ON DERIVED EQUIVALENCES OF CATEGORIES OF SHEAVES OVER FINITE POSETS

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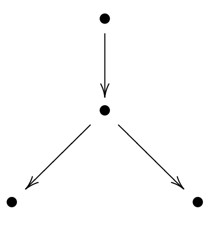
Throughout this note, the term *poset* will mean a finite partially ordered set. Given a poset $(X, \leq)$, one can define on X a structure of a T_0 topological space by saying that a subset $Y \subseteq X$ is closed if whenever $y \in Y$ and $y' \leq y$ we have $y' \in Y$. In this way, posets are identified with finite T_0 topological spaces. Such spaces have been studied in the past ([2], [5]) and have been shown to be quite general in terms of their homotopy and homology properties.

The *Hasse diagram* G_X of X is a directed graph whose set of vertices is X, with edges $x \to y$ whenever $x < y$ and there is no $z \in X$ with $x < z < y$. Given an abelian category $\mathcal{A}$, the category of sheaves on X (considered as a topological space) with values in $\mathcal{A}$ will be denoted by $\mathcal{A}^X$. It can be shown that the category $\mathcal{A}^X$ is equivalent to the category of (commutative) diagrams over $\mathcal{A}$ with shape G_X ([3], [4]), with the global sections functor corresponding to the limit functor.

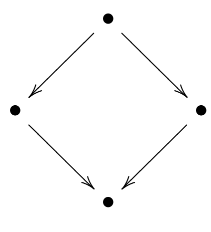
Fix a field k and let $\mathcal{A}$ be the category of finite dimensional vector spaces over k. In this case $\mathcal{A}^X$ is equivalent to the category of finite dimensional right modules over the incidence algebra of X, which is a finite dimensional algebra over k. One can also think of $\mathcal{A}^X$ as the category of representations of the quiver G_X with all the commutativity relations.

Given a category $\mathcal{A}^X$ with $\mathcal{A}$ as before, the graph G_X, hence X, can be recovered up to isomorphism by reconstructing the vertices as the simple objects of $\mathcal{A}^X$, and the edges by considering Ext^1 between the simple objects.

However, the situation is different for $\mathcal{D}^b(X)$, the bounded derived category of $\mathcal{A}^X$, as there are equivalences as triangulated categories $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ (*derived equivalences*) for non-isomorphic posets X, Y. Two well-known examples are the BGP reflection functors [1] and the equivalence between the derived categories of diagrams over



and



This leads to the natural question:

- When $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ for two posets X, Y?

We present a general construction that gives many derived equivalences between posets, and generalizes the above examples.

Let S be a poset, and let $\mathfrak{X} = \{X_s\}_{s \in S}$ be a collection of posets indexed by the elements of S. The *lexicographic sum of the X_s along S*, denoted $\oplus_S \mathfrak{X}$, is the poset $(X, \leq)$ where $X = \coprod_{s \in S} X_s$ is the disjoint union of the X_s and for $x \in X_s$, $y \in X_t$ we have $x \leq y$ if either $s < t$ (in S) or $s = t$ and $x \leq y$ (in X_s).

A poset S is called a *bipartite graph* if its Hasse diagram is a bipartite graph. This is equivalent to the condition that one can write $S = S_0 \coprod S_1$ as a disjoint union with S_0, S_1 discrete (anti-chains) such that $s < s'$ in S implies $s \in S_0$, $s' \in S_1$.

Given a poset S, we denote by S^{op} the *opposite poset*, with $S^{op} = S$ and $s \leq s'$ in S^{op} if and only if $s \geq s'$ in S.

Theorem. *If S is a bipartite graph and $\mathfrak{X} = \{X_s\}_{s \in S}$ is a collection of posets, then $\mathcal{D}^b(\oplus_S \mathfrak{X}) \simeq \mathcal{D}^b(\oplus_{S^{op}} \mathfrak{X})$ as triangulated categories.*

In particular, we get that $\mathcal{D}^b(X \oplus Y) \simeq \mathcal{D}^b(Y \oplus X)$ where $\oplus$ is the ordinal sum, without any restriction on the posets X, Y.

The construction is based on ideas from sheaf theory. We note that the technique used in the proof can also give derived equivalences between categories of sheaves of posets and representations of quivers with additional relations.

By considering the Euler bilinear form over the Grothendieck group of $\mathcal{D}^b(X)$, questions on derived equivalence can be transformed to questions on congruency of integral matrices.

If time permits, the question when $\mathcal{D}^b(X_1 \oplus X_2 \oplus X_3) \simeq \mathcal{D}^b(X_2 \oplus X_1 \oplus X_3)$ for three posets will be addressed. While there are examples showing that this is not true in general, one can still have such derived equivalences for certain classes of posets.

References

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