Preprojective Algebras

and

Cluster algebras

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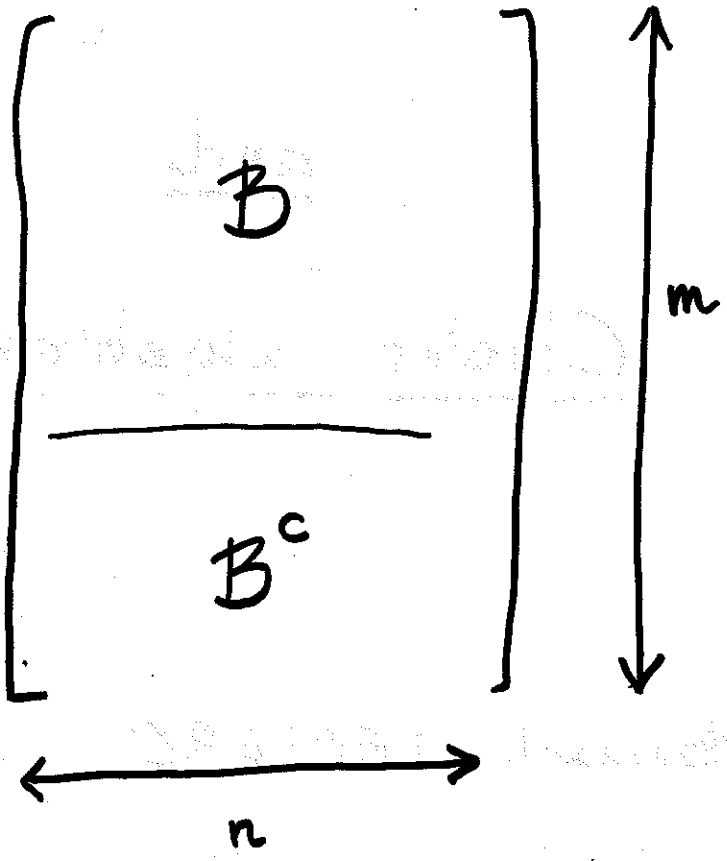
1 "Abstract" cluster algebras

$$\mathcal{F} = \mathbb{Q}(x_1, \dots, x_m); \quad 1 \leq n \leq m.$$

$$\tilde{B} = \begin{bmatrix} B \\ \hline B^c \end{bmatrix}$$



- B is anti-symmetric

$\tilde{B}$ $\longleftrightarrow$ quiver

$$\widetilde{x} := \{x_1, \dots, x_m\}$$

$$x := \{x_1, \dots, x_n\} \subset \widetilde{x} \quad \text{(cluster)}$$

$$\mathcal{Y} := (\widetilde{x}, \widetilde{B}) \quad \text{(initial seed)}$$

For $k \leqslant n$, seed mutation in direction k :

$$\mu_k(\mathcal{Y}) = (\mu_k(\widetilde{x}), \mu_k(\widetilde{B}))$$

where :

$$\left| \begin{array}{l} \mu_k(\widetilde{B}) \quad \longleftrightarrow \quad \text{quiver mutation} \\ \\ x_k^* = \dfrac{\prod\limits_{b_{ik}>0} x_i^{b_{ik}} + \prod\limits_{b_{ik}<0} x_i^{-b_{ik}}}{x_k} \end{array} \right.$$

$\mathcal{A}_{\mathcal{Y}} :=$ subalgebra of $\mathcal{F}$ generated by all mutations of $\widetilde{x}$.

2 "Concrete" cluster algebras

$\mathfrak{g}$ = complex simple Lie algebra of type A, D, E

$\mathfrak{n}$ = maximal nilpotent subalgebra

N = unipotent complex Lie group with $\mathrm{Lie}(N) = \mathfrak{n}$

$\mathbb{C}[N]$ = ring of polynomial functions $f : N \longrightarrow \mathbb{C}$

r = $\dim N$

n = $\mathrm{rk}\ \mathfrak{g}$

Berenstein–Fomin–Zelevinsky

$\rightsquigarrow$ cluster algebra structure on $\mathbb{C}[N]$

initial seed ??

Factorization problem

- e_i = Chevalley generators of $\mathfrak{n}$
- $x_i(t) = \exp(t e_i) \in N$
- $\underline{i} = (i_1, \cdots, i_r)$ reduced word for w_0

- $\Psi_{\underline{i}} : \mathbb{C}^r \longrightarrow N$

$$(t_1, \cdots, t_r) \longmapsto x_{i_1}(t_1) \cdots x_{i_r}(t_r)$$

is injective and the image is dense.

Problem: For $x \in \mathrm{Im}\, \Psi_{\underline{i}}$, write $t_1, \cdots, t_r$ as functions of x.

Solution: B-Z 1997

Ex: A_3 $\quad \underline{i} = (3,2,1,3,2,3)$

$$x = x_3(t_1)\, x_2(t_2)\, x_1(t_3)\, x_3(t_4)\, x_2(t_5)\, x_3(t_6)$$

$$= \begin{pmatrix} 1 & t_3 & t_3 t_5 & t_3 t_5 t_6 \\ & 1 & t_2 + t_5 & t_2 t_4 + t_2 t_6 + t_5 t_6 \\ & & 1 & t_1 + t_3 + t_6 \\ & & & 1 \end{pmatrix}$$

$$t_3 = \Delta_2 \qquad t_5 = \frac{\Delta_3}{\Delta_2} \qquad t_6 = \frac{\Delta_4}{\Delta_3}$$

$$t_2 = \frac{\Delta_{23}}{\Delta_2} \qquad t_4 = \frac{\Delta_{34}\ \Delta_2}{\Delta_{23}\ \Delta_3} \qquad t_1 = \frac{\Delta_{234}}{\Delta_{23}}$$

are rational functions in

$$\mathcal{C} = \{\Delta_2, \Delta_3, \Delta_{23}, \Delta_4, \Delta_{34}, \Delta_{234}\}$$

Ex: $A_3 \qquad \underline{i} = (3,2,1,2,3,2)$

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$$= \begin{bmatrix} 1 & t_3 & t_3t_4 + t_3t_6 & t_3t_4t_5 \\ & 1 & t_2 + t_4 + t_6 & t_2t_5 + t_4t_5 \\ & & 1 & t_1 + t_5 \\ & & & 1 \end{bmatrix}$$

$$t_3 = \Delta_2 \qquad t_2 = \frac{\Delta_{23}}{\Delta_2} \qquad t_5 = \frac{\Delta_{24}}{\Delta_{23}}$$

$$t_4 = \frac{\Delta_4\, \Delta_{23}}{\Delta_2\, \Delta_{24}} \qquad t_6 = \frac{\Delta_{34}}{\Delta_{24}} \qquad t_1 = \frac{\Delta_{234}}{\Delta_{23}}$$

are rational functions in

( )

$$\mathcal{C} \xleftrightarrow{\text{mutation}} \mathcal{C}'$$

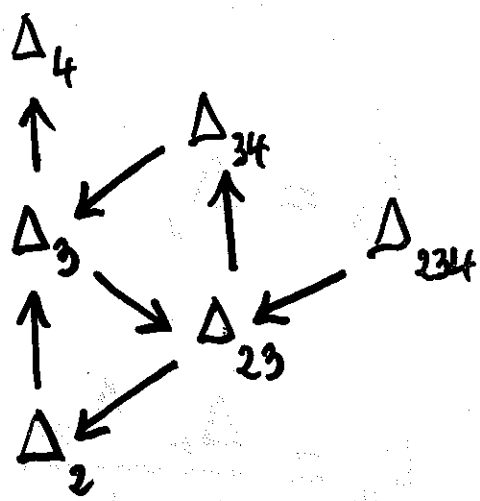
$$\Delta_3 \longleftrightarrow \Delta_{24}$$

$$\Delta_3 \Delta_{24} = a_{13} \begin{vmatrix} a_{12} & a_{14} \\ 1 & a_{24} \end{vmatrix}$$

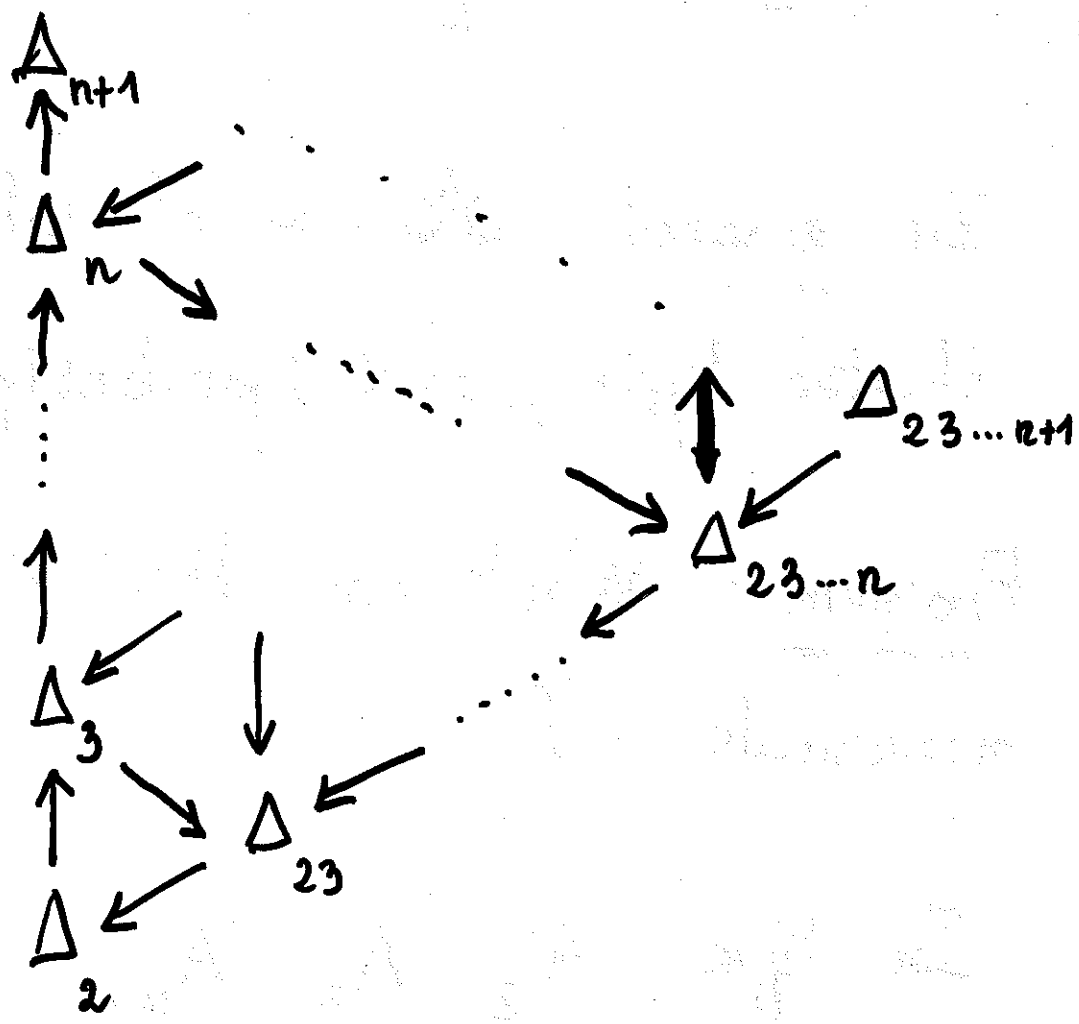
$$= a_{12} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} + a_{14} \begin{vmatrix} a_{12} & a_{13} \\ 1 & a_{23} \end{vmatrix}$$

$$= \Delta_2 \Delta_{34} + \Delta_{23} \Delta_4$$

$$\tilde{B}(\mathcal{C}) = \begin{array}{c} \\ \Delta_2 \\ \Delta_3 \\ \Delta_{23} \\ \cdots \\ \Delta_4 \\ \Delta_{34} \\ \Delta_{234} \end{array} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \\ \cdots & \cdots & \cdots \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$



In general in type A_n :



is an initial seed $\mathcal{S}$.

Notation : $\mathcal{A} = \mathcal{A}_{\mathcal{S}} \subset \mathbb{C}(N)$

Th [BFZ] The cluster algebra $\mathcal{A} = \mathbb{C}[N]$.

In general $\mathcal{A}$ is of infinite cluster type, and (probably) not acyclic.

Problem: What are the cluster monomials ?

In type A_2, A_3, A_4,

{cluster monomials} = dual canonical basis of $\mathbb{C}[N]$

3 Preprojective algebras

Q : orientation of the Dynkin diagram of $\mathfrak{g}$

$\overline{Q}$: double quiver

$\Lambda := \mathbb{C}\overline{Q}/(\rho)$ preprojective algebra

$$\rho = \sum_{a \in Q} a^* a - a a^*$$

(Gelfand–Ponomarev 1979)

- Λ is finite-dimensional, self-injective, of infinite representation type (except $A_{2,3,4}$)
- $\underline{\text{mod}}\,\Lambda$ is 2-Calabi-Yau.
- But prefer to work with $\text{mod}\,\Lambda$.

$\rightsquigarrow$ "concrete" cluster algebra $\mathbb{C}[N]$ with coefficients.

Th: For $M \in \operatorname{mod} \Lambda$, there exists a unique $\varphi_M \in \mathbb{C}[N]$ such that:

$$\forall\, \underline{i} = (i_1, \dots, i_k)\,,\ \forall\, (t_1, \dots, t_k) \in \mathbb{C}^k,$$

$$\varphi_M\left(x_{i_1}(t_1) \cdots x_{i_k}(t_k)\right) = \sum_{\substack{\underline{a} = (a_1, \dots, a_k) \\ \in \mathbb{N}^k}} \chi\left(\Phi_{\underline{i}^{\underline{a}}, M}\right) \frac{\underline{t}^{\underline{a}}}{\underline{a}!}$$

where: $\Phi_{\underline{i}^{\underline{a}}, M}$ = variety of composition series of M of type

$$(\underbrace{i_1, \dots, i_1}_{a_1}, \dots\dots, \underbrace{i_k, \dots, i_k}_{a_k})$$

χ = topological Euler characteristic

$$\frac{\underline{t}^{\underline{a}}}{\underline{a}!} = \frac{t_1^{a_1} \cdots t_k^{a_k}}{a_1! \cdots a_k!}$$

Proof:

- Geometric construction of $U(\mathfrak{n})$ in terms of constructible functions on varieties of Λ-modules (Lusztig).
- Dualize.

Ex: All (generalized) minors of BFZ are of the form φ_M, where M is a subquotient of an indecomposable projective Λ-module.

Aim: Try to understand the cluster structure on $\mathbb{C}[N]$ via the map:

$$\begin{aligned} \operatorname{mod}\Lambda &\longrightarrow \mathbb{C}[N] \\ M &\longmapsto \varphi_M \end{aligned}$$

- Similar to [BMRRT], [CC], [CK]

$$\begin{aligned} \text{cluster cat.} &\longrightarrow \text{acyclic cluster alg.} \\ M &\longmapsto X_M \end{aligned}$$

4 Rigid modules

Def: $M \in \mathrm{mod}\,\Lambda$ is rigid $\Leftrightarrow \mathrm{Ext}^1_\Lambda(M,M) = 0$

$\Leftrightarrow \mathcal{O}_M$ is open in its module variety

$\Rightarrow \varphi_M$ belongs to the dual of Lusztig's semi-canonical basis of $\mathbb{C}[N]$

Th (Geiss-Schröer) M rigid.
M has at most r non isomorphic indecomp. summands.

Df: M is maximal rigid if it has r summands.

Note: Every maximal rigid module contains the n indecomposable projective.

Does there exist a maximal rigid module ??

Th: Let A_Q = Auslander alg. of Q

$\widehat{A}_Q$ = repetitive algebra
= Galois covering of Λ

F_λ : mod $\widehat{A}_Q \longrightarrow$ mod Λ

$T_Q := F_\lambda(A_Q)$

- Then - T_Q is maximal rigid stable
- quiver of $\mathrm{End}_\Lambda T_Q$ is the AR-quiver of Q + additional arrows : $M \to \tau(M)$
- If $T_Q = T_1 \oplus \cdots \oplus T_{r-n} \oplus P_1 \oplus \cdots \oplus P_n$

$\widetilde{B}$ = $r \times (r-n)$ matrix constructed from the quiver of $\mathrm{End}_\Lambda T_Q$

then $\left((\varphi_{T_1}, \cdots, \varphi_{P_n}), \widetilde{B}\right)$ is one of the initial seeds of BFZ for $\mathbb{C}[N]$.

Consequences : Quiver of $\mathrm{End}_\Lambda T_Q$ has no loop

$\Rightarrow$ $\mathrm{End}_\Lambda T_Q$ has global dimension 3

(approximation theory)

$\Rightarrow$ (Iyama) T_Q is a maximal 1-orthogonal module

Then using tilting theory and imitating some arguments of Iyama :

Th : For every basic maximal rigid Λ-module T, we have :

- $\mathrm{gl\,dim}\,(\mathrm{End}_\Lambda T) = 3$
- quiver of $\mathrm{End}_\Lambda T$ has no loop, no 2-cycle, no source.

$\Rightarrow$ mutation theory for basic complete rigid modules

Essential ingredient: multiplication formula

$$T = T_1 \oplus \cdots \oplus T_{r-n} \oplus P_1 \oplus \cdots \oplus P_n$$

$$\mu_k(T) = T_1 \oplus \cdots \oplus T_k^* \oplus \cdots \oplus P_n$$

$$0 \to T_k \to T' \to T_k^* \to 0$$

$$0 \to T_k^* \to T'' \to T_k \to 0$$

Then $\varphi_{T_k}\varphi_{T_k^*} = \varphi_{T'} + \varphi_{T''}$

Proof: inspired and adapted from Caldero–Keller

Conclusion: The cluster algebra structure on $\mathbb{C}[N]$ can be lifted to mod Λ

clusters $\rightsquigarrow$ basic maximal rigid modules

cluster variables $\rightsquigarrow$ indecomposable non projective rigid modules

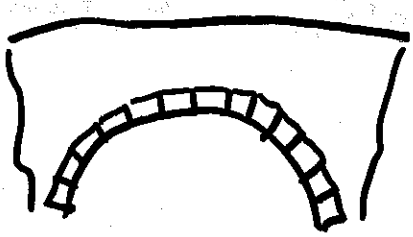
"coefficients" $\rightsquigarrow$ projective modules

cluster monomials $\rightsquigarrow$ rigid modules

$\Rightarrow$ $\{$ cluster monomials $\} \subsetneq S^*$

dual semi canonical basis

BFZ



Lusztig