Derived equivalence induced by infinitely generated n-tilting modules

Silvana Bazzoni

(Joint work with Francesca Mantese and Alberto Tonolo)

Università di Padova

Trieste, February 1-5, 2010

Outline

- Why Infinitely generated n-tilting modules?
- Equivalences induced by a classical n-tilting module.
- Derived equivalences in the infinitely generated case.
- Application to module categories.

Why infinitely generated modules?

- "Generic modules"

"Generic" modules appear in the Ziegler closure of direct limits of finitely generated modules.
They parametrize families of finite dimensional modules, (Crawley-Boevey, Ringel, Krause, Herzog)

- Approximation theory

Classical notion: covariantly or contravariantly finite classes of finitely generated modules

approximations via preenvelopes or precovers allowing infinitely generated modules are somehow easier to handle.

Application: tilting classes are always preenveloping.

Why infinitely generated modules?

- Finitistic dimension conjectures

(Angeleri, Trlifaj '02)
The little finitistic dimension of a noetherian ring is finite if and only if there is a tilting module representing the category of finitely generated modules of finite projective dimension.

Even in the case of finite dimensional algebra it may happen that such a tilting module cannot be chosen to be finitely generated.

- n-tilting classes are of finite type

(B, Herbera, Šťovíček '07)
Every tilting class is determined by finitely presented data: it is the right Ext-orthogonal of a set of finitely presented modules.

Infinitely generated n-tilting modules

R associative ring with 1.

Definition

A right R-module T is *n-tilting module*, if

(T1) there exists a projective resolution of right R-modules $0 \to P_n \to \ldots \to P_1 \to P_0 \to T \to 0$;

(T2) $\mathrm{Ext}^i_R(T, T^{(\alpha)}) = 0$ for each $i > 0$ and each cardinal α;

(T3) there exists a coresolution of right R-modules $0 \to R \to T_0 \to T_1 \to \ldots \to T_m \to 0$, with T_i' in $\mathrm{Add}\ T$.

- T is a *classical n-tilting module* if P_i's in (T1) are finitely generated.

$\mathcal{T} = \{M \in \mathrm{Mod}\text{-}R \mid \mathrm{Ext}^i_R(T, M) = 0, \forall i > 0\}$
is called the *n-tilting class*.

Classical equivalences for the case $n = 1$

Theorem [Brenner-Butler '80, Colby-Fuller '90]

T_R classical 1-tilting module. $S = \mathrm{End}_R(T)$

$\mathcal{T} = \mathrm{Gen}\, T = \mathrm{Ker}(\mathrm{Ext}^1_R(T, -)), \quad \mathcal{F} = \mathrm{Ker}(\mathrm{Hom}_R(T, -)).$

$(\mathcal{T}, \mathcal{F})$ a torsion pair in $\mathrm{Mod}\text{-}R$

$\mathcal{Y} = \mathrm{Ker}(\mathrm{Tor}^S_1(-, T)) \quad \mathcal{X} = \mathrm{Ker}(- \otimes_S T)$

$(\mathcal{X}, \mathcal{Y})$ torsion pair in $\mathrm{Mod}\text{-}S$

$$\begin{tikzcd} \mathcal{T} \arrow[r, bend left, "\mathrm{Hom}_R(T,-)"] & \mathcal{Y} \arrow[l, bend left, "-\otimes_S T"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F} \arrow[r, bend left, "\mathrm{Ext}^1_R(T,-)"] & \mathcal{X} \arrow[l, bend left, "\mathrm{Tor}^S_1(-,T)"] \end{tikzcd}$$

Classical equivalences for $n > 1$

T_R a classical n-tilting module. $S = \mathrm{End}_R(T)$

$$KE_i = \bigcap_{0 \leq j \neq i} \mathrm{Ker}(\mathrm{Ext}^j_R(T, -)) \qquad 0 \leq i \leq n$$

$$KT_i = \bigcap_{0 \leq j \neq i} \mathrm{Ker}(\mathrm{Tor}^S_j(-, T)) \qquad 0 \leq i \leq n$$

Theorem [Miyashita, '86]

There are equivalences:

$$\begin{tikzcd} KE_i \arrow[r, bend left, "\mathrm{Ext}^i_R(T,-)"] & KT_i \arrow[l, bend left, "\mathrm{Tor}^S_i(-,T)"] \end{tikzcd} \qquad 0 \leq i \leq n$$

If T_R is a infinitely generated, the equivalences can be generalized at the cost of intersecting with particular subcategories of Mod-S.

The classical derived equivalences

Theorem [Happel '87, Cline-Parshall-Scott '87]

T_R a classical n-tilting module with endomorphism ring S. There is a derived equivalence:

$$\begin{tikzcd} \mathcal{D}^b(R) \arrow[r, bend left, "\mathbb{R}\mathrm{Hom}_R(T,-)"] & \mathcal{D}^b(S) \arrow[l, bend left, "-\overset{\mathbb{L}}{\otimes}_S T"] \end{tikzcd}$$

Good n-tilting modules

T_R and T'_R n-tilting modules are equivalent if they induce the same n-tilting class, or if $\mathrm{Add}\ T' = \mathrm{Add}\ T$.

Definition

An n-tilting module T_R with endomorphism ring S is good if condition (T3) can be replaced by

$$[(T3')] \quad 0 \to R \to T_0 \to T_1 \to \ldots \to T_n \to 0$$

where the T_i's are in $\mathrm{add}\ T$.

Each classical n-tilting module is good.

Proposition

Every n-tilting module admits an equivalent good n-tilting module.

Proposition

Let T_R be a good n-tilting module, $S = \mathrm{End}_R(T)$. Then,

(T1) there exists $0 \to Q_n \to \ldots \to Q_0 \to {}_S T \to 0$
Q_i finitely generated projective left S-modules,

(T2) $\mathrm{Ext}^i_S(T, T) = 0$ for each $i \geq 0$, and $R \cong \mathrm{End}({}_S T)$.

Thus, ${}_S T$ is a partial classical n-tilting S-module.

Lemma Miyashita

Let T_R be a good n-tilting module with endomorphism ring S.
Then, for each injective module I_R

- $\mathrm{Hom}_R(T, I) \otimes_S T \cong I$;
- $\mathrm{Hom}_R(T, I)$ is an $(- \otimes_S T)$-acyclic right S-module;

For each projective right S-module P_S

- $P \otimes_S T$ is an $\mathrm{Hom}_R(T, -)$-acyclic right R-module.

Generalization of the derived equivalence

- T_R R-module, $\mathrm{End}(T) = S$.
- $\mathcal{D}(R)$, $\mathcal{D}(S)$ derived categories of $\mathrm{Mod}\text{-}R$ and $\mathrm{Mod}\text{-}S$.
- The adjoint pair

$$H = \mathrm{Hom}_R(T, -) \colon \mathrm{Mod}\text{-}R \rightleftarrows \mathrm{Mod}\text{-}S \colon G = - \otimes_S T$$

induces an adjoint pair of total derived functors

$$\mathbb{R}H = \mathbb{R}\mathrm{Hom}_R(T, -) \colon \mathcal{D}(R) \rightleftarrows \mathcal{D}(S) \colon \mathbb{L}G = - \overset{\mathbb{L}}{\otimes}_S T$$

Theorem

T_R a good n-tilting module, $\mathrm{End}(T) = S$.

$\mathbb{R}H = \mathbb{R}\mathrm{Hom}_R(T, -)$, $\quad \mathbb{L}G = - \overset{\mathbb{L}}{\otimes}_S T$

The following hold:

(1) The counit of the adjunction $\psi \colon \mathbb{L}G \circ \mathbb{R}H \to Id_{\mathcal{D}(R)}$ is invertible.

(2) There is a triangle equivalence $\Theta \colon \mathcal{D}(S)/\mathrm{Ker}(\mathbb{L}G) \to \mathcal{D}(R)$

$$\begin{tikzcd}
\mathcal{D}(S) \arrow[r, "\mathbb{L}G"] \arrow[d, "q"'] & \mathcal{D}(R) \\
\mathcal{D}(S)/\mathrm{Ker}(\mathbb{L}G) \arrow[ur, "\Theta\ \cong"'] &
\end{tikzcd}$$

(3) Σ : system of morphisms $u \in \mathcal{D}(S)$ such that $\mathbb{L}G(u)$ is invertible in $\mathcal{D}(R)$. Σ admits a calculus of left fractions and

$$\mathcal{D}(S)[\Sigma^{-1}] \cong \mathcal{D}(S)/\mathrm{Ker}(\mathbb{L}G)$$

The key fact is that the counit of the adjunction

- $\psi\colon \mathbb{L}G \circ \mathbb{R}H \to Id_{\mathcal{D}(R)}$ is invertible.
- Obtained by using:
- the functors $\mathrm{Hom}_R(T, -)$ and $- \otimes_S T$ have finite homological dimension
  and their total derived functors can be computed on complexes with acyclic components.
- $\mathbb{R}\mathrm{Hom}(T, I^\bullet) \overset{\mathbb{L}}{\otimes}_S T = \mathrm{Hom}(T, I^\bullet) \otimes_S T$,
  $I^\bullet$ complex whose terms are injective right R-modules.
  $\mathbb{R}\mathrm{Hom}(T, P^\bullet \overset{\mathbb{L}}{\otimes}_S T) = \mathrm{Hom}(T, P^\bullet \otimes_S T)$.
  $P^\bullet$ complex whose terms are projective right S-modules.
- The rest follows by Proposition 1.3 in Gabriel-Zisman's book.

The perpendicular subcategory

$\phi : 1_{\mathcal{D}(S)} \to \mathbb{R}H \circ \mathbb{L}G$ the unit of the adjunction

$\psi : \mathbb{L}G \circ \mathbb{R}H \to 1_{\mathcal{D}(R)}$ the counit of the adjunction is invertibe.

Proposition

The functor $L := \mathbb{R}H \circ \mathbb{L}G \colon \mathcal{D}(S) \to \mathcal{D}(S)$ is a Bousfield localization

So the kernel L, i.e. $\mathcal{E} = \mathrm{Ker}\mathbb{L}G$ is a localizing subcategory, and if $\mathcal{E}_\perp$ is the perpendicular category

$$\mathcal{E}_\perp := \{X \in \mathcal{D}(S) : \mathrm{Hom}_{\mathcal{D}(S)}(\mathcal{E}, X) = 0\}$$

L factorizes as

$$\mathcal{D}(S) \xrightarrow{q} \mathcal{D}(S)/\mathrm{Ker}\mathbb{L}G \xrightarrow[\cong]{\rho} \mathcal{E}_\perp \xhookrightarrow{j} \mathcal{D}(S)$$

where q is the canonical quotient functor and ρ is an equivalence.

Theorem

Let T_R be a good n-tilting R-module and $S = \mathrm{End}(T)$.
Let $\mathcal{E}$ be the kernel of $\mathbb{L}G$ we have triangle equivalence:

$$\begin{tikzcd} \mathcal{D}(R) \arrow[r, bend left, "\mathbb{R}H"] & \mathcal{E}_\perp \arrow[l, bend left, "\mathbb{L}G"] \end{tikzcd}$$

($\mathbb{R}H$ and $\mathbb{L}G$ corestriction and restriction) and
we have a commutative diagram:

$$\begin{tikzcd} \mathcal{D}(R) \arrow[rr, shift left, "\mathbb{R}H \cong"] \arrow[dr, "\cong\ \Theta"'] & & \mathcal{E}_\perp \arrow[ll, shift left, "\mathbb{L}G"] \\ & \dfrac{\mathcal{D}(S)}{\mathrm{Ker}\mathbb{L}G} \arrow[ur, "\rho\ \cong"'] & \end{tikzcd}$$

Back to the classical derived equivalence

Proposition

The following are equivalent.

- T_R is a classical n-tilting module;
- $\mathcal{E} = 0$ or equivalently $\mathcal{E}_\perp = \mathcal{D}(S)$;
- the class $\mathcal{E}$ is smashing, i.e. $\mathcal{E}_\perp$ is closed under direct sums.

Back to module categories

Using the canonical embeddings
$\mathrm{Mod}\text{-}R \to \mathcal{D}(R) \qquad \mathrm{Mod}\text{-}S \to \mathcal{D}(S)$

we have a generalization to infinitely generated n-tilting modules of Brenner-Blutler, Colby-Fuller and Miyashita equivalences:

$$\begin{tikzcd} KE_i \arrow[r, bend left, "\mathrm{Ext}^i_R(T,-)"] & KT_i \cap \mathcal{E}_\perp \arrow[l, bend left, "\mathrm{Tor}^S_i(-,T)"] \end{tikzcd}$$