DERIVED CATEGORIES AND TILTING

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Abstract. We review the basic definitions of derived categories and derived functors. We illustrate them on simple but non trivial examples. Then we explain Happel's theorem which states that each tilting triple yields an equivalence between derived categories. We establish its link with Rickard's theorem which characterizes derived equivalent algebras. We then examine invariants under derived equivalences. Using t-structures we compare two abelian categories having equivalent derived categories. Finally, we briefly sketch a generalization of the tilting setup to differential graded algebras.

Contents

1. Introduction

1.1. **Motivation: Derived categories as higher invariants.** Let k be a field and A a k-algebra (associative, with 1). We are especially interested in the case where A is a non commutative algebra. In order to study A, one often looks at various invariants associated with A, for example its Grothendieck group $K_0(A)$, its center $Z(A)$, its higher K-groups $K_i(A)$, its Hochschild cohomology groups $HH^*(A, A)$, its cyclic cohomology groups .... Of course, each isomorphism of algebras $A \to B$ induces an isomorphism in each of these invariants. More generally, for each of them, there is a fundamental theorem stating that the invariant is preserved not only under isomorphism but also under passage from A to a matrix ring $M_n(A)$, and, more generally, that it is preserved under *Morita equivalence.* This means that it only depends on the category $\mathsf{Mod}\, A$ of (right) A-modules so that one can say that the map taking A to any of the above invariants factors through the map which takes A to its module category:

$$\begin{array}{ccc} A & \longmapsto & K_0(A), Z(A), K_i(A), HH^*(A,A), HC_*(A), \ldots \\ & \searrow & \nearrow \\ & \mathsf{Mod}\, A & \end{array}$$

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Now it turns out that for each of these invariants, there is a second fundamental theorem to the effect that the invariant does not depend so much on the module category but only on its *derived category* $\mathsf{D}\,\mathsf{Mod}\,A$ in the sense that each (triangle) equivalence between derived categories allows us to construct an isomorphism in the invariant. So we obtain a second factorization:

$$\begin{array}{ccc} A & \longmapsto & K_0(A), Z(A), K_i(A), HH^*(A,A), HC_*(A), \ldots \\ & \searrow & \uparrow \\ & \mathsf{Mod}\,A \longmapsto & \mathsf{D}\,\mathsf{Mod}\,A \end{array}$$

In this picture, the derived category appears as a kind of *higher invariant*, an invariant which, as we will see, is much coarser than the module category (at least in the non commutative case) but which is still fine enough to determine all of the classical homological and homotopical invariants associated with A.

Tilting theory enters the picture as a rich source of derived equivalences. Indeed, according to a theorem by D. Happel, if B is an algebra and T a tilting module for B with endomorphism ring A, then the total derived tensor product by T is an equivalence from $\mathsf{D}\,\mathsf{Mod}\,A$ to $\mathsf{D}\,\mathsf{Mod}\,B$. In particular, A and B then share all the above-mentioned invariants. But an equivalence between the derived categories of $\mathsf{Mod}\,A$ and $\mathsf{Mod}\,B$ also yields strong links between the abelian categories $\mathsf{Mod}\,A$ and $\mathsf{Mod}\,B$ themselves: often, it allows one to identify suitable 'pieces' of $\mathsf{Mod}\,A$ with 'pieces' of $\mathsf{Mod}\,B$. This has proved to be an extremely useful method in representation theory.

1.2. **Contents.** We will recall the definition of the derived category of an abelian category. We will make this abstract construction more intuitive by considering the quivers of module categories and their derived categories in several examples. These examples will suggest the existence of certain equivalences between derived categories. We will construct these equivalences using D. Happel's theorem: the derived functor of tensoring by a tilting module is an equivalence. We will then proceed to a first crude analysis of the relations between module categories with equivalent derived categories and examine some examples. In the next section, we generalize Happel's theorem to Rickard's Morita theorem for derived categories. Here, the key notion is that of a tilting complex. This generalizes the notion of a tilting module. Tilting modules over selfinjective algebras are always projective, but there may exist non trivial tilting complexes. We illustrate this by exhibiting the action of a braid group on the derived category of a selfinjective algebra following Rouquier-Zimmermann. Then we proceed to a more sophisticated analysis of the links between two abelian categories with equivalent derived categories. We use aisles (=t-structures) and also sketch the link with the spectral sequence approach due to Vossieck and Brenner-Butler. Finally, we show how the notion of a tilting complex can be weakened even more if, instead of algebras, we consider differential graded algebras. We present the description of suitable 'algebraic' triangulated categories via derived categories of differential graded algebras. As an illustration, we present D. Happel's description of the derived category of a finite-dimensional algebra via the category of graded modules over its trivial extension.

2. Derived categories

2.1. **First definition.** Let $\mathcal{A}$ be an additive category. For example, $\mathcal{A}$ could be the category $\mathsf{Mod}\,A$ of (right) modules over a ring A or the category $\mathsf{Mod}\,O_X$ of sheaves of O_X-modules on a ringed space (X, O_X). A *complex over* $\mathcal{A}$ is a diagram

$$\cdots \longrightarrow M^p \xrightarrow{d^p} M^{p+1} \longrightarrow \cdots \;,\; p \in \mathbb{Z}\,,$$

such that $d^p \circ d^{p-1} = 0$ for all $p \in \mathbb{Z}$. A *morphism of complexes* is a morphism of diagrams. We obtain the *category of complexes* $\mathsf{C}\,\mathcal{A}$.

Now suppose that $\mathcal{A}$ is abelian. This is the case for the above examples. For $p \in \mathbb{Z}$, the *pth homology* H^pM of a complex M is $\ker d^p / \operatorname{im} d^{p-1}$. A morphism of complexes is a *quasi-isomorphism* if it induces isomorphisms in all homology groups. Ignoring a set-theoretic problem, we define the *derived category* $\mathsf{D}\,\mathcal{A}$ as the localization of the category of complexes with respect to the class of quasi-isomorphisms. This means that the objects of the derived category are all complexes. And morphisms in the derived category between two complexes are given by paths composed of morphisms of complexes and formal inverses of quasi-isomorphisms, modulo a suitable equivalence relation *cf.* [28].

This quick definition is not very explicit but it immediately yields an important universal property of the derived category: The canonical functor $\mathsf{C}\,\mathcal{A} \to \mathsf{D}\,\mathcal{A}$ makes all quasi-isomorphisms invertible and is universal among all functors $F : \mathsf{C}\,\mathcal{A} \to \mathcal{C}$ with this property. More precisely, for each category $\mathcal{C}$, the canonical functor $\mathsf{C}\,\mathcal{A} \to \mathsf{D}\,\mathcal{A}$ yields an isomorphism of functor categories

$$\mathsf{Fun}(\mathsf{D}\,\mathcal{A}, \mathcal{C}) \xrightarrow{\sim} \mathsf{Fun}_{qis}(\mathsf{C}\,\mathcal{A}, \mathcal{C}) ,$$

where the category on the right is the full subcategory on the functors making all quasi-isomorphisms invertible. We deduce that a pair of exact adjoint functors between two abelian categories induces a pair of adjoint functors between their derived categories.

2.2. **Second definition.** We keep the notations of paragraph 2.1. A morphism of complexes $f : L \to M$ is *null-homotopic* if there are morphisms $r^p : L^p \to M^{p-1}$ such that $f^p = d^{p-1} \circ r^p + r^{p+1} d^p$ for all $p \in \mathbb{Z}$. Null-homotopic morphisms form an *ideal* in the category of complexes. We define the *homotopy category* $\mathsf{H}\,\mathcal{A}$ to be the quotient of $\mathsf{C}\,\mathcal{A}$ by this ideal. Thus, the objects of $\mathsf{H}\,\mathcal{A}$ are all complexes, and morphisms between two objects are classes of morphisms of complexes modulo null-homotopic morphisms. Note that the homology functors $M \mapsto H^pM$ descend to functors defined on the homotopy category. A *quasi-isomorphism in* $\mathsf{H}\,\mathcal{A}$ is a morphism whose image under the homology functors is invertible. Let Σ be the class of quasi-isomorphisms in $\mathsf{H}\,\mathcal{A}$. The following lemma states that the analogues of the Ore conditions in the localization theory of rings hold for the class Σ (the assumption that the elements to be made invertible be non-zero divisors is weakened into condition c).

Lemma. a) *Identities are quasi-isomorphisms and compositions of quasi-isomorphisms are quasi-isomorphisms.*

b) *Each diagram*

$$L' \xleftarrow{s} L \xrightarrow{f} M \qquad (\text{(resp. } L' \xrightarrow{f'} M' \xleftarrow{s'} M \text{)}$$

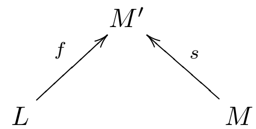
of $\mathsf{H}\,\mathcal{A}$*, where s (resp. s') is a quasi-isomorphism, may be embedded into a square*

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ L' & \xrightarrow{f'} & M' \end{array}$$

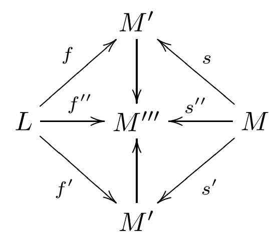
which commutes in $\mathsf{H}\,\mathcal{A}$*.*

c) *Let f be a morphism of* $\mathsf{H}\,\mathcal{A}$*. Then there is a quasi-isomorphism s such that $sf = 0$ in* $\mathsf{H}\,\mathcal{A}$ *if and only if there is a quasi-isomorphism t such that $ft = 0$ in* $\mathsf{H}\,\mathcal{A}$*.*

The lemma is proved for example in [44, 1.6.7]. Clearly condition a) would also be true for the pre-image of Σ in the category of complexes. However, for b) and c) to hold, it is essential to pass to the homotopy category. Historically [39], this observation was the main reason for inserting the homotopy category between the category of complexes and the derived category. We now obtain a second, equivalent, definition [82] of the *derived category* $\mathsf{D}\,\mathcal{A}$: it is the category of fractions of the homotopy category with respect to the class of quasi-isomorphisms. This means that the derived category has the same objects as the homotopy category (namely all complexes) and that morphisms in the derived category from L to M are given by 'left fractions' "$s^{-1}f$", *i.e.* equivalence classes of diagrams



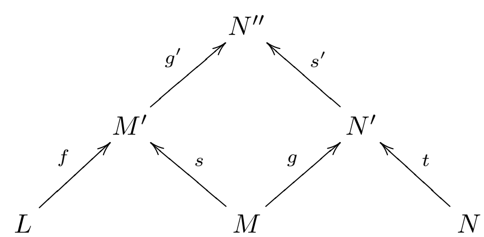
where s is a quasi-isomorphism, and a pair (f, s) is equivalent to (f', s') iff there is a commutative diagram of $\mathsf{H}\,\mathcal{A}$



where s'' is a quasi-isomorphism. Composition is defined by

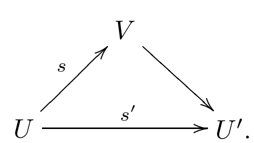
$$\text{“}t^{-1}g\text{”} \circ \text{“}s^{-1}f\text{”} = \text{“}(s't)^{-1} \circ g'f\text{”} ,$$

where $s' \in \Sigma$ and g' are constructed using condition b) as in the following commutative diagram of $\mathsf{H}\,\mathcal{A}$



One can then check that composition is associative and admits the obvious morphisms as identities. Using 'right fractions' instead of left fractions we would have obtained an isomorphic category (use lemma 2.2 b). The universal functor $\mathsf{C}\,\mathcal{A} \to \mathsf{D}\,\mathcal{A}$ of paragraph 2.1 descends to a canonical functor $\mathsf{H}\,\mathcal{A} \to \mathsf{D}\,\mathcal{A}$. It sends a morphism $f : L \to M$ to the fraction "$\mathbf{1}_M^{-1}f$". It makes all quasi-isomorphisms invertible and is universal among functors with this property.

2.3. **Cofinal subcategories.** A subcategory $\mathcal{U} \subset \mathsf{H}\,\mathcal{A}$ is *left cofinal* if, for each quasi-isomorphism $s : U \to V$ with $U \in \mathcal{U}$ and $V \in \mathsf{H}\,\mathcal{A}$, there is a quasi-isomorphism $s' : U \to U'$ with $U' \in \mathcal{U}$ and a commutative diagram



Dually, one defines the notion of a right cofinal subcategory.

Lemma. *If $\mathcal{U} \subset \mathcal{HA}$ is left or right cofinal, then the essential image of $\mathcal{U}$ in $\mathsf{D}\,\mathcal{A}$ is equivalent to the localization of $\mathcal{U}$ at the class of quasi-isomorphisms $s : U \to U'$ with $U, U' \in \mathcal{U}$.*

For example, the category $\mathsf{H}^-(\mathcal{A})$ of complexes U with $U^n = 0$ for all $n \gg 0$ is easily seen to be left cofinal in $\mathcal{HA}$. The essential image of $\mathsf{H}^-(\mathcal{A})$ in $\mathsf{D}\,\mathcal{A}$ is the *right bounded derived category* $\mathsf{D}^-\,\mathcal{A}$, whose objects are all complexes U with $H^nU = 0$ for all $n \gg 0$. According to the lemma, it is equivalent to the localization of the category $\mathsf{H}^-\,\mathcal{A}$ with respect to the class of quasi-isomorphisms it contains. Similarly, the category $\mathsf{H}^+\,\mathcal{A}$ of all complexes U with $U^n = 0$ for all $n \ll 0$ is right cofinal in $\mathcal{HA}$ and we obtain an analogous description of the *left bounded derived category* $\mathsf{D}^+\,\mathcal{A}$. Finally, the category $\mathsf{H}^b\,\mathcal{A}$ formed by the complexes U with $U^n = 0$ for all $|n| \gg 0$ is left cofinal in $\mathsf{H}^+\,\mathcal{A}$ and right cofinal in $\mathsf{H}^-\,\mathcal{A}$. We infer that the *bounded derived category* $\mathsf{D}^b\,\mathcal{A}$, whose objects are the U with $H^nU = 0$ for all $|n| \gg 0$, is equivalent to the localization of $\mathsf{H}^b(\mathcal{A})$ with respect to its quasi-isomorphisms.

2.4. **Morphisms and extension groups.** The following lemma yields a more concrete description of some morphisms of the derived category. We use the following notation: An object $A \in \mathcal{A}$ is identified with the complex

$$\ldots \to 0 \to A \to 0 \to \ldots$$

having A in degree 0. If M is an arbitrary complex, we denote by S^nM or $M[n]$ the complex with components $(S^nM)^p = M^{n+p}$ and differential $d_{S^nM} = (-1)^n d_M$. A complex I (resp. P) is *fibrant* (resp. *cofibrant*) if the canonical map

$$\mathsf{Hom}_{\mathsf{H}\,\mathcal{A}}(L, I) \to \mathsf{Hom}_{\mathsf{D}\,\mathcal{A}}(L, I) \text{ resp. } \mathsf{Hom}_{\mathsf{H}\,\mathcal{A}}(P, L) \to \mathsf{Hom}_{\mathsf{D}\,\mathcal{A}}(P, L)$$

is bijective for each complex L.

Lemma. a) *The category $\mathsf{D}\,\mathcal{A}$ is additive and the canonical functors $\mathsf{C}\,\mathcal{A} \to \mathsf{H}\,\mathcal{A} \to \mathsf{D}\,\mathcal{A}$ are additive.*

b) *If I is a left bounded complex (*i.e. $I^n = 0$ for all $n \ll 0$*) with injective components then I is fibrant. Dually, if P is a right bounded complex with projective components, then P is cofibrant.*

c) *For all $L, M \in \mathcal{A}$, there is a canonical isomorphism*

$$\partial : \mathsf{Ext}^n_{\mathcal{A}}(L, M) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{D}\,\mathcal{A}}(L, S^nM)$$

valid for all $n \in \mathbb{Z}$ if we adopt the convention that Ext^n vanishes for $n < 0$. In particular, the canonical functor $\mathcal{A} \to \mathsf{D}\,\mathcal{A}$ is fully faithful.

The calculus of fractions yields part a) of the lemma (cf. [28]). Part b) follows from [38, I, Lemma 4.5]. Part c) is in [38, I, §6]. Let us prove c) in the case where $\mathcal{A}$ has enough injectives (*i.e.* each object admits a monomorphism into an injective). In this case, the object M admits an injective resolution, i.e. a quasi-isomorphism $s : M \to I$ of the form

$$\begin{array}{ccccccccccc} \ldots & \to & 0 & \to & M & \to & 0 & \to & 0 & \to & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \to & 0 & \to & I^0 & \to & I^1 & \to & I^2 & \to & \ldots \end{array}$$

where the I^p are injective. Then, since s becomes invertible in $\mathsf{D}\,\mathcal{A}$, it induces an isomorphism

$$\mathsf{Hom}_{\mathsf{D}\,\mathcal{A}}(L, S^nM) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{D}\,\mathcal{A}}(L, S^nI).$$

By part b) of the lemma, we have the isomorphism

$$\mathsf{Hom}_{\mathsf{D}\,\mathcal{A}}(L, S^nI) \xleftarrow{\sim} \mathsf{Hom}_{\mathsf{H}\,\mathcal{A}}(L, S^nM).$$

Finally, the last group is exactly the nth homology of the complex

$$\mathsf{Hom}_{\mathcal{A}}(L, I)\ ,$$

which identifies with $\mathsf{Ext}_{\mathcal{A}}(L,M)$ by (the most common) definition.

2.5. **Derived categories of semi-simple or hereditary categories.** In two very special cases, we can directly describe the derived category in terms of the module category: First suppose that $\mathcal{A}$ is *semi-simple*, *i.e.* $\mathsf{Ext}^1_{\mathcal{A}}(A,B)=0$ for all $A,B\in\mathcal{A}$. For example, this holds for the category of vector spaces over a field. Then it is not hard to show that the functor $M\mapsto H^*M$ establishes an equivalence between $\mathsf{D}\,\mathcal{A}$ and the category of $\mathbb{Z}$-graded k-vector spaces. In the second case, suppose that $\mathcal{A}$ is *hereditary* (*i.e.* $\mathsf{Ext}^2_{\mathcal{A}}(A,B)=0$ for all $A,B\in\mathcal{A}$). We claim that each object M of $\mathsf{D}\,\mathcal{A}$ is quasi-isomorphic to the sum of the $(H^nM)[-n]$, $n\in\mathbb{Z}$. To prove this, let us denote by Z^n the kernel of $d^n : M^n\to M^{n+1}$, and put $H^n=H^n(M)$. Then we have an exact sequence

$$0 \longrightarrow Z^{n-1} \longrightarrow M^{n-1} \xrightarrow{\delta} Z^n \longrightarrow H^n \longrightarrow 0$$

for each $n\in\mathbb{Z}$, where δ is induced by d. Its class in $\mathsf{Ext}^2_{\mathcal{A}}$ vanishes by the assumption on $\mathcal{A}$. Therefore, there is a factorization

$$M^{n-1} \rightarrowtail^{\varepsilon} E^n \xrightarrow{\zeta} \!\!\!\!\to Z^n$$

of δ where ε is a monomorphism, ζ an epimorphism, Z^{n-1} identifies with the kernel of ζ and H^n with the cokernel of ε. It follows that the direct sum H of the complexes $H^n[-n]$ is quasi-isomorphic to the direct sum S of the complexes

$$\cdots \longrightarrow 0 \longrightarrow M^{n-1} \xrightarrow{\varepsilon} E^n \longrightarrow 0 \longrightarrow \cdots .$$

There is an obvious quasi-isomorphism $S\to M$. Thus we have a diagram of quasi-isomorphisms

$$M \longleftarrow S \longrightarrow H$$

and the claim follows. Note that the direct sum of the $(H^nM)[-n]$, $n\in\mathbb{Z}$, identifies with their direct product. Therefore, if L and M are two complexes, then the morphisms from L to M in $\mathsf{D}\,\mathcal{A}$ are in bijection with the families (f_n,ε_n), $n\in\mathbb{Z}$, of morphisms $f_n : H^nL\to H^nM$ and extensions $\varepsilon_n\in\mathsf{Ext}^1_{\mathcal{A}}(H^nL,H^{n-1}M)$.

2.6. **The quiver of a k-linear category.** We briefly sketch the definition of this important invariant (*cf.* [27, Ch. 9] and [1, Ch. VII] for thorough treatments). It will enable us to visualize the abelian categories and derived categories appearing in the examples below. Let k be a field and $\mathcal{A}$ a k-linear category such that all morphism spaces $\mathcal{A}\,(A,B)$, $A,B\in\mathcal{A}$, are finite-dimensional. Recall that an object U of $\mathcal{A}$ is *indecomposable* if it is non zero and is not the direct sum of two non zero objects. We suppose that $\mathcal{A}$ *is multilocular* [27, 3.1], *i.e.*

a) each object of $\mathcal{A}$ decomposes into a finite sum of indecomposables and
b) the endomorphism ring of each indecomposable object is local.

Thanks to condition b), the decomposition in a) is then unique up to isomorphism and permutation of the factors [27, 3.3].

For example, the category $\mathsf{mod}\,A$ of finite-dimensional modules over a finite-dimensional algebra A is multilocular, *cf.* [27, 3.1], and so is the category $\mathsf{coh}\,X$ of coherent sheaves on a projective variety X, *cf.* [72]. The bounded derived categories of these abelian categories are also multilocular.

A multilocular category $\mathcal{A}$ is determined by its full subcategory $\mathsf{ind}\,\mathcal{A}$ formed by the indecomposable objects. Condition b) implies that the sets

$$\mathsf{rad}(U,V)=\{f:U\to V \mid f \text{ is not invertible}\}$$

form an ideal in $\operatorname{ind}\mathcal{A}$. For $U, V \in \operatorname{ind}\mathcal{A}$, we define the *space of irreducible maps* to be

$$\operatorname{irr}(U, V) = \operatorname{rad}(U, V)/\operatorname{rad}^2(U, V).$$

The *quiver* $\Gamma(\mathcal{A})$ is the quiver (=oriented graph) whose vertices are the isomorphism classes $[U]$ of indecomposable objects U of $\mathcal{A}$ and where, for two vertices $[U]$ and $[V]$, the number of arrows from $[U]$ to $[V]$ equals the dimension of the space of irreducible maps $\operatorname{irr}(U, V)$.

For example, the quiver of the category of finite dimensional vector spaces $\operatorname{mod} k$ has a single vertex (corresponding to the one-dimensional vector space) and no arrows. The quiver of the bounded derived category $\mathsf{D}^b \operatorname{mod} k$ has vertex set $\mathbb{Z}$, where $n \in \mathbb{Z}$ corresponds to the isoclass of $k[n]$, and has no arrows. The quiver of the category of finite-dimensional modules over the algebra of lower triangular 5×5-matrices is depicted in the top part of figure 1. This example and several others are discussed below in section 2.8.

2.7. **Algebras given by quivers with relations.** Interesting but accessible examples of abelian categories arise as categories of modules over non semi-simple algebras. To describe a large class of such algebras, we use quivers with relations. We briefly recall the main construction: A *quiver* is an oriented graph. It is thus given by a set Q_0 of points, a set Q_1 of arrows, and two maps $s, t : Q_1 \to Q_0$ associating with each arrow its source and its target. A simple example is the quiver

$$\vec{A}_{10} \ : \quad 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \to \ldots \to 8 \xrightarrow{\alpha_8} 9 \xrightarrow{\alpha_9} 10.$$

A *path* in a quiver Q is a sequence $(y|\beta_r|\beta_{r-1}|\ldots|\beta_1|x)$ of composable arrows β_i with $s(\beta_1) = x$, $s(\beta_i) = t(\beta_{i-1})$, $2 \leq i \leq r$, $t(\beta_r) = y$. In particular, for each point $x \in Q_0$, we have the *lazy path* $(x|x)$. It is neutral for the obvious *composition* of paths. The *quiver algebra* kQ has as its basis all paths of Q. The product of two basis elements equals the composition of the two paths if they are composable and 0 otherwise. For example, the quiver algebra of $Q = \vec{A}_{10}$ is isomorphic to the algebra of lower triangular 10×10 matrices.

The construction of the quiver algebra kQ is motivated by the (easy) fact that the category of left kQ-modules is equivalent to the category of all diagrams of vector spaces of the shape given by Q. It is not hard to show that each quiver algebra is hereditary. It is finite-dimensional iff the quiver has no oriented cycles. Gabriel [26] has shown that the quiver algebra of a finite quiver has only a finite number of k–finite-dimensional indecomposable modules (up to isomorphism) iff the underlying graph of the quiver is a disjoint union of Dynkin diagrams of type A, D, E.

The above example has underlying graph of Dynkin type A_{10} and thus its quiver algebra has only a finite number of finite-dimensional indecomposable modules.

An ideal I of a finite quiver Q is *admissible* if, for some N, we have

$$(kQ_1)^N \subseteq I \subseteq (kQ_1)^2,$$

where (kQ_1) is the two-sided ideal generated by all paths of length 1. A *quiver Q with relations R* is a quiver Q with a set R of generators for an admissible ideal I of kQ. The algebra kQ/I is then the *algebra associated with* (Q, R). Its category of left modules is equivalent to the category of diagrams of vector spaces of shape Q obeying the relations in R. The algebra kQ/I is finite-dimensional (since I contains all paths of length at least N), hence artinian and noetherian. By induction on the number of points one can show that if the quiver Q contains no oriented cycle, then the algebra kQ/I is of finite global dimension.

One can show that every finite-dimensional algebra over an algebraically closed field is Morita equivalent to the algebra associated with a quiver with relations and that the quiver is unique (up to isomorphism).

2.8. **Example: Quiver algebras of type A_n.** Let k be a field, $n \geq 1$ an integer and $\mathcal{A}$ the category of k-finite-dimensional (right) modules over the quiver algebra A of the quiver $\vec{A}_n$ given by

$$\vec{A}_n : \ 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n \ .$$

The quiver $\Gamma(\mathcal{A})$ is triangle-shaped with $n(n+1)/2$ vertices. For $n = 5$, it is given in the top part of figure 1: There are n (isomorphism classes of) indecomposable projective modules given by the $P_i = e_{ii}A$, $1 \leq i \leq n$. They occur in increasing order on the left rim of the triangle. There are n simple modules $S_i = P_i/P_{i-1}$, $1 \leq i \leq n$ (where $P_0 := 0$). They are represented in increasing order by the vertices at the bottom. There are n injective modules $I_i = \mathsf{Hom}_k(Ae_{ii}, k)$, $1 \leq i \leq n$. They are represented in decreasing order by the vertices on the right rim. Note that each simple module has a projective resolution of length 1 which confirms that $\mathcal{A}$ is hereditary.

Using 2.5 we see that the indecomposable objects of $\mathsf{D}^b\,\mathcal{A}$ are precisely the $U[n]$, $n \in \mathbb{Z}$, $U \in \mathsf{ind}(\mathcal{A})$. Thus the quiver $\mathsf{D}^b\,\mathcal{A}$ has the vertices $[S^nU]$, $n \in \mathbb{Z}$, where $[U]$ is a vertex of $\Gamma(\mathcal{A})$. Arrows from $[S^nU]$ to $[S^mV]$ can occur only if m equals n or $n+1$, again by 2.5. Now Lemma 2.4 shows that the functor

$$\mathsf{ind}\,\mathcal{A} \to \mathsf{ind}\,\mathsf{D}\,\mathcal{A}\ ,\ M \mapsto S^nU$$

preserves the spaces of irreducible maps. So the arrows $[S^nU] \to [S^nV]$, where U and V are indecomposable in $\mathcal{A}$, are in bijection with the arrows $[U] \to [V]$ in $\Gamma(\mathcal{A})$. The additional arrows $[S^nU] \to [S^{n+1}V]$ are described in [34, 5.5] for $\mathcal{A} = \mathsf{mod}\,A$, where A is an arbitrary finite-dimensional path algebra of a quiver. For $A = k\vec{A}_n$, the quiver $\Gamma(\mathsf{D}^b\,\mathcal{A})$ is isomorphic to the infinite stripe $\mathbb{Z}\vec{A}_n$ depicted in the middle part of figure 1. The objects $[U]$, $U \in \mathsf{ind}\,\mathcal{A}$, correspond to the vertices (g, h) in the triangle

$$g \geq 0\ ,\ h \geq 1\ ,\ g + h \leq n.$$

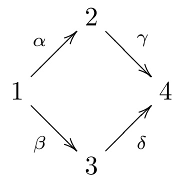
The translation $U \mapsto SU$ corresponds to the glide-reflection $(g, h) \mapsto (g+h, n+1-h)$. Remarkably, this quiver was actually discovered twenty years before D. Happel's work appeared in R. Street's Ph. D. Thesis [77] [76], *cf.* also [78] [79] [75].

The quiver

$$Q : \ 1 \longleftarrow 2 \longleftarrow 3 \longrightarrow 4 \longrightarrow 5$$

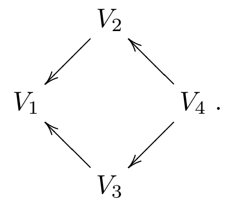
is obtained from $\vec{A}_5$ by changing the orientation of certain arrows. The quiver of $\mathsf{mod}\,kQ$ is depicted in the lower part of figure 1. The quiver of $\mathsf{D}^b(\mathsf{mod}\,kQ)$ turns out to be isomorphic to that of $\mathsf{D}^b(\mathsf{mod}\,k\vec{A}_5)$! The isomorphism commutes with the shift functor $U \mapsto SU$. In fact, the isomorphism between the quivers of $\mathsf{D}^b(\mathsf{mod}\,\vec{A}_5)$ and $\mathsf{D}^b(\mathsf{mod}\,kQ)$ comes from an equivalence between the derived categories themselves, as we will see below. However, this equivalence does not respect the module categories embedded in the derived categories. This is also visible in figure 1: Some modules for $k\vec{A}_5$ correspond to *shifted* modules for kQ and vice versa. Note that the module categories of kQ and $k\vec{A}_5$ cannot be equivalent, since the quivers of the module categories are not isomorphic.

2.9. **Example: Commutative squares and representations of $\vec{D}_4$.** Let A be the algebra given by the quiver with relations

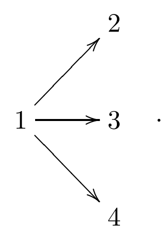


$$, \ \gamma\alpha = \delta\beta.$$

A (right) A-module is the same as a commutative diagram of vector spaces



The quivers of $\mathsf{mod}\, A$ and $\mathsf{D}^b(\mathsf{mod}\, A)$ are depicted in figure 2. Their computation is due to D. Happel [33] and, independently, R. Street, *cf.* p. 118 of [75]. The shift functor $U \mapsto SU$ corresponds to the map $(g, h) \mapsto (g + 3, h)$. Note that the algebra A is not hereditary. Therefore, some indecomposable objects of the derived category are not isomorphic to shifted modules. In the notations of the figure, these are the translates of Y. Let Q be the quiver $\vec{D}_4$:



The quiver of $\mathsf{mod}\, kQ$ is depicted in the lower part of figure 2. The quiver of the derived category $\mathsf{D}^b(\mathsf{mod}\, kQ)$ turns out to be isomorphic to that of $\mathsf{D}^b(\mathsf{mod}\, A)$! Moreover, the isomorphism respects the shift functors. The isomorphism between the quivers of the bounded derived categories of A and kQ comes from an equivalence between the categories themselves, as we will see below.

2.10. **Example: Kronecker modules and coherent sheaves on the projective line.** Let Q be the *Kronecker quiver*

$$1 \rightrightarrows 2 \ .$$

The quiver of the category $\mathsf{mod}\, kQ$ is depicted in the top part of figure 3, *cf.* [67]. It is a disjoint union of infinitely many connected components: one *postprojective* component containing the two (isoclasses of) indecomposable projective modules P_1, P_2, one *preinjective* component containing the two indecomposable injective modules I_1, I_2 and an infinity of components containin the *regular* modules $R(t, n)$ indexed by a point $(t_0 : t_1)$ of the projective line $\mathbb{P}^1(k)$ and an integer $n \geq 1$. Explicitly, the module $R(x, n)$ is given by the diagram

$$V_{n+1} \underset{x_1}{\overset{x_0}{\leftleftarrows}} V_n \ ,$$

where V_n is the nth homogeneous component of the graded space $k[x_0, x_1]/((t_1 x_0 - t_0 x_1)^n)$. The category $\mathsf{mod}\, kQ$ is hereditary. Thus the indecomposables in its derived category are simply shifted copies of indecomposable modules. The quiver of the derived category is depicted in the middle part of figure 3. Remarkably, it is isomorphic to the quiver of the derived category of the category $\mathsf{coh}\,\mathbb{P}^1$ of coherent sheaves on the projective line. The quiver of the category $\mathsf{coh}\,\mathbb{P}^1$ is depicted in the bottom section of figure 3. It contains one component whose vertices are the line bundles $O(n)$, $n \in \mathbb{Z}$, and an infinity of components containing the skyscraper sheaves O_{nx} of length $n \geq 1$ concentrated at a point $x \in \mathbb{P}^1$. Note that via the isomorphism of the quivers of the derived categories, these correspond to the indecomposable regular modules over kQ, while the line bundles correspond to postprojective modules and to preinjective modules shifted by one degree. We will see that the isomorphism between the quivers of the derived categories of the categories $\mathsf{mod}\,kQ$ and $\mathsf{coh}\,\mathbb{P}^1$ comes from an equivalence between the derived categories themselves.

3. Derived functors

3.1. **Deligne's definition.** The difficulty in finding a general definition of derived functors is to establish a framework which allows one to prove, in full generality, as many as possible of the pleasant properties found in the examples. This seems to be best achieved by Deligne's definition [23], which we will give in this section (compare with Grothendieck-Verdier's definition in [82]).

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$ an additive functor. A typical example is the fixed point functor

$$\mathsf{Fix}_G : \mathsf{Mod}\,\mathbb{Z}G \to \mathsf{Mod}\,\mathbb{Z}$$

which takes a module M over a group G to the abelian group of G-fixed points in M. The additive functor F clearly induces a functor $\mathsf{C}\,\mathcal{A} \to \mathsf{C}\,\mathcal{B}$ between the categories of complexes (obtained by applying F componentwise) and a functor $\mathsf{H}\,\mathcal{A} \to \mathsf{H}\,\mathcal{B}$ between the homotopy categories. By abuse of notation, both will be denoted by F as well. We are looking for a functor $? : \mathsf{D}\,\mathcal{A} \to \mathsf{D}\,\mathcal{B}$ which should make the following square commutative

$$\begin{array}{ccc} \mathsf{H}\,\mathcal{A} & \xrightarrow{F} & \mathsf{H}\,\mathcal{B} \\ \downarrow & & \downarrow \\ \mathsf{D}\,\mathcal{A} & \xrightarrow{?} & \mathsf{D}\,\mathcal{B} \end{array}$$

However, if F is not exact, it will not transform quasi-isomorphisms to quasi-isomorphisms and the functor in question cannot exist. What we will define then is a functor $\mathbf{R}F$ called the 'total right derived functor', which will be a 'right approximation' to an induced functor. More precisely, for a given $M \in \mathsf{D}\,\mathcal{A}$, we will not define $\mathbf{R}F(M)$ directly but only a functor

$$(\mathbf{r}F)(?, M) : (\mathsf{D}\,\mathcal{B})^{op} \to \mathsf{Mod}\,\mathbb{Z}$$

which, if representable, will be represented by $\mathbf{R}F(M)$. For $L \in \mathsf{D}\,\mathcal{B}$, we define $(\mathbf{r}F)(L, M)$ to be the set of 'left F-fractions', i.e. equivalence classes of diagrams

$$L \xrightarrow{f} FM' \qquad M' \xleftarrow{s} M$$

where f is a morphism of $\mathsf{D}\,\mathcal{B}$ and s a quasi-isomorphism of $\mathsf{H}\,\mathcal{A}$. Equivalence is defined in complete analogy with section 2.1. We say that $\mathbf{R}F$ is *defined at* $M \in \mathsf{D}\,\mathcal{A}$

if the functor $(\mathbf{r}F)(?, M)$ is representable and if this is the case, then the value $\mathbf{R}FM$ is defined by the isomorphism

$$\mathsf{Hom}_{\mathsf{D}\,\mathcal{B}}(?, (\mathbf{R}F)(M)) \xrightarrow{\sim} (\mathbf{r}F)(?, M).$$

The link between this definition and more classical constructions is established by the

Proposition. *Suppose that $\mathcal{A}$ has enough injectives and M is left bounded. Then $\mathbf{R}F$ is defined at M and we have*

$$\mathbf{R}FM = FI$$

where $M \to I$ is a quasi-isomorphism with a left bounded complex with injective components.

Under the hypotheses of the proposition, the quasi-isomorphism $M \to I$ always exists [44, 1.7.7]. Viewed in the homotopy category $\mathsf{H}\,\mathcal{A}$ it is functorial in M since it is in fact the universal morphism from M to a fibrant (2.4) complex. For example, if M is concentrated in degree 0, then I may be chosen to be an injective resolution of M and we find that

(3.1.1) $$H^n \mathbf{R}FM = (\mathbf{R}^n F)(M)\ ,$$

the nth right derived functor of F in the sense of Cartan-Eilenberg [19].

The above definition works not only for functors induced from functors $F : \mathcal{A} \to \mathcal{B}$ but can also be applied without any changes to arbitrary functors $F : \mathsf{H}\,\mathcal{A} \to \mathsf{H}\,\mathcal{B}$. One obtains $\mathbf{R}F$ (defined in general only on a subcategory). Dually, one defines the functor $\mathbf{L}F : \mathsf{D}\,\mathcal{A} \to \mathsf{D}\,\mathcal{B}$: For each $M \in \mathsf{D}\,\mathcal{A}$, where $\mathbf{L}F(M)$ is defined, it represents the functor

$$L \mapsto \mathbf{l}F(M, L)\ ,$$

where $\mathbf{l}F(M, L)$ is the set of equivalence classes of diagrams

$$M \xleftarrow{\ s\ } M' \qquad FM' \xrightarrow{\ g\ } L$$

As an exercise, the reader can prove the isomorphism of functors

$$\mathbf{R}\mathsf{Fix}_{G/H} \circ \mathbf{R}\mathsf{Fix}_H = \mathbf{R}\mathsf{Fix}_G$$

for a group G and a normal subgroup H of G. Here, all derived functors are defined on the full subcategory of left bounded complexes $\mathsf{D}^+ \,\mathsf{Mod}\,\mathbb{Z}G$ of $\mathsf{D}\,\mathsf{Mod}\,\mathbb{Z}G$. This isomorphism replaces the traditional Lyndon-Hochschild-Serre spectral sequence:

(3.1.2) $$E_2^{pq} = H^p(G/H, H^q(H, M)) \Rightarrow H^{p+q}(G, M)$$

for a G-module M. In fact, using the methods of section 7 one can obtain the spectral sequence from the isomorphism of functors.

Equation 3.1.1 shows that in general, derived functors defined on $\mathsf{D}^b(\mathcal{A})$ will take values in the unbounded derived categories. It is therefore useful to work with unbounded derived categories from the start. The following theorem ensures the existence of derived functors in all the cases we will need: Let A be a k-algebra and $\mathcal{B}$ a Grothendieck category (*i.e.* an abelian category having a generator, such that all set-indexed colimits exist and all filtered colimits are exact).

Theorem. a) *Every functor with domain $\mathsf{H}(\mathcal{B})$ admits a total right derived functor.*

b) *Every functor with domain $\mathsf{H}(\mathsf{Mod}\,A)$ admits a total right derived functor and a total left derived functor.*

c) *If* (F, G) *is a pair of adjoint functors from* $\mathsf{H}(\mathsf{Mod}\, A)$ *to* $\mathsf{H}(\mathcal{B})$*, then* $(\mathbf{L}F, \mathbf{R}G)$ *is a pair of adjoint functors from* $\mathsf{D}(\mathsf{Mod}\, A)$ *to* $\mathsf{D}(\mathcal{B})$*.*

We refer to [25] and [80] for a) and to [74] and [49] for b). Statement c) is a special case of the following easy

Lemma. *Let* (F, G) *be an adjoint pair of functors between the homotopy categories* $\mathsf{H}(\mathcal{A})$ *and* $\mathsf{H}(\mathcal{B})$ *of two abelian categories* $\mathcal{A}$ *and* $\mathcal{B}$*. Suppose that* $\mathbf{L}F$ *and* $\mathbf{R}G$ *are defined everywhere. Then* $(\mathbf{L}F, \mathbf{R}G)$ *is an adjoint pair between* $\mathsf{D}(\mathcal{A})$ *and* $\mathsf{D}(\mathcal{B})$*.*

4. Tilting and derived equivalences

4.1. **Tilting between algebras.** Let A and B be associative unital k-algebras and T an A-B-bimodule. Then we have adjoint functors

$$? \otimes_A T : \mathsf{Mod}\, A \rightleftarrows \mathsf{Mod}\, B : \mathsf{Hom}_B(T, ?)$$

(and in fact each pair of adjoint functors between module categories is of this form). One variant of Morita's theorem states that these functors are quasi-inverse equivalences iff

a) T_B is finitely generated projective,
b) the canonical map $A \to \mathsf{Hom}_B(T_B, T_B)$ is an isomorphism, and
c) the free B-module of rank one B_B is a direct factor of a finite direct sum of copies of T.

If, in this statement, we replace the module categories by their derived categories, and adapt the conditions accordingly, we obtain the statement of the

Theorem (Happel [33])**.** *The total derived functors*

$$\mathbf{L}(? \otimes_A T) : \mathsf{D}(\mathsf{Mod}\, A) \rightleftarrows \mathsf{D}(\mathsf{Mod}\, B) : \mathbf{R}\mathsf{Hom}_B(T, ?)$$

are quasi-inverse equivalences iff

a) *As a* B*-module,* T *admits a finite resolution*

$$0 \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to T \to 0$$

by finitely generated projective B*-modules* P_i*,*

b) *the canonical map*

$$A \to \mathsf{Hom}_B(T, T)$$

is an isomorphism and for each $i > 0$*, we have* $\mathsf{Ext}^i_B(T, T) = 0$*, and*

c) *there is a long exact sequence*

$$0 \to B \to T^0 \to T^1 \to \ldots \to T^{m-1} \to T^m \to 0$$

where B *is considered as a right* B*-module over itself and the* T^i *are direct factors of finite direct sums of copies of* T*.*

If these conditions hold and, moreover, A *and* B *are right noetherian, then the derived functors restrict to quasi-inverse equivalences*

$$\mathsf{D}^b(\mathsf{mod}\, A) \rightleftarrows \mathsf{D}^b(\mathsf{mod}\, B) \ ,$$

where $\mathsf{mod}\, A$ *denotes the category of finitely generated* A*-modules.*

4.2. **First links between the module categories.** Now assume that (A,T,B) is a *tilting triple*, *i.e.* that the conditions of Happel's theorem 4.1 hold. Note that we make no assumption on the dimensions over k of A, B, or T. Let w be the maximum of the two integers n and m occuring in conditions a) and c). Put

$$F = ?\otimes^{\mathsf{L}}_A T\ ,\ G = \mathbf{R}\mathsf{Hom}_B(T,?)$$

and, for $n \in \mathbb{Z}$, put

$$F_n = H^{-n}\circ F|\,\mathsf{Mod}\,A\ ,\ G^n = H^n \circ G|\,\mathsf{Mod}\,B.$$

These functors are homological, *i.e.* each short exact sequence of modules will give rise to a long exact sequence in these functors. This makes it clear that the subcategories

$$\mathcal{A}_n = \{M \in \mathsf{Mod}\,A \mid F_i(M) = 0\ ,\ \forall i \neq n\}$$
$$\mathcal{B}_n = \{N \in \mathsf{Mod}\,B \mid G^i(N) = 0\ ,\ \forall i \neq n\}$$

are closed under extensions, that they vanish for $n < 0$ or $n > w$ and that $\mathcal{A}_w$ and $\mathcal{B}_0$ are closed under submodules and $\mathcal{A}_0$ and $\mathcal{B}_w$ under quotients. Moreover, since

$$F_n|\mathcal{A}_n \xrightarrow{\sim} S^{-n}F|\mathcal{A}_n \quad \text{and} \quad G^n|\mathcal{B}_n \xrightarrow{\sim} S^nG|\mathcal{B}_n$$

the functors F_n and G_n induce quasi-inverse equivalences between $\mathcal{A}_n$ and $\mathcal{B}_n$. Let us now assume that $\mathsf{Mod}\,A$ is hereditary. Then each indecomposable of $\mathsf{D}(\mathsf{Mod}\,A)$ is concentrated in precisely one degree. Thus, for each indecomposable N of $\mathsf{Mod}\,A$, FN will have non-vanishing homology in exactly one degree and so N will lie in precisely one of the $\mathcal{B}_n$. Thus, as an additive category, $\mathsf{Mod}\,B$ is made up of 'pieces' of the hereditary category $\mathsf{Mod}\,A$. Whence the terminology that $\mathsf{Mod}\,B$ is *piecewise hereditary*. The algebras in the examples below are all hereditary or piecewise hereditary.

This first analysis of the relations between abelian categories with equivalent derived categories will be refined in section 7.

4.3. **Example:** $k\vec{A}_5$**.** We continue example 2.8. In $\mathsf{mod}\,k\vec{A}_5$, we consider the module T given by the sum of the indecomposables T_i, $1 \leq i \leq 5$, marked in the top part of figure 1. The endomorphism ring of T over $k\vec{A}_5$ is isomorphic to kQ so that T becomes a kQ-$k\vec{A}_5$-bimodule. It is not hard to check that $(kQ, T, k\vec{A}_5)$ is a tilting triple. The resulting equivalence between the derived categories gives rise to the identification of their quivers depicted in figure 1. For two indecomposables U and V, let us write $U \leq V$ if there is a path from U to V in the quiver of the module category. Then we can describe the indecomposables of the subcategories $\mathcal{A}_n$ and $\mathcal{B}_n$ of 4.2 as follows:

$$\mathcal{B}_0 : U \in \mathsf{ind}(k\vec{A}_5) \text{ such that } U \geq T_i \text{ for some i}$$
$$\mathcal{A}_0 : U \in \mathsf{ind}(kQ) \text{ such that } U \leq GI_i \text{ for some i}$$
$$\mathcal{B}_1 : P_1, P_2, S_2 \in \mathsf{ind}(k\vec{A}_5)$$
$$\mathcal{A}_1 : S_2', I_2', I_1' \in \mathsf{ind}(kQ)$$

In terms of representations of Q and $\vec{A}_5$, the functor $G_0 = \mathsf{Hom}_{k\vec{A}_5}(T,?)$ corresponds to the 'reflection functor' [10] which sends a representation

$$V_1 \xleftarrow[\alpha_1]{} V_2 \xleftarrow[\alpha_2]{} V_3 \longleftarrow V_4 \longleftarrow V_5$$

to

$$\mathsf{ker}(\alpha_2) \longrightarrow \mathsf{ker}(\alpha_1\alpha_2) \longrightarrow V_3 \longleftarrow V_4 \longleftarrow V_5 \quad .$$

The functor $G_1 = \mathsf{Ext}_{k\vec{A}_5}(T, ?)$ corresponds to the functor which sends a representation

$$V_1 \xleftarrow{\alpha_1} V_2 \xleftarrow{\alpha_2} V_3 \longleftarrow V_4 \longleftarrow V_5$$

to

$$\mathsf{cok}(\alpha_2) \longrightarrow \mathsf{cok}(\alpha_1\alpha_2) \longrightarrow 0 \longleftarrow 0 \longleftarrow 0\ .$$

To describe the total right derived functor $G = \mathbf{R}\mathsf{Hom}_{k\vec{A}_5}(T, ?)$, we need the *mapping cone*: recall that if $f : K \to L$ is a morphism of complexes, the mapping cone $C(f)$ is the complex with components $L^p \oplus K^{p+1}$ and with the differential

$$\begin{bmatrix} d_L & f \\ 0 & -d_K \end{bmatrix}.$$

We view complexes of $k\vec{A}_5$-modules as representations of $\vec{A}_5$ in the category of complexes of vector spaces and similarly for complexes of kQ-modules. Then the functor $\mathbf{R}\mathsf{Hom}_{k\vec{A}_5}(T, ?)$ is induced by the exact functor which sends

$$V_1 \xleftarrow{\alpha_1} V_2 \xleftarrow{\alpha_2} V_3 \longleftarrow V_4 \longleftarrow V_5$$

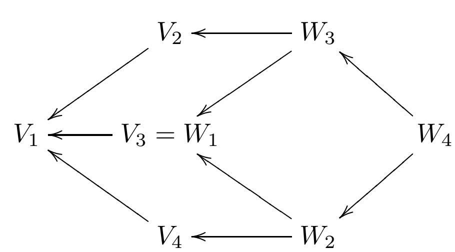
(where the V_i are complexes of vector spaces) to

$$C(\alpha_2) \longrightarrow C(\alpha_1\alpha_2) \longrightarrow V_3 \longleftarrow V_4 \longleftarrow V_5\ .$$

4.4. **Example: Commutative squares and representations of $\vec{D}_4$.** We continue example 2.9. Let T be the $k\vec{D}_4$-module which is the direct sum of the indecomposables T_i, $1 \le i \le 4$, marked in the lower part of figure 2. It is not hard to see that the endomorphism ring of T is isomorphic to A and that $(A, T, k\vec{D}_4)$ is a tilting triple. The resulting equivalence of derived categories leads to the identification of their quivers depicted in figure 2. The indecomposables of the subcategories $\mathcal{A}_n$ and $\mathcal{B}_n$ of 4.2 are as follows

$$\begin{aligned}
&\mathcal{A}_0 : U \in \mathsf{ind}\, A \text{ such that } U < \tau I_4 \\
&\mathcal{B}_0 : U \in \mathsf{ind}\, kQ \text{ such that } U \neq Y, Z \text{ and } U \geq T_1 \\
&\mathcal{A}_1 : U \in \mathsf{ind}\, A \text{ such that } U \geq \tau I_4 \\
&\mathcal{B}_1 : P_1', P_2', P_4', Z \in \mathsf{ind}\, A.
\end{aligned}$$

Note that GY has homology in degrees 0 and 1 so that Y belongs neither to $\mathcal{B}_0$ nor to $\mathcal{B}_1$. In terms of representations of quivers, the functor $G_0 = \mathsf{Hom}_{k\vec{D}_4}(T, ?)$ is constructed as follows: Given a diagram V, we form



where all ‘squares’ are cartesian. Then the image of V is the commutative square W.

4.5. **Historical remarks.** Happel's theorem 4.1, the links between module categories described in section 4.2 and examples like the above form the theory of tilting as it was developped in the representation theory of finite-dimensional algebras in the 1970s and 80s. Important precursors to the theory were: Gelfand-Ponomarev [31] [30], Bernstein-Gelfand-Ponomarev [10], Auslander-Platzeck-Reiten [2], Marmaridis [56], ... The now classical theory (based on homological algebra but avoiding derived categories) is due to: Brenner-Butler [14], who first proved the 'tilting theorem', Happel-Ringel [37], who improved the theorem and defined tilted algebras, Bongartz [13], who streamlined the theory, and Miyashita [58], who generalized it to tilting modules of projective dimension > 1. The use of derived categories goes back to D. Happel [33]. Via the work of Parshall-Scott [21], this lead to J. Rickard's Morita theory for derived categories [65], which we present below.

4.6. **Tilting from abelian categories to module categories.** Let $\mathcal{B}$ be a k-linear abelian Grothendieck category. An object T of $\mathcal{B}$ is a *tilting object* if the functor

$$\mathbf{R}\mathsf{Hom}(T, ?) : \mathsf{D}(\mathcal{B}) \to \mathsf{D}(\mathsf{Mod}\,\mathsf{End}(T))$$

is an equivalence.

Proposition ([8], [12])**.** *Suppose that $\mathcal{B}$ is locally noetherian of finite homological dimension and that $T \in \mathcal{B}$ has the following properties:*

a) *T is noetherian.*
b) *We have $\mathsf{Ext}^n(T,T) = 0$ for all $n > 0$.*
c) *Let $\mathsf{add}(T)$ be the closure of T under forming finite direct sums and direct summands. The closure of $\mathsf{add}(T)$ under kernels of epimorphisms contains a set of generators for $\mathcal{B}$.*

Then T is a tilting object. If, moreover, $\mathsf{End}(T)$ is noetherian, the functor $\mathbf{R}\mathsf{Hom}(T, ?)$ induces an equivalence

$$\mathsf{D}^b(\mathcal{B}_{noe}) \to \mathsf{D}^b(\mathsf{mod}\,\mathsf{End}(T)) ,$$

where $\mathcal{B}_{noe}$ is the subcategory of noetherian objects of $\mathcal{B}$.

An analysis of the links between $\mathcal{B}$ and $\mathsf{mod}\,\mathsf{End}(T)$ analogous to 4.2 can be carried out. The more refined results of section 7 also apply in this situation.

4.7. **Example: Coherent sheaves on the projective line.** We continue example 2.10. Let $\mathcal{A}$ be the category of quasi-coherent sheaves on the projective line $\mathbb{P}^1(k)$. Let T be the sum of $O(-1)$ with $O(0)$. Then the conditions of the above proposition hold: Indeed, $\mathcal{A}$ is locally noetherian and hereditary and T is noetherian. So condition a) holds. Condition b) is a well-known computation. The sheaves $O(-n)$, $n \in \mathbb{N}$, form a system of generators for $\mathcal{A}$. Therefore condition c) follows from the existence of the short exact sequences

$$0 \to O(-n-1) \to O(-n) \oplus O(-n) \to O(-n+1) \to 0 \, , \; n \in \mathbb{Z}.$$

The endomorphism ring of T is isomorphic to the Kronecker algebra of example 2.10. The resulting equivalence

$$\mathbf{R}\mathsf{Hom}(T, ?) : \mathsf{D}^b(\mathsf{coh}(\mathbb{P}^1)) \to \mathsf{D}^b(\mathsf{End}(T))$$

induces the identification of the quivers depicted in figure 3. With notations analogous to 4.2, the indecomposables of $\mathcal{A}_0$ are those of the postprojective and the regular components. Those of $\mathcal{A}_1$ are the ones in the preinjective component. The indecomposables in $\mathcal{B}_0$ are the line bundles $O(n)$ with $n \geq 0$ and the skyscraper sheaves. Those of $\mathcal{B}_1$ are the line bundles $O(n)$ with $n < 0$.

This example is a special case of Beilinson's [11] description of the derived category of coherent sheaves on $\mathbb{P}^n(k)$. It was generalized to other homogeneous varieties by Kapranov [40] [41] [42] [43] and to weighted projective lines by Geigle and Lenzing [29] and Baer [8].

5. Triangulated categories

5.1. **Definition and examples.** Let $\mathcal{A}$ be an abelian category (for example, the category $\mathsf{Mod}\, R$ of modules over a ring R). One can show that the derived category $\mathsf{D}\,\mathcal{A}$ is abelian only if all short exact sequences of $\mathcal{A}$ split. This deficiency is partly compensated by the so-called triangulated structure of $\mathsf{D}\,\mathcal{A}$, which we are about to define. Most of the material of this section first appears in [82].

A *standard triangle* of $\mathsf{D}\,\mathcal{A}$ is a sequence

$$X \xrightarrow{Qi} Y \xrightarrow{Qp} Z \xrightarrow{\partial\varepsilon} X[1]\,,$$

where $Q : \mathsf{C}\,\mathcal{A} \to \mathsf{D}\,\mathcal{A}$ is the canonical functor,

$$\varepsilon : 0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$$

a short exact sequence of complexes, and $\partial\varepsilon$ a certain morphism of $\mathsf{D}\,\mathcal{A}$, functorial in ε, and which lifts the connecting morphism $H^*Z \to H^{*+1}X$ of the long exact homology sequence associated with ε. More precisely, $\partial\varepsilon$ is the fraction $''s^{-1} \circ j''$ where j is the inclusion of the subcomplex Z into the complex $X'[1]$ with components $Z^n \oplus Y^{n+1}$ and differential

$$\begin{bmatrix} d_Z & p \\ 0 & -d_Y \end{bmatrix},$$

and $s : X[1] \to X'[1]$ is the morphism $[0, i]^t$. A *triangle* of $\mathsf{D}\,\mathcal{A}$ is a sequence (u', v', w') of $\mathsf{D}\,\mathcal{A}$ *isomorphic* to a standard triangle, i.e. such that we have a commutative diagram

$$\begin{array}{ccccccc} X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \\ {\scriptstyle x}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle x[1]} \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array},$$

where the vertical arrows are isomorphisms of $\mathsf{D}\,\mathcal{A}$ and the bottom row is a standard triangle.

Lemma. T1 *For each object X, the sequence*

$$0 \to X \xrightarrow{\mathbf{1}} X \to S0$$

is a triangle.

T2 *If (u, v, w) is a triangle, then so is $(v, w, -Su)$.*

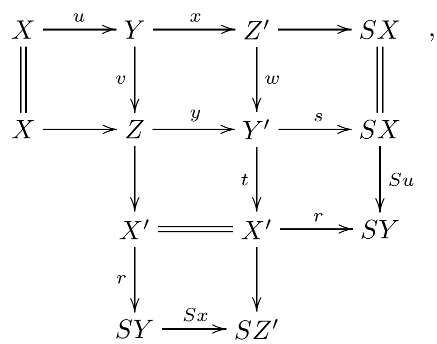
T3 *If (u, v, w) and (u', v', w') are triangles and x, y morphisms such that $yu = u'x$, then there is a morphism z such that $zv = v'y$ and $(Sx)w = w'z$.*

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & SX \\ {\scriptstyle x}\downarrow & & {\scriptstyle y}\downarrow & & {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle Sx} \\ X' & \xrightarrow[u']{} & Y' & \xrightarrow[v']{} & Z' & \xrightarrow[w']{} & SX' \end{array}.$$

T4 *For each pair of morphisms*

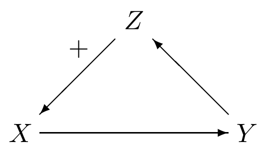
$$X \xrightarrow{u} Y \xrightarrow{v} Z$$

there is a commutative diagram

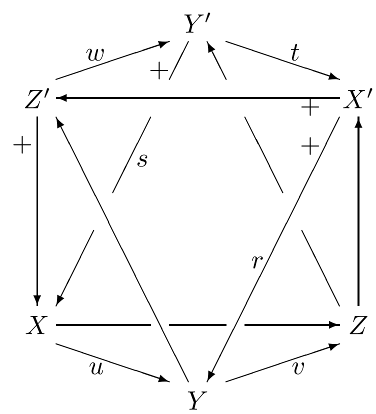


where the first two rows and the two central columns are triangles.

Property T4 can be given a more symmetric form if we represent a morphism $X \to SY$ by the symbol $X \overset{+}{\to} Y$ and write a triangle in the form



With this notation, the diagram of T4 can be written as an octahedron in which 4 faces represent triangles. The other 4 as well as two of the 3 squares 'containing the center' are commutative.



A *triangulated category* is an additive category $\mathcal{T}$ endowed with an autoequivalence $X \mapsto X[1]$ and a class of sequences (called triangles) of the form

$$X \to Y \to Z \to X[1]$$

which is stable under isomorphisms and satisfies properties T1–T4.

Note that 'being abelian' is a property of an additive category, whereas 'being triangulated' is the datum of extra structure.

A whole little theory can be deduced from the axioms of triangulated categories. This theory is nevertheless much poorer than that of abelian categories. The main reason for this is the non-uniqueness of the morphism z in axiom T3.

We mention only two consequences of the axioms: a) They are actually self-dual, in the sense that the opposite category $\mathcal{T}^{op}$ also carries a canonical triangulated structure. b) For each $U \in \mathcal{T}$, the functor $\mathsf{Hom}_{\mathcal{T}}(U, ?)$ is *homological*, *i.e.* it takes triangles to long exact sequences. Dually, the functor $\mathsf{Hom}_{\mathcal{T}}(?, V)$ is *cohomological* for each V of $\mathcal{T}$. By the 5-lemma, this implies for example that if in axiom T3, two of the three vertical morphisms are invertible, then so is the third.

For later use, we record a number of examples of triangulated categories: If $\mathcal{A}$ is abelian, then not only the derived category $\mathsf{D}\,\mathcal{A}$ is triangulated but also the homotopy category $\mathsf{H}\,\mathcal{A}$. Here the triangles are constructed from componentwise split short exact sequences of complexes.

If $\mathcal{T}$ is a triangulated category, a *full triangulated subcategory of* $\mathcal{T}$ is a full subcategory $\mathcal{S} \subset \mathcal{T}$ such that $\mathcal{S}[1] = \mathcal{S}$ and that whenever we have a triangle (X, Y, Z) of $\mathcal{T}$ such that X and Z belong to $\mathcal{T}$ there is an object Y' of $\mathcal{S}$ isomorphic to Y. For example, the full subcategories $\mathsf{H}^*\,\mathcal{A}$, $* \in \{-, +, b\}$, of $\mathsf{H}\,\mathcal{A}$ are full triangulated subcategories. Note that the categories $\mathsf{H}^*\,\mathcal{A}$, $* \in \{\emptyset, +, -, b\}$, are in fact defined for any additive category $\mathcal{A}$.

If $\mathcal{T}$ is a triangulated category and $\mathcal{X}$ a class of objects of $\mathcal{T}$, there is a smallest *strictly* (=closed under isomorphism) full triangulated subcategory $\mathsf{tria}(\mathcal{X})$ of $\mathcal{T}$ containing $\mathcal{X}$. It is called the *triangulated subcategory generated by* $\mathcal{X}$. For example, the category $\mathsf{D}^b\,\mathcal{A}$ is generated by $\mathcal{A}$ (identified with the category of complexes concentrated in degree 0).

If R is a ring, a very important triangulated subcategory is the full subcategory $\mathsf{per}\,R \subset \mathsf{D}\,\mathsf{Mod}\,R$ formed by the *perfect* complexes, i.e. the complexes quasi-isomorphic to bounded complexes with components in $\mathsf{proj}\,R$, the *category* of finitely generated projective R-modules. The subcategory $\mathsf{per}\,R$ may be intrinsically characterized [65, 6.3] as the subcategory of *compact objects* of $\mathsf{D}\,\mathsf{Mod}\,R$, i.e. objects X whose associated functor $\mathsf{Hom}(X, ?)$ commutes with arbitrary set-indexed coproducts. Note that by lemma 2.4, the canonical functor

$$\mathsf{H}^b\,\mathsf{proj}\,R \to \mathsf{per}\,R$$

is an equivalence so that the category $\mathsf{per}\,R$ is relatively accessible to explicit computations.

5.2. **Auslander-Reiten sequences and triangles.** How are short exact sequences or triangles reflected in the quiver of a multilocular abelian or triangulated category ? The problem is that the three terms of a triangle, like that of a short exact sequence, are only very rarely all indecomposable. The solution to this problem is provided by Auslander-Reiten theory, developed in [3] [4] [5] [6] [7] and presented, for example, in [27] and [1]. The typical 'mesh structure' which we observe in the quivers in figures 1 to 3 is produced by the 'minimal non split' exact sequences (resp. triangles), *i.e.* the Auslander-Reiten sequences (resp. triangles).

Let $\mathcal{A}$ be a multilocular abelian category and let X and Z be indecomposable objects of $\mathcal{A}$. An *almost split sequence (or Auslander-Reiten sequence) from* Z *to* X is a non-split exact sequence

$$0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$$

having the two equivalent properties

i) each non isomorphism $U \to Z$ with indecomposable U factors through p;
ii) each non isomorphism $X \to V$ with indecomposable V factors through i.

In this case, the sequence is determined up to isomorphism by Z (as well as by X) and X is the *translate* of Z (resp. Z the *cotranslate* of X). Moreover, if an indecomposable U occurs in Y with multiplicity μ, then there are μ arrows from X to U and μ arrows from U to Z in the quiver of $\mathcal{A}$. We write $X = \tau Z$ resp. $Z = \tau^- X$. This yields the following additional structure on the quiver $\Gamma(\mathcal{A})$:

- a bijection τ from set of 'non-projective' vertices to the set of 'non-injective' vertices;

- for each non projective vertex $[Z]$ and each indecomposable U, a bijection σ from the set of arrows from $[U]$ to $[Z]$ to the set of arrows from $\tau[Z]$ to $[U]$.

Auslander-Reiten have shown, *cf.* [27] or [1], that if $\mathcal{A}$ is the category $\mathsf{mod}\,A$ of finite-dimensional modules over a finite-dimensional algebra, each non-projective indecomposable Z occurs as the right hand term of an almost split sequence and each non-injective indecomposable X as the left hand term.

Analogously, if $\mathcal{A}$ is a multilocular triangulated category, an *almost split triangle (or Auslander-Reiten triangle)* is defined as a triangle

$$X \xrightarrow{i} Y \xrightarrow{p} Z \xrightarrow{e} X[1]$$

such that X and Z are indecomposable and the equivalent conditions i) and ii) above hold. Almost split triangles have properties which are completely analogous to those of almost split sequences. D. Happel [33] has shown that in the derived category of the category of finite-dimensional modules over a finite-dimensional algebra, an object Z occurs as the third term of an almost split triangle iff it is isomorphic to a bounded complex of finitely generated projectives. For example, for the quiver of $\mathsf{D}^b(\mathsf{mod}\,k\vec{A}_n)$ in the middle part of figure 1, the translation $X \mapsto \tau X$ is given by $(g,h) \mapsto (g-1,h)$. In the quivers of the two module categories, it is the 'restriction' (where defined) of this map. Similarly, in the middle part of figure 2, the translation τ is given by $(g,h) \mapsto (g-1,h)$ and in the lower part of the figure by the restriction (where defined) of this map. The analogous statement is true for the category of 'commutative squares' in the top part of the figure except for $\tau^- P_1$, whose translate is P_1 and I_4, whose translate is τI_4 (such exceptions are to be expected because the category of commutative squares is not hereditary).

5.3. **Grothendieck groups.** Then *Grothendieck group* $K_0(\mathcal{T})$ of a triangulated category $\mathcal{T}$ is defined [32] as the quotient of the free abelian group on the isomorphism classes $[X]$ of objects of $\mathcal{T}$ divided by the subgroup generated by the relators

$$[X] - [Y] + [Z]$$

where (X, Y, Z) runs through the triangles of $\mathcal{T}$.

For example, if R is a right coherent ring, then the category $\mathsf{mod}\,R$ of finitely presented R-modules is abelian and the K_0-group of the triangulated category $\mathsf{D}^b\,\mathsf{mod}\,R$ is isomorphic to $G_0 R = K_0(\mathsf{mod}\,R)$ via the Euler characteristic:

$$[M] \mapsto \sum_{i\in\mathbb{Z}} (-1)^i [H^i M].$$

If R is any ring, the K_0-group of the triangulated category $\mathsf{per}\,R$ is isomorphic to $K_0 R$ via the map

$$[P] \mapsto \sum_{i\in\mathbb{Z}} (-1)^i [P^i]\ ,\ P \in \mathsf{H}^b\,\mathsf{proj}\,R.$$

Note that this shows that any two rings with the 'same' derived category, will have isomorphic K_0-groups. To make this more precise, we need the notion of a triangle equivalence (cf. below).

5.4. **Triangle functors.** Let $\mathcal{S}, \mathcal{T}$ be triangulated categories. A *triangle functor* $\mathcal{S} \to \mathcal{T}$ is a pair (F, φ) formed by an additive functor $F : \mathcal{S} \to \mathcal{T}$ and a functorial isomorphism

$$\varphi X : F(X[1]) \xrightarrow{\sim} (FX)[1]\ ,$$

such that the sequence

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{(\varphi X)\,Fw} (FX)[1]$$

is a triangle of $\mathcal{T}$ for each triangle (u, v, w) of $\mathcal{S}$.

For example, if $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \to \mathcal{B}$ is an additive functor, one can show [23] that the domain of definition of the right derived functor $\mathbf{R}F$ is a strictly full triangulated subcategory $\mathcal{S}$ of $\mathsf{D}\,\mathcal{A}$ and that $\mathbf{R}F : \mathcal{S} \to \mathsf{D}\,\mathcal{B}$ becomes a triangle functor in a canonical way.

A triangle functor (F, φ) is a *triangle equivalence* if the functor F is an equivalence. We leave it to the reader as an exercise to define 'morphisms of triangle functors', and 'quasi-inverse triangle functors', and to show that a triangle functor admits a 'quasi-inverse triangle functor' if and only if it is a triangle equivalence [53].

6. Morita theory for derived categories

6.1. **Rickard's theorem.** Let k be a commutative ring. One version of the Morita theorem states that for two k-algebras A and B the following statements are equivalent:

(i) There is a k-linear equivalence $F : \mathsf{Mod}\,A \to \mathsf{Mod}\,B$.
(ii) There is an A-B-bimodule X (with central k-action) such that the tensor product $? \otimes_A X$ is an equivalence from $\mathsf{Mod}\,A$ to $\mathsf{Mod}\,B$.
(iii) There is a finitely generated projective B-module P which generates $\mathsf{Mod}\,B$ and whose endomorphism ring is isomorphic to A.

This form of the Morita theorem carries over to the context of derived categories. The following theorem is due to J. Rickard [65] [66]. A direct proof can be found in [49] (with a more didactical version in [52]).

Theorem (Rickard)**.** *Let A and B be k-algebras which are flat as modules over k. The following are equivalent*

i) *There is a k-linear triangle equivalence $(F, \varphi) : \mathsf{D}\,\mathsf{Mod}\,A \to \mathsf{D}\,\mathsf{Mod}\,B$.*
ii) *There is a complex of A-B-modules X such that the total left derived functor*

$$\mathbf{L}(? \otimes_A X) : \mathsf{D}\,\mathsf{Mod}\,A \to \mathsf{D}\,\mathsf{Mod}\,B$$

is an equivalence.

iii) *There is a complex T of B-modules such that the following conditions hold*
  a) *T is perfect,*
  b) *T generates $\mathsf{D}\,\mathsf{Mod}\,B$ as a triangulated category with infinite direct sums,*
  c) *we have*

$$\mathsf{Hom}_{\mathsf{D}\,B}(T, T[n]) = 0 \textit{ for } n \neq 0 \textit{ and } \mathsf{Hom}\,\mathsf{D}\,B(T, T) \cong A \;\; ;$$

Condition b) in iii) means that $\mathsf{D}\,\mathsf{Mod}\,B$ coincides with its smallest strictly full triangulated subcategory stable under forming arbitrary (set-indexed) coproducts. The implication from ii) to i) is clear. To prove the implication from i) to iii), one puts $T = FA$ (where A is regarded as the free right A-module of rank one concentrated in degree 0). Since F is a triangle equivalence, it is then enough to check that the analogues of a), b), and c) hold for the object A of $\mathsf{D}\,\mathsf{Mod}\,A$. Properties a) and c) are clear. Checking property b) is non-trivial [51]. The hard part of the proof is the implication from iii) to ii). Indeed, motivated by the proof of the classical Morita theorem we would like to put $X = T$. The problem is that although A acts on T as an object of the derived category, it does not act on the individual components of T, so that T is not a complex of bimodules as required in ii). We refer to [48] for a direct solution of this problem.

Condition b) of iii) may be replaced by the condition that the direct summands of T generate $\mathsf{per}\,B$ as a triangulated category, which is easier to check in practice.

If the algebras A and B are even projective as modules over k, then the complex X may be chosen to be bounded and with components which are projective from both sides. In this case, the tensor product functor $? \otimes_A X$ is exact and the total left derived functor $? \otimes^{\mathsf{L}}_A X$ is isomorphic to the one induced by $? \otimes_A X$.

By definition [66], the algebra A is *derived equivalent* to B if the conditions of the theorem hold. In this case, T is called a *tilting complex*, X a *two-sided tilting complex* and $\mathbf{L}(? \otimes_A X)$ a *standard equivalence*.

We know that any equivalence between module categories is given by the tensor product with a bimodule. Strangely enough, in the setting of derived categories, it is an open question whether all k-linear triangle equivalences are (isomorphic to) standard equivalences.

An important special case of the theorem is the one where the equivalence F in (i) takes the free A-module A_A to an object $T = F(A_A)$ whose homology is concentrated in degree 0. Then T becomes an A-B-bimodule in a natural way and we can take $X = T$ in (ii). The equivalence between (ii) and (iii) then specializes to Happel's theorem (4.1). In particular, this yields many non-trivial examples of derived equivalent algebras which are not Morita equivalent.

Derived equivalence is an equivalence relation, and if two algebras A and B are related by a tilting triple, then they are derived equivalent. One may wonder whether derived equivalence coincides with the smallest equivalence relation containing all pairs of algebras related by a tilting triple. Let us call this equivalence relation *T-equivalence*. It turns out that T-equivalence is strictly stronger than derived equivalence. For example, any T-equivalence between self-injective algebras comes from a Morita-equivalence but there are many derived equivalent self-injective algebras which are not Morita equivalent, *cf.* below. On the other hand, two *hereditary* finite-dimensional algebras are T-equivalent iff they are derived equivalent, by a result of Happel-Rickard-Schofield [36].

In the presence of an equivalence $\mathsf{D}(A) \to \mathsf{D}(B)$, strong links exist between the abelian categories $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, B$. They can be analyzed in analogy with 4.2 (where w now becomes the width of an interval containing all non-zero homology groups of T). The more refined results of section 7 also apply in this situation.

6.2. **Example: A braid group action.** To illustrate theorem 6.1, let $n \geq 2$ and consider the algebra A given by the quiver

with the relators

$$\alpha_{i+1}\alpha_i\ ,\ \beta_i\beta_{i+1}\ ,\ \alpha_i\beta_i - \beta_{i+1}\alpha_{i+1} \text{ for } 1 \leq i < n-1 \text{ and } \alpha_1\beta_1\alpha_1\ ,\ \beta_{n-1}\alpha_{n-1}\beta_{n-1}.$$

Note that the bilinear form

$$< [P], [Q] >= \dim \mathsf{Hom}(P, Q)$$

defined on $K_0(A)$ is symmetric and non degenerate. In fact, its matrix in the basis given by the $P_i = e_i A$, $1 \leq i \leq n$, is the Cartan matrix of the root system of type A_n. For $1 \leq i \leq n$, let X_i be the complex of A-A-bimodules

$$0 \to Ae_iA \to A \to 0\,,$$

where A is concentrated in degree 0. It is not very hard to show that X_i is a two-sided tilting complex. Note that the automorphism σ_i of $K_0(A)$ induced by $? \otimes^{\mathsf{L}}_A X_i$ is the reflection at the *ith* simple root $[P_i]$ so that the group generated by these automorphisms is the Weyl group of A_n (*i.e.* the symmetric group of degree $n + 1$). Rouquier-Zimmermann [69] (*cf.* also [55]) have shown that the functors

$F_i = ? \otimes^{\mathsf{L}} X_i$ themselves satisfy the braid relations (up to isomorphism of functors) so that we obtain a (weak) action of the braid group on the derived category $\mathsf{D}\, A$.

6.3. **The simplest form of Broué's conjecture.** A large number of derived equivalent (and Morita non equivalent) algebras is provided by Broué's conjecture [17], [16], which, in its simplest form, is the following statement

Conjecture (Broué)**.** *Let k be an algebraically closed field of characteristic p and let G be a finite group with abelian p-Sylow subgroups. Then $B_{pr}(G)$ (the principal block of of kG) is derived equivalent to $B_{pr}(N_G(P))$, where P is a p-Sylow subgroup.*

We refer to [64] for a proof of the conjecture for blocks of group algebras with cyclic p-Sylows and to J. Chuang and J. Rickard's contribution to this volume [20] for much more information on the conjecture.

6.4. **Rickard's theorem for bounded derived categories.** Often, it makes sense to consider subcategories of the derived category defined by suitable finiteness conditions. The following theorem shows, among other things, that this yields the same derived equivalence relation:

Theorem ([65])**.** *If A is derived equivalent to B, then*

a) *there is a triangle equivalence $\mathsf{per}\, A \xrightarrow{\sim} \mathsf{per}\, B$ (and conversely, if there is such an equivalence, then A is derived equivalent to B);*
b) *if A and B are right coherent, there is a triangle equivalence $\mathsf{D}^b \mathsf{mod}\, A \xrightarrow{\sim} \mathsf{D}^b \mathsf{mod}\, B$ (and conversely, if A and B are right coherent and there is such an equivalence, then A is derived equivalent to B).*

6.5. **Subordinate invariants.** One of the main motivations for considering derived categories is the fact that they contain a large amount of information about classical homological invariants. Suppose that A and B are k-algebras, projective as modules over k and that there is a complex of A-B-bimodules X such that $? \otimes^{\mathsf{L}} X$ is an equivalence.

a) The algebra A is of finite global dimension iff this holds for the algebra B and in this case, the difference of their global dimensions is bounded by $r - s + 1$ where $[r, s]$ is the smallest interval containing the indices of all non vanishing homology groups of X, *cf.* [27, 12.5]. Note that the homological dimensions may actually differ, as we see from example 4.4.
a) There is a canonical isomorphism $K_0 A \xrightarrow{\sim} K_0 B$ and, if A and B are right coherent, an isomorphism $G_0 A \xrightarrow{\sim} G_0 B$, *cf.* [65].
b) There is a canonical isomorphism between the centers of A and of B, *cf.* [65]. More generally, there is a canonical isomorphism between the Hochschild cohomology algebras of A and B, *cf.* [35] [66]. Moreover, this isomorphism is compatible with the Gerstenhaber brackets, *cf.* [47].
c) There is a canonical isomorphism between the Hochschild homologies of A and B, *cf.* [66], as well as between all variants of their cyclic homologies (in fact, the mixed complexes associated with A and B are linked by a quasi-isomorphism of mixed complexes, *cf.* [50]).
d) There is a canonical isomorphism between $K_i(A)$ and $K_i(B)$ for all $i \geq 0$. In fact, Thomason-Trobaugh have shown [81] how to deduce this from Waldhausen's results [84], *cf.* [22] or [24]. If A and B are right noetherian of finite global dimension, so that $K_i(A) = G_i(A)$, $i \geq 0$, it also follows from Neeman's description of the K-theory of an abelian category $\mathcal{A}$ purely in terms of the *triangulated category* $\mathsf{D}^b(\mathcal{A})$, *cf.* [60] [61] [62].
e) The topological Hochschild homologies and the topological cyclic homologies of A and B are canonically isomorphic. This follows from work of Schwede-Shipley, *cf.* [70].

7. Comparison of t-structures, spectral sequences

The reader is advised to skip this section on a first reading.

7.1. **Motivation.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and suppose that there is a triangle equivalence

$$\Phi : \mathsf{D}^b(\mathcal{A}) \to \mathsf{D}^b(\mathcal{B})$$

between their derived categories. Our aim is to obtain relations between the categories $\mathcal{A}$ and $\mathcal{B}$ themselves. We will refine the analysis which we performed in section 4.2. For this, we will use the fact that the derived category $\mathsf{D}^b(\mathcal{A})$ is 'glued together' from countably many copies of $\mathcal{A}$. The gluing data are encoded in the natural t-structure on $\mathsf{D}^b(\mathcal{A})$. On the other hand, thanks to the equivalence Φ, we may also view $\mathsf{D}^b(\mathcal{A})$ as glued together from copies of $\mathcal{B}$. This is encoded in a second t-structure on $\mathsf{D}^b(\mathcal{A})$, the pre-image under Φ of the natural t-structure on $\mathcal{B}$. We now have two t-structures on $\mathsf{D}^b(\mathcal{A})$. The sought for relations between $\mathcal{A}$ and $\mathcal{B}$ will be obtained by comparing the two t-structures. We will see how spectral sequences arise naturally in this comparison. This generalizes an idea first used in tilting theory by Vossieck [83] and developped in this volume by Brenner-Butler [15]. Finally, we will review the relatively subtle results [54] which are obtained by imposing compatibility conditions between the two t-structures. These compatibility conditions (strictly) imply the vanishing of 'half' the E_2-terms of the spectral sequences involved.

Note that to obtain the second t-structure on $\mathsf{D}^b(\mathcal{A})$ we could equally well have started from a duality

$$\Psi : \mathsf{D}^b(\mathcal{A}) \to (\mathsf{D}^b(\mathcal{B}))^{op}.$$

Indeed, both tilting theory and Grothendieck-Roos duality theory [68] yield examples which fit into the framework which we are about to sketch.

7.2. **Aisles and t-structures.** Let $\mathcal{T}$ be a triangulated category with suspension functor S. A full additive subcategory $\mathcal{U}$ of $\mathcal{T}$ is called an *aisle* in $\mathcal{T}$ if

a) $S\mathcal{U} \subset \mathcal{U}$,
b) $\mathcal{U}$ is stable under extensions, i.e. for each triangle $X \to Y \to Z \to SX$ of $\mathcal{T}$ we have $Y \in \mathcal{U}$ whenever $X, Z \in \mathcal{U}$,
c) the inclusion $\mathcal{U} \to \mathcal{T}$ admits a right adjoint $\mathcal{T} \to \mathcal{U}$, $X \mapsto X_{\mathcal{U}}$.

For each full subcategory $\mathcal{V}$ of $\mathcal{T}$ we denote by $\mathcal{V}^{\perp}$ (resp. ${}^{\perp}\mathcal{V}$) the full additive subcategory consisting of the objects $Y \in \mathcal{T}$ satisfying $\mathsf{Hom}(X, Y) = 0$ (resp. $\mathsf{Hom}(Y, X) = 0$) for all $X \in \mathcal{V}$.

Proposition ([46]). *A strictly (=closed under isomorphisms) full subcategory $\mathcal{U}$ of $\mathcal{T}$ is an aisle iff it satisfies a) and c')*

c') *for each object X of $\mathcal{T}$ there is a triangle $X_{\mathcal{U}} \to X \to X^{\mathcal{U}^{\perp}} \to S(X_{\mathcal{U}})$ with $X_{\mathcal{U}} \in \mathcal{U}$ and $X^{\mathcal{U}^{\perp}} \in \mathcal{U}^{\perp}$.*

Moreover, a triangle as in c') is unique.

Given an aisle $\mathcal{U} \subset \mathcal{T}$ and $n \in \mathbb{Z}$, we define

$$\mathcal{U}_{\leq n} = \mathcal{U}_{<n+1} = S^n\mathcal{U}\ ,\ \mathcal{U}_{>n} = \mathcal{U}_{\geq n+1} = (\mathcal{U}_{\leq n})^{\perp}\ ,$$
$$\tau^{\mathcal{U}}_{\leq n}X = \tau^{\mathcal{U}}_{<n+1}X = X_{\mathcal{U}_{\leq n}}\ ,\ \tau^{\mathcal{U}}_{>n}X = \tau^{\mathcal{U}}_{\geq n+1}X = X^{\mathcal{U}_{>n}}.$$

Then the proposition above shows that $(\tau^{\mathcal{U}}_{\leq n}, \tau^{\mathcal{U}}_{>n})_{n\in\mathbb{Z}}$ is a *t-structure* [9] on $\mathcal{T}$ and that we have a bijection between aisles in $\mathcal{T}$ and t-structures on $\mathcal{T}$.

For example, let $\mathcal{T}$ be the derived category $\mathsf{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$. Then the full subcategory $\mathcal{U}$ formed by the complexes X such that $H^n(X) = 0$ for all $n > 0$ is the *natural aisle* on $\mathsf{D}(\mathcal{A})$. Its right orthogonal is formed by the complexes

Y with $H^n(Y) = 0$ for all $n \leq 0$. The corresponding truncation functors $\tau_{\leq 0}$ and $\tau_{>0}$ are given by

$$\tau_{\leq 0}(X) = (\ldots \to X^{-1} \to Z^0(X) \to 0 \to \ldots)$$
$$\tau_{>0}(X) = (\ldots 0 \to X^0/Z^0(X) \to X^1 \to \ldots).$$

The corresponding t-structure is the *natural t-structure* on $\mathsf{D}(\mathcal{A})$. Let $\mathcal{U} \subset \mathcal{T}$ be an aisle. Its *heart* is the full subcategory

$$\mathcal{U} \cap S(\mathcal{U}^{\perp}) = \mathcal{U}_{\leq 0} \cap \mathcal{U}_{\geq 0}.$$

It equals the heart of the corresponding t-structure [9]. For example, the heart of the natural t-structure on $\mathsf{D}(\mathcal{A})$ equals $\mathcal{A}$ (identified with the full subcategory of the complexes with homology concentrated in degree 0). In general, the heart $\mathcal{H}$ of an aisle $\mathcal{U}$ is always abelian, each short exact sequence (i,p) of $\mathcal{H}$ fits into a unique triangle

$$A \xrightarrow{i} B \xrightarrow{p} C \xrightarrow{e} SA$$

and the functor $H^0_{\mathcal{U}} = \tau_{\leq 0}\tau_{\geq 0}$ is a homological functor. We put $H^n_{\mathcal{U}} = H^0_{\mathcal{U}} \circ S^n$.

7.3. **Example: classical tilting theory.** Let (A, T, B) be a tilting triple. In $\mathcal{T} = \mathsf{D}^b(\mathsf{Mod}\, A)$, we consider the natural aisle $\mathcal{U}$ and the aisle $\mathcal{V}$ which is the image of the natural aisle of $\mathsf{D}^b(\mathsf{Mod}\, A)$ under the functor $\mathbf{R}\mathsf{Hom}_B(T, ?)$. Then the heart $\mathcal{A}$ of $\mathcal{U}$ identifies with $\mathsf{Mod}\, A$, the heart $\mathcal{B}$ of $\mathcal{V}$ with $\mathsf{Mod}\, B$, the functor $H^n_{\mathcal{V}}|\mathcal{A}$ with $\mathsf{Tor}^A_{-n}(?, T)$ and the functor $H^n_{\mathcal{U}}|\mathcal{B}$ with $\mathsf{Ext}^n_B(T, ?)$.

7.4. **Example: duality theory.** Let R be a commutative ring which is noetherian and regular, *i.e.* of finite homological dimension. Recall [57, 17.4] that

a) For each finitely generated R-module M, the codimension

$$c\,(M) = \inf \{\dim R_p : p \in \mathrm{Spec}\;(R)\;, M_p \neq 0\}$$

equals the grade

$$g\,(M) = \inf\{i : \mathrm{Ext}^i_R\,(M, R) \neq 0\}.$$

b) We have $c\,(\mathrm{Ext}^n_R\,(M, R)) \geq n$ for all finitely generated R-modules M and N and each n.

The derived functor $D = \mathbf{R}\mathsf{Hom}_R(?, R)$ induces a duality

$$\mathsf{D}^b(\mathsf{mod}\, R) \xrightarrow{\sim} (\mathsf{D}^b(\mathsf{mod}\, R))^{op}.$$

In $\mathcal{T} = \mathsf{D}^b(\mathsf{mod}\, R)$, we consider the natural aisle $\mathcal{V}$ and the aisle $\mathcal{U}$ which is the image of the natural co-aisle under D. The heart $\mathcal{B}$ of $\mathcal{V}$ identifies with $\mathsf{mod}\, R$ and the heart $\mathcal{A}$ of $\mathcal{U}$ with $(\mathsf{mod}\, R)^{op}$. The functors $H^n_{\mathcal{U}}|\mathcal{B}$ and $H^n_{\mathcal{V}}|\mathcal{A}$ are given by $\mathsf{Ext}^{-n}_R(?, R)$ and $\mathsf{Ext}^n_R(?, R)$.

7.5. **Spectral sequences.** Let $\mathcal{T}$ be a triangulated category and H^0 a homological functor defined on $\mathcal{T}$ with values in an abelian category. Put $H^n = H^0 \circ S^n$, $n \in \mathbb{Z}$. Let

$$\ldots \to X_{q-1} \xrightarrow{i_q} X_q \to \ldots\;,\; q \in \mathbb{Z}$$

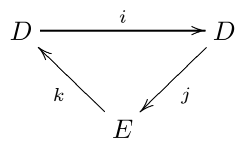
be a diagram in $\mathcal{T}$ such that $X_q = 0$ for all $q \ll 0$ and i_q is invertible for all $q \gg 0$. Let X be the colimit (=direct limit) of this diagram. Let us choose a triangle

$$X_{q-1} \xrightarrow{i_q} X_q \to X^q_{q-1} \to SX_q$$

for each $q \in \mathbb{Z}$. Then the sequences

$$\ldots \to H^{p+q}(X_{q-1}) \to H^{p+q}(X_q) \to H^{p+q}(X^q_{q-1}) \to \ldots\;,\; p, q \in \mathbb{Z}\,,$$

combine into an exact couple



where $D^{pq} = H^{p+q}(X_q)$ and $E^{pq} = H^{p+q}(X^q_{q-1})$. The associated spectral sequence has $E_2^{pq} = E^{pq}$ and its rth differential is of degree $(r, 1-r)$. It converges after finitely many pages to $H^{p+q}(X)$

$$E_2^{pq} = H^{p+q}(X_q) \Longrightarrow H^{p+q}(X).$$

The qth term of the corresponding filtration of $H^{p+q}(X)$ is the image of $H^{p+q}(X_q)$ under canonical map ι so that we have canonical isomorphisms

$$E_\infty^{pq} \xrightarrow{\sim} \iota(H^{p+q}(X_q))/\iota(H^{p+q}(X_{q-1})).$$

Now suppose that in $\mathcal{T}$, we are given two aisles $\mathcal{U}$ and $\mathcal{V}$ with hearts $\mathcal{A}$ and $\mathcal{B}$. We suppose that $\mathcal{A}$ generates $\mathcal{U}$ as a triangulated category and that the same holds for $\mathcal{B}$. This entails that for each $X \in \mathcal{T}$, the sequence

$$\ldots \to \tau^{\mathcal{V}}_{\leq q-1} X \to \tau^{\mathcal{V}}_{\leq q} X \to \ldots$$

satisfies the assumptions made above. We choose the canonical triangles

$$\tau^{\mathcal{V}}_{\leq q-1} X \to \tau^{\mathcal{V}}_{\leq q} X \to S^{-q} H^q_{\mathcal{V}}(X) \to S\tau^{\mathcal{V}}_{\leq q-1} X.$$

If we apply the above reasoning to these data and to the homological functor $H^0_{\mathcal{U}}$, we obtain a spectral sequence, convergent after finitely many pages, with

(7.5.1) $$E_2^{pq} = H^{p+q}_{\mathcal{U}}(S^{-q} H^q_{\mathcal{V}}(X)) = H^p_{\mathcal{U}} H^q_{\mathcal{V}}(X) \Longrightarrow H^{p+q}(X).$$

Of course, if we exchange $\mathcal{U}$ and $\mathcal{V}$, we also obtain a spectral sequence

(7.5.2) $$E_2^{pq} = H^p_{\mathcal{V}} H^q_{\mathcal{U}}(X) \Longrightarrow H^{p+q}(X).$$

In example 7.3, the two sequences become

$$E_2^{pq} = \mathsf{Ext}^p_B(T, \mathsf{Tor}^A_{-q}(M, T)) \Longrightarrow M \quad \text{and} \quad E_2^{pq} = \mathsf{Tor}^A_{-p}(\mathsf{Ext}^q_B(T, N)) \Longrightarrow N\,,$$

where we suppose that $M \in \mathsf{Mod}\, A$ and $N \in \mathsf{Mod}\, B$. They lie respectively in the second and in the fourth quadrant and have their non zero terms inside a square of width equal to the projective dimension of T. Thus, we have $E_\infty = E_{r+1}$ if r is the projective dimension of T. In particular, if $r = 1$, then $E_2 = E_\infty$. For the first sequence, the qth term of the corresponding filtration of

$$M = \mathbf{R}\mathsf{Hom}_B(T, M \otimes^{\mathsf{L}}_A T)$$

equals the image of the map $H^0_{\mathcal{U}}(\tau^{\mathcal{V}}_{\leq q} M) \to M$, *i.e.* of

$$H^0(\mathbf{R}\mathsf{Hom}_B(T, \tau_{\leq q}(M \otimes^{\mathsf{L}}_A T)) \to M.$$

In example 7.4, the first sequence becomes

$$E_2^{pq} = \mathsf{Ext}^p_R(\mathsf{Ext}^{-q}_R(M, R), R) \Longrightarrow M\,,$$

where we suppose that $M \in \mathsf{mod}\, R$.

7.6. **Compatibility of t-structures.** Let $\mathcal{T}$ be a triangulated category with suspension functor S. Let $\mathcal{U}$ and $\mathcal{V}$ be aisles in $\mathcal{T}$. We use the notations of 7.5 for the t-structures associated with the aisles. Moreover, we put $\mathcal{A}_{\geq n} = \mathcal{A} \cap \mathcal{V}_{\geq n}$ and $\mathcal{B}_{\leq n} = \mathcal{B} \cap \mathcal{U}_{\leq n}$. Thus we have filtrations

$$\mathcal{A} \supset \ldots \supset \mathcal{A}_{\geq n} \supset \mathcal{A}_{\geq n+1} \supset \ldots \quad \text{and} \quad \ldots \subset \mathcal{B}_{\leq n} \subset \mathcal{B}_{\leq n+1} \subset \ldots \subset \mathcal{B} \ .$$

Note that $M \in \mathcal{A}$ belongs to $\mathcal{A}_{\geq n}$ iff $H^q_{\mathcal{V}}(M) = 0$ for all $q < n$. This occurs iff all the lines below $q = n$ vanish in the spectral sequence 7.5.1. Similarly, $N \in \mathcal{B}$ belongs to $\mathcal{B}_{\leq n}$ iff all the lines above $q = n$ vanish in the spectral sequence 7.5.2.

The co-aisle $\mathcal{U}^{\perp}$ is *compatible* with the aisle $\mathcal{V}$ if $\mathcal{U}^{\perp}$ is stable under the truncation functors $\tau^{\mathcal{V}}_{\geq n}$ for all $n \in \mathbb{Z}$. Dually, the aisle $\mathcal{U}$ is *compatible* with the co-aisle $\mathcal{V}^{\perp}$, if $\mathcal{U}$ is stable under the truncation functors $\tau^{\mathcal{V}}_{\leq n}$ for all $n \in \mathbb{Z}$, *cf.* [54]. If $\mathcal{U}$ is compatible with $\mathcal{V}^{\perp}$, it is not hard to check that $\mathcal{U}$ is also stable under $\tau^{\mathcal{V}}_{>n}$ and we have $H^n_{\mathcal{V}}(\mathcal{U}_{\leq 0}) \subset \mathcal{B}_{\leq -n}$. Thus we obtain

$$H^p_{\mathcal{U}} H^q_{\mathcal{V}} | \mathcal{U}_{\leq 0} = 0$$

for all $p + q > 0$. Thus, if X belongs to $\mathcal{U}_{\leq 0}$, then in the spectral sequence 7.5.1, all terms above the codiagonal $p + q = 0$ vanish. The following proposition shows that the converse often holds:

Proposition ([54]). *Suppose that $\mathcal{A}$ generates $\mathcal{T}$ as a triangulated category and that the same holds for $\mathcal{B}$. Then the following are equivalent*

(i) *$\mathcal{U}$ is compatible with $\mathcal{V}^{\perp}$.*
(ii) *$\mathcal{U} = \{X \in \mathcal{T} \mid H^n_{\mathcal{V}}(X) \in \mathcal{B}_{\leq -n}$ for all $n \in \mathbb{Z}\}$.*
(iii) *We have*
  a) *$H^p_{\mathcal{U}} H^q_{\mathcal{V}} | \mathcal{A} = 0$ for all $p + q > 0$ and*
  b) *for each morphism $g : N \to N'$ of $\mathcal{B}$ with $N \in \mathcal{B}_{\leq n}$ and $N' \in \mathcal{B}_{\leq n+1}$, we have $\mathsf{ker}(g) \in \mathcal{B}_{\leq n}$ and $\mathsf{cok}(g) \in \mathcal{B}_{\leq n+1}$.*

It is not hard to show that in example 7.4, the aisle $\mathcal{U}$ is compatible with $\mathcal{V}^{\perp}$ and $\mathcal{V}^{\perp}$ is compatible with $\mathcal{U}$. In example 7.3 these properties are satisfied if T is of projective dimension 1. They are not always satisfied for tilting modules T of higher projective dimension.

7.7. **Links between the hearts of compatible t-structures.** Keep the notations of 7.6 and suppose moreover that the *t-structures defined by $\mathcal{U}$ and $\mathcal{V}$ are compatible*, *i.e.* that $\mathcal{U}$ is compatible with $\mathcal{V}^{\perp}$ and $\mathcal{V}^{\perp}$ compatible with $\mathcal{U}$. Then for each object $N \in \mathcal{B}$, one obtains a short exact sequence

$$0 \to H^0_{\mathcal{V}}(\tau^{\mathcal{U}}_{\leq q} N) \to N \to H^0_{\mathcal{V}}(\tau^{\mathcal{U}}_{>q} N) \to H^1_{\mathcal{V}}(\tau^{\mathcal{U}}_{\leq q} N) \to 0.$$

Its terms admit intrinsic descriptions: First consider $N_{\leq q} = H^0_{\mathcal{V}}(\tau^{\mathcal{U}}_{\leq q} N)$. One shows that for each $N \in \mathcal{B}$, the morphism $N_{\leq q} \to N$ is the largest subobject of N contained in $\mathcal{B}_{\leq q}$. It follows that $\mathcal{B}_{\leq q}$ is stable under quotients. Note that $N_{\leq q}$ is also the qth term of the filtration on N given by the spectral sequence

$$E_2^{pq} = H^p_{\mathcal{V}} H^q_{\mathcal{U}}(N) \Longrightarrow N.$$

Now consider $N_{>q} = H^0_{\mathcal{V}}(\tau^{\mathcal{U}}_{>q} N)$. Call a morphism $t : N \to N'$ of $\mathcal{B}$ a *q-quasi-isomorphism* if its kernel belongs to $\mathcal{B}_{\leq q}$ and its cokernel to $\mathcal{B}_{\leq q-1}$; call an object of B of $\mathcal{B}$ *q-closed* if the map

$$\mathsf{Hom}(t, B) : \mathsf{Hom}(N', B) \to \mathsf{Hom}(N, B)$$

is bijective for each q-quasi-isomorphism $t : N \to N'$. It is easy to see that the morphism $N \to N_{>q}$ is a q-quasi-isomorphism. Moreover, one shows that $N_{>q}$ is q-closed. Thus the functor $N \mapsto N_{>q}$ is left adjoint to the inclusion of the subcategory of q-closed objects in $\mathcal{B}$.

Dually, one defines q-co-quasi-isomorphisms and q-co-closed objects in $\mathcal{A}$. Let $\underline{\mathcal{A}}_q \subset \mathcal{A}_{\geq q}$ be the full subcategory of $(q+1)$-co-closed objects and $\underline{\mathcal{B}}_q \subset \mathcal{B}_{\leq q}$ the full subcategory of $(q+1)$-closed objects. Then we have the

Proposition. *The functors $H^q_{\mathcal{U}}$ and $H^q_{\mathcal{V}}$ induce a pair of adjoint functors*

$$H^q_{\mathcal{U}} : \mathcal{B}_{\leq -q} \rightleftarrows \mathcal{A}_{\geq q} : H^q_{\mathcal{V}}$$

and inverse equivalences

$$\underline{\mathcal{B}}_q \rightleftarrows \underline{\mathcal{A}}_{-q} \ .$$

8. Algebraic triangulated categories and dg algebras

8.1. **Motivation.** One form of Morita's theorem characterizes module categories among abelian categories: if $\mathcal{A}$ is an abelian category admitting all set-indexed coproducts and P is a compact (*i.e.* $\mathsf{Hom}(P, ?)$ commutes with all set-indexed coproducts) projective generator of $\mathcal{A}$, then the functor

$$\mathsf{Hom}(P, ?) : \mathcal{A} \to \mathsf{Mod}\,\mathsf{End}(P)$$

is an equivalence. Is there an analogue of this theorem for triangulated categories ? Presently, it is not known whether such an analogue exists for arbitrary triangulated categories. However, for triangulated categories obtained as homotopy categories of Quillen model categories, there are such analogues. The most far-reaching ones are due to Schwede-Shipley, *cf.* [71] and [73]. The simplest, and historically first [49], case is the one where the triangulated category is the stable category of a Frobenius category. It turns out that all triangulated categories arising in algebra are actually of this form. In this case, the rôle of the module category $\mathsf{Mod}\,\mathsf{End}(P)$ is played by the derived category of a differential graded algebra. In this section, we will review the definition of differential graded algebras and their derived categories, state the equivalence theorem and illustrate it with Happel's description of the derived category of an ordinary algebra.

8.2. **Differential graded algebras.** Let k be a commutative ring. Following Cartan [18] a *differential graded (=dg) k-algebra* is a $\mathbb{Z}$-graded associative k-algebra

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$

endowed with a differential, *i.e.* a homogeneous k-linear endomorphism $d : A \to A$ of degree $+1$ such that $d^2 = 0$ and the Leibniz rule holds: we have

$$d(ab) = d(a)\,b + (-1)^p a\,d(b)$$

for all $a \in A^p$ and all $b \in A$. Let A be a dg algebra. A *differential graded A-module* is a $\mathbb{Z}$-graded A-module M endowed with a differential $d : M \to M$ homogeneous of degree $+1$ such that the Leibniz rule holds:

$$d(ma) = d(m)\,a + (-1)^p m\,d(a)$$

for all $m \in M^p$ and all $a \in A$. Note that the homology $H^*(A)$ is a $\mathbb{Z}$-graded algebra and that $H^*(M)$ becomes a graded $H^*(A)$-module for each dg A-module M.

If $A^p = 0$ for all $p \neq 0$, then A is given by the ordinary algebra A^0. In this case, a dg A-module is nothing but a complex of A^0-modules. In the general case, A becomes a dg module over itself: the free A-module of rank one. If M is an arbitrary A-module and n is an integer, then the shifted complex $M[n]$ carries a natural dg A-module structure (no additional sign changes here).

To give a more interesting example of a dg algebra, let us recall the morphism complex: Let B be an ordinary associative k-algebra. For two complexes

$$M = (\ldots \to M^p \overset{d_M^p}{\to} M^{p+1} \to \ldots)$$

and N of B-modules, the *morphism complex* $\mathsf{Hom}^\bullet{}_B(M,N)$ has as its nth component the k-module of B-linear maps $f : M \to N$, homogeneous of degree n (which need not satisfy any compatibility condition with the differential). The differential of the morphism complex is defined by

$$d(f) = d_N \circ f - (-1)^n f \circ d_M \ ,$$

where f is of degree n. Note that the zero cycles of the morphism complex identify with the morphisms of complexes $M \to N$ and that its 0th homology identifies with the set of homotopy classes of such morphisms. Then the composition of graded maps yields a natural structure of dg algebra on the endomorphism complex $\mathsf{Hom}^\bullet{}_B(M,M)$ and for each complex N, the morphism complex $\mathsf{Hom}^\bullet{}_B(M,N)$ becomes a natural dg module over $\mathsf{Hom}^\bullet(M,M)$. Note that even if M is concentrated in degrees ≥ 0, the dg algebra $\mathsf{Hom}^\bullet{}_B(M,M)$ may have non-zero components in positive and negative degrees.

8.3. **The derived category.** Let A be a dg algebra. A morphism $s : L \to M$ of dg A-modules is a *quasi-isomorphism* if it induces a quasi-isomorphism in the underlying complexes. By definition, the *derived category* $\mathsf{D}(A)$ is the localization of the category of dg A-modules at the class of quasi-isomorphisms. If A is concentrated in degree 0, *i.e.* $A = A^0$, then $\mathsf{D}(A)$ equals the ordinary derived category $\mathsf{D}(A^0)$. Note that, for arbitrary dg algebras A, homology yields a well defined functor

$$H^* : \mathsf{D}(A) \to \mathsf{Grmod}(H^*A) \ , \ M \mapsto H^*(M) \ ,$$

where $\mathsf{Grmod}(H^*A)$ denotes the category of graded $H^*(A)$-modules. To compute morphism spaces in the derived category, it is useful to introduce the homotopy category: a morphism of dg modules $f : L \to M$ is *nullhomotopic* if there is a morphism $r : L \to M$ of *graded* A-modules (not compatible with the differential) such that

$$f = d_M \circ r + r \circ d_L.$$

The nullhomotopic morphisms form an ideal in the category of dg A-modules and the quotient by this ideal is the homotopy category $\mathsf{H}(A)$. We have a canonical functor $\mathsf{H}(A) \to \mathsf{D}(A)$. A dg module M is *cofibrant* (resp. *fibrant*) if the map

$$\mathsf{Hom}_{\mathsf{H}(A)}(M,L) \to \mathsf{Hom}_{\mathsf{D}(A)}(M,L) \quad \text{resp.} \quad \mathsf{Hom}_{\mathsf{H}(A)}(L,M) \to \mathsf{Hom}_{\mathsf{D}(A)}(L,M)$$

is bijective for all dg A-modules L. We have the

Proposition ([49]). a) *The derived category $\mathsf{D}(A)$ admits a canonical triangulated structure whose suspension functor is $M \to M[1]$. Moreover, it admits all set-indexed coproducts and these are computed as coproducts of dg A-modules.*

b) *For each dg A-module M, there are quasi-isomorphisms*

$$\mathbf{p}M \to M \quad \textit{and} \quad M \to \mathbf{i}M$$

such that $\mathbf{p}M$ is cofibrant and $\mathbf{i}M$ is fibrant.

c) *The free A-module A_A is cofibrant. We have*

$$\mathsf{Hom}_{\mathsf{D}(A)}(A, M[n]) \xrightarrow{\sim} H^n(M)$$

for all dg A-modules M. In particular, the functor $\mathsf{Hom}_{\mathsf{D}(A)}(A, ?)$ commutes with coproducts and we have

$$H^n(A) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{H}(A)}(A, A[n]) \xrightarrow{\sim} \mathsf{Hom}_{\mathsf{D}(A)}(A, A[n]).$$

Part b) of the proposition shows in particular that

$$\mathsf{Hom}_{\mathsf{D}(A)}(L, M)$$

is actually a set (not just a class) for all dg A-modules L and M. We deduce from the proposition that the object $A \in \mathsf{D}(A)$ is *compact* (*i.e.* its covariant Hom-functor commutes with coproducts) and *generates* $\mathsf{D}(A)$, in the sense that an object M vanishes iff we have $\mathsf{Hom}_{\mathsf{D}(A)}(A, M[n]) = 0$ for all $n \in \mathbb{Z}$.

The objects $\mathbf{p}M$ and $\mathbf{i}M$ are functorial in $M \in \mathsf{D}(A)$. They yield a left and a right adjoint of the canonical functor $\mathsf{H}(A) \to \mathsf{D}(A)$. For each functor $F : \mathsf{H}(A) \to \mathcal{C}$, one defines the total right and left derived functors via

$$\mathbf{R}F = F \circ \mathbf{i} \quad \text{and} \quad \mathbf{L}F = F \circ \mathbf{p}.$$

The *perfect derived category* $\mathsf{per}(A)$ is the full subcategory of $\mathsf{D}(A)$ whose objects are obtained from the free A-module of rank one by forming extensions, shifts (in both directions) and direct factors. Clearly it is a triangulated subcategory consisting of compact objects. We have the following important

Proposition ([59]). *The perfect derived category* $\mathsf{per}(A)$ *equals the subcategory of compact objects of* $\mathsf{D}(A)$.

An explicit proof, based on [59], can be found in [49]. If A is an ordinary algebra, $\mathsf{per}(A)$ is equivalent to $\mathsf{H}^b(\mathsf{proj}\, A)$, the homotopy category of bounded complexes with finitely generated projective components.

8.4. **Stalk algebras.**

Proposition. *Let $f : A \to B$ be a morphism of dg algebras which is a quasi-isomorphism of the underlying complexes. Then the restriction functor*

$$\mathsf{D}(B) \to \mathsf{D}(A)$$

is an equivalence.

It A is a dg algebra, then the complex

$$\tau_{\leq 0}(A) = (\ldots \to A^{-2} \to A^{-1} \to Z^0(A) \to 0 \to \ldots)$$

becomes a dg subalgebra and the canonical map $\tau_{\leq 0}(A) \to H^0 A$ a morphism of dg algebras (where we consider $H^0 A$ as a dg algebra concentrated in degree 0). Thus, if $H^*(A)$ is concentrated in degree 0, then A is linked to $H^0(A)$ by two quasi-isomorphisms. Thus we get the

Corollary. *If A is a dg algebra such that $H^*(A)$ is concentrated in degree 0, then there is a canonical triangle equivalence*

$$\mathsf{D}(A) \xrightarrow{\sim} \mathsf{D}(H^0 A)$$

8.5. **Example: mixed complexes.** Let Λ be the exterior k-algebra on one generator x of degree -1. Endow Λ with the zero differential. Then a dg Λ-module is given by a $\mathbb{Z}$-graded k-module M endowed with $b = d_M$ and with the map

$$B : M \to M\ ,\ m \mapsto (-1)^{\deg(m)} m.x\ ,$$

which is homogeneous of degree -1. We have

$$b^2 = 0\ ,\ B^2 = 0\ ,\ bB + Bb = 0.$$

By definition, the datum of the $\mathbb{Z}$-graded k-module M together with b and B satisfying these relations is a *mixed complex*, *cf.* [45]. The augmentation of Λ yields the Λ-bimodule k. The tensor product over Λ by k yields a functor

$$? \otimes_\Lambda k : \mathsf{H}(\Lambda) \to \mathsf{H}(k)$$

and if we compose its derived functor $? \otimes^{\mathsf{L}}_{\Lambda} k : \mathsf{D}(\Lambda) \to \mathsf{D}(k)$ with H^{-n}, we obtain the cyclic homology:

$$HC_n(M) = H^{-n}(M \otimes^{\mathsf{L}}_{\Lambda} k).$$

Moreover, the negative cyclic homology groups identify with morphism spaces in the derived category:

$$HN_n(M) = \mathsf{Hom}_{\mathsf{D}(\Lambda)}(k, M[n]).$$

8.6. **Frobenius categories.** A *Frobenius category* is an exact category in the sense of Quillen [63] which has enough injectives, enough projectives and where the class of the injectives coincides with that of the projectives. Let $\mathcal{E}$ be a Frobenius category. The morphisms factoring through a projective-injective form an ideal and the quotient by this ideal is the associated *stable category* $\underline{\mathcal{E}}$. The stable category admits a canonical structure of triangulated category [33] whose suspension functor S is defined by choosing admissible short exact sequences

$$0 \to L \to I \to S(L) \to 0$$

with projective-injective I for each object L. The triangles are constructed from the admissible exact sequences of $\mathcal{E}$.

For example, let A be a dg algebra (*e.g.* an ordinary algebra). Then the category of dg A-modules becomes a Frobenius category if we define a short exact sequence

$$0 \to L \to M \to N \to 0$$

of dg A-modules to be admissible exact if it splits in the category of graded A-modules. Then the morphisms factoring through projective-injectives are precisely the nullhomotopic morphisms and the associated stable category is the category $\mathsf{H}(A)$. Now let $\mathcal{C}_c(A)$ be the full subcategory of the category of dg A-modules whose objects are the cofibrant dg A-modules. It is not hard to see that it inherits the structure of a Frobenius category and that its associated stable category is equivalent to $\mathsf{D}(A)$ as a triangulated category.

8.7. **Algebraic triangulated categories and dg algebras.** Let $\mathcal{T}$ be an algebraic triangulated category, *i.e.* a triangulated category which is triangle equivalent to the stable category of some Frobenius category. As we have seen at the end of section 8.6, all derived categories of dg algebras are of this form.

Theorem ([49]). *Let T be an object of $\mathcal{T}$.*

a) *There is a dg algebra $\mathbf{R}\mathsf{Hom}(T,T)$ with homology*

$$H^*(\mathbf{R}\mathsf{Hom}(T,T)) = \bigoplus_{p\in\mathbb{Z}} \mathsf{Hom}_{\mathcal{T}}(T, T[p])$$

and a k-linear triangle functor

$$F : \mathcal{T} \to \mathsf{D}(\mathbf{R}\mathsf{Hom}(T,T))$$

which takes T to the free module of rank one and whose composition with homology is given by

$$\mathcal{T} \to \mathsf{Grmod}(H^*(\mathbf{R}\mathsf{Hom}(T,T)))\ ,\ X \mapsto \bigoplus_{p\in\mathbb{Z}}(T, X[p]).$$

b) *Suppose that $\mathcal{T}$ admits all set-indexed coproducts and that T is a compact generator for $\mathcal{T}$. Then the functor F is a k-linear triangle equivalence*

$$\mathcal{T} \xrightarrow{\sim} \mathsf{D}(\mathbf{R}\mathsf{Hom}(T,T)).$$

c) *Suppose that $\mathcal{T}$ is the closure of T under forming extensions, shifts (in both directions) and direct factors. Then F is a k-linear triangle equivalence*

$$\mathcal{T} \xrightarrow{\sim} \mathsf{per}(\mathbf{R}\mathsf{Hom}(T,T)).$$

If we take $\mathcal{T}$ to be the derived category of a k-algebra B and T a tilting complex, we can deduce the implication from iii) to i) in Rickard's theorem 6.1.

8.8. **Illustration: Happel's theorem.** Let k be a field and A a finite-dimensional k-algebra. Put $DA = \mathsf{Hom}_k(A, k)$. We view DA as an A-A-bimodule. Let B be the graded algebra with $B^p = 0$ for $p \neq 0, 1$, $B^0 = A$ and $B^1 = DA$. Consider the category $\mathsf{Grmod}\, B$ of $\mathbb{Z}$-graded B-modules and its subcategory $\mathsf{grmod}\, B$ of graded B-modules of total finite dimension. If we endow them with all exact sequences, both become abelian Frobenius categories. We would like to apply theorem 8.7 to the stable category $\mathcal{T} = \underline{\mathsf{Grmod}} B$ and the B-module T given by A considered as a graded B-module concentrated in degree 0. A straightforward computation shows that T is compact in $\mathcal{T}$, that

$$\mathsf{Hom}_{\mathcal{T}}(T, T[n]) = 0$$

for all $n \neq 0$ and that $\mathsf{Hom}_{\mathcal{T}}(T, T)$ is canonically isomorphic to A (beware that the suspension in $\mathcal{T}$ has nothing to do with the shift functor of $\mathsf{Grmod}\, B$). By theorem 8.7 and corollary 8.4, we get a triangle functor

$$F : \underline{\mathsf{Grmod}} B \to \mathsf{D}(A).$$

Now by proving the hypotheses of b) and c) of theorem 8.7, one obtains the

Theorem (Happel [33]). *If A is of finite global dimension, then F is a triangle equivalence*

$$\underline{\mathsf{Grmod}} B \xrightarrow{\sim} \mathsf{D}(A).$$

and induces a triangle equivalence

$$\underline{\mathsf{grmod}} B \xrightarrow{\sim} \mathsf{per}(A) \xrightarrow{\sim} \mathsf{D}^b(\mathsf{mod}\, A).$$

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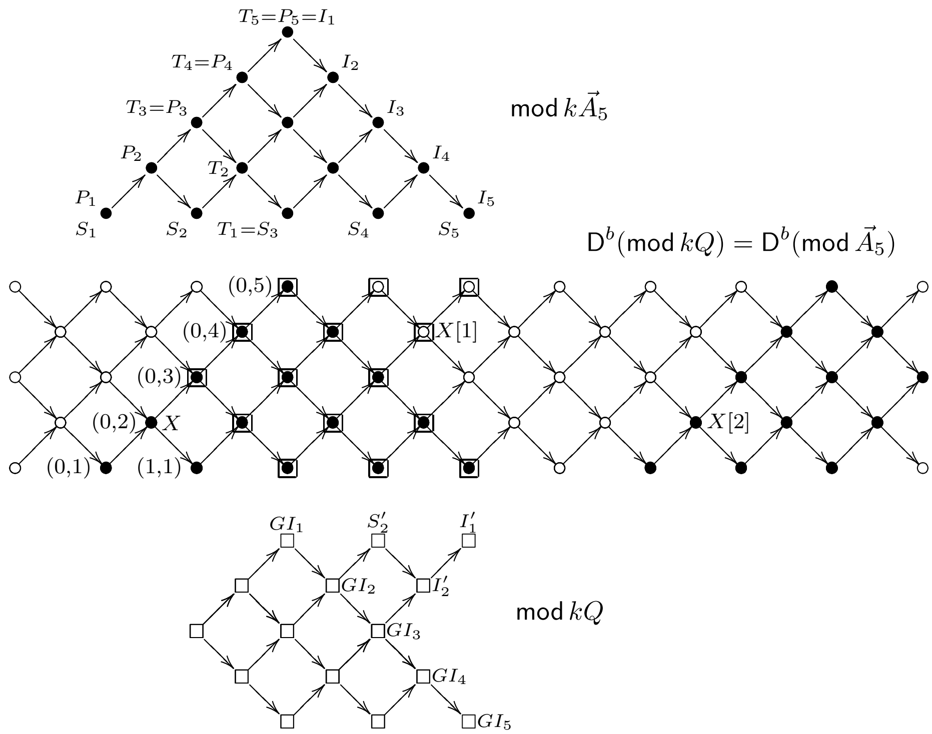

FIGURE 1. Quivers of categories associated with algebras of type A_n

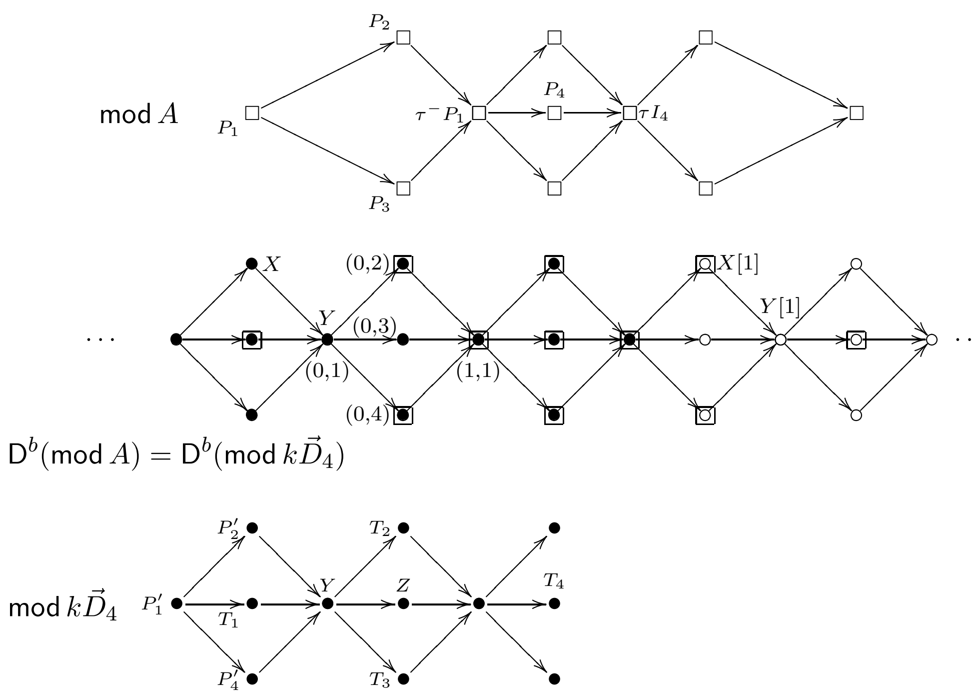

FIGURE 2. Two module categories with the same derived category of type D_4

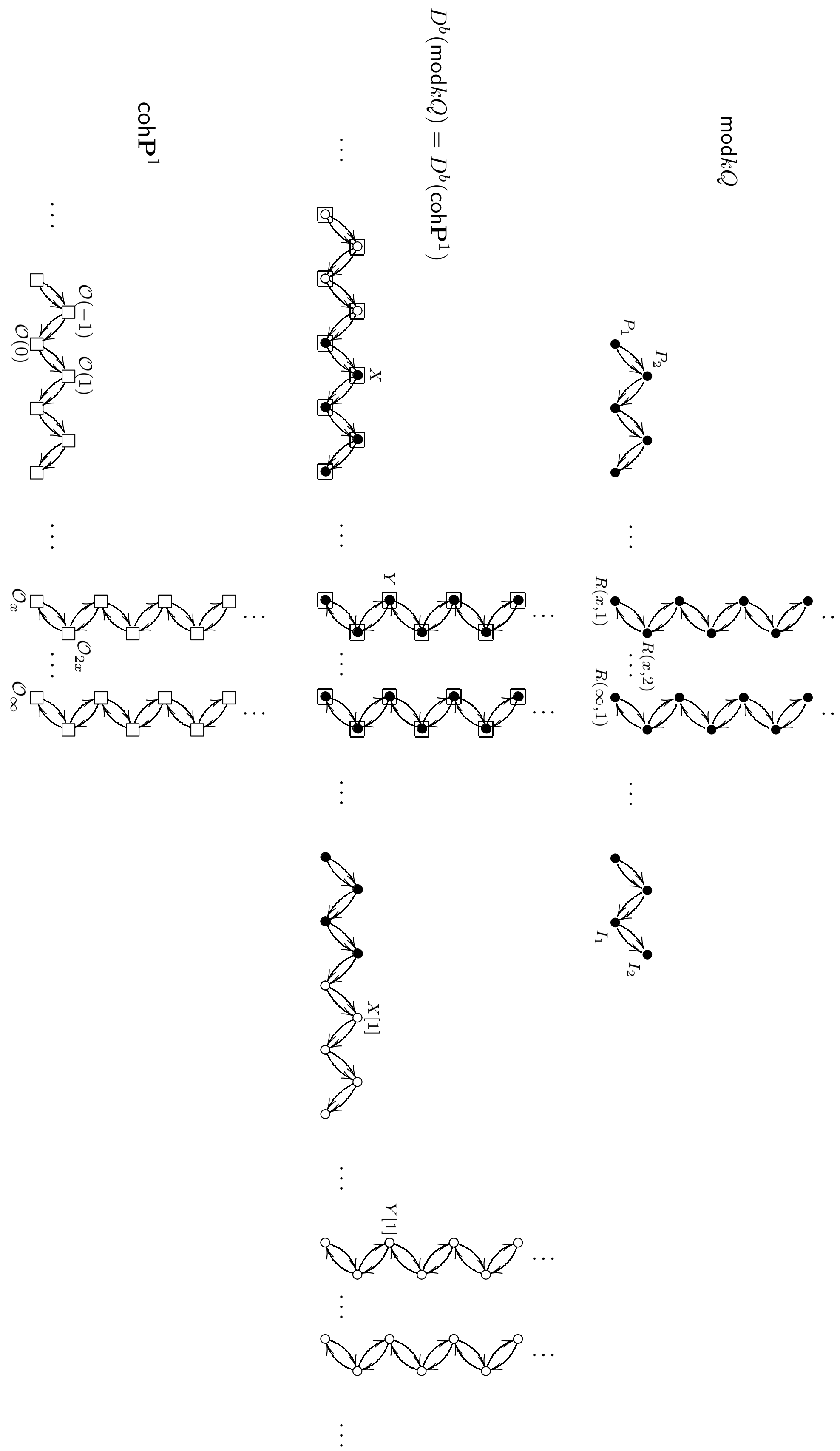

FIGURE 3. Modules over the Kronecker quiver and coherent sheaves on $\mathbb{P}^1$