TILTING THEORY AND CLUSTER ALGEBRAS

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Introduction

The purpose of this paper to give an introduction to the theory of cluster categories and cluster-tilted algebras, with some background on the theory of cluster algebras, which motivated these topics. We will also discuss some of the interplay between cluster algebras on one side and cluster categories/ cluster-tilted algebras on the other, as well as feedback from the latter theory to cluster algebras.

The theory of cluster algebras was initiated by Fomin and Zelevinsky [FZ1], and further developed by them in a series of papers, including [FZ2], some involving other coauthors. This theory has in recent years had a large impact on the representation theory of algebras. The first connection with quiver representations was given in [MRZ]. Then the cluster categories were introduced in [BMRRT] in order to model some of the ingredients in the definition of a cluster algebra. For this purpose a tilting theory was developed in the cluster category. (See [CCS1] for the independent construction of a category in the A_n case which turned out to be equivalent to the cluster category [CCS2]). This further led to the theory of cluster-tilted algebras initiated in [BMR1] and further developed in many papers by various authors.

The theory of cluster-tilted algebras (and cluster categories) is closely connected with ordinary tilting theory. Much of the inspiration comes from usual tilting theory, and features missing in tilting theory when trying to model clusters from the theory of cluster algebras made it necessary to replace the module category $\mod H$ for a finite dimensional hereditary algebra H with a related category which is the cluster category. On the other hand, the theory of cluster-tilted algebras provides a new point of view on the old tilting theory.

The Bernstein-Gelfand-Ponomarev (BGP) reflection functors were an important source of inspiration for the development of tilting theory, which provided a major generalization of the work in [BGP]. The Fomin-Zelevinsky (FZ) mutation, which is an essential ingredient in the definition of cluster algebras, gives a generalization of these reflections in another direction.

We start with introducing cluster algebras in the first section. We illustrate the essential concepts with an example, which will be used throughout the paper. We give main results and conjectures about cluster algebras which are relevant for our further discussion. In Section 2 we introduce and investigate cluster categories, followed by cluster-tilted algebras in Section 3 . In Section 4 we discuss the interplay between cluster algebras and cluster categories/cluster-tilted algebras, and we also give applications to cluster algebras. The cluster categories are a special case of the more general class of Hom-finite triangulated Calabi-Yau categories of dimension 2 (2-CY categories), and much of the theory generalizes to this setting. An important case is the stable category $\underline{\mod}\Lambda$, where Λ is the preprojective algebra of a Dynkin

quiver. The closely related category mod Λ has been studied extensively by Geiss-Leclerc-Schröer, who extended results from cluster categories to this setting, and gave applications to cluster algebras [GLS1][GLS2]. We treat this in Section 5 .

We suppose that the reader is familiar with the basic theory of quiver representations and almost split sequences (see [Rin1],[ARS],[ASS] and other papers in this volume). We also presuppose some background from ordinary tilting theory (see [Rin1],[ASS],[AHK]), but here we shall nevertheless recall relevant definitions and results when they are needed. We generally do not give proofs, but sometimes we include some indication of proofs in order to stress some ideas. Instead we give examples to illustrate the theory, and we try to give some motivation for the work. We should also emphasize that the selection of the material reflects our personal interests.

For each section we add some historical notes with references at the end, rather than giving too many references as we go along. We also refer to the surveys [BM][Rin2]. We assume throughout that we work over a field k which is algebraically closed.

These notes are based on the series of lectures I gave in Trieste in January 2006. I would like to thank I. Muchtadi Alumsayh and G. Bobinski for providing me with a copy of their notes, and Aslak Bakke Buan for helpful comments.

1. CLUSTER ALGEBRAS

In this section we introduce a special class of cluster algebras and illustrate the underlying concepts through a concrete example. We also give a selection of main results and conjectures of Fomin-Zelevinsky which provide an appropriate background for our further discussion.

1.1. **FOMIN-ZELEVINSKY MUTATION.** Let Q be a finite connected quiver with vertices $1, 2, \cdots, n$. We say that Q is a *cluster quiver* if it has no loops $\cdot\,\circlearrowright$ and no 2-cycles $\cdot\,\overset{\frown}{\underset{\smile}{}}\,\cdot$ For each vertex $i = 1, 2, \cdots, n$, we define a new quiver $\mu_i(Q)$ obtained by *mutating* Q, and we call the process Fomin-Zelevinsky mutation, or FZ-mutation for short.

The quiver $\mu_i(Q)$ is obtained from Q as follows.
(i) Reverse all arrows starting or ending at i.
(ii) If in Q we have $n > 0$ arrows from t to i and $m > 0$ arrows from i to s and r arrows from t to s (interpreted as $-r$ arrows from s to t if $r < 0$), then in the new quiver $\mu_i(Q)$ we have $nm - r$ arrows from s to t (interpreted as $r - nm$ arrows from s to t if $nm - r < 0$).

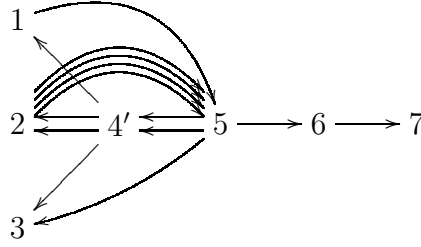An important easily verified property of the mutation is the following.

Proposition 1.1. *For a cluster quiver Q, we have $\mu_i(\mu_i(Q)) = Q$ for each vertex i of Q.*

We illustrate with some examples.
(a) Let Q be the quiver
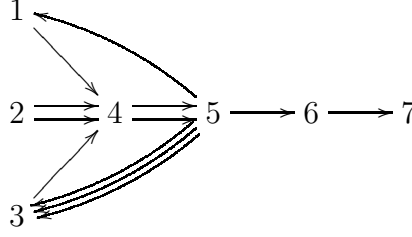
Then $\mu_4(Q)$ it the following quiver:

For example there are now 2·2=4 arrows from vertex 2 to vertex 5, and considering the paths between 3 and 5, we have $1 \cdot 2 - 3 = -1$, so that there is one arrow from 5 to 3.

(b) Let Q be the quiver $1 \to 2 \to 3$. Then $Q' = \mu_3(Q)$ is the quiver $1 \to 2 \leftarrow 3$ obtained by reversing the arrows involving 3. For in this case, since 3 is a sink in the quiver, there is no path of length two with middle vertex 3. (The same thing happens when we mutate at a vertex which is a source).

Hence we see that when we mutate at a sink or a source, the procedure coincides with the BGP-reflections.

When we have a BGP reflection, like the above example, there is an equivalence between the subcategories of the categories of finite dimensional representations $\operatorname{rep} Q$ and $\operatorname{rep} Q'$ obtained by "removing" in each case the simple representation at the vertex 3 [BGP].

1.2. **DEFINITION OF CLUSTER ALGEBRAS.** Let Q be a cluster quiver with vertices $1, 2, \cdots, n$ and let $F = \mathbb{Q}(x_1, \cdots, x_n)$ be the function field in n indeterminates over $\mathbb{Q}$. Consider the pair $(\underline{x}, Q)$, where $\underline{x} = \{x_1, \cdots, x_n\}$. The cluster algebra $C(\underline{x}, Q)$ will be defined to be a subring F. The main ingredients involved in the definition are the following concepts: *cluster, cluster variable, seed, mutation of seeds*.

The pair $(\underline{x}, Q)$ consisting of a transcendence basis $\underline{x}$ for F over the rational numbers $\mathbb{Q}$, together with a quiver with n vertices, is called a *seed*. For $i = 1, \cdots, n$ we define a mutation μ_i taking the seed $(\underline{x}, Q)$ to a new seed $(\underline{x}', Q')$, where $Q' = \mu_i(Q)$ as discussed in 1.1, and $\underline{x}'$ is obtained from $\underline{x}$ by replacing x_i by a new element x_i' in F. Here x_i' is defined by $x_i x_i' = m_1 + m_2$, where m_1 is a monomial in the variables $x_1, \cdots, x_n$, where the power of x_j is the number of arrows from j to i, and m_2 is the monomial where the power of x_j is the number of arrows from i to j. (If there is no arrow from j to i, then $m_1 = 1$, and if there is no arrow from i to j, then $m_2 = 1$.)

Note that while in the new seed the quiver Q' only depends on the quiver Q, then $\underline{x}'$ depends on both $\underline{x}$ and Q. We have $\mu_i^2(\underline{x}, Q) = (\underline{x}, Q)$.

We perform this operation for all $i = 1, \cdots, n$, then perform it on the new seeds etc. or we get back to one of the seeds already computed. The n-element subsets $\underline{x}, \underline{x}', \underline{x}'', \cdots$ occurring are by definition the *clusters*, the elements in the clusters are the *cluster variables*, and the *seeds* are all pairs $(\underline{x}', Q')$ occurring. The corresponding *cluster algebra* $C(\underline{x}, Q)$ which as an algebra only depends on Q, is the subring of F generated by the cluster variables.

When we are given the cluster algebra only, the information on the clusters, cluster variables and seeds may be lost, and also the rule for mutation of seeds. We want to keep all this information in mind, in addition to the cluster algebra itself, which is determined by this information.

We remark that the more general definition of cluster algebras includes the possibility of having the so-called coefficients, and also allows valued quivers. In the language of [FZ1] the last generalization means to consider skew symmetrizable matrixes rather than just skew symmetric ones. The correspondence between quivers and matrices is illustrated by the following example: The quiver $Q \colon 1 \to 2 \to 3$ is sent to $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

1.3. AN EXAMPLE.

Let Q be the quiver $1 \to 2 \to 3$ and $\underline{x} = \{x_1, x_2, x_3\}$, where x_1, x_2, x_3 are indeterminates, and $F = \mathbb{Q}(x_1, x_2, x_3)$. We have $\mu_1(\underline{x}, Q) = (\underline{x}', Q')$, where $Q' = \mu_1(Q)$ is the quiver $1 \leftarrow 2 \to 3$ and $\underline{x}' = \{x_1', x_2, x_3\}$, where $x_1 x_1' = 1 + x_2$, so that $x_1' = \frac{1+x_2}{x_1}$. Further $\mu_2(\underline{x}, Q) = (\underline{x}'', Q'')$, where $Q'' = \mu_2(Q)$ is the quiver $1 \leftrightarrows 2 \leftrightarrows 3$, and $\underline{x}'' = \{x_1, x_2'', x_3\}$, where $x_2 x_2'' = x_1 + x_3$, so that $x_2'' = \frac{x_1 + x_3}{x_2}$. Continuing , we have $\mu_3(\underline{x}, Q) = (\underline{x}''', Q''')$ where Q''' is the quiver $1 \to 2 \leftarrow 3$ and $\underline{x}''' = \{x_1, x_2, x_3'''\}$, where $x_2 x_3''' = x_2 + 1$, so that $x_3''' = \frac{x_2 + 1}{x_3}$.

Hence we have the diagram shown in Figure 1. In this diagram, $Q_1 = \bullet \longrightarrow \bullet \longrightarrow \bullet$, $Q_2 = \bullet \longleftarrow \bullet \longrightarrow \bullet$, $Q_3 = \bullet \leftrightarrows \bullet \leftrightarrows \bullet$, $Q_4 = \bullet \longrightarrow \bullet \longleftarrow \bullet$ and $y = \frac{(1+x_2)x_1 + (1+x_2)x_3}{x_1 x_2 x_3}$. The clusters are: $\{x_1, x_2, x_3\}$, $\{\frac{1+x_2}{x_1}, x_2, x_3\}$, $\{x_1, \frac{x_1+x_3}{x_2}, x_3\}$, $\{x_1, x_2, \frac{1+x_2}{x_3}\}$, $\{\frac{1+x_2}{x_1}, \frac{x_1+(1+x_2)x_3}{x_1 x_2}, x_3\}$, $\{\frac{1+x_2}{x_1}, x_2, \frac{1+x_2}{x_3}\}$, $\{\frac{x_1+(1+x_2)x_3}{x_1 x_2}, \frac{x_1+x_3}{x_2}, x_3\}$, $\{x_1, \frac{x_1+x_3}{x_2}, \frac{(1+x_2)x_1+x_3}{x_2 x_3}\}$, $\{x_1, \frac{(1+x_2)x_1+x_3}{x_2 x_3}, \frac{1+x_2}{x_3}\}$, $\{\frac{1+x_2}{x_1}, \frac{x_1+(1+x_2)x_3}{x_1 x_2}, \frac{(1+x_2)x_1+(1+x_2)x_3}{x_1 x_2 x_3}\}$, $\{\frac{1+x_2}{x_1}, \frac{(1+x_2)x_1+(1+x_2)x_3}{x_1 x_2 x_3}, \frac{1+x_2}{x_3}\}$, $\{\frac{x_1+(1+x_2)x_3}{x_1 x_2}, \frac{x_1+x_3}{x_2}, \frac{(1+x_2)x_1+(1+x_2)x_3}{x_1 x_2 x_3}\}$, $\{\frac{(1+x_2)x_1+(1+x_2)x_3}{x_1 x_2 x_3}, \frac{x_1+x_3}{x_2}, \frac{(1+x_2)x_1+x_3}{x_2 x_3}\}$, $\{\frac{(1+x_2)x_1+(1+x_2)x_3}{x_1 x_2 x_3}, \frac{(1+x_2)x_1+x_3}{x_2 x_3}, \frac{1+x_2}{x_3}\}$, and the cluster variables are: $x_1, x_2, x_3, \frac{1+x_2}{x_1}, \frac{x_1+x_3}{x_2}, \frac{1+x_2}{x_3}, \frac{x_1+(1+x_2)x_3}{x_1 x_2}, \frac{(1+x_2)x_1+x_3}{x_2 x_3}, \frac{(1+x_2)x_1+(1+x_2)x_3}{x_1 x_2 x_3}$.

If Q' is a quiver mutation equivalent to Q, then the cluster algebras $C(Q')$ and $C(Q)$ are isomorphic.

1.4. SOME MAIN RESULTS.

There is a large number of interesting results in the theory of cluster algebras. Here we give some of the main theorems and open problems which are of special interest for these notes.

(a) Finiteness conditions. The cluster algebra $C(Q) = C(\underline{x}, Q)$ is said to be of *finite type* if there is only a finite number of cluster variables. This is equivalent to saying that there is only a finite number of clusters, and also to the fact that there is
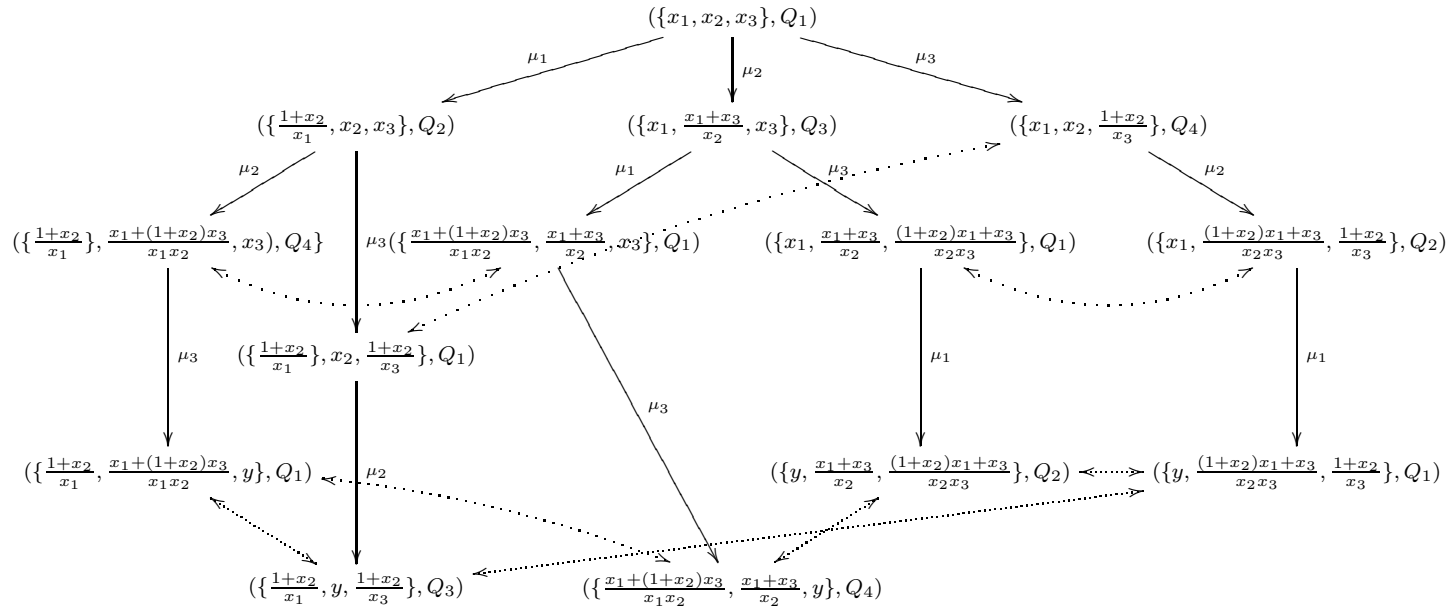
FIGURE 1. Example.

only a finite number of seeds. But as we shall see later, it is not equivalent to having only a finite number of quivers. There is the following description of finite type.

Theorem 1.2. *Let Q be a cluster quiver. Then the cluster algebra $C(Q)$ is of finite type if and only if Q is mutation equivalent to a Dynkin quiver.*

Note that this result is similar to Gabriel's classification theorem of the quivers of finite representation type.

In Section 4 we consider the following problem problem posed by Seven:

For which quivers Q is the mutation class of Q finite?

(b) Laurent phenomenon. Observe that in the example in 1.3 we see that all denominators of the cluster variables (when written in reduced form) are monomials. Surprisingly enough, this is a special case of the following general result.

Theorem 1.3. *Let $C(Q)$ be a cluster algebra with initial seed $(\underline{x}, Q)$. Then for any cluster variable in reduced form, the denominator is a monomial in $x_1, \cdots, x_n$.*

(c) The monomial in the denominators of cluster variables. Taking a closer look at the monomials in the denominators in the example in 1.3, we see that interpreting the factors x_i as the simple modules S_i corresponding to vertex i, the denominators correspond to indecomposable modules via the composition factors. This was already proved in [FZ2] for the case of a Dynkin quiver with no paths of length greater than two. As we shall see later, there are more general results in this direction, obtained as application of the theory of cluster categories and cluster-tilted algebras.

(d) Positivity. Considering again our example, we see that in the numerator, all monomials have positive coefficients. This has been conjectured to be true in general.

(e) Clusters and seeds. Another interesting problem is the following, proved for finite type in [FZ2]
Problem: Is a seed $(\underline{x}', Q')$ expressed in terms of the initial seed $(\underline{x}, Q)$ uniquely determined its cluster $\underline{x}'$?

We shall see that this in the case when the quiver Q in the initial seed has no oriented cycles.

(f) Clusters differing only at one cluster variable. When applying mutation of seeds, the new cluster $\underline{x}''$ has exactly one cluster variable different from the old cluster $\underline{x}'$. If we again consider the example in 1.3, we see that if removing a cluster variable from a cluster, there is a unique other cluster variable which can replace it to give a new cluster. More generally, the following was shown in [FZ2].

Theorem 1.4. *Let $C(Q)$ be the cluster algebra associated with a Dynkin quiver Q. Then there is a unique way to replace a cluster variable in a cluster by another cluster variable to give a new cluster.*

In general, there is the following open problem.

Question: For any cluster algebra, is there a unique way of replacing any cluster variable in a cluster by another cluster variable to give a new cluster?

We shall discuss this problem for acyclic cluster quivers, that is, quivers without oriented cycles. We remark that in the Dynkin case it is known that the cluster variables are in bijection with the almost positive roots, that is, the positive roots together with the negative simple roots.

1.5. **POSSIBLE MODELLING.** The theory of cluster algebras has many nice features, and it is an interesting problem to see if one can find good analogs of the main ingredients involved in their definition, in some appropriate category.

We want the additive category $\mathcal{C}$ to have the following properties.

(i) To have an analog of clusters we want a special class of objects, all having the same number n of nonisomorphic indecomposable summands.

(ii) To imitate the process of seed mutation, we would want that each indecomposable summand of an object in the class can be replaced by a (unique) nonisomorphic indecomposable object such that we still get an object in our class.

(iii) To get a categorical interpretation of the definition of the new cluster variable x_i'' coming from x_i', we would want that when an indecomposable object M is replaced by an indecomposable object M^*, then there are exact sequences or triangles connecting M and M^*, in both directions.

(iv) We would want an interpretation of the FZ-mutation.

The hope would be that this point of view should lead to an interesting theory in itself, and at the same time, or instead, give a better understanding of the cluster algebras.

Notes: The material in 1.1,1.2,1.4 is taken from [FZ1][FZ2][FZ3][BFZ]; see [BIRS] for material related to 1.5.

2. CLUSTER CATEGORIES

Associated with a given cluster algebra we want to find some category $\mathcal{C}$ having a set of objects which we can view as analogs of clusters and which satisfy some or all of the requirements listed in 1.5.

A cluster algebra is said to be acyclic if in the mutation class of the associated cluster quivers there is some quiver Q with no oriented cycles. Then we have an associated finite dimensional hereditary k-algebra kQ. So the category $\mathrm{mod}\,kQ$ of finite dimensional kQ-modules might be a natural choice of category for modelling acyclic cluster algebras.

2.1. **TILTING MODULES OVER HEREDITARY ALGEBRAS.** If we consider $\mathcal{C} = \mathrm{mod}\,kQ$ as the category we are looking for, then a natural choice of objects would be the tilting kQ-modules. On one hand the reason is that they have n nonisomorphic indecomposable summands, where n is the number of vertices in Q. On the other hand there is a special tilting module associated with a BGP-reflection of a quiver, and as we have seen, BGP-reflection is a special case of FZ-mutation.

It will be instructive to first discuss this connection with tilting. Recall that for a hereditary algebra H, an H-module T is *tilting* if $\operatorname{Ext}^1_H(T,T) = 0$ and T has exactly n nonisomorphic indecomposable summands up to isomorphism.

**Example:** Let Q be the quiver $1 \to 2 \to 3$.

(a) We first do mutation at the vertex 3. Then $\mu_3(Q) = Q'\colon 1 \to 2 \leftarrow 3$. If $H = kQ$, the H-module H is clearly a tilting H-module. Write $H = P_1 \oplus P_2 \oplus P_3$, where P_i is the indecomposable projective module associated with the vertex i, and let S_i denote the simple top of P_i. Let $T = P_1 \oplus P_2 \oplus \tau^{-1}S_3$, where τ denotes the translation associated with almost split sequences, so that $\tau^{-1}S_3 = S_2$. We then have the following AR-quiver.
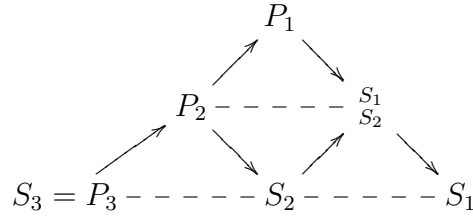
Note that $\operatorname{End}_H(T)^{op} \simeq kQ' = H'$, and $\operatorname{End}_H(H)^{op} \simeq kQ$. So we can pass from kQ to kQ', and hence from Q to Q', by replacing the indecomposable summand P_3 of the tilting module H by $\tau^{-1}S_3$ to get another tilting H-module, and then taking endomorphism algebras. Note that $\tau^{-1}P_3$ is the only indecomposable H-module which can replace P_3 to give a new tilting module. Also $\operatorname{Hom}_H(T,\ )\colon \operatorname{mod} H \to \operatorname{mod} H'$ induces the equivalence between subcategories as discussed in 1.1.

This example illustrates the module theoretical interpretation of the BGP-reflection functors. Hence we also get a close connection between the AR-quivers, and the AR-quiver for H' is the following
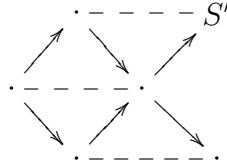
(b) We now do FZ-mutation at vertex 2 in Q, and get $\mu_2(Q) = Q''\colon 1 \overset{\frown}{\leftarrow} 2 \overset{\frown}{\leftarrow} 3$. Then it is natural to try to replace P_2 in $H = P_1 \oplus P_2 \oplus P_3$ to see if we get a nonisomorphic tilting module, and if there is a unique one. This is indeed the case, and the new tilting module is $T = P_1 \oplus S_1 \oplus P_3$. But here we have maps $P_3 \to P_1 \to S_1$ with zero composition, so that $\operatorname{End}_H(T)^{op}$ is given by the quiver with relations $1 \overset{\frown}{\leftarrow} 2 \cdots\cdots 3$, where an arrow $3 \to 2$ is missing compared to Q'.

So our procedure does not work from the point of view of getting a model for the FZ-mutation, but it is quite close to working. What we would need is to have more maps in our category than what we have in $\operatorname{mod} H$, in particular we would like to have nonzero maps from S_1 to $P_3 = S_3$.

(c) We also consider $\mu_1(Q) = 1 \leftarrow 2 \to 3$ from the same point of view. Now we would like to replace P_1 in $H = P_1 \oplus P_2 \oplus P_3$ with another indecomposable H-module to obtain a tilting module. But here we encounter a problem at an earlier

stage. This is actually not possible. The general explanation is that a projective injective module has to be a direct summand of any tilting module. So here we get a problem which indicates that the category $\mathrm{mod}\,H$ is not large enough for being able to replace P_1.

In conclusion, as illustrated by this example, there are the following problems with using the tilting modules over hereditary algebras as a model for clusters.

(1) There are not enough objects in order to replace any indecomposable summand of a tilting module with a nonisomorphic indecomposable module to get a new tilting module.

(2) The quiver of the endomorphism algebra of a tilting module is not the desired one, the problem being that there are not enough maps.

Actually, the following is known for tilting H-modules.

Theorem 2.1. *(a) If T is a tilting H-module, then each indecomposable summand M can be replaced by at most one indecomposable H-module to get a new tilting module.*

(b) There is exactly one if and only if T/M is sincere, that is, each simple H-module occurs as a composition factor.

Note that in our example $P_1 \oplus P_2$ and $P_1 \oplus P_3$ are sincere, whereas $P_2 \oplus P_3$ is not. We call an H-module $\overline{T}$ with $\mathrm{Ext}_H^1(\overline{T}, \overline{T}) = 0$ and with $n-1$ nonisomorphic indecomposable summands an *almost complete tilting H-module*.

In the case when an almost complete tilting module $\overline{T}$ has two complements, that is, there are two ways of completing it to a tilting module, they are connected as follows:

Theorem 2.2. *Let $\overline{T}$ be an almost complete tilting H-module and M and M^* nonisomorphic indecomposable modules such that $\overline{T} \oplus M$ and $\overline{T} \oplus M^*$ are tilting modules. Then, by possibly exchanging M and M^*, there is an exact sequence $0 \to M^* \to B \to M \to 0$ where $f\colon B \to M$ is a minimal right $\mathrm{add}\,\overline{T}$-approximation and $g\colon M^* \to B$ is a minimal left $\mathrm{add}\,\overline{T}$-approximation.*

There is an important class of algebras associated with tilting modules over hereditary algebras. An algebra is said to be *tilted* if it is of the form $\mathrm{End}_H(T)^{op}$, where T is a tilting module over a finite dimensional hereditary algebra H. These algebras are general enough to appear a lot in a natural way in representation theory, and they are close enough to hereditary algebras to inherit nice properties.

For an H-module T, denote by $\mathrm{Fac}\,T$ the subcategory of $\mathrm{mod}\,H$ whose objects are factors of finite direct sums of copies of T. Recall also that a subcategory of $\mathrm{mod}\,H$ is a torsion class if it is closed under factors and extensions, and a torsionfree class if it is closed under submodules and extensions. Then we have the following relationship between hereditary algebras and tilted algebras.

Theorem 2.3. *Let H be a hereditary finite dimensional algebra and T a tilting H-module, and $\Lambda = \mathrm{End}_H(T)^{op}$.*
(a) $\mathcal{T} = \mathrm{Fac}\,T$ is a torsion class in $\mathrm{mod}\,H$, with associated torsionfree class $\mathcal{F} = \{X; \mathrm{Hom}_H(T, X) = 0\}$.
(b) There exists a torsion pair $(\mathcal{X}, \mathcal{Y})$ in $\mathrm{mod}\,\Lambda$, where $\mathcal{X}$ is a torsion class and $\mathcal{Y}$ a torsionfree class, such that

(i) $\mathrm{Hom}_H(T, \ )\colon \mathrm{mod}\, H \to \mathrm{mod}\, \Lambda$ *induces an equivalence between* $\mathcal{T}$ *and* $\mathcal{Y}$
(ii) $\mathrm{Ext}_H^1(T, \ )\colon \mathrm{mod}\, H \to \mathrm{mod}\, \Lambda$ *induces an equivalence between* $\mathcal{F}$ *and* $\mathcal{X}$
(iii) *each indecomposable object in* $\mathrm{mod}\, \Lambda$ *is in* $\mathcal{X}$ *or* $\mathcal{Y}$.

An important homological property which can be proved for a tilted algebra Λ is that Λ has global dimension at most 2.

2.2. DEFINITION AND EXAMPLES.

The question is now how to modify the category $\mathrm{mod}\, H$ to take care of the shortcomings discussed in 2.1. In addition we know from Section 1 that for cluster algebras given by Dynkin quivers, the cluster variables are in one-one correspondence with the almost positive roots. Hence there are n more cluster variables than the number of indecomposable modules for the Dynkin quiver, where n is the number of vertices.

We now explain how to modify $\mathrm{mod}\, H$ in view of of the above remarks.

Let $\mathbf{D}^b(H)$ be the bounded derived category of the finite dimensional hereditary k-algebra $H = kQ$, where Q is a finite quiver without oriented cycles. Then the indecomposable objects are all isomorphic to stalk complexes. The translation τ, which in this case gives an equivalence from the category $\mathrm{mod}_{\mathcal{P}}\, H$ whose indecomposable H-modules are not projective to the category $\mathrm{mod}_{\mathcal{I}}\, H$ whose indecomposable H-modules are not injective, induces an equivalence $\tau\colon \mathbf{D}^b(H) \to \mathbf{D}^b(H)$. Then $\tau(C)$ is the left hand term of the almost split triangle with right hand term C. Note that under the embedding $\mathrm{mod}\, H \to \mathbf{D}(H)$, almost split sequences go to almost split triangles.

Let now F be the equivalence $\tau^{-1}[1]$ from $\mathbf{D}^b(H)$ to $\mathbf{D}^b(H)$, where $[1]$ is the shift functor. Then we define the cluster category $\mathcal{C}_H$ to be the orbit category $\mathbf{D}^b(H)/F$. Here the objects are the F-orbits of objects in $\mathbf{D}^b(H)$. In order to recall the definition of maps, we consider the *fundamental domain* of indecomposable objects given by $\mathrm{ind}\, H \vee \{P_i[1]; i = 1, \cdots, n\}$, where $P_1, \cdots, P_n$ are the nonisomorphic indecomposable projective H-modules. As is easy to see, any F-orbit of indecomposable objects contains exactly one of the indecomposable objects listed. Let A, B be indecomposable objects in this fundamental domain, and denote by A and B also the orbits in which they lie. Then by definition $\mathrm{Hom}_{\mathcal{C}_H}(A, B) = \oplus_{i \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{C}_H}(A, F^i B)$,which in our case equals $\mathrm{Hom}_{\mathbf{D}^b(H)}(A, B) \oplus \mathrm{Hom}_{\mathbf{D}^b(H)}(A, FB)$. We illustrate with the following.

Example: Let Q be the quiver $1 \to 2 \to 3$, and let S_i and P_i be the simple and indecomposable projective H-modules corresponding to the vertex i, where $H = kQ$. We then have the following AR-quiver for H, and for $\mathbf{D}^b(H)$
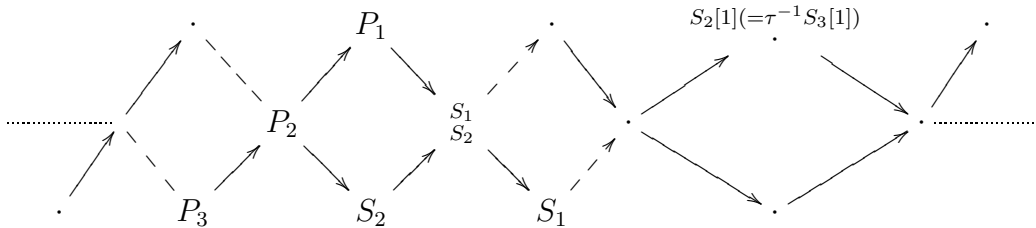
Then we have

$$\operatorname{Hom}_{\mathcal{C}_H}(S_1, S_3) = \operatorname{Hom}_{\mathbf{D}^b(H)}(S_1, S_3) \oplus \operatorname{Hom}_{\mathbf{D}^b(H)}(S_1, \tau^{-1}S_3[1]) = \operatorname{Ext}^1_H(S_1, S_2) \simeq k$$

.

Note that considering again Example (b) in 2.1, we see that $\operatorname{End}_{\mathcal{C}_H}(P_1 \oplus S_1 \oplus S_3)^{op}$ has indeed quiver $\rightrightarrows \rightarrow$, due to the extra maps from S_1 to S_3.

Also the problem about complements in Example (c) in 2.1 can now be solved, with an appropriate notion of "tilting" objects. We have that $T = P_1[1] \oplus P_2 \oplus P_3$ is an object in $\mathcal{C}_H$ with $\operatorname{Ext}^1_{\mathcal{C}_H}(T, T) = 0$.

We next give an example to show that for $\operatorname{Hom}_{\mathcal{C}_H}(A, B) = \operatorname{Hom}_{\mathbf{D}^b(H)}(A, B) \oplus \operatorname{Hom}_{\mathbf{D}^b(H)}(A, FB)$, where A and B are in the fundamental domain, there can be nonzero maps in both summands.

Example Let Q be the quiver

$$2 \leftarrow 1 \longrightarrow 4 \qquad \begin{array}{c} 3 \\ \nearrow \\ \searrow \\ 5 \end{array}$$
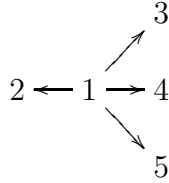
and $H = kQ$ the associated path algebra. Let M be the indecomposable module $\left(\begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix} \right)$. Then M lies in a tube of rank two, and we have $\tau \left(\begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix} \right) = \left(\begin{smallmatrix} & 1 & \\ 4 & & 5 \end{smallmatrix} \right)$. For computing $\operatorname{Hom}_{\mathcal{C}_H}(M, M)$, we have $\operatorname{Hom}_H(M, M) \neq 0$ and $\operatorname{Hom}_{\mathbf{D}^b(H)}(M, \tau^{-1}M[1]) = \operatorname{Ext}^1_H(M, \tau^{-1}M) \simeq D \operatorname{Hom}_H(\tau^{-2}M, M) \simeq D \operatorname{Hom}_H(M, M) \neq 0$.

The following properties of cluster categories will turn out to be important.

Theorem 2.4. *(a) The cluster categories are triangulated categories, and the natural functor from $\mathbf{D}^b(H)$ to $\mathcal{C}_H$ is triangulated.*

(b) The cluster category $\mathcal{C}_H$ has almost split triangles, and they are induced by almost split triangles in $\mathbf{D}^b(H)$.

Note that part (a) is highly nontrivial, and it is not true in general that nice orbit categories of $\mathbf{D}^b(\Lambda)$ for any finite dimensional algebra Λ are triangulated.

While any almost split triangle in $\mathcal{C}_H$ is induced by an almost split triangle in $\mathbf{D}^b(H)$, it is not true that any triangle in $\mathcal{C}_H$ comes from a triangle in $\mathbf{D}^b(H)$. This is for example not the case for the triangle induced by a map $(f, g) \colon M \to M$ in the above example, where f and g are nonzero. This in one reason why it is difficult to show that $\mathcal{C}_H$ is triangulated.

There are orbit categories of $\mathbf{D}^b(H)$ which were previouly known to be triangulated, namely the stable categories $\underline{\operatorname{mod}}\Lambda$ for selfinjective algebras of finite type in [Rie]. This gave an indication that the same thing might be true for cluster categories, but did not cover the examples of cluster categories. In addition the orbit categories $\mathbf{D}^b(H)/\langle \tau^2 \rangle$ were known to be triangulated [PX].

2.3. **CLUSTER-TILTING OBJECTS.** We need to define the objects in $\mathcal{C}_H$ which should replace tilting H-modules. It would be desirable if the tilting modules when viewed in $\mathcal{C}_H$ would belong to this class.

It turns out to be natural to consider the condition $\operatorname{Ext}^1_{\mathcal{C}_H}(T, T) = 0$, that is, T is *exceptional*, with T being maximal with this property. This was called *cluster-tilting* object in $\mathcal{C}_H$, and the corresponding property for module categories has been called maximal rigid in [GLS1]. The relationship to tilting modules is given by the following.

Theorem 2.5. *The cluster tilting objects in the cluster category $\mathcal{C}_H$ are exactly those coming from tilting modules over some hereditary algebra H' derived equivalent to H.*

Another concept related to cluster-tilting objects is that of Ext-configurations, defined for some $\mathbf{D}^b(H)$, or some orbit category like $\mathcal{C}_H$. The concept is motivated by the Hom-configurations from [Rie], defined by a set of indecomposable objects $\{C_i\}_{i \in I}$ where $\operatorname{Hom}(C_i, C_j) = 0$ for C_i not isomorphic to C_j, and for any indecomposable X there is a nonzero map $g \colon C_j \to X$ for some C_j. Similarly we say that $\{C_i\}_{i \in I}$ is an *Ext-configuration* if $\operatorname{Ext}^1(C_i, C_j) = 0$ for $C_i \not\simeq C_j$, and for any indecomposable X there is some C_i such that $\operatorname{Ext}^1(C_i, X) \neq 0$. It turns out that for cluster categories this concept coincides with cluster-tilting. Here the formula $D \operatorname{Ext}^1(A, B) \simeq \operatorname{Ext}^1(B, A)$ valid in cluster categories is important.

Proposition 2.6. *An object T in the cluster-category $\mathcal{C}_H$ is cluster-tilting if and only if it is an* Ext-*configuration.*

Proof. Clearly an Ext-configuration must be cluster-tilting. For the converse, assume that T is cluster-tilting. By Theorem 2.3 we can assume that T is a tilting H-module. If M is indecomposable exceptional in $\mathcal{C}_H$ and $\operatorname{Ext}^1_{\mathcal{C}_H}(T, M) = 0$, then $T \oplus M$ is exceptional, and hence M is in $\operatorname{add} T$. If M is an indecomposable object not in $\operatorname{add} T$, we can assume that M is an H-module since all $P[1]$ with P indecomposable projective are exceptional. Since $\operatorname{Ext}^1_H(T, M) = 0 = \operatorname{Ext}^1_H(M, T)$, then tilting theory gives that M is in $\operatorname{Fac} T \cap \operatorname{Sub} T = \operatorname{add} T$. $\qquad\square$

The notion of Ext-configuration is closely related to Iyama's definition of maximal 1-orthogonal modules or subcategories, which he introduced in connection with his generalizations of the theory of almost split sequences, originally in the abelian case, where the notion was essential [I]. For a cluster category the triangulated version of maximal 1-orthogonal objects [IY] is the same as objects which give Ext-configurations, and hence concides with cluster-tilting objects.

We say that $\overline{T}$ is an *almost complete cluster-tilting object* if there is some indecomposable object M not in $\operatorname{add} T$ such that $\overline{T} \oplus M$ is a cluster-tilting object. Then M is said to be a complement of $\overline{T}$. There is a graph where the vertices are the (non-isomorphic) cluster-tilting objects and there is an edge between two vertices if the corresponding cluster-tilting objects have a common almost complete cluster-tilting summand. This graph turns out to be connected.

It follows from 2.3 that all cluster-tilting objects in a given $\mathcal{C}_H$ for $H = kQ$ have the same number of nonisomorphic indecomposable summands as the vertices in the

quiver Q. Also we now get a better result on exchanging indecomposable summands of cluster-tilting objects, more closely related to clusters.

Theorem 2.7. *Let $\overline{T}$ be an almost complete cluster-tilting object in $\mathcal{C}_H$. Then $\overline{T}$ has exactly two nonisomorphic complements in $\mathcal{C}_H$.*

Note that if we start with an almost complete tilting H-module $\overline{T}$ which is not sincere, there is only one complement in mod H. But then some simple H-module S is not a composition factor of $\overline{T}$, so that $\mathrm{Hom}_H(P, \overline{T}) = 0$ where P is the projective cover of S. Hence we have $\mathrm{Ext}^1_{\mathcal{C}_H}(P[1], \overline{T}) = \mathrm{Hom}_{\mathcal{C}_H}(P, \overline{T}) = 0$, and so $P[1]$ is a complement.

2.4. **EXCHANGE PAIRS.** Let $\overline{T}$ be an almost complete cluster-tilting object in a cluster category $\mathcal{C}_H$, and let M and M^* be the nonisomorphic complements for $\overline{T}$. We shall now investigate the relationship between M and M^*.

We have the following connection.

Theorem 2.8. *Let the notation be as above. Then we have the triangles $M^* \xrightarrow{f} B \xrightarrow{g} M \to$ and $M \xrightarrow{s} B' \xrightarrow{t} M^* \to$, where $g\colon B \to M$ and $t\colon B' \to M^*$ are minimal right $\mathrm{add}\,\overline{T}$-approximations and $f\colon M^* \to B$ and $s\colon M \to B'$ are minimal left $\mathrm{add}\,\overline{T}$-approximations.*

We illustrate with the following.

Example: Consider again $\mathcal{C}_H$ for $H = kQ$, where $Q\colon 1 \to 2 \to 3$. Let $\overline{T} = P_3 \oplus P_1$. Then the two complements are $M = P_2$ and $M^* = S_1$. The triangles connecting M and M^* are:
$$S_1 \to S_3 \to P_2 \to \quad \text{and} \quad P_2 \to P_1 \to S_1 \to$$
where $\mathrm{Hom}_{\mathcal{C}_H}(S_1, S_3) = \mathrm{Hom}(S_1, \tau^{-1}S_3[1]) = \mathrm{Hom}(S_1, S_2[1]) \simeq k$.

When M and M^* are complements to a common almost complete cluster-tilting object, we call (M, M^*) *an exchange pair*. There is the following characterization of such a pair.

Theorem 2.9. *A pair of indecomposable objects (M, M^*) in a cluster category $\mathcal{C}_H$ is an exchange pair if and only if $\mathrm{Ext}^1_{\mathcal{C}_H}(M, M^*)$ is one dimensional over $\mathrm{End}(M)/\mathrm{rad}\,\mathrm{End}(M)$ and over $\mathrm{End}(M^*)/\mathrm{rad}\,\mathrm{End}(M^*)$.*

2.5. **ANALOGS.** We have seen that in the cluster category $\mathcal{C}_H$ we have identified a collection of objects, which all have the same number of nonisomorphic indecomposable summands, the same way as all clusters have the same number of elements. In both cases these numbers coincide with the number of vertices in the quiver.

We note a slight difference with respect to exchange. For cluster-tilting objects there is a unique way of exchanging an indecomposable summand. For clusters there is by definition at least one way of exchanging a cluster variable to get a new cluster, but it is not clear that it is unique.

The analog of cluster variables is now clearly the indecomposable summands of the cluster-tilting objects, which are the indecomposable exceptional objects.

As analogs of the seeds $(\underline{x}', Q')$ we have *tilting seeds* (T, Q_T), where T is a cluster-tilting object and Q_T is the quiver of $\mathrm{End}_{\mathcal{C}_H}(T)^{op}$. Note that also here there is a slight difference, since a tilting seed is by definition determined by the cluster-tilting object, while the corresponding result is not known in general in the context of cluster algebras. Actually, these differences in behaviour also indicate some strength, and can be used to prove new results on cluster algebras.

The exchange triangles $M^* \to B \to M \to$ and $M \to B' \to M^* \to$ are connected with the exchange multiplication $x_i x_i^* = m_1 + m_2$ in cluster algebras. Here the monomials m_1 and m_2 correspond to the objects B and B'.

We point out that for almost all results in this chapter we could instead deal with hereditary abelian categories with finite dimensional homomorphism and extension spaces and which have a tilting object. By [Ha2] it is known that the only additional categories we the have to deal with are the categories coh $\mathbb{X}$ of coherent sheaves on weighted projective lines [GL]

The only result which remains open in this setting is whether the cluster tilting graph is connected.

Notes: The results from tilting theory are taken from [APR][BB][HR][HU1][HU2][RS][U]. It was proved in [Kel] that the cluster categories are triangulated. Otherwise the material in this section is taken from [BMRRT][BMR2].

3. CLUSTER-TILTED ALGEBRAS

In the same way as the class of tilted algebras is defined as endomorphism algebras of tilting modules over hereditary algebras, we consider endomorphism algebras of cluster-tilting objects in cluster categories. These algebras have been called *cluster-tilted* algebras. They have several interesting properties, ranging from homological properties to description in terms of quivers with relations. In particular there are nice relationships with the associated hereditary algebras.
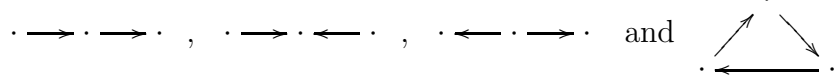
3.1. **THE QUIVERS OF THE CLUSTER-TILTED ALGEBRAS.** We first note that the hereditary algebra H is itself a cluster-tilted algebra since $\mathrm{Hom}_{\mathcal{C}_H}(H, H) = \mathrm{Hom}_H(H, H) \oplus \mathrm{Hom}_{\mathbf{D}^b(H)}(H, \tau^{-1}H[1]$, where the last term is clearly zero. The basis for information on the quivers of cluster-tilted algebras comes from comparing the cluster-tilted algebras $\Gamma = \mathrm{End}_{\mathcal{C}_H}(T)^{op}$ and $\Gamma' = \mathrm{End}_{\mathcal{C}_H}(T')^{op}$, where T and T' are nonisomorphic cluster-tilting objects having a common almost complete cluster-tilting object, that is, T and T' are neighbours in the cluster-tilting graph.

Theorem 3.1. *With the above notation, let Q_T be the quiver of $\Gamma = \mathrm{End}_{\mathcal{C}_H}(T)^{op}$ and $Q_{T'}$ the quiver of $\Gamma' = \mathrm{End}_{\mathcal{C}_H}(T')^{op}$. Write $T = T_1 \oplus \cdots \oplus T_n$, where T_i are nonisomorphic indecomposable objects, and T_k is not a summand of T'. Then $\mu_k(Q_T) = Q_{T'}$.*

Using this, we get the following consequence, where we use that the cluster-tilting graph is connected.

Theorem 3.2. *Let Q be a finite quiver without oriented cycles, and let $\mathcal{C}_H$ be the cluster category associated with $H = kQ$. Then the quivers of the cluster-tilted algebras associated with $\mathcal{C}_H$ are the quivers in the mutation class of Q.*

Example. The quivers of the cluster-tilted algebras of type A_3 are

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \ , \quad \cdot \longrightarrow \cdot \longleftarrow \cdot \ , \quad \cdot \longleftarrow \cdot \longrightarrow \cdot \quad \text{and} \quad \begin{array}{c} \cdot \\ \nearrow \quad \searrow \\ \cdot \longleftarrow \cdot \end{array}$$

as is seen directly or by considering the example in 1.2.

The following result on cluster-tilted algebras, also of interest in itself, is useful for proving Theorem 3.1.

Proposition 3.3. *If Γ is a cluster-tilted algebra, then for any sum e of vertices in the quiver, we have that $\Gamma/\Gamma e\Gamma$ is also cluster-tilted.*

We note that there is a difference as compared to tilted algebras, where the corresponding result is not true in general. On the other hand the class of tilted algebras is closed under taking endomorphism algebras of projective modules, while this is not in general the case for cluster-tilted algebras.

The above result makes it possible to reduce the proof of Theorem 3.1 to the case of cluster-tilted algebras with 3 simple modules. In this case there is a description of the possible quivers in terms of Markov chains [BBH], and there is more information on the algebras in [Ker2].

Theorem 3.2 establishes a nice connection between cluster-tilting theory and cluster algebras. We shall see some applications in the next section.

3.2. **RELATIONS.** It is of course also of interest to describe the relations for a cluster-tilted algebra once the quiver is given. The following may be true.

Conjecture 3.4. *A cluster-tilted algebra is uniquely determined by its quiver.*

The conjecture has been verified in the case of finite representation type. In this case an explicit description is given of a set of minimal relations. Note that the quiver Q has only single arrows. For each arrow $\alpha : i \to j$ in the quiver which lies on a full cycle, that is, a cycle where there are no other arrows in Q between the vertices of the cycle, take the sum of the paths from j to i, which together with α give a full cycle. Then these relations determine the corresponding cluster-tilted algebra.
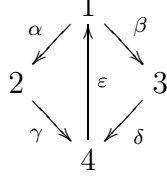
We give some examples to illustrate.

Example Let Q be the quiver

$$\begin{array}{ccc} & 2 & \\ \nearrow & & \searrow \\ 1 & & 3 \\ \nwarrow & & \swarrow \\ 5 & \longleftarrow & 4 \end{array}$$
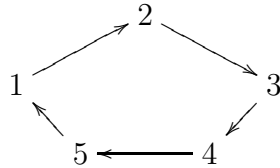
which is the quiver of a cluster-tilted algebra Γ of type $\mathbf{D}_5$. This can be seen by finding a sequence of mutations from Q to a quiver of type $\mathbf{D}_5$. Then Γ is of finite

type, and is hence determined by the relations given by all paths of length 4 being zero.

Example Let Q be the quiver

$$
\begin{array}{ccc}
 & 1 & \\
{\scriptstyle\alpha}\swarrow & \uparrow\;\big|{\scriptstyle\varepsilon} & \searrow{\scriptstyle\beta} \\
2 & & 3 \\
{\scriptstyle\gamma}\searrow & \big|\swarrow{\scriptstyle\delta} & \\
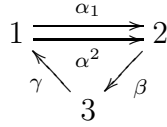 & 4 &
\end{array}
$$

Then Q is mutation equivalent to the quiver of type $\mathbf{D}_4$, and the corresponding cluster-tilted algebra is hence of finite type. Hence the relations are $\gamma\alpha + \delta\beta = 0$, $\varepsilon\gamma = 0$, $\varepsilon\delta = 0$, $\alpha\varepsilon = 0$, $\beta\varepsilon = 0$.

Some of these results about the relations hold more generally, as given in the following.
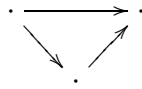
Proposition 3.5. *Let Γ be a cluster-tilted algebra and S_1, S_2 simple Γ-modules. Then we have* $\dim \operatorname{Ext}^1(S_1, S_2) \geq \dim \operatorname{Ext}^2(S_2, S_1)$.

It is known that $\dim \operatorname{Ext}^2_\Gamma(S_2, S_1)$ gives the number of relations from the vertex of S_1 to the vertex of S_2, in a minimal set of relations [Bo]. Hence we see that we have relations from j to i in a minimal set only if there are arrows from i to j. But there may not be such a relation for each arrow lying on a cycle, and if there is some relation from j to i, it might go through a cycle.

Example The quiver

$$
\begin{array}{ccc}
1 & \overset{\alpha_1}{\underset{\alpha^2}{\rightrightarrows}} & 2 \\
 & {\scriptstyle\gamma}\nwarrow\;\searrow{\scriptstyle\beta} & \\
 & 3 &
\end{array}
$$

is mutation equivalent to

$$
\begin{array}{ccc}
\cdot & \longrightarrow & \cdot \\
 \searrow & \nearrow & \\
 & \cdot &
\end{array}
$$

and is hence the quiver of a cluster-tilted algebra. A minimal set of relations is given by $\alpha_2\gamma = 0 = \beta\alpha_2 = \gamma\beta$. Here we have $\dim \operatorname{Ext}^2(S_2, S_1) = 1 < 2 = \dim \operatorname{Ext}^1(S_1, S_2)$.

3.3. **RELATIONSHIP WITH HEREDITARY ALGEBRAS.** There is a close relationship between cluster-tilted algebras and the associated hereditary algebras, as we shall now discuss.

For tilted algebras Λ there is a close relationship with the corresponding hereditary algebras H, where two subcategories of modΛ, coming from a torsion pair, are also equivalent to subcategories of mod H belonging to a torsion pair. But here we may have that H is of infinite type, while Λ is of finite type. Roughly speaking, in the case of cluster-tilted algebras we have the "same number" of indecomposables. Here we first enlarge the category mod$H$ by passing to $\mathcal{C}_H$ and hence adding n indecomposable objects, where n is the number of nonisomorphic simple H-modules. Then we "remove" n other indecomposable objects from $\mathcal{C}_H$ to obtain modΓ.

When T is a tilting H-module and $\Lambda = \operatorname{End}_H(T)^{op}$, then, as we have pointed out, the functor $\operatorname{Hom}_H(T, \ ): \operatorname{mod} H \to \operatorname{mod} \Lambda$ induces an equivalence between the torsion class $\operatorname{Fac} T$ in $\operatorname{mod} H$ and the torsionfree class in $\operatorname{mod} \Lambda$, actually $\operatorname{Sub} D(T)$. So this functor is far from being dense in general. But the situation is quite different if we replace $\operatorname{mod} H$ by the cluster category $\mathcal{C}_H$, and the tilted algebra Λ by the cluster-tilted algebra $\Gamma = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$. Here the triangulated structure is important. While for an H-module C there is usually no exact sequence $T_1 \to T_0 \to C \to 0$ in $\operatorname{mod} H$, with T_0 and T_1 in $\operatorname{add} T$, we have the following crucial property for $\mathcal{C}_H$.

Lemma 3.6. *Let $H = kQ$ be a finite dimensional hereditary k-algebra, and $\mathcal{C}_H$ the associated cluster category. Then for any C in $\mathcal{C}_H$ there is a triangle $T_1 \to T_0 \to C \to T_1[1]$ with T_0, T_1 in $\operatorname{add} T$.*

Proof. Let $f: T_0 \to C$ be a right $\operatorname{add} T$-approximation in $\mathcal{C}_H$, and complete to a triangle $X \to T_0 \to C \to X[1]$. Apply $G = \operatorname{Hom}(T, \ )$ to get the exact sequence $(T, X) \to (T, T_0) \xrightarrow{(T,f)} (T, C) \to (T, X[1]) \to (T, T_0[1]) = 0$. Since (T, f) is surjective, it follows that $\operatorname{Ext}^1(T, X) = \operatorname{Hom}(T, X[1]) = 0$, so that $X \in \operatorname{add} T$. $\qquad\square$

Using this lemma one can show the following close relationship between $\mathcal{C}_H$ and the cluster-tilted algebra $\Gamma = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$.

Theorem 3.7. *Let H be a finite dimensional hereditary k-algebra, T a cluster-tilting object in the cluster category $\mathcal{C}_H$, and $\Gamma = \operatorname{End}_{\mathcal{C}_H}(T)^{op}$ the associated cluster-tilted algebra. Then $G = \operatorname{Hom}_{\mathcal{C}_H}(T, \ ): \mathcal{C}_H \to \operatorname{mod} \Gamma$ induces an equivalence of categories $\overline{G}: \operatorname{Hom}_{\mathcal{C}_H}(T, \ ): \mathcal{C}_H / \operatorname{add} \tau T \to \operatorname{mod} \Gamma$.*

Proof. We illustrate the use of Lemma 3.6 by showing that the functor G is dense. So let C be in $\operatorname{mod} \Gamma$, and consider a (minimal) projective presentation $(T, T_1) \xrightarrow{(T,f)} (T, T_0) \to C \to 0$, where $f: T_1 \to T_0$ is a map in $\operatorname{add} T$. Complete to a triangle $T_1 \to T_0 \to X \to T_1[1]$, and apply $\operatorname{Hom}_{\mathcal{C}_H}(T, \ )$ to get the exact sequence $(T, T_1) \to (T, T_0) \to (T, X) \to (T, T_1[1]) = 0$, so that $C \cong (T, X)$. This shows that G is dense.

Then we show that G is full. Let X and Y be in $\mathcal{C}_H$, and consider a map $\varphi: G(X) \to G(Y)$. We want to find a map $f: X \to Y$ such that $G(f) = \varphi$. By Lemma 3.6 we have triangles $T_1 \to T_0 \to X \to T_1[1]$ and $T_1' \to T_0' \to Y \to T_1'[1]$ with T_1, T_0, T_1', T_0' in $\operatorname{add} T$. Then we have exact sequences $(T, T_1) \to (T, T_0) \to (T, X) \to 0$ and $(T, T_1') \to (T, T_0') \to (T, Y) \to 0$, and it is easy to see that G is full.

Since $G(\tau T) = (T, \tau T) = (T, T[1]) = 0$, there is an induced functor $\overline{G}: \mathcal{C}_H / \operatorname{add} \tau T \to \operatorname{mod} \Gamma$. The following short proof of $\overline{G}$ being faithful is due to Keller. So let $f: X \to Y$ be a map in $\mathcal{C}_H$, and assume $G(f) = 0$ in $\operatorname{mod} \Gamma$. Consider the diagram

$$T_1 \longrightarrow T_0 \longrightarrow X \xrightarrow{\ i\ } T_1[1]$$
$$\Big\downarrow f$$
$$Y$$

where the triangle, with T_1, T_0 in $\operatorname{add} T$, exists by Lemma 3.6. Since the compostion $T_0 \to X \xrightarrow{f} Y$ is zero because $G(f) = 0$, there exists $g: T_1[1] \to Y$ such that $gi = f$. So f factors through $T_1[1] = \tau T_1$. $\qquad\square$

This result has some similarity with the equivalences associated with the BGP reflection functors, which induce equivalences between subcategories obtained by leaving out only one indecomposable object, and where the AR-quivers are closely related. And actually in this context of cluster-tilted algebras there is also a surprisingly close connection between the AR-quivers for H and for Γ, via $\mathcal{C}_H$. This can be used to rule out the possibility for a given algebra to be cluster-tilted.

Theorem 3.8. *Let the notation be as before. Then the AR-quiver for Γ is obtained by dropping the vertices corresponding to the objects τT_i for the indecomposable summands T_i of the cluster-tilting object T.*

This has the following direct consequence.

Corollary 3.9. *Let T be a cluster-tilting object in the cluster category $\mathcal{C}_H$, and $\Gamma = \mathrm{End}_{\mathcal{C}_H}(T)^{op}$ the corresponding cluster tilted algebra.*
  (a)Γ is selfinjective if and only if $\tau^2 T \simeq T$.
  (b)The projective Γ-module P corresponding to an indecomposable summand T_i of T, and the injective Γ-module I corresponding to $\tau^2 T_i$ are related by $P/radP \simeq \mathrm{soc}\, I$.

Proof. (a)Clearly the indecomposable projective Γ-modules correspond to the indecomposable summands T_i of T, and the indecomposable injective ones to the $\tau^2 T_i$, so the claim follows.
 (b) We have $\mathrm{Hom}_\Gamma(P, I) \simeq \mathrm{Hom}_{\mathcal{C}_H}(T_i, \tau^2 T_i)$ and also $\mathrm{Hom}_\Gamma(\Gamma, D(\Gamma)) \simeq \mathrm{Hom}_{\mathcal{C}_H}(T, \tau^2 T)$, using Theorem 3.8 and $\mathrm{Hom}_{\mathcal{C}_H}(T, \tau T) = 0$. Further $\mathrm{Hom}_{\mathcal{C}_H}(T_i, \tau^2 T_i) = \mathrm{Hom}_{\mathbf{D}^b(H)}(T_i, \tau^2 T_i) \oplus \mathrm{Hom}_{\mathbf{D}^b(H)}(T_i, \tau T_i[1])$. We have $\mathrm{Hom}_{\mathbf{D}(H)}(T_i, \tau T_i[1]) = \mathrm{Ext}^1_{\mathbf{D}^b(H)}(T_i, \tau T_i)$, and we choose a nonzero element α in $\mathrm{Ext}^1_{\mathbf{D}^b(H)}(T_i, \tau T_i)$ corresponding to an almost split triangle $\tau T_i \to E \to T_i$. Then for a nonisomorphism $f\colon T_j \to T_i$, the composition $\alpha f\colon T_j \to \tau^2 T_i$ is zero. Hence for the corresponding map $\alpha\colon P \to I$, we have $\mathrm{Im}\,\alpha' = \mathrm{soc}\, I$, which shows the claim. $\qquad\square$

The above gives a way of constructing $\mathrm{mod}\Gamma$ from $\mathrm{mod}H$, via $\mathcal{C}_H$. There is also another way, going instead via the tilted algebra $\Lambda = \mathrm{End}_H(T)^{op}$, when T is a tilting H-module.

Theorem 3.10. *Let T be a tilting module over the hereditary algebra H, and $\Lambda = \mathrm{End}_H(T)^{op}$.*
  (1) *The cluster-tilted algebra $\Gamma = \mathrm{End}_{\mathcal{C}_H}(T)^{op}$ is isomorphic to the trivial extension algebra $\Lambda \ltimes \mathrm{Ext}^2(D\Lambda, \Lambda)$.*
  (2) *The quiver of Γ is obtained from the quiver with relations of Λ, by adding an arrow from j to i for each relation in a minimal set of relations.*
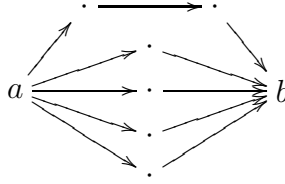
Example If we have the tilted algebra
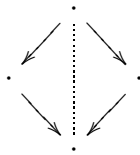
the associated cluster-tilted algebra has quiver
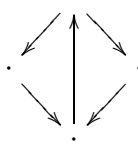
The same rule applies when we start with a canonical algebra instead of a tilted algebra.

Example Let Λ be a canonical algebra over k given by the quiver

where $\Lambda = \mathrm{End}_{\mathrm{coh}\,\mathbb{X}}(T)^{op}$ for a tilting object T in the associated category $\mathrm{coh}\,\mathbb{X}$ of coherent sheaves on a weighted projective line. Then we obtain the quiver for the algebra $\mathrm{End}_{\mathcal{C}_{\mathrm{coh}\,\mathbb{X}}}(T)^{op}$ by adding 5-2=3 arrows from b to a in the above quiver.

3.4. **HOMOLOGICAL PROPERTIES.** While the tilted algebras have global dimension at most two, it turns out that the cluster-tilted algebras typically have infinite global dimension. But they have other homological similarities with hereditary algebras.

Recall that a finite dimensional algebra Γ is *Gorenstein* of dimension at most one if $\mathrm{id}\,_\Lambda\Lambda \leq 1$ and $\mathrm{id}\,\Lambda_\Lambda \leq 1$. (Using tilting theory, one knows that the last condition can be dropped.) Clearly hereditary algebras satisfy this property, and we also have the following.

Theorem 3.11. *The cluster-tilted algebras are Gorenstein of dimension at most one.*

We shall give another homological property of cluster-tilted algebras.

Recall that for a Gorenstein algebra Γ of dimension at most one the category $\mathrm{Sub}\,\Gamma$, which is the category $\mathrm{CM}\,\Gamma$ of Cohen-Macaulay modules in this case, is functorially finite [AS] and extension closed since Γ is a cotilting Γ-module. Hence $\mathrm{Sub}\,\Gamma$ has almost split sequences [AS]. $\mathrm{Sub}\,\Gamma$ is also a Frobenius category, that is, $\mathrm{Sub}\,\Gamma$ has enough projectives and enough injectives, and the projectives and injectives coincide. Hence the stable category $\underline{\mathrm{Sub}}\Gamma$ is a triangulated category [Ha1]. We have the following necessary condition on cluster-tilted algebras.

Theorem 3.12. *With the above notation, for a cluster-tilted algebra Γ, the stable category $\underline{\mathrm{Sub}}\Gamma$ is 3-CY, that is $D\,\mathrm{Hom}(A,B) \simeq \mathrm{Hom}(B,A[3])$ for A, B in $\underline{\mathrm{Sub}}\Gamma$.*

We note that there are algebras satisfying both the above homological conditions, but which are not cluster-tilted.

Example. Let Q be the quiver

$$
\begin{array}{ccc}
1 & \xrightarrow{\;\alpha\;} & 2 \\
\delta\uparrow & & \downarrow\beta \\
4 & \xleftarrow{\;\gamma\;} & 3
\end{array}
$$

and let Λ be the path algebra kQ modulo the relations given by all paths of length 7. This is a Nakayama algebra, which is selfinjective and hence Gorenstein. The indecomposable projectives have length 7, and we have $\underline{\text{Sub}}\Lambda = \underline{\text{mod}}\Lambda$.

As we shall see in Section 5 , in order to show that $\underline{\text{mod}}\Lambda$ is 3-CY, it is enough to show that $\tau = \Omega^{-2}$. We have $\tau(S_1) = S_2$, and $\Omega^{-1}(S_1) = P_3/S_1$ and $\Omega^{-1}(P_3/S_1) = S_2$, so $\tau(S_1) \cong \Omega^{-2}(S_1)$. Calculating further, we then see that $\text{mod}\Lambda$ is 3-CY. But Λ is not cluster-tilted since the relations in the unique cluster-tilted algebra with this quiver are paths of length 3.

Notes. Most of the material in this section is taken from [BMR1], [BMR2], [BMR3]. Proposition 3.5 is taken from Assem-Brüstle-Schiffler and Reiten-Todorov (see [BMR3]) and [KR1], Theorem 3.10 is taken from [ABS], Theorem 3.11 from [KR1], Theorem 3.12 from [KR1], while for selfinjective cluster tilted algebras it is due to Geiss-Keller. See also [KZ].

4. Interplay and applications

In this section we give some illustration of how the theory of cluster categories and cluster-tilted algebras has had some feedback on the theory of cluster algebras, and we also give examples of nice interplay.

4.1. **FINITE MUTATION CLASSES.** Recall from Section 1 that there is a finite number of cluster variables, equivalently a finite number of clusters, equivalently a finite number of seeds, if and only if one of the seeds contains a Dynkin quiver. However, there may be a finite number of quivers occurring even if none of the quivers is Dynkin. Actually we have the following answer to a question of Seven.

Theorem 4.1. *If the cluster quiver Q has no oriented cycles, then there is only a finite number of quivers in the mutation class of Q if and only if Q is Dynkin or extended Dynkin or has two vertices and is a generalized Kroncker quiver* $\cdot\rightrightarrows\cdot$.

An essential point to use is that the quivers occurring are exactly the quivers of cluster-tilted algebras by Theorem 3.2. To investigate this we study tilting theory for tame and wild hereditary algebras.

Examples: 1) $\cdot\rightrightarrows\cdot$ is the only quiver in its mutation class.
2) The mutation class of $\cdot\rightrightarrows\cdot\rightrightarrows\cdot$ has in addition only $\cdot\leftleftarrows\cdot\leftleftarrows\cdot$
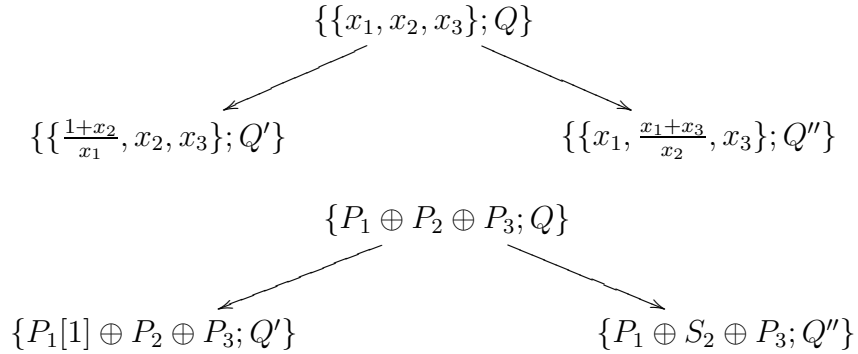
4.2. **LISTS OF HAPPEL-VOSSIECK AND SEVEN.** There is a famous *Happel-Vossieck list* in representation theory, which consists of quivers with relations for the minimal tilted algebras of infinite representation type, that is the tilted algebras Λ which are of infinite type, but where $\Lambda/\Lambda e\Lambda$ is of finite type for any vertex e in the quiver [HV]. On the other hand there is the list of Seven of minimal infinite cluster quivers, namely the quivers not mutation equivalent to a Dynkin quiver, but if one vertex is removed, the quiver is mutation equivalent to a Dynkin quiver. As pointed out in [S], there is a close connection between these lists, as the list of Seven is obtained from the Happel-Vossieck list by inserting arrows in the opposite direction whenever there is a dotted arrow indicating a minimal relation.

This is explained using that the list of Seven gives the quivers of the minimal cluster-tilted algebras of infinite type, using Theorem 3.2. Then we also use how to obtain the quiver of a cluster-tilted algebra from the quiver with relations for the corresponding tilted algebra, as we have discussed in Theorem 3.2 For example one passes from

$$\text{(quiver diagram)} \quad \text{to} \quad \text{(quiver diagram)}$$
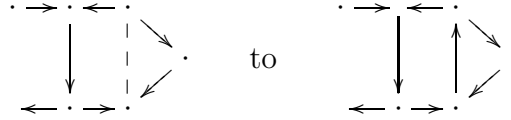
4.3. **DENOMINATORS.** We have seen that for acyclic cluster algebras there are many similarities between the ingredients in the definition of a cluster algebra and the cluster-tilting theory in the corresponding cluster category, for example we have a cluster graph and a cluster-tilting graph, with seeds or tilting seeds at each vertex.

The next step is to try to define a map from cluster variables to indecomposable exceptional objects, which takes clusters to tilting objects and seeds to tilting seeds (or in the other direction), which is 1-1 or surjective or both. When Q is a cluster quiver without oriented cycles, we have the initial seed $(\underline{x}, Q)$, where $\underline{x} = \{x_1, \cdots, x_n\}$. The natural initial tilting seed is (H, Q), where $H = kQ = P_i \oplus \cdots \oplus P_n$, and the P_i are the indecomposable projectives. We can start with defining $\varphi(x_i) = P_i$. When we do the exchange of cluster variables, we have a corresponding exchange of exceptional objects, and it is natural to send the new cluster variable to the new exceptional object, as illustrated in the following example, where Q is $1 \to 2 \to 3$.

$$\{\{x_1, x_2, x_3\}; Q\}$$

$$\{\{\tfrac{1+x_2}{x_1}, x_2, x_3\}; Q'\} \qquad\qquad \{\{x_1, \tfrac{x_1+x_3}{x_2}, x_3\}; Q''\}$$

$$\{P_1 \oplus P_2 \oplus P_3; Q\}$$

$$\{P_1[1] \oplus P_2 \oplus P_3; Q'\} \qquad\qquad \{P_1 \oplus S_2 \oplus P_3; Q''\}$$

Here Q' and Q'' are as in 1.3. We define $\varphi(x_i) = P_i$, $\varphi(\tfrac{1+x_2}{x_1}) = P_1[1]$, $\varphi(\tfrac{x_1+x_3}{x_2}) = S_3$ etc.

Note that by the theory we have already discussed, we have the same corresponding quiver on the bottom diagram, which by definition is the quiver of Q_T for the tilting object T.

If we follow the same fixed path from the initial seed on both sides, the procedure for defining φ is unique. But if we reach the same cluster variable via a different path, then we might get a different indecomposable object. If there is a map from cluster variables to indecomposable exceptional objects sending clusters to tilting objects and seeds to tilting seeds, then it has to be given by φ, so the problem is to prove that φ is well defined.

As we have discussed in Section 1 , for our example $Q : 1 \to 2 \to 3$, the denominators of the cluster variables, when written in reduced form , are given by the composition factors of an indecomposable exceptional module, which is known to be uniquely determined by its composition factors (see [Ker1]). And for this example, the map φ is well defined and is a bijection.

Note that if we know that the denominator of any cluster variable is given by the composition factors of some indecomposable exceptional object, and that for any choice of path from the initial seed the map φ takes the cluster variable to this corresponding exceptional object, then the definition of φ would not depend on the choice of paths. Actually, these two statements can be proved simultaneously.

Theorem 4.2. *Let Q be a cluster quiver without oriented cycles.*

(a)For each cluster variable f/g different from $x_1, \cdots, x_n$, in reduced form, there is a unique indecomposable exceptional H-module whose composition factors are given by g.

(g)The map φ from cluster variables to indecomposable exceptional objects discussed above is well defined and surjective and takes clusters to tilting objects and seeds to tilting seeds.

One of the problems dealing with cluster variables is to decide when a given expression f/g is in reduced form. There is a surprisingly elementary positivity condition to deal with this problem.

We say that $f = f(x_1, \cdots, x_n)$ satisfies the *positivity condition* if $f(e_i) > 0$ for $e_i = (1, 1 \cdots, 0, 1, \cdots, 1)$, where 0 is in the i-th position, for $i = 1, \cdots, n$. The following result is crucial.

Lemma 4.3. *If f satisfies the positivity condition and m is a monomial, then f/m is in reduced form.*

Proof. Assume $f = f_1 x_i$. Then $f(e_i) = f_1(e_i) \cdot 0 = 0$, so that f does not satisfy the positivity condition. Hence f/m is in reduced form. $\qquad\square$
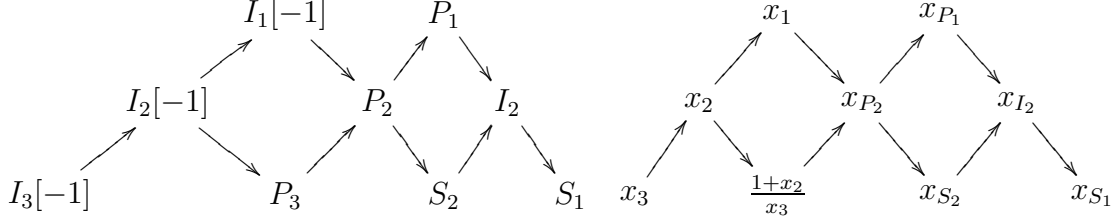
As a consequence of Theorem 4.2, we get the following , answering a conjecture from Section 1 in the acyclic case.

Theorem 4.4. *For an acyclic cluster algebra, a seed is determined by its cluster.*

We illustrate some of these ideas on our standard example.

Example Let Q be the quiver $1 \to 2 \to 3$ and $H = kQ$ the corresponding path algebra. We define a map $\psi \colon \operatorname{ind} H \to$ cluster variables for the cluster algebra

$C(Q)$, where the composition factors of X in ind H determine the denominator in $\psi(X)$. We here use the AR-quiver, and it is natural to define a map in the opposite direction from what we considered above. The map ψ is given by the following pictures

$$\begin{array}{ccccccccc}
& & I_1[-1] & & P_1 & & & x_1 & & x_{P_1} \\
& I_2[-1] & & P_2 & & I_2 & & x_2 & & x_{P_2} & & x_{I_2} \\
I_3[-1] & & P_3 & & S_2 & & S_1 \quad x_3 & & \frac{1+x_2}{x_3} & & x_{S_2} & & x_{S_1}
\end{array}$$

Note that $I_j[-1] = P_j[1]$ in the cluster category $\mathcal{C}_H$ and we send $I_j[-1]$ to x_j. Then the cluster-tilting object $T = I_1[-1] \oplus I_2[-1] \oplus I_3[-1]$ is sent to the cluster $\underline{x} = \{x_1, x_2, x_3\}$. Exchanging $I_2[-1]$ in T gives S_3, and mutating $\underline{x}$ at vertex 3 amounts to replacing x_3 by $x_3' = \frac{1+x_3}{x_3}$. So we define $\psi(S_3) = \frac{1+x_3}{x_3}$. Denote $\psi(M) = x_M = f_M/m_M$, in reduced form, where m_M is a monomial. We then get $x_{P_2} = (x_1 + \frac{1+x_2}{x_3}) \cdot \frac{1}{x_2} = \frac{1+x_2+x_1x_3}{x_1x_3}$ and $x_{P_1} = (1 + x_{P_2})1/x_1 = \frac{1+x_2+x_2x_3+x_1x_3}{x_1x_2x_3}$. We see that the denominators for the x_{P_i} correspond to the compostion factors of the P_i. Further we have $x_{S_3} = \frac{1+x_{P_2}}{x_{S_3}} = (1 + f_{P_2}/m_{P_2}) \cdot m_{S_3}/f_{S_3} = \frac{m_{P_2}+f_{P_2}}{f_{S_3}} \cdot \frac{1}{m_{P_2}/m_{S_3}}$. The monomial in the denominator is $m_{P_2}/m_{S_3} = m_{S_2}$. We do however need that such an expression is in reduced form, and here the positivity condition is important. We have that f_{p_2} and f_{S_3} satisfy the positivity condition, and it follows from this that $\frac{m_{P_2}+f_{P_2}}{f_{S_3}}$ also satisfies the same condition, hence this is OK. Continuing this "knitting" procedure we get a map ψ as indicated on the picture, where the denominators of the cluster variables correspond to the composition factors of the H-module they come from.

We can use the same procedure for any H of finite type, and we get in this elementary way a one-one map from indecomposable objects in $\mathcal{C}_H$ to cluster variables. To show that it is a bijection one can for example use the fact mentioned in Section 1 that the numbers are the same.

In a similar way we define for any $H = kQ$ a one-one map ψ from the indecomposable preprojective modules to cluster variables. This gives an alternative proof of the fact that if Q is a connected quiver which is not Dynkin, then there in an infinite number of cluster variables. Note however that the important Laurent phenomenon is used in all considerations.

4.4. **BIJECTION AND FURTHER APPLICATIONS.** In this section we mention some further improvements, as a consequence of using some more advanced techniques, in particular a beautiful formula of Caldero-Chapoton involving Euler characteristics, generalized by Caldero-Keller. This allows one to get a natural map ψ from indecomposable exceptional objects to cluster variables. As a consequence we have the following.

Theorem 4.5. *The map ψ gives a bijection from indecomposable exceptional objects to cluster variables, taking cluster tilting objects to clusters and tilting seeds to seeds.*

There are also further results on cluster variables, answering conjectures from Section 1 .

Theorem 4.6. *For any cluster variable f/m in reduced form for an acyclic cluster algebra, all coefficients of f are positive.*

Theorem 4.7. *For an acyclic cluster algebra there is a unique way of replacing a cluster variable in a cluster by another cluster variable.*

There are additional results, so far only proved for finite representation type.

Theorem 4.8. *For any acyclic cluster algebra of finite type, the image under ψ of the exceptional objects in the cluster category give a $\mathbb{Z}$-basis for the cluster algebra.*

Notes: 4.1 is taken from [BR], 4.2 from [BRS]. For 4.3 and 4.4 see [BMR2], [BMRT] (with appendix), [CC][CK1],[CK2][CR], unpublished work of Seven.

5. 2-Calabi Yau categories

Many of the results on cluster categories and cluster-tilted algebras have a natural generalization to the more general class of Hom-finite triangulated 2-CY categories over k, with an appropriate choice of special objects and associated algebras. In this section we give a brief account of this development.

5.1. **CONNECTION WITH ALMOST SPLIT SEQUENCES/TRIANGLES.**
Recall that a Hom-finite triangulated k-category $\mathcal{C}$ is 2-CY if and only if $D \operatorname{Ext}^1_{\mathcal{C}}(A, B) \cong \operatorname{Ext}^1_{\mathcal{C}}(B, A)$ for A and B in $\mathcal{C}$. Since the symmetry property for $\operatorname{Ext}^1_{\mathcal{C}}(,)$ plays a crucial role in the investigation of cluster catgories, in particular the fact that $\operatorname{Ext}^1_{\mathcal{C}}(A, B) = 0$ if and only if $\operatorname{Ext}^1_{\mathcal{C}}(B, A) = 0$, it is natural to look for generalizations to 2-CY categories.

We clearly have that $\mathcal{C}$ is 2-CY if and only if $D \operatorname{Hom}_{\mathcal{C}}(A, B) \simeq \operatorname{Hom}_{\mathcal{C}}(B, A[2])$, if and only if $D \operatorname{Ext}^1_{\mathcal{C}}(A, B) \simeq \operatorname{Hom}(B, A[1])$. The last formula shows the close connection with $\mathcal{C}$ having almost split triangles, and in fact $\mathcal{C}$ being 2-CY is equivalent to $\mathcal{C}$ having almost split triangles with the corresponding translate τ being [1] (see [RV]).

The original formulas from which existence of almost split sequences was deduced, were valid for module categories or subcategories of module categories. In some cases there is however a direct reformulation in closely associated triangulated categories. For example, let Λ be a selfinjective finite dimensional algebra. Then the stable category $\underline{\mathrm{mod}}\Lambda$ of the category $\mathrm{mod}\Lambda$ of finitely generated (left) Λ-modules is known to be triangulated with shift $[1] = \Omega^{-1}$, the first inverse syzygy [Ha1]. We have the formula $D \operatorname{Ext}^1_{\Lambda}(A, B) \cong \underline{\operatorname{Hom}}(B, \tau A)$ in $\mathrm{mod}\Lambda$, on which the existence of almost split sequences is based [AR1]. Let $0 \to B \to I \to \Omega^{-1}B \to 0$ be an exact sequence, where I is an injective envelope of B, and hence I is also projective. There is induced an isomorphism $\underline{\operatorname{Hom}}(A, \Omega^{-1}B) \cong \operatorname{Ext}^1_{\Lambda}(A, B)$, and $\underline{\operatorname{Hom}}(A, \Omega^{-1}B) \cong \underline{\operatorname{Hom}}(A, B[1]) = \operatorname{Ext}^1_{\underline{\mathrm{mod}}\Lambda}(A, B)$. So the formula is at the same time a Serre duality formula in the stable category $\underline{\mathrm{mod}}\Lambda$. Hence we have the following, since we know that the formula holds for $\mathrm{mod}\Lambda$.

Proposition 5.1. *Let Λ be a finite dimensional selfinjective algebra. Then the stable category* $\underline{\mathrm{mod}}\Lambda$ *is 2-CY if and only if* $\tau = \Omega^{-1}$.

Also for a commutative complete local isolated Gorenstein singularity R we have a similar result, for the same reason, since we have a corresponding formula for the category $\mathrm{CM}(R)$ of maximal Cohen-Macaulay R-modules [A], and $\underline{\mathrm{CM}}(R)$ is triangulated since $\mathrm{CM}(R)$ is a Frobenius category, that is, has enough projectives and injectives, and the projective and injective objects coincide [Ha1].

Proposition 5.2. *With R as above,* $\underline{\mathrm{CM}}(R)$ *is 2-CY if and only if* $\tau = \Omega^{-1}$.

Similarly we have that for a Frobenius subcategory $\mathcal{C}$ of $\mathrm{mod}\Lambda$, for a finite dimensional algebra Λ, where $\mathcal{C}$ is extension closed and functorially finite, then $\underline{\mathcal{C}}$ is d-CY (that is $D\,\mathrm{Hom}_{\mathcal{C}}(A,B) \simeq \mathrm{Hom}_{\mathcal{C}}(B,A[d])$ if and only if $\tau_{\mathcal{C}} = \Omega^{-(d-1)}$.

5.2. **CHOICE OF CLUSTER-TILTING OBJECT.** When generalizing from cluster categories to 2-CY categories, one needs the appropriate notion of "tilting" object. For cluster categories we had what we called cluster-tilting object T, which satisfied $\mathrm{Ext}^1_{\mathcal{C}}(T,T) = 0$, where T is maximal with this property. This was equivalent to what we called Ext-configurations, or to being maximal 1.orthogonal. The property of Ext-configuration (equivalently, maximal 1-orthogonal) was essential in some of the proofs, hence it is natural to choose this condition, which we call a *cluster-tilting* object in the general case of 2-CY categories. It also turns out that the two concepts do not coincide in general [BIKR].

In a 2-CY category it may happen that there is no cluster-tilting object, actually even no nonzero object M with $\mathrm{Ext}^1(M,M) = 0$. Sometimes there is instead what is called a cluster-tilting subcategory, which may have an infinite number of indecomposable objects. Here one requires in addition that the category is functorially finite, as done for maximal 1-orthogonal subcategories, but which was not required for Ext-configurations (see [KR1], [BIRS]for such examples).

5.3. **ANALOGOUS RESULTS.** When comparing with section 3 basically all the results still hold in the more general context of 2-CY categories. In order to deal with cluster quivers, we want to assume that the algebras associated with the cluster-tilting objects in 2-CY categories, called *2-CY tilted algebras*, have no loops or 2-cycles in their quivers. Note that there is no known analog of the description of cluster-tilted algebras as trivial extensions of tilted algebras. Contrary to the case of cluster categories, there may now be both loops and 2-cycles [BIKR].

5.4. **PREPROJECTIVE ALGEBRAS OF DYNKIN QUIVERS.** Important examples of 2-CY categories are the stable module categories $\underline{\mathrm{mod}}\Lambda$, where Λ is a preprojective algebra of a Dynkin quiver over a field k. Recall that if for example Q

is the quiver

$$1 \xrightarrow{\alpha} 2 \quad \substack{\beta \nearrow \; 4 \\ \gamma \searrow \; 3}$$
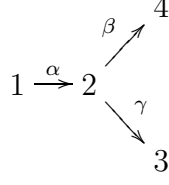
then the preprojective algebra $\Pi(Q)$ is given by the quiver

$$1 \underset{\alpha^+}{\overset{\alpha}{\rightleftarrows}} 2 \quad \substack{\beta \; 4 \; \beta^+ \\ \gamma \; 3 \; \gamma^+}$$
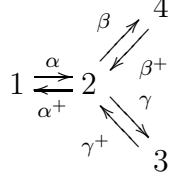
with the relations $\alpha\alpha^+ + \alpha^+\alpha + \beta\beta^+ + \beta^+\beta + \gamma\gamma^+ + \gamma^+\gamma = 0$.

For the case of Dynkin quivers, the preprojective algebras are known to be selfinjective, and we have the following.

Proposition 5.3. *When Q is Dynkin, with associated preprojective algebra $\Lambda = \Pi(Q)$, then the stable category $\underline{\mathrm{mod}}\Lambda$ is 2-CY.*

Proof. In view of Proposition 5.1, we only need to see that $\tau = \Omega^{-1}$. This follows from [AR,3.2,2.1]. We here give an outline of the proof, specialized to the case of interest here. The proof is based on some facts about the category $\mathrm{CM}(R)$ of maximal Cohen-Macaulay modules over two-dimensional simple hypersurface singularities R, which are of finite Cohen-Macaulay type and correspond to Dynkin diagrams. We have that τ_R is the identity, and $\Omega^2 \cong \mathbf{1}$ on the stable category $\underline{\mathrm{CM}}(R)$, and we have the formula $D\,\mathrm{Ext}^1(A,B) \cong \mathrm{Hom}(B,\tau A)$ [A]. In addition, if M is the direct sum of one copy of each indecomposable object in $\underline{\mathrm{CM}}(R)$ up to isomorphism, then $\Lambda = \underline{\mathrm{End}}(M)^{op}$ is isomorphic to $\Pi(Q)$ where the underlying graph of Q is the Dynkin diagram corresponding to R. We view the category $\mathrm{mod}\Gamma$ as $\mathrm{mod}(\underline{\mathrm{CM}}(R))$, the category of finitely presented contravariant functors from $\underline{\mathrm{CM}}(R)$ to $\mathrm{mod}\,k$. Denote by $\nu_{\mathcal{C}} = D((\ ,\underline{C})^*)$ the Nakayama functor, where C is in $\underline{\mathrm{CM}}(R)$. Then we have $\tau_{\mathcal{C}} = \Omega^2\nu_{\mathcal{C}}$. Since Γ is selfinjective, we have $D(\ ,\underline{C}) \cong D\,\mathrm{Ext}^1_R(\ ,\Omega^1 C)$, which is isomorphic to $\underline{\mathrm{Hom}}(\Omega^1 C,\ )$ since $\tau_R = \mathbf{1}$. Since $(\underline{Y},\ )^* = (\ ,\underline{Y})$ for Y in $\mathrm{CM}(R)$, where $(\ )^*$ denotes $\mathrm{Hom}_R(\ ,R)$, we have $\nu^{-1}(\ ,\underline{C}) = (D(\ ,\underline{C}))^* = (\ ,\Omega^1_R C)$ hence $\nu(\ ,\underline{C}) = (\ ,\Omega^{-1}_R C)$, for C in $\mathrm{CM}(R)$. One can show that $\Omega^{-1}_R \colon \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ induces in a natural way a functor α from $\mathcal{C}$ to $\mathcal{C}$, hence from $\underline{\mathcal{C}}$ to $\underline{\mathcal{C}}$, which is isomorphic to $\Omega^{-3}_{\mathcal{C}}$. It follows that $\tau_{\mathcal{C}} = \Omega^2_{\mathcal{C}}\nu_{\mathcal{C}}$ is isomorphic to $\Omega^{-1}_{\mathcal{C}}$ from $\underline{\mathcal{C}}$ to $\underline{\mathcal{C}}$. $\square$

This case of preprojective algebras has been investigated extensively in a series of papers by Geiss-Leclerc-Schröer. They work in the category $\mathrm{mod}\Lambda$, rather that in the 2-CY category $\underline{\mathrm{mod}}\Lambda$, but the categories $\mathrm{mod}\Lambda$ and $\underline{\mathrm{mod}}\Lambda$ are closely related, and one can go back and forth between exact sequences and triangles.

As for cluster categories, also in this setting the concepts of Ext-configuration maximal 1-orthogonal and the original definition of cluster-tilting object, here called maximal rigid, coincide. And also one has that the associated 2-CY tilted algebras

have no loops or 2-cycles in their quiver. Also in this case there are many interesting connections with cluster algebras [GLS2], [GLS1].

5.5. **FURTHER EXAMPLES.** We have already indicated that examples of 2-CY categories may be found amongst the stable categories $\underline{\mathrm{CM}}(R)$, where R is a complete local commutative noetherian isolated Gorenstein singularity. In view of Proposition 5.1, all we have to check is that $\tau = \Omega^{-1}$. But it is known from the work of Auslander [A] that if $d = \dim R$, then $\tau = \Omega^{2-d}$, hence $\underline{\mathrm{CM}}(R)$ is 2-CY if and only if $d = 3$.

A concrete example is the following: Let $S = k[[X, Y, Z]]$ and let G be the subgroup $\left\langle \begin{pmatrix} \xi\ 0\ 0 \\ 0\ \xi\ 0 \\ 0\ 0\ \xi \end{pmatrix} \right\rangle$ of the special linear group $\mathrm{SL}(3, k)$ where ξ is a primitive third root of 1. Then the invariant ring $R = S^G$ is a 3-dimensional ring with the desired properties, so that $\underline{\mathrm{CM}}(R)$ is 2-CY.

For 2-CY categories of the form $\underline{\mathrm{CM}}(R)$ it may happen that there are loops and 2-cycles, and also that the maximal rigid objects are not cluster-tilting [BIKR].

There is a large class of 2-CY categories associated with preprojective algebras of quivers which are not Dynkin [BIRS]. They contain both the cluster categories and the stable categories $\underline{\mathrm{mod}}\Lambda$ where Λ is the preprojective algebra of a Dynkin quiver as special cases. Also there are examples arising from taking stable categories of appropriate subcategories of $\mathrm{mod}\Lambda$.

5.6. **RECOGNIZING CLUSTER CATEGORIES.** A natural question is whether one can tell from a 2-CY tilted algebra which 2-CY category it came from. In particular, we can tell when it comes from a cluster category.

Theorem 5.4. *Let Γ be a 2-CY tilted algebra whose quiver Q has no oriented cycles. Then Γ comes from a cluster category $\mathcal{C}_Q$, and is hence cluster-tilted.*

We illustrate with the example from 5.5. Here S turns out to be a cluster-tilting object [I], and the quiver of $\underline{\mathrm{End}}(S)^{op}$ turns out to be $\bullet \rightrightarrows \bullet$. Hence $\underline{\mathrm{CM}}(R)$ is equivalent to $\mathcal{C}_{k(\rightrightarrows)}$.

Notes. The generalization from cluster categories to 2-CY categories in 5.2 and 5.3 is taken from [KR1], and the recognition theorem in 5.6 is given in [KR2].

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