LECTURES ON HALL ALGEBRAS

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Abstract

Let $A$ be an abelian category which satisfies the following finiteness conditions: for any two objects $M, N \in \text{Obj}(A)$ and any $j \geq 1$ the sets $\text{Hom}(M, N)$ and $\text{Ext}^j(M, N)$ are finite, and $A$ is itself of finite global dimension (that is there exists $i \geq 0$ such that $\text{Ext}^i(M, N) = 0$ for any $M, N \in \text{Obj}(A)$). Such categories abound: for example, one may take $A$ to be the category of representations of a quiver over some finite field $\mathbb{F}_q$, or the category of coherent sheaves over some smooth projective curve $C$ again defined over some finite field $\mathbb{F}_q$. Following an idea of Ringel, who was himself inspired by Hall, Steinitz and others, one associates to such a category $A$ an associative algebra $H_A$, called the Hall, or Ringel-Hall algebra of $A$, defined as follows: as a vector space, $H_A$ has a $C$-basis $\{[M]\}_{M \in \mathcal{I}}$ parametrized by the set $\mathcal{I} = \text{Obj}(A)/\sim$ of objects of $A$, counted up to isomorphism, and with product

$$[M] \cdot [N] = \sum_{Q \in \mathcal{I}} P^Q_{M,N}[Q],$$

where $P^Q_{M,N} = \# \{L \subset Q \mid L \simeq N, Q/L \simeq M\}$ is the number of subobjects of $Q$ of type $N$ and cotype $M$. Thus the Hall algebra encodes the structure of extensions between objects of $A$, and many properties of the category $A$ can be read off from its Hall algebra. Note that $H_A$ is in general not commutative (as the set of objects which may be obtained as an extension of $M$ by $N$ is usually different from the set of objects which may be obtained as an extension of $N$ by $M$).

The aim of the course is to study various properties of the algebras $H_A$, both in general, and for particular choices of abelian category $A$ (such as the above-mentioned examples related to quivers and curves). It turns out that these are often isomorphic to the quantum groups associated to certain (Kac-Moody or loop) Lie algebras. Hence Hall algebras throw a bridge between representations of quivers or vector bundles on curves on the one hand, and the structure of Kac-Moody or loop Lie algebras on the other hand. Such a link has proven to be extremely fruitful for both fields of mathematics, as the course will attempt to show.

References


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