CLUSTER ALGEBRAS AND CLUSTER CATEGORIES

ICTP, TRIESTE, JANUARY 2010

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Cluster algebras were invented by S. Fomin and A. Zelevinsky [FZ02] in the spring of the year 2000 in a project whose aim it was to develop a combinatorial approach to the results obtained by G. Lusztig concerning total positivity in algebraic groups [Lus94] on the one hand and canonical bases in quantum groups [Lus90] on the other hand (let us stress that canonical bases were discovered independently and simultaneously by M. Kashiwara [Kas90]). Despite great progress during the last few years [FZ03a] [BFZ05] [FZ07], we are still relatively far from these initial aims. Presently, the best results on the link between cluster algebras and canonical bases are probably those of C. Geiss, B. Leclerc and J. Schröer [GLS05] [GLS06] [GLS08a] [GLS] [GLS08b] but even they cannot construct canonical bases from cluster variables for the moment. Despite these difficulties, the theory of cluster algebras has witnessed spectacular growth thanks notably to the many links that have been discovered with a wide range of subjects including

- Poisson geometry [GSV03] [GSV05] ...,
- integrable systems [FZ03c] ...,
- higher Teichmüller spaces [FGa] [FGb] [FGc] [FG06] ...,
- combinatorics and the study of combinatorial polyhedra like the Stasheff associahedra [CFZ02] [Cha05] [Kra06] [FR05] [Mus] [MSW] [FST] ...,
- commutative and non commutative algebraic geometry, in particular the study of stability conditions in the sense of Bridgeland [Bri06] [Bri] [Bri07], Calabi-Yau algebras [Gin], Donaldson-Thomas invariants [Sze] [KS] ...,
- and last not least the representation theory of quivers and finite-dimensional algebras, cf. for example the surveys [BBM06] [Rei] [Rin07] [GLS08b] [Kel].

We refer to the introductory papers [FZ03b] [Zel] [Zel02] [Zel05] [Zel07] and to the cluster algebras portal [Fom] for more information on cluster algebras and their links with other parts of mathematics.

The link between cluster algebras and quiver representations follows the spirit of categorification: One tries to interpret cluster algebras as combinatorial (perhaps K-theoretic) invariants associated with categories of representations. Thanks to the rich structure of these categories, one can then hope to prove results on cluster algebras which seem beyond the scope of the purely combinatorial methods. It turns out that the link becomes especially beautiful if we use *triangulated categories* constructed from categories of quiver representations.

In these lectures, we will start with an informal presentation of Fomin-Zelevinsky's classification theorem and of the cluster algebras of rank 2 (without coefficients) associated with Dynkin diagrams. Then we will introduce quiver mutations and the cluster algebra associated with a quiver. We will state the main properties of this construction following Fomin-Zelevinsky [FZ03a]. Then we will present the link to representation theory in the case of cluster algebras associated with acyclic quivers using the cluster category of [BBMR$^+$06]. In order to treat quivers with oriented cycles, we will introduce quivers with potentials and their mutations following Derksen-Weyman-Zelevinsky [DWZ08]. We will then describe how these allow one to categorify cluster algebras following work by Amiot [Ami], Palu [Pal08] [Pal] and Derksen-Weyman-Zelevinsky [DWZ]. If time permits, we will also sketch the link to Kontsevich-Soibelman's interpretation of cluster transformations [KS].

Date: October 2009, last modified on October 8, 2009.

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