Triangulated categories versus module categories

ICTP, Trieste, January 2010

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Let Λ be a finite dimensional algebra over an algebraically closed field k and let $\operatorname{mod}\Lambda$ be the category of finite dimensional left $\Lambda-$modules. If we assume that the algebra Λ is basic, then Λ is uniquely determined by $\operatorname{mod}\Lambda$. The main aim of representation theory is to give a rather complete description of $\operatorname{mod}\Lambda$. Depending on the complexity of $\operatorname{mod}\Lambda$ this can be achieved to some extent. One aspect of representation theory is to investigate homological properties of Λ or of $\operatorname{mod}\Lambda$. The main goal of this series of lectures is to show how triangulated categories can be used to study these questions.

Triangulated categories were introduced by Grothendieck and Verdier [V] around 1965. The most prominent example is the bounded derived category $D^b(\mathcal{A})$ of an abelian category $\mathcal{A}$. The category $\mathcal{A}$ can be identified with the full subcategory of $D^b(\mathcal{A})$ consisting of complexes concentrated in degree zero. We denote by $D^b(\Lambda)$ the bounded derived category of $\operatorname{mod}\Lambda$ for a finite dimensional $k-$algebra Λ. In general, the structure of $D^b(\Lambda)$ is quite complicated and usually Λ is not determined by $D^b(\Lambda)$, but contains the basic homological information of Λ. For example $D^b(\Lambda)$ detects whether or not Λ is of finite global dimension, but does not contains the information of the actual value. $D^b(\Lambda)$ also contains the information of the Hochschild cohomology algebra as a $\mathbb{Z}-$graded algebra.

The lectures will cover besides some basic information on triangulated and derived categories the following material:

- Structure of $D^b(H)$ for a finite dimensional hereditary algebra
- Tilting and piecewise hereditary algebras
- Frobenius categories and repetitive algebras
- Hochschild cohomology

The following list of references contains basic references for triangulated categories and derived categories. Also it contains references to the more specific topics dealt with in the lectures.

References

[H1] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras,* London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988.

[H2] D. Happel, *Hochschild cohomology of finite-dimensional algebras,* Sminaire d'Algbre Paul Dubreil et Marie-Paul Malliavin, 39me Anne (Paris, 1987/1988), 108–126, Lecture Notes in Math., 1404, Springer, Berlin, 1989.

[H3] D. Happel, *Hochschild cohomology of piecewise hereditary algebras,* Colloq. Math. 78 (1998), no. 2, 261–266.

[H4] D. Happel. *Auslander-Reiten triangles in derived categories of finite-dimensional algebras,* Proc. Amer. Math. Soc. 112 (1991), no. 3, 641–648.

[H5] D. Happel, *On the derived category of a finite-dimensional algebra,* Comment. Math. Helv. 62 (1987), no. 3, 339–389.

[HKR] D. Happel, B. Keller, I. Reiten, *Bounded derived categories and repetitive algebras,* J. Algebra 319 (2008), no. 4, 1611–1635.

[HRS] D. Happel, J. Rickard, A. Schofield, *Piecewise hereditary algebras*, Bull. London Math. Soc. 20 (1988), no. 1, 23–28.

[HZ1] D. Happel, D. Zacharia, *A homological characterization of piecewise hereditary algebras,* Math. Z. 260 (2008), no. 1, 177–185.

[HZ2] D. Happel, D. Zacharia, *Homological properties of piecewise hereditary algebas,* preprint.

[Ha] R. Hartshorne, *Residues and duality* Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20 Springer-Verlag, Berlin-New York 1966.

[I] B. Iversen, *Cohomology of sheaves,* Universitext. Springer-Verlag, Berlin, 1986.

[K1] B. Keller, *Deriving DG categories,* Ann. Sci. cole Norm. Sup. (4) 27 (1994), no. 1, 63–102.

[K2] B. Keller, *Derived categories and universal problems,* Comm. Algebra 19 (1991), no. 3, 699–747.

[K3] B. Keller, *A remark on tilting theory and DG algebras,* Manuscripta Math. 79 (1993), no. 3-4, 247–252.

[R1] J. Rickard, *Derived equivalences as derived functors,* J. London Math. Soc. (2) 43 (1991), no. 1, 37–48.

[R2] J. Rickard, *Derived categories and stable equivalence,* J. Pure Appl. Algebra 61 (1989), no. 3, 303–317.

[R3] J. Rickard, *Morita theory for derived categories,* J. London Math. Soc. (2) 39 (1989), no. 3, 436–456.

[V] J.L. Verdier, *Catégories deérivée, état 0,* Springer Lecture Notes in Mathematics 569, 1977.

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