall that descent for quasi coherent sheaves on schemes says that given a scheme X and $U \twoheadrightarrow X$ a smooth surjective morphism (one could replace smooth by fppf here) then we have an equivalence of categories

$$Qcoh(X) \xrightarrow{\cong} \left\langle \begin{array}{c} \mathcal{F} \text{ quasicoherent sheaf on } U \text{ together with a descent datum:} \\ \phi : pr_1^*\mathcal{F} \xrightarrow{\cong} pr_2^*\mathcal{F} \text{ on } U \times_X U + \text{cocycle cond.} \end{array} \right\rangle .$$

(See [16] Exopsé VIII, Théorème 1.1)

Definition 2.8. A *quasi coherent* sheaf $\mathcal{F}$ on an algebraic stack $\mathcal{M}$ is the datum consisting of:

1. For all smooth maps $x : X \to \mathcal{M}$, where X is a scheme, a quasi-coherent sheaf $\mathcal{F}_{X,x}$ on X.
2. For all diagrams $$\begin{array}{ccc} V & \xrightarrow{f} & U \\ & {\scriptstyle v}\searrow \quad \swarrow{\scriptstyle u} & \\ & \mathcal{M} & \end{array}$$ together with an isomorphism $\phi : u \circ f \to v$ an isomorphism $\theta_{f,\phi} : f^*\mathcal{F}_{U,u} \to \mathcal{F}_{V,v}$ compatible under composition.

Remark 2.9.

1. The category of quasi coherent sheaves on $\mathcal{M}$ can also be described as the category of sheaves on some atlas together with a descent datum.
2. Since the functors f^*, f_* commute with flat base change, we immediately get such functors for representable $F : \mathcal{M} \to \mathcal{N}$.
3. We can always define F_* as a limit, i.e.,
$$\Gamma(\mathcal{M}, \mathcal{F}) := \left\{ (s_{U,u} \in H^0(U, \mathcal{F}_{U,u}) | \theta_{f,\phi}(s_{U,u}) = s_{V,v} \right\}.$$
And again this can also be computed on a single atlas.

Example 2.10.

1. The structure sheaf $\mathcal{O}_\mathcal{M}$ of an algebraic stack is given by $\mathcal{O}_{\mathcal{M},U,u} = \mathcal{O}_U$. Similarly we can define ideal sheaves of closed substacks, by defining it to be given by the ideal sheaf of the preimage of the substack in any atlas.
2. Continuing the first example, given a smooth, closed substack $\mathcal{N} \subset \mathcal{M}$ of a smooth algebraic stack the normal bundle of $\mathcal{N}$ is a vector bundle, given by the normal bundle computed on any presentation.
3. There is a universal vector bundle $\mathcal{E}_{\mathrm{univ}}$ on $C \times \mathrm{Bun}_n$, simply because any morphism $T \to \mathrm{Bun}_n$ defines a bundle on $C \times T$.
4. To give a line bundle on Bun_n is the same as a functorial assignment of a line bundle to any family of vector bundles. An example is given by the determinant of cohomology $\det(H^*(C, \mathcal{E}))$. (See G. Hein's lecture).

Remark 2.11. Using coherent sheaves we can also perform local constructions on stacks, i.e., blowing up substacks, taking the projective bundle of a vector bundle, taking normalizations or the reduced substack: We can do this on any smooth atlas and then use descent to define the corresponding object over any $T \to \mathcal{M}$.

For quotient stacks $[X/G]$ this just means to do the corresponding construction on X and observe that the G-action extends to the scheme obtained.

To put this definition in the general framework one should of course spell out an explicit Grothendieck topology in order to obtain all the standard functorialities. This requires some careful work. Also some natural sheaves (like the cotangent bundle) do not satisfy the condition that the $\theta_{f,\phi}$ are isomorphisms so it is natural to drop this condition. The first written results in this direction appeared in the language of simplicial schemes [12]. A reference for the results on stacks is [20] together with corrections by Olsson [23].

Exercise 2.12 (A normalization). The group $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{A}^2$ by interchanging the coordinates. Let $X \subset \mathbb{A}^2$ denote the union of the two coordinate axis and consider the stack $\mathcal{M} := [X/(\mathbb{Z}/2\mathbb{Z})]$. Note that the inclusion of $\{0\} \subset X$ defines a closed embedding $B\mathbb{Z}/2\mathbb{Z} \subset [X/(\mathbb{Z}/2\mathbb{Z})]$. Show that the normalization $\mathcal{M}^{norm} \cong \mathbb{A}^1$ is a scheme.

Exercise 2.13 (A blow-up). Let the multiplicative group $\mathbb{G}_m$ act on $\mathbb{A}^2$ via $t.(x,y) = (tx, t^{-1}y)$ and consider the quotient stack $[\mathbb{A}^2/\mathbb{G}_m]$. Again the inclusion $\{0\} \subset \mathbb{A}^2$ defines a closed substack $B\mathbb{G}_m \subset \mathbb{A}^2$.

Calculate the blow-up $Bl_{\mathcal{M}}(B\mathbb{G}_m)$ of $B\mathbb{G}_m$ in $\mathcal{M}$. Show that an open subset of the exceptional fibre is isomorphic to $B\mu_2$ (here $\mu_2 \subset \mathbb{G}_m$ is the subgroup of elements of square 1, so this is just ± 1 if we are not in characteristic 2).

Exercise 2.14. Let ζ be a primitive $6-$th root of unity. Let $\mathbb{Z}/6$ act on $\mathbb{A}^2$ by $n.(x,y) := (\zeta^{2n}x, \zeta^{3n}y)$. Calculate the inertia stack $I([\mathbb{A}^2/(\mathbb{Z}/6)])$ and describe its irreducible components. (These are sometimes called sectors in physics-related literature.)

3. Lecture: Relation with coarse moduli spaces

Before describing some applications of stacks to classical questions I want to give a sample theorem on sheaves on algebraic stack which appeared in [20]. This theorem might look completely innocuous, or even boring at first sight. However it turns out to have surprising applications:

Theorem 3.1 ([20], Prop. 15.4.). *Let $\mathcal{M}$ be a noetherian algebraic stack. Then any quasi-coherent sheaf on $\mathcal{M}$ is the filtered limit of its coherent subsheaves.*

Corollary 3.2. *Any representation of a smooth noetherian algebraic group is the union of its finite dimensional subrepresentations.*

Proof of corollary. Take $\mathcal{M} = BG$. Then by definition a quasicoherent sheaf on BG is the same as a sheaf on $\mathrm{Spec}(k)$, i.e. a vector space, together with an action of G. □

Proof. Let $\mathcal{F}$ be a quasi coherent sheaf on $\mathcal{M}$. Choose an atlas $u : U \twoheadrightarrow \mathcal{M}$ of $\mathcal{M}$. In particular $\mathcal{F}_{u,U} = u^*\mathcal{F}$ is quasi-coherent on U. In particular this sheaf is the union of its quasi-coherent subsheaves $u^*\mathcal{F} = \varinjlim \mathcal{G}_i$, where $\mathcal{G}_i$ are coherent on U. In particular we have $\mathcal{F} \hookrightarrow u_*u^*\mathcal{F} = \varinjlim u_*\mathcal{G}_i$.

Define $\mathcal{F}_i := \mathcal{F} \cap u_*\mathcal{G}_i$ so that $\mathcal{F} = \varinjlim \mathcal{F}_i$. To see that $\mathcal{F}_i$ are coherent consider the diagram:

$$\begin{array}{ccc} \mathcal{F}_i := \mathcal{F} \cap u_*\mathcal{G}_i & \hookrightarrow & u_*\mathcal{G}_i \\ \downarrow & & \downarrow \\ \mathcal{F} & \hookrightarrow & u_*u^*\mathcal{F}. \end{array}$$

By adjunction we get:

$$\begin{array}{ccc} u^*\mathcal{F}_i & \longrightarrow & \mathcal{G}_i \\ \big\downarrow & & \big\downarrow \\ u^*\mathcal{F} & \xrightarrow{id} & u^*\mathcal{F}. \end{array}$$

In particular $u^*\mathcal{F}_i$ is a subsheaf of $\mathcal{G}_i$. Thus $\mathcal{F}_i$ is coherent. ☐

Exercise 3.3. (If you know the proof that any representation of an algebraic group G over a field is the union of its finite-dimensional subrepresentations.) Rewrite the proof that any quasi-coherent sheaf on a noetherian stack is the filtered inductive limit of coherent subsheaves in the case of BG explicitly on the standard presentation $pt \to BG$ in order to see that the argument is a generalization of the argument you know.

Corollary 3.4. *Let $\mathcal{M}$ be a smooth, noetherian algebraic stack and $\mathcal{U} \subset \mathcal{M}$ be an open substack. Let $\mathcal{L}_{\mathcal{U}}$ be a line bundle on $\mathcal{U}$. Then there exists a line bundle $\mathcal{L}$ on $\mathcal{M}$ such that $\mathcal{L}|_{\mathcal{U}} \cong \mathcal{L}_{\mathcal{U}}$.*

I learnt this corollary from Lieblich's thesis [22]. Note that we cannot argue with divisors here, because the example of BG already shows that a line bundle on a stack does not necessarily have meromorphic sections.

Proof. The sheaf $j_*\mathcal{L}_{\mathcal{U}}$ is quasi coherent. Thus we can write $j_*\mathcal{L}_{\mathcal{U}} = \varinjlim \mathcal{F}_i$ for some coherent sheaves $\mathcal{F}_i$. This implies that we can even find $\mathcal{F}_i$ such that $\mathcal{F}_i|_U = \mathcal{L}_{\mathcal{U}}$. Then the double dual $(\mathcal{F}_i^\vee)^\vee$ is a reflexive sheaf of rank 1 on a smooth stack. So it has to be a line bundle. (Again, this result holds for stacks, because we can check it on a smooth atlas.) ☐

3.1. Coarse moduli spaces

Definition 3.5. Let $\mathcal{M}$ be an algebraic stack. An algebraic space M together with a map $p : \mathcal{M} \to M$ is called *coarse moduli space* for $\mathcal{M}$ if

1. For all schemes X and morphisms $q : \mathcal{M} \to X$ there exist a unique morphism $M \to X$ making

$$\begin{array}{ccc} \mathcal{M} & \longrightarrow & X \\ & \searrow & \nearrow \\ & M & \end{array}$$

commutative.

2. For all algebraically closed fields $\overline{K}$ we have $\mathcal{M}(\overline{K})/\text{isomorphism} = M(\overline{K})$.

If M only satisfies the first condition, it is called categorial coarse moduli space.

Remark 3.6. Categorial coarse moduli spaces can be very small. If $\mathcal{M} = [\mathbb{A}^n/\mathbb{G}_m]$, the quotient of the affine space by the multiplication by non-zero scalars, then $\mathcal{M} \to pt$ is a categorial quotient.

Example 3.7. Let Bun_n^{stable} be the moduli stack of stable bundles on a curve. Then the coarse moduli space of stable bundles M^{stable} constructed by geometric invariant theory (see G. Hein's lectures) is a coarse moduli space for Bun_n^{stable}. The GIT construction shows that $\mathrm{Bun}_n^{stable} = [X/\mathsf{GL}_N]$ (for some scheme X) and constructs $M^{stable} = X/\mathsf{PGL}_N$. In particular we get a map $\mathrm{Bun}_n^{stable} \to M^{stable}$ which satisfies the stronger property that for any $T \to M^{stable}$ there exists an etale covering $T' \twoheadrightarrow T$ such that the map $T' \to M^{stable}$ lifts to $T' \to X$ and therefore it lifts to Bun_n^{stable}.

Note that in the above example all geometric fibres of the map $\mathrm{Bun}_n^{stable} \to M^{stable}$ are isomorphic to $B\mathbb{G}_m$. This corresponds to the fact that the automorphism group of a stable bundle consist only of scalars. Such a morphism is called a gerbe. Let us give the definition:

Definition 3.8. A morphism $F : \mathcal{M} \to \mathcal{N}$ of algebraic stacks is called a *gerbe* over $\mathcal{N}$ if

1. F is locally surjective, i.e., for any $T \to \mathcal{N}$ there exists a covering $T' \twoheadrightarrow T$ such that the morphism $T' \to \mathcal{N}$ can be lifted to $\mathcal{M}$. (In other words, any object of $\mathcal{N}$ locally comes from an object of $\mathcal{M}$.)
2. All objects in a fibre are locally isomorphic: if $t_1, t_2 : T \to \mathcal{M}$ are two objects with $F(t_1) \cong F(t_2)$ then there exists a covering $T' \twoheadrightarrow T$ such that $t_1|_{T'} \cong t_2|_{T'}$.

$F : \mathcal{M} \to \mathcal{N}$ is called a $\mathbb{G}_m$-gerbe if for all $t : T \to \mathcal{M}$ the relative automorphism group $\mathrm{Aut}(t/\mathcal{N})$ is canonically isomorphic to $\mathbb{G}_m(T)$ (equivalently: $I(\mathcal{M}) \times_{I(\mathcal{N})} \mathcal{N} \cong \mathbb{G}_m \times \mathcal{N}$).

Example 3.9. We have just seen that $\mathrm{Bun}_n^{stable} \to M^{stable}$ is a $\mathbb{G}_m$-gerbe. More generally if $\mathcal{N} = [X/\mathsf{PGL}_n]$ then $\mathcal{M} := [X/\mathsf{GL}_n] \to [X/\mathsf{PGL}_n]$ is a $\mathbb{G}_m$-gerbe.

So a $\mathbb{G}_m$-gerbe on a scheme X can be thought of as a $B\mathbb{G}_m$-bundle over X. A notion of triviality of such a bundle is useful:

Lemma 3.10. *For a $\mathbb{G}_m$-gerbe $F : \mathcal{M} \to \mathcal{N}$ the following are equivalent:*

1. *The morphism $F : \mathcal{M} \to \mathcal{N}$ has a section.*
2. *$\mathcal{M} \cong B\mathbb{G}_m \times \mathcal{N}$.*
3. *There is a line bundle of weight*[4] *1 on $\mathcal{M}$.*

A gerbe satisfying the above conditions is called neutral.

Remark 3.11 (Weight). Since for any $u : U \to \mathcal{M}$ we have $\mathbb{G}_m \subset \mathrm{Aut}(U \to \mathcal{M})$, the $\theta_{Id,\alpha}$ for $\alpha \in \mathbb{G}_m(U)$ define a $\mathbb{G}_m$ action on $\mathcal{F}_{U,u}$, i.e. a direct sum decomposition $\mathcal{F}_{U,u} = \oplus_{n\in\mathbb{Z}} \mathcal{F}^n_{U,u}$ such that $\mathcal{F}^i$ is the subsheaf on which $\mathbb{G}_m$ acts via multiplication with the $i-th$ power.

Since this decomposition is canonical it defines $\mathcal{F} = \oplus \mathcal{F}^n$. A sheaf is called of weight i if $\mathcal{F} = \mathcal{F}^i$.

[4] see below

Proof. By the remark, 2. implies 3., because the universal bundle on $B\mathbb{G}_m$ is of weight 1. The implication 2. $\Rightarrow$ 1. is also clear.

Let us show that 1. $\Rightarrow$ 3.. Let $s : \mathcal{N} \to \mathcal{M}$ be a section. Then $\mathcal{N} \times_{\mathcal{M}} \mathcal{N} = \mathrm{Aut}(s) = \mathbb{G}_m \times \mathcal{N}$, i.e. the section $s : \mathcal{N} \to \mathcal{M}$ makes $\mathcal{N}$ into a $\mathbb{G}_m$-bundle of weight one on $\mathcal{M}$.

Finally 3. $\Rightarrow$ 2.: Let $\mathcal{L}$ be a line bundle of weight 1 on $\mathcal{M}$ and denote by $\mathcal{L}^\circ$ the corresponding $\mathbb{G}_m$-bundle. Since $B\mathbb{G}_m$ is the classifying stack of line bundles this defines a morphism $\mathcal{M} \to B\mathbb{G}_m$ such that $\mathcal{L}^\circ = pt \times_{B\mathbb{G}_m} \mathcal{M}$. So we get a cartesian diagram:

$$\begin{array}{ccc} \mathcal{L}^\circ & \longrightarrow & \mathcal{N} = \mathcal{N} \times pt \\ \downarrow & & \downarrow \\ \mathcal{M} & \longrightarrow & \mathcal{N} \times B\mathbb{G}_m. \end{array}$$

This implies that $\mathcal{M} \to \mathcal{N} \times B\mathbb{G}_m$ is locally surjective, all objects in a fibre are locally isomorphic, because this already holds for $\mathcal{M} \to \mathcal{N}$ so this is a gerbe. However the map is also an isomorphism on automorphism groups, so it must be an isomorphism. □

Let us apply these notions in order to study when a Poincaré-family exists on the coarse moduli spaces $M_n^{d,stable}$ of stable vector bundles of rank n and degre d. Recall that a Poincaré-family is a vector bundle on $C \times M_n^{d,stable}$ such that the fibre over every point of $M_n^{d,stable}$ lies in the isomorphism class of bundles defined by this point. So such a bundle is the same thing as a section of the map $\mathrm{Bun}_n^{d,stable} \to M_n^{d,stable}$.

With these preparations we can now show the following result of Ramanan ([24]), reproven by Drezet and Narasimhan ([10] Théorème G).

Corollary 3.12. *Assume that the genus g of our curve C is bigger than* 1.

1. *If $(n, d) = 1$ then there exists a Poincaré-family on the coarse moduli space $M_n^{d,stable}$ of stable vector bundles on C.*
2. *If $(n, d) \neq 1$ then there is no open subset $U \subset M_n^{d,stable}$ (with $U \neq \emptyset$) such that there exists a Poincaré family on $C \times U$.*

Proof. The first part is well known: For any point $c \in C$ the restriction of the universal bundle $\mathcal{E}$ on $C \times \mathrm{Bun}_n^d$ to $c \times \mathrm{Bun}_n^d$ has weight 1, so $\det(\mathcal{E}|_{c \times \mathrm{Bun}_n^d})$ has weight n.

Next, note that by transport of structure for any bundle $\mathcal{E}$ the scalar automorphisms act as scalar automorphisms on the cohomology groups $H^i(C, \mathcal{E})$. Thus, the Riemann-Roch theorem the bundle $\det(H^*(C, \mathcal{E}))$ has weight $d + (g-1)n$ so that there exist a product of these two bundles having weight 1. By the Lemma this means that the map $\mathrm{Bun}_n^{d,stable} \to M_n^d$ has a section, so that we can pull back the universal bundle by this section to obtain a Poincaré bundle on M_n^d.

For the second part let $(n, d) = (k) \neq 1$. Assume there was a Poincaré family on some non-empty open subset $U \subset M_n^d$. This would mean that there is a section

$U \to \mathrm{Bun}_n^d$. By the Lemma there would then exist a line bundle of weight 1 on $\mathcal{U} := U \times_{M_n^d} \mathrm{Bun}_n^d \subset \mathrm{Bun}_n^d$. Since Bun_n^d is smooth, we can apply the corollary from the beginning of the lecture to extend this line bundle to any noetherian substack of Bun_n^d. And this would still have weight 1.

Now pick a stable bundle $\mathcal{E}$ of rank n/k and degree d/k (the assumption on the genus of C assures that stable bundles exist, see remark following this proof). Then the bundle $\mathcal{E}^{\oplus k}$ defines a point of Bun_n^d. We have $\mathrm{Aut}(\mathcal{E}^{\oplus k}) = \mathsf{GL}_k$ and thus we find $\mathrm{BGL}_k \subset \mathrm{Bun}_n^d$. However there is no line bundle of weight 1 on $B\mathsf{GL}_k \to B\mathsf{PGL}_k$, because this would imply that $B\mathsf{GL}_k = B\mathsf{PGL}_k \times B\mathbb{G}_m$ but $\mathsf{GL}_k \ncong \mathsf{PGL}_k \times \mathbb{G}_m$. So we found a contradiction. $\square$

A much more general statement of this type can be found in an article of Biswas and Hoffmann [8].

Remark 3.13. In the course of the above proof we used the existence of (semi-)stable bundles on curves of genus > 1. This is a classical result (*see also* fr example [25]), but again one can also rephrase this in terms of the moduli stack. Namely in the next lecture we will see that the substack of instable bundles has positive codimension in the stack of all bundles if the genus of C is > 0, so in particular semistable bundles have to exist. A similar argument works for the strictly semistable locus if the genus of C is > 1.

Exercise 3.14. Let M be a scheme and $\mathcal{M} \to M$ be a $\mathbb{G}_m$-gerbe. Let $E \to \mathcal{M}$ be a vector bundle of weight 1. Denote the complement of the zero-section of E by $E^\circ \subset E$. Show that $E^\circ \to M$ is a bundle of projective spaces, i.e. there is a smooth covering $U \to M$ such that $E^\circ|_U \cong U \times \mathbb{P}^{n-1}$.

4. Lecture: Cohomology of Bun_n^d

Next we want to calculate the cohomology of some étale sheaves on stacks. (If you prefer to work in an analytic category these would just correspond to constructable sheaves in the analytic topology.) The aim of this lecture is on the one hand to give an impression of some techniques which help to do such computations and on the other hand to show that the results are often much nicer than the corresponding results for coarse moduli spaces.

To avoid to introducing more theory, we use the same working-definition as before:

Definition 4.1. A sheaf on an algebraic stack $\mathcal{M}$ is a collection of sheaves $\mathcal{F}_{U,u}$ for all $u : U \to \mathcal{M}$ together with compatible morphisms $\theta_{f,\phi} : f^{-1}\mathcal{F}_{U,u} \to \mathcal{F}_{V,v}$ for all $v : V \to \mathcal{M}$, $f : V \to U$ and $\phi : u \circ f \to v$.

A sheaf is called *cartesian* if all $\theta_{f,\phi}$ are isomorphisms.

Remark 4.2. As before it turns out that the cohomology groups $H^*(\mathcal{M}, \mathcal{F})$ can be calculated from an atlas. Given an atlas $u : U \twoheadrightarrow \mathcal{M}$ we denote by $U_p :=$

$U \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} U$ the $p+1$-fold product of U over $\mathcal{M}$. Then there exists a spectral sequence (see e.g., [23]) with

$$E_1^{p,q} := H^q(U_p, \mathcal{F}_{U_p}) \Rightarrow H^{p+q}(\mathcal{M}, \mathcal{F}).$$

From this one can immediately conclude that the Künneth formula and Gysin-sequences ([9]) also exist for stacks, simply by applying the formulas for the U_p.

Example 4.3. Let us compute $H^*(B\mathbb{G}_m, \mathbb{Q}_\ell)$. The quotient map $\mathbb{A}^n - 0 \to \mathbb{P}^{n-1}$ is a $\mathbb{G}_m$-bundle. Thus we get a cartesian diagram:

$$\begin{array}{ccc} \mathbb{A}^n - 0 & \longrightarrow & pt \\ \downarrow & & \downarrow \\ \mathbb{P}^{n-1} & \xrightarrow{p} & B\mathbb{G}_m. \end{array}$$

From this we see that the fibres of p are $\mathbb{A}^n - 0$. Thus we get that

$$\mathbf{R}^i p_* \mathbb{Q}_\ell = \begin{cases} \mathbb{Q}_\ell & i = 0 \\ 0 & i < 2n-1. \end{cases}$$

The Leray spectral sequence for p therefore implies that $H^i(B\mathbb{G}_m) \cong H^i(\mathbb{P}^{n-1})$ for $i < 2n-1$. Thus we find that $H^*(B\mathbb{G}_m, \mathbb{Q}_\ell) = \mathbb{Q}_\ell[c_1]$ is a polynomial ring in one generator of degree 2.

Example 4.4. Replacing $\mathbb{A}^n - 0$ in the previous example by the space $\mathsf{Mat}(n, N)_{rk=n}$ of $n \times N$ matrices of rank n one can show that $H^*(B\mathsf{GL}_n, \mathbb{Q}_\ell) = \mathbb{Q}_\ell[c_1, \ldots, c_n]$ is a polynomial ring, with generators c_i(the universal Chern classes) of degree $2i$.

As an application of this computation one can define the Chern classes of a vector bundle as the pull back of the c_i under the morphism to $B\mathsf{GL}_n$ which is defined by the vector bundle.

Although we will not use it, I would like to mention Behrend's generalization of the Lefschetz trace formula for algebraic stacks over finite fields. This gives another way to interpret the above examples. Recall that for a smooth variety X over a finite field $k = \mathbb{F}_q$ there is a natural action of the Frobenius on the cohomology groups $H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$. And the Lefschetz trace formula (see [9] p.88),says[5] that

$$\#X(\mathbb{F}_q) = q^{\dim(X)} \sum_i (-1)^i \mathrm{Trace}(Frob, H^i(X_{\overline{k}}, \mathbb{Q}_\ell)).$$

Furthermore, the Weil conjectures imply that for a smooth, proper variety the eigenvalues of Frob on $H^i(X_{\overline{k}}, \mathbb{Q}_\ell)$ have absolute value $q^{-i/2}$ and that this is enough to recover the dimension of the cohomology groups from the knowledge of $\#X(\mathbb{F}_{q^n})$ for sufficiently many n.

[5]This is usually formulated for cohomology with compact supports. In that case the leading factor $q^{\dim(X)}$ disappears.

Behrend showed [5],[6] that a similar trace formula also holds for a large class of smooth algebraic stacks $\mathcal{M}$ over $\mathbb{F}_q$, including quotient stacks and Bun_n, namely that

$$\sum_{x\in\mathcal{M}(\mathbb{F}_q)/iso} \frac{1}{\#\operatorname{Aut}(x)(\mathbb{F}_q)} = q^{\dim(\mathcal{M})}\sum_i (-1)^i \mathrm{Trace}(Frob, H^i(\mathcal{M}_{\overline{k}}, \mathbb{Q}_\ell)).$$

For example for $B\mathbb{G}_m$ the left hand side is $1/(q-1) = q^{-1}(1+q^{-1}+q^{-2}+\dots)$. The fact that the coefficients of the powers of q in this expansion are all equal to 1 already suggests that the cohomology should have a single generator in each even degree.

Exercise 4.5. Check Behrend's trace formula for the stacks $B(\mathbb{Z}/2)$ and $B(\mathbb{Z}/n)$ over the finite field $\mathbb{F}_p$.

Remark 4.6. In contrast to the preceeding exercise the proof of the trace formula uses a reduction to quotients by GL_n. This trick is quite useful in other contexts as well, because GL_n-bundles are locally trivial for the Zariski topology.

Our aim of this lecture is to compute the cohomology $H^*(\mathrm{Bun}_n^d, \mathbb{Q}_\ell)$. First note that the Gysin sequence implies that the cohomology in low degrees does not change if one removes a substack of high codimension. Therefore

$$H^*(\mathrm{Bun}_n^d, \mathbb{Q}_\ell) = \lim_{\mathcal{U}\subset \mathrm{Bun}_n^d \text{ finite type}} H^*(\mathcal{U}, \mathbb{Q}_\ell).$$

And for each fixed degree $*$ the limit on the right hand side becomes stationary for sufficiently large $\mathcal{U}$, i.e. $H^i(\mathrm{Bun}_n^d, \mathbb{Q}_\ell) = H^i(\mathcal{U}, \mathbb{Q}_\ell)$ if the codimension of the complement of $\mathcal{U}$ is larger than $i/2$.

Let us first recall the construction of the so called *Atiyah-Bott classes* in the cohomology of Bun_n^d:

We have already seen that by definition of sheaves on stacks, there is a universal family of vector bundles $\mathcal{E}_{\mathrm{univ}}$ on $C\times \mathrm{Bun}_n^d$. In particular this bundle defines a morphism $\mathrm{univ} : C\times \mathrm{Bun}_n^d \to B\mathsf{GL}_n$ and we set

$$c_i(\mathcal{E}_{\mathrm{univ}}) := \mathrm{univ}^* c_i \in H^*(C\times \mathrm{Bun}_n^d, \mathbb{Q}_\ell) = H^*(C, \mathbb{Q}_\ell)\otimes H^*(\mathrm{Bun}_n^d, \mathbb{Q}_\ell).$$

We choose a basis $1 \in H^0(C,\mathbb{Q}_\ell), (\gamma_j)_{j=1,\dots,2g} \in H^1(C,\mathbb{Q}_\ell), [pt] \in H^2(C,\mathbb{Q}_\ell)$ in order to decompose these Chern classes:

$$c_i(\mathcal{E}_{\mathrm{univ}}) = 1\otimes a_i + \sum_{j=1}^{2g} \gamma_j \otimes b_i^j + [pt]\otimes f_i$$

for some $a_i, b_i^j, f_i \in H^*(\mathrm{Bun}_n^d, \mathbb{Q}_\ell)$.

Using these classes we can give the main theorem of this lecture. This was first proved by Atiyah-Bott in an analytic setup using equivariant cohomology. An algebraic proof was first given by Biffet-Ghione-Laetizia [7]. The proof we will explain here is a variant of [18]. To simplify notations we will write $H^*(X, \overline{\mathbb{Q}}_\ell)$ for the cohomology of X computed over the algebraic closure of the ground field.

Theorem 4.7. *The cohomology of* Bun_n^d *is freely generated by the Atiyah-Bott classes:*

$$H^*(\mathrm{Bun}_n^d, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell[a_1, \dots, a_n] \otimes \bigwedge[b_i^j]_{\substack{i=1,\dots n\\ j=1,\dots 2g}} \otimes \overline{\mathbb{Q}}_\ell[f_2, \dots, f_n].$$

A similar result holds for moduli spaces of principal bundles (see [2] for an analytic proof and [19] for an algebraic version).

We want to indicate a proof of the above result.

4.1. First step: Independence of the generators

We first want show that the Atiyah-Bott classes generate a free subalgebra of the cohomology. Let us consider the simplest case $n = 1$. Denote by Pic_C^d the Picard scheme of C, which is a coarse moduli space for Bun_1^d. Over an algebraically closed field there exists a Poincaré bundle on $C \times \mathrm{Pic}_C^d$ and thus 3.10 implies that:

$$\mathrm{Bun}_1^d \cong \mathrm{Pic}_C^d \times B\mathbb{G}_m.$$

Furthermore we know that Pic_C^d is isomorphic to the Jacobian of C, which is an abelian variety and its cohomology is the exterior algebra on $H^1(C, \overline{\mathbb{Q}}_\ell)$. Thus

$$H^*(\mathrm{Bun}_1^d, \overline{\mathbb{Q}}_\ell) \cong H^*(Pic_C^d, \overline{\mathbb{Q}}_\ell) \otimes H^*(B\mathbb{G}_m, \overline{\mathbb{Q}}_\ell) \cong \bigwedge[b_1^j] \otimes \overline{\mathbb{Q}}_\ell[a_1].$$

For $n > 1$ for any partition $d = \sum_{i=1^n} d_i$ with $d_i \in \mathbb{Z}$ consider the map

$$\oplus_{\underline{d}} : \prod_{i=1}^n \mathrm{Bun}_1^{d_i} \to \mathrm{Bun}_n^d$$
$$(\mathcal{L}_i) \mapsto \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_n$$

We know that the Chern classes of a direct sum of line bundles are given by the elementary symmetric polynomials σ_i in the Chern classes of the line bundles. Write $c_1(\mathcal{L}_{\mathrm{univ}}^{d_i}) := 1 \otimes A_i + \sum \gamma_j \otimes B_i^j + [pt] \otimes d_i$. Then we have:

$$\begin{aligned}(\oplus_{\underline{d}})^*(c_i(\mathcal{E}_{\mathrm{univ}})) &= \sigma_i(c_1(\mathcal{L}_{\mathrm{univ}}^{d_1}), \dots, c_1(\mathcal{L}_{\mathrm{univ}}^{d_n}))\\ &= \sigma_i(A_i, \dots, A_n) + \sum_{j,k} \gamma_j \otimes \partial_k \sigma_i(A_1, \dots, A_n) B_k^j\\ &+ \sum_{j+m=2g+1,k,l} [pt] \otimes \partial_k \partial_l \sigma(A_1, \dots, A_n) B_k^j B_l^m\\ &+ \sum_{j,k} [pt] \otimes \partial_k \sigma_i(A_1, \dots, A_n) d_k\end{aligned}$$

Thus taking the union over all $\underline{d} = (d_1, \dots, d_n)$ we get a commutative diagram:

$$\begin{array}{ccc} H^*(\mathrm{Bun}_n^d, \overline{\mathbb{Q}}_\ell) & \longrightarrow & \prod_{\underline{d}, \sum d_i = d} \overline{\mathbb{Q}}_\ell[A_1, \dots, A_n] \otimes \bigwedge[B_i^j]_{i,j} \\ \uparrow & & \uparrow {\scriptstyle D_i \mapsto (d_i)_{\underline{d}}} \\ \overline{\mathbb{Q}}_\ell[a_i, f_i] \otimes \bigwedge[b_i^j] & \hookrightarrow & \overline{\mathbb{Q}}_\ell[A_i] \otimes \bigwedge[B_i^j] \otimes \overline{\mathbb{Q}}_\ell[D_1, D_2, \dots, D_n] / \sum D_i = d \end{array}$$

Here the lower horizontal map is given by

$$a_i \mapsto \sigma_i(A_1, \dots, A_n), b_i^j \to \sum_k \partial_k \sigma_i(A_1, \dots, A_n) B_k^j$$

and f_i is mapped to the last two sumands of our computation above, replacing the constants d_k by variables D_k. This map is injective, because the $\partial_k \sigma_i$ are linearly independent (this is equivalent to the fact that the map $\mathbb{A}^n \to \mathbb{A}^n/S_n$ is generically étale).

The right vertical arrow is the evaluation of D_i. This is injective, because we evaluate at all integers simultaneously.

This shows that the left vertical arrow must be injective as well.

4.2. Second step: Why is it the whole ring?

One way to see this is to use Beauville's trick (I think he quotes Ellingsrud and Strømme) to show that the Atiyah-Bott classes generate the cohomology of some coarse moduli spaces: If X is a smooth projective scheme then the Künneth components of the diagonal $[\Delta] \subset H^*(X \times X) \cong H^*(X) \otimes H^*(X)$ generate $H^*(X)$. (This is not difficult. Note however that this does not seem to make sense for stacks, because the diagonal morphism is not an embedding - look at the example of $B\mathbb{G}_m$.)

Let again $\mathcal{E}_{\text{univ}}$ denote the universal bundle on $C \times \text{Bun}_n^d$ and consider the sheaf $\mathcal{H}om(p_{12}^*\mathcal{E}_{\text{univ}}, p_{13}^*\mathcal{E}_{\text{univ}})$ on $C \times \text{Bun}_n^{d,stable} \times \text{Bun}_n^{d,stable}$.

The complex $\mathbf{R}p_{23,*}\mathcal{H}om(p_{12}^*\mathcal{E}_{\text{univ}}, p_{13}^*\mathcal{E}_{\text{univ}})$ can be represented by a complex $[K_0 \xrightarrow{d^1} K_1]$ of vector bundles on $\text{Bun}_n^{d,stable} \times \text{Bun}_n^{d,stable}$. Since there are no homomorphisms between non-isomorphic stable vector bundles of the same rank and degree we know that the map d^1 has maximal rank outside the diagonal $\Delta \subset \text{Bun}_n^{d,stable} \times \text{Bun}_n^{d,stable}$.

Thus we can apply the Porteous formula ([14], Chapter 14.4) (if we know that $\operatorname{codim} \Delta = \chi(K_0 \to K_1) + 1$) to see that the top Chern class $c_{top}(K_0 \to K_1) = [\Delta]$. On the other hand we can use the Riemann Roch theorem to compute

$$ch(\mathbf{R}p_{23,*}\mathcal{H}om(p_{12}^*\mathcal{E}_{\text{univ}}, p_{13}^*\mathcal{E}_{\text{univ}})) = pr_{23,*}(pr_C^*(Todd(C)) \cdot ch(\mathcal{E}_{\text{univ}})ch(\mathcal{E}_{\text{univ}}^\vee)).$$

The right hand side of this formula is given in terms of the Atiyah-Bott classes. Together with Porteous formula, this gives an expression of $[\Delta]$ in terms of the Atiyah-Bott classes. However this does only work for stable bundles and to use the trick one also needs to know that $\text{Bun}_n^{d,stable}$ is a $\mathbb{G}_m$-gerbe over a smooth projective variety.

To get into such a situation one can use parabolic bundles: Pick a finite set of points $S = \{p_1, \dots, p_N\} \in C$. Then one defines the stack of parabolic bundles:

$$\text{Bun}_{n,S}^d(T) := \langle \mathcal{E} \in \text{Bun}_n^d(T), (\mathcal{E}_{1,p} \subsetneq \cdots \subsetneq \mathcal{E}_{n,p} = \mathcal{E}|_{p \times T})_{p \in S} \text{ a full flag of subspaces} \rangle$$

Forgetting the flags defines a morphism *forget* : $\text{Bun}_{n,S}^d \to \text{Bun}_n^d$, the fibres of which are products of flag manifolds $\prod_{p \in S} \text{Flag}_n$. The theorem of Leray-Hirsch says that for such a fibration we have:

$$H^*(\mathrm{Bun}_{n,S}^d) = H^*(\mathrm{Bun}_n^d) \otimes \bigotimes_{p\in S} H^*(\mathrm{Flag}_n).$$

In particular $H^*(\mathrm{Bun}_n^d)$ is generated by the Atiyah-Bott classes if and only if $H^*(\mathrm{Bun}_{n,S}^d)$ is generated by the Atiyah-Bott classes and the Chern classes defined by the flags $\mathcal{E}_{i,p}$ (we call this collection of classes the canonical classes).

Now one can argue as follows (all these steps require some care):

1. There exist open substacks $\mathrm{Bun}_{n,S}^{d,\alpha-stable} \subset \mathrm{Bun}_{n,S}^d$ of α-stable bundles, depending on some parameter α. If α is chosen well, this substack has a projective coarse moduli space $M_{n,S}^d$ and the map $\mathrm{Bun}_{n,S}^{d,\alpha-stable} \to M_{n,S}^d$ is a $\mathbb{G}_m$-gerbe.
2. We can do Beauville's trick for parabolic bundles using homomorphisms respecting the flags instead of arbitrary homomorphisms of vector bundles. Thus $H^*(\mathrm{Bun}_{n,S}^{d,stable})$ is generated by the canonical classes.
3. The codimension of the instable bundles $\mathrm{Bun}_{n,S}^{d,inst} \subset \mathrm{Bun}_{n,S}^d$ goes to ∞ for $N \to \infty$ (and well-chosen stability parameters).

Putting these results together we get a proof of the theorem.

5. Lecture: The cohomology of the coarse moduli space (coprime case)

In this lecture we want to continue our study of geometric properties of the stack Bun_n^d in order to give some more phenomena that can occur when studying algebraic stacks. As aim of the lecture we also want to explain why the results of the previous lecture are useful, even if one is only interested in coarse moduli spaces. Namely, we want to deduce a description of the cohomology of the coarse moduli space from the results of the previous lecture. To do this we will study the part of the moduli stack that parameterizes instable bundles. This stack has a natural stratification. We will see what the strata look like and we will analyze the Gysin-sequence for this stratification.

We begin with the Harder-Narasimhan "stratification"[6] of Bun_n^d:

Proposition 5.1 (Harder-Narasimhan filtration). *Let $\mathcal{E}$ be a vector bundle on C, defined over an algebraically closed field. Then there exists a canonical filtration $0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_s = \mathcal{E}$ such that for all i we have:*

1. $\mu(\mathcal{E}_i) := \frac{\deg(\mathcal{E}_i)}{rank(\mathcal{E}_i)} > \mu(\mathcal{E}_{i+1})$. *($\mu(\mathcal{E})$ is called the slope of $\mathcal{E}$)*
2. *$\mathcal{E}_{i+1}/\mathcal{E}_i$ is a semistable vector bundle.*

We denote by $t(\mathcal{E}) := ((n_i, d_i))_i$ the type of instability *of $\mathcal{E}$.*

[6] We put quotation marks here, in order to warn the reader that the closure of a stratum need not be a union of strata if $n > 2$, see Example 5.11.

The proof of the proposition proceeds by induction, observing that a subsheaf of maximal slope has to be semistable and that for two such subsheaves, their sum also satisfied this condition.

Remark 5.2. 1. The type of instability $t(\mathcal{E})$ defines a convex polygon, with vertices (n_i, d_i). Here convexity is guaranteed by the first condition above. This is called the Harder-Narasimhan polygon of $\mathcal{E}$.
2. For any $\mathcal{F} \subset \mathcal{E}$ the point $(rank(\mathcal{F}), \deg(\mathcal{F}))$ lies below the polygon of $\mathcal{E}$.
3. If one has a family of vector bundles on C, then the HN-polygon can only get bigger under specialization, because the closure of a subsheaf in the generic fibre defines a subsheaf in the special fibre.

In particular for any $T \to \mathrm{Bun}_n^d$ given by a family $\mathcal{E}$ we get a canonical decomposition $T = \cup_{t \text{ polygon}} T^t$ into locally closed subschemes such that T^t consists of those points such that the Harder-Narasimhan polygon of the corresponding bundle is of type t. Since this is canonical it defines a decomposition of $\mathrm{Bun}_n^d = \cup_t \mathrm{Bun}_n^{d,t}$ and by the 3rd point of the above remark, the substack $\mathrm{Bun}_n^{d,\leq t} = \cup_{t' \leq t} \mathrm{Bun}_n^{d,t} \subset \mathrm{Bun}_n^d$ is open.

We can describe this more precisely:

1. Given a type $t = (n_i, d_i)_i$ we define the stack of filtered bundles
$$\mathrm{Filt}_{\underline{n}}^{\underline{d}}(T) := \langle \mathcal{E}_1 \subset \mathcal{E}_2 \cdots \subset \mathcal{E}_s | \deg(\mathcal{E}_i) = d_i, \mathrm{rk}(\mathcal{E}_i) = n_i \rangle..$$
This is an algebraic stack and the forgetful morphism
$$\mathrm{Filt}_{\underline{n}}^{\underline{d}} \to \mathrm{Bun}_n^d$$
is representable, by the theory of Quot-schemes.
2. There is a morphism $\mathrm{Filt}_{\underline{n}}^{\underline{d}} \to \prod_i \mathrm{Bun}_{n_i - n_{i-1}}^{d_i - d_{i-1}}$ that maps the filtered bundle $\mathcal{E}_\bullet$ to its subquotients $\mathcal{E}_i / \mathcal{E}_{i-1}$.
3. There is an open substack $\mathrm{Filt}_{\underline{n},\underline{d}}^{ss} \subset \mathrm{Filt}_{\underline{n},\underline{d}}$ defined by the condition that the subquotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ are semistable.

Given a filtered bundle $\mathcal{E}_\bullet$ we define $\mathcal{E}\mathrm{nd}(\mathcal{E}_\bullet) \subset \mathcal{E}\mathrm{nd}(\mathcal{E}_s)$ to be the subgroup of those endomorphisms respecting the filtration, i.e. those ϕ such that $\phi(\mathcal{E}_i) \subset \mathcal{E}_i$ for all i.

Proposition 5.3. *If a type $t = (\underline{n}, \underline{d})$ is a convex polygon, then the forgetful map $\mathrm{Filt}_{\underline{n}}^{\underline{d},ss} \to \mathrm{Bun}_n^d$ is an immersion.*

The normal bundle $\mathcal{N}_{forget}$ to forget is given by $\mathbf{R}^1 p_*(\mathcal{E}nd(\mathcal{E}_{univ})/\mathcal{E}nd(\mathcal{E}_{\bullet,univ}))$.

The first point has a nice corollary:

Corollary 5.4. *The Harder-Narasimhan filtration of a vector bundle $\mathcal{E}$ on $C \times \mathrm{Spec}(K)$ is defined over K and not only after passing to the algebraic closure.*

Remark 5.5. The above proposition implies that all Harder-Narasimhan strata are smooth stacks. However, for any $k \in \mathbb{Z}$ there are only finitely many strata of dimension $\geq k$. Moreover, any bundle $\mathcal{E}$ of rank $n > 1$ admits subbundles of rank

1. However given an extension $\mathcal{L} \to \mathcal{E} \to \mathcal{E}/\mathcal{L}$ we can find a family over $\mathbb{A}^1 \times C$ such that the restriction to $\mathbb{G}_m$ is the constant family $\mathcal{E} \times \mathbb{G}_m$, but the fibre over 0 is $\mathcal{L} \oplus \mathcal{E}/\mathcal{L}$. In particular no point of Bun_n is closed.

To prove the proposition we need to introduce the tangent stack of an algebraic stack:

Recall that for a scheme X the tangent space TX can be defined as the scheme representing the functor given on affine schemes by: $TX(\mathrm{Spec}(A)) := X(\mathrm{Spec}(A[\epsilon]/\epsilon^2))$. We can do the same for stacks:

Definition 5.6. The tangent stack $T\mathcal{M}$ to an algebraic stack $\mathcal{M}$ is the stack given on affine schemes by

$$T\mathcal{M}(\mathrm{Spec}(A)) := \mathcal{M}(\mathrm{Spec}(A[\epsilon]/\epsilon^2)).$$

Remark 5.7. $T\mathcal{M}$ is an algebraic stack, given an atlas $u : U \twoheadrightarrow \mathcal{M}$ the canonical map $TU \to T\mathcal{M}$ is an atlas for $T\mathcal{M}$.

Example 5.8. $TBG = [pt/TG]$. Note that the tangent space to a group is again a group. This is immediate from the above definition of TG.

Example 5.9. The fibre of the tangent stack $T\,\mathrm{Bun}_n^d$ at a bundle $\mathcal{E}$ on C is by definition the groupoid of extensions $\tilde{\mathcal{E}}$ of $\mathcal{E}$ to $C \times \mathrm{Spec}(k[\epsilon]/\epsilon^2)$. As in our proof of smoothness of Bun_n^d this can be described as follows: We have an exact sequence of $k[\epsilon]/\epsilon^2$-modules $k \to k[\epsilon]/\epsilon^2 \to k$. Thus an extension $\tilde{\mathcal{E}}$ of $\mathcal{E}$ gives an extension

$$0 \to \mathcal{E} \to \tilde{\mathcal{E}} \to \mathcal{E} \to 0.$$

And conversely such an extension of vector bundles defines a $k[\epsilon]/\epsilon^2$-module structure on $\tilde{\mathcal{E}}$, multiplication by ϵ being given by the compositon $\tilde{\mathcal{E}} \to \mathcal{E} \to \tilde{\mathcal{E}}$. The automorphisms of such an extension are given by $Hom(\mathcal{E}, \mathcal{E}) = H^0(C, \mathcal{E}\mathrm{nd}(\mathcal{E}))$. Thus we see that:

$$T_{\mathcal{E}}\,\mathrm{Bun}_n^d = [H^1(C, \mathcal{E}\mathrm{nd}(\mathcal{E}))/H^0(C, \mathcal{E}\mathrm{nd}(\mathcal{E}))]$$

where the quotient is taken by letting the additive group H^0 act trivially on H^1.

The same computation holds for $\mathrm{Filt}_{\underline{n}}^{\underline{d}}$ if one replaces $\mathcal{E}\mathrm{nd}(\mathcal{E})$ by $\mathcal{E}\mathrm{nd}(\mathcal{E}_\bullet)$.

Now note:

1. The uniqueness of the Harder-Narasimhan flag of a bundle $\mathcal{E}$ is equivalent to the statement that the fibre of *forget* over $\mathcal{E}$ consists of a single point.
2. The map $T\,\mathrm{Filt}_{\underline{n}}^{\underline{d}} \to T\,\mathrm{Bun}_n^d$ at $\mathcal{E}$ can be computed from the cohomology sequence:

$$H^0(C, \mathcal{E}\mathrm{nd}(\mathcal{E})) \hookrightarrow H^0(C, \mathcal{E}\mathrm{nd}(\mathcal{E}_\bullet)) \to 0 = H^0(C, \mathcal{E}\mathrm{nd}(\mathcal{E})/\mathcal{E}\mathrm{nd}(\mathcal{E}_\bullet))$$
$$H^1(C, \mathcal{E}\mathrm{nd}(\mathcal{E})) \hookrightarrow H^1(C, \mathcal{E}\mathrm{nd}(\mathcal{E}_\bullet)) \to H^1(C, \mathcal{E}\mathrm{nd}(\mathcal{E})/\mathcal{E}\mathrm{nd}(\mathcal{E}_\bullet))$$

Here we used that $H^0(C, \mathcal{E}\mathrm{nd}(\mathcal{E})/\mathcal{E}\mathrm{nd}(\mathcal{E}_\bullet)) = 0$ because there are no homomorphisms from a semistable bundle to a semi stable bundle of smaller slope.

This implies the proposition (pointwise), because an unramified map (i.e., inducing an injection on the tangent spaces) whose fibres are points is an immersion.

Remark 5.10. In the preceding computation we could replace the bundle $\mathcal{E}$ by any family parametrized by an affine scheme $\mathrm{Spec}(A)$. For any family $T \to \mathrm{Bun}_n^d$ (of finite type) write $\mathbf{R}p_*\mathcal{E}nd(\mathcal{E}) = [K_0 \to K_1]$ as a complex of vector bundles on T. Using the computation for affine schemes it is easy to see that the pull back of $T\,\mathrm{Bun}_n^d$ to T is given by the quotient stack $[K_0/K_1]$. This then proves the last part of the proposition.

Example 5.11. We briefly consider the cae $n = 3, d = 1$, in order to indicate, why the closure of a HN-stratum does not need to bee a union of strata. (See [13] for a similar example and a complete analysis in the case of elliptic curves.) We consider strata of bundles such that the HN-filtration consists of a single subsheaf, namely the strata of type $t_1 = (n_1 = 1, d_1 = 1)$ and $t_2 = (n_1 = 2, d_1 = 2)$. Since the HN-polygon of t_1 lies below t_2, the closure of $\mathrm{Bun}_{3,1}^{t_1}$ can contain elements of $\mathrm{Bun}_{3,1}^{t_2}$, but by Remark 5.2 point 3. any such specialization $\mathcal{E}$ will contain a subsheaf $\mathcal{L}$ of rank 1 and degree 1. Since the destabilizing subbundle $\mathcal{E}_1$ of $\mathcal{E}$ is a vector bundle of rank 2 and degree 2, $\mathcal{L}$ will be contained in $\mathcal{E}_1$, so that $\mathcal{E}_1$ is semi-stable but not stable. Thus, in case there exist stable bundles of rank 2 and degee 2 on C, the closure of $\mathrm{Bun}_{3,1}^{t_1}$ cannot contain the whole of $\mathrm{Bun}_{3,1}^{t_2}$.

Let us give a concrete example, to show that the closure indeed intersects $\mathrm{Bun}_{3,1}^{t_2}$: The stratum $\mathrm{Bun}_{3,1}^{t_2}$ is non-empty because it contains direct sums of line bundles $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ with $\deg(\mathcal{L}_1) = \deg(\mathcal{L}_2) = 1$ and $\deg(\mathcal{L}_3) = -1$.

Moreover, $\mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_3) = H^1(C, \mathcal{L}_3 \otimes \mathcal{L}_2^{-1})$ and by Riemann-Roch this is a vector space of dimension $2 - 1 + g > 0$. Thus there exist non-trivial extensions $\mathcal{L}_3 \to \mathcal{E}_2 \to \mathcal{L}_2$ and such an extension $\mathcal{E}_2$ cannot contain subbundles of positive degree, since such a subbundle would have to split the extension. So we find a bundle $\mathcal{L}_1 \oplus \mathcal{E}_2$ in $\mathrm{Bun}_{3,0}^{t_1}$ that can be degenerated into $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$.

Corollary 5.12. *There is a Gysin sequence:*

$$\cdots \to H^{*-codim}(\mathrm{Filt}^{t,ss}) \to H^*(\mathrm{Bun}_n^{d,\leq t}) \to H^*(\mathrm{Bun}_n^{d,<t}) \to \dots .$$

To prove that this sequence splits we need a lemma:

Lemma 5.13. *Let $p : \widetilde{\mathcal{M}} \to \mathcal{M}$ be a $\mathbb{G}_m$ gerbe and $\mathcal{E}$ be a vector bundle of weight $w \neq 0$ on $\widetilde{\mathcal{M}}$. Then*

$$H^*(\widetilde{\mathcal{M}}) = H^*(\mathcal{M})[c_1(\mathcal{E})]$$

and the top Chern class $c_{top}(\mathcal{E})$ is not a zero divisor in $H^*(\widetilde{\mathcal{M}})$.

Proof. First note that the map $(p, \det(\mathcal{E})) : \widetilde{\mathcal{M}} \to \mathcal{M} \times B\mathbb{G}_m$ is a $\mu_{\mathrm{rk}(\mathcal{E})\cdot w}$-gerbe. For finite groups G the cohomology $H^*(BG, \overline{\mathbb{Q}}_\ell)$ vanishes. Therefor the Leray spectral sequence for $(p, \det(\mathcal{E}))$ shows that $H^*(\widetilde{\mathcal{M}}, \overline{\mathbb{Q}}_\ell) \cong H^*(\mathcal{M})[c_1]$. Which is the first claim.

Now let $x : \mathrm{Spec}(k) \to \mathcal{M}$ be any geometric point and $\tilde{x} : \mathrm{Spec}(k) \to \widetilde{\mathcal{M}}$ a lift of x. Then $\mathrm{Spec}(k) \times_{\mathcal{M}} \widetilde{\mathcal{M}} = B\mathbb{G}_m$ canonically. Using this we get a map:

$$m : B\mathbb{G}_m \xrightarrow{i} \widetilde{\mathcal{M}} \xrightarrow{\det(\mathcal{E})} B\mathbb{G}_m.$$

And this composition is given by raising to the $\mathrm{rk}(\mathcal{E}) \cdot w$'th power.

Thus $i^*(c_{top}(\mathcal{E})) = (w \cdot c_1)^n$. Writing $c_{top}(\mathcal{E}) = \sum_{i=0}^{rk(\mathcal{E})} \beta_i c_1^i$ we see that $\beta_{\mathrm{rk}(\mathcal{E})}$ is a non-zero constant and this proves the second claim. □

Corollary 5.14. *The Gysin sequence:*

$$\cdots \to H^{*-codim}(\mathrm{Filt}^{t,ss}) \to H^*(\mathrm{Bun}_n^{d,\leq t}) \to H^*(\mathrm{Bun}_n^{d,<t}) \to \dots$$

splits.

In particular $H^*(\mathrm{Bun}_n^{d,ss})$ *is a quotient of* $H^*(\mathrm{Bun}_n^d)$ *and thus generated by the Atiyah-Bott classes.*

Proof. The composition $H^{*-codim}(\mathrm{Filt}^{t,ss}) \to H^*(\mathrm{Bun}_n^{d,\leq t}) \to H^*(\mathrm{Filt}^{t,ss})$ is given by the cup product with $c_{top}(\mathcal{N}_{forget})$, which is injective. □

Remark 5.15. The cohomology of Filt^t can also be computed:

$$H^*(\mathrm{Filt}_{\underline{n}}^{\underline{d}}) = \otimes_i H^*(\mathrm{Bun}_{n_i}^{d_i}).$$

The same holds for the semistable part.

This remark implies, that the cohomology of the instable part which occurs in the Gysin sequence can be described in terms of the cohomology of moduli stacks of bundles of smaller rank. This gives an inductive procedure to compute $H^*(\mathrm{Bun}_n^{d,ss})$ for all n, d. Furthermore, in case that $(n, d) = 1$ this space is a $\mathbb{G}_m$ gerbe over the coarse moduli space, so we get a recursive formula for the cohomology of the coarse moduli space as well, by Lemma 5.13.

However, since the recursive formula contains a sum over all possible types of instability, the result will not look very pleasant and we will not write it down. To resolve the recursive formula is a quite difficult combinatorical problem. This was first solved by Zagier [26], and in the more general situation of $G-$bundles this was solved by Laumon and Rapoport [21].

In the special case of vector bundles of rank 2 these difficulties disappear, so let us give the result in this simple case.

In order to cope with the formulae, let us introduce the Poincaré series of an algebraic stack $\mathcal{X}$ with finite dimensional cohomology groups:

$$P(\mathcal{X}, t) := \sum_{i=0}^{\infty} \dim(H^i(\mathcal{X}))t^i.$$

More generally, one can also use this formula to define $P(H^*, t)$ for any graded algebra H^* such that the graded pieces H^i are finite dimensional. For example

$P(\overline{\mathbb{Q}}_\ell[z],t) = \frac{1}{1-t^{\deg(z)}}$ and $P(H_1^* \otimes H_2^*, t) = P(H_1^*,t)P(H_2^*,t)$. This implies that

$$P(\mathrm{Bun}_{n,d},t) = \frac{\prod_{i=1}^n (1+t^{2i-1})^{2g}}{\prod_{i=1}^n (1-t^{2i}) \prod_{i=1}^{n-1}(1-t^{2i})}.$$

Theorem 5.16. *The Poincaré series of the moduli stacks of semi-stable bundles of rank 2 are:*

$$P(\mathrm{Bun}^{ss}_{2,d},t) = \begin{cases} \frac{(1+t)^{2g}}{(1-t^2)^2(1-t^4)}((1+t^3)^{2g} - t^{2g}(1+t)^{2g}) & \text{if } d \text{ is odd} \\ \frac{(1+t)^{2g}}{(1-t^2)^2(1-t^4)}((1+t^3)^{2g} - t^{2g+2}(1+t)^{2g}) & \text{if } d \text{ is even} \end{cases}$$

For odd d we have:

$$P(M_2^{d,stable}),t) = \frac{(1+t)^{2g}}{(1-t^2)(1-t^4)}((1+t^3)^{2g} - t^{2g}(1+t)^{2g}).$$

Remark 5.17. It is a nice exercise to check these formulae for $g = 0$ using an explicit description of $\mathrm{Bun}^{ss}_{2,d}$.

Proof. Remark 5.15 implies that

$$P(\mathrm{Filt}^{(i,d-i)}_{(1,2)},t) = P(\mathrm{Bun}^i_1,t)P(\mathrm{Bun}^{d-i}_1,t) = \frac{((1+t)^{2g})^2}{(1-t^2)^2},$$

which is independent of i and d.

To compute the codimension of a HN-stratum, we recall the fibre of the normal bundle to the stratum $\mathrm{Bun}^{(1,i)}_{2,d}$ at $\mathcal{E} \in \mathrm{Bun}^{(1,i)}_{2,d}$ is $H^1(C, \mathcal{E}\mathrm{nd}(\mathcal{E})/\mathcal{E}\mathrm{nd}(\mathcal{E}_\bullet))$. By the Riemann-Roch theorem the dimension of this vector space is $-((d-2i)+1-g) = g-1+2i-d$. Now we apply Corollary 5.14:

$$\begin{aligned} P(\mathrm{Bun}^{ss}_{n,d},t) &= P(\mathrm{Bun}_{n,d},t) - \sum_{i>\frac{d}{2}} t^{2(g-1+2i-d)} P(\mathrm{Filt}^{(i,d-i)}_{(1,2)},t) \\ &= P(\mathrm{Bun}_{n,d},t) - P(\mathrm{Bun}^0_1,t)^2 t^{2g-2} \sum_{i>\frac{d}{2}} t^{2(2i-d)}. \end{aligned}$$

Thus for odd d we find:

$$P(\mathrm{Bun}^{ss}_{n,d},t) = \frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^2)^2(1-t^4)} - \frac{((1+t)^{2g})^2}{(1-t^2)^2}\frac{t^{2g}}{1-t^4},$$

and for even d we have:

$$P(\mathrm{Bun}^{ss}_{n,d},t) = \frac{(1+t)^{2g}(1+t^3)^{2g}}{(1-t^2)^2(1-t^4)} - \frac{((1+t)^{2g})^2}{(1-t^2)^2}\frac{t^{2g+2}}{1-t^4},$$

To deduce the statement for the coarse moduli space we note that for $(2,d) = 1$ we have seen (Corollary 3.12 and Lemma 3.10) that $\mathrm{Bun}^{ss}_{2,d} = M_2^{d,\mathrm{stable}} \times B\mathbb{G}_m$. And we know $P(B\mathbb{G}_m,t) = P(\overline{\mathbb{Q}}_\ell[c_1],t) = \frac{1}{1-t^2}$. This proves the theorem. $\square$

Exercise 5.18. In the lecture we used the following construction: Let $E_0 \to E_1$ be a map of vector bundles on a stack $\mathcal{M}$, then the quotient stack $[E_1/E_0]$ is an algebraic stack. Prove this by giving a presentation.

Now if $E_0' \to E_1'$ is another map of vector bundles and $(E_0 \to E_1) \to (E_0' \to E_1')$ a morphism of complexes, which is a quasi-isomorphism, then this map induces an isomorphism $[E_1/E_0] \to [E_1'/E_0']$.

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