Derived equivalence induced by infinitely generated $n$-tilting modules

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Outline

- Why Infinitely generated $n$-tilting modules?
- Equivalences induced by a classical $n$-tilting module.
- Derived equivalences in the infinitely generated case.
- Application to module categories.
Why infinitely generated modules?

- “Generic modules”

“Generic” modules appear in the Ziegler closure of direct limits of finitely generated modules. They parametrize families of finite dimensional modules, (Crawley-Boevey, Ringel, Krause, Herzog)

- Approximation theory

Classical notion: covariantly or contravariantly finite classes of finitely generated modules approximations via preenvelopes or precovers allowing infinitely generated modules are somehow easier to handle.

Application: tilting classes are always preenveloping.
Why infinitely generated modules?

- Finitistic dimension conjectures
  (Angeleri, Trlifaj '02)
  The little finitistic dimension of a noetherian ring is finite if and only if there is a tilting module representing the category of finitely generated modules of finite projective dimension.

  Even in the case of finite dimensional algebra it may happen that such a tilting module cannot be chosen to be finitely generated.

- \( n \)-tilting classes are of finite type
  (B, Herbera, Šťovíček '07)
  Every tilting class is determined by finitely presented data: it is the right Ext-orthogonal of a set of finitely presented modules.
Infinitely generated \( n \)-tilting modules

\( R \) associative ring with 1.

**Definition**
A right \( R \)-module \( T \) is \( n \)-tilting module, if

1. **(T1)** there exists a projective resolution of right \( R \)-modules
   \[ 0 \to P_n \to \ldots \to P_1 \to P_0 \to T \to 0; \]
2. **(T2)** \( \text{Ext}^i_R(T, T^{(\alpha)}) = 0 \) for each \( i > 0 \) and each cardinal \( \alpha \);
3. **(T3)** there exists a coresolution of right \( R \)-modules
   \[ 0 \to R \to T_0 \to T_1 \to \ldots \to T_m \to 0, \text{ with } T_i' \text{ in } \text{Add } T. \]

- \( T \) is a classical \( n \)-tilting module if \( P_i \)'s in (T1) are finitely generated.

\[ \mathcal{T} = \{ M \in \text{Mod-}R \mid \text{Ext}^i_R(T, M) = 0, \forall i > 0 \} \]

is called the \( n \)-tilting class.
**Classical equivalences for the case $n = 1$**

**Theorem [Brenner-Butler ’80, Colby-Fuller ’90]**

$T_R$ classical 1-tilting module. $S = \text{End}_R(T)$

$T = \text{Gen} T = \text{Ker}(\text{Ext}^1_R(T, -))$, \quad $\mathcal{F} = \text{Ker}(\text{Hom}_R(T, -))$.

$(T, \mathcal{F})$ a torsion pair in $\text{Mod-} R$

$\mathcal{Y} = \text{Ker}(\text{Tor}_1^S(-, T))$ \quad $\mathcal{X} = \text{Ker}(- \otimes_S T)$

$(\mathcal{X}, \mathcal{Y})$ torsion pair in $\text{Mod-} S$

\[
\begin{array}{ccc}
T & \overset{\text{Hom}_R(T, -)}{\leftarrow} & \mathcal{Y} \\
& \downarrow{\otimes _S T} & \\
& \mathcal{Y} & \overset{\text{Ext}^1_R(T, -)}{\leftarrow} \mathcal{F} \\
& \downarrow{\text{Tor}_1^S(-, T)} & \\
\mathcal{F} & \overset{\text{Hom}_R(T, -)}{\leftarrow} & \mathcal{X}
\end{array}
\]
Classical equivalences for $n > 1$

$T_R$ a classical $n$-tilting module. $S = \text{End}_R(T)$

$$KE_i = \bigcap_{0 \leq j \neq i} \text{Ker}(\text{Ext}^j_R(T, -)) \quad 0 \leq i \leq n$$

$$KT_i = \bigcap_{0 \leq j \neq i} \text{Ker}(\text{Tor}^j_S(-, T)) \quad 0 \leq i \leq n$$

**Theorem** [Miyashita, ’86]

There are equivalences:

$$\begin{align*}
\text{Ext}^i_R(T, -) & \quad \text{KE}_i & \quad \text{KT}_i & \quad 0 \leq i \leq n \\
\text{Tor}^i_S(-, T) & \quad & & 
\end{align*}$$

If $T_R$ is infinitely generated, the equivalences can be generalized at the cost of intersecting with particular subcategories of $\text{Mod-}S$. 
The classical derived equivalences

**Theorem** [Happel ’87, Cline-Parshall-Scott ’87]

$T_R$ a classical $n$-tilting module with endomorphism ring $S$. There is a derived equivalence:

$$
\mathcal{D}^b(R) \xrightarrow{\mathbb{R}\text{Hom}_R(T, -)} \mathcal{D}^b(S) \\
\mathcal{D}^b(S) \xleftarrow{- \otimes_S T} \mathcal{D}^b(R)
$$
Good $n$-tilting modules

$T_R$ and $T'_R$ $n$-tilting modules are equivalent if they induce the same $n$-tilting class, or if $\text{Add } T' = \text{Add } T$.

**Definition**

An $n$-tilting module $T_R$ with endomorphism ring $S$ is good if condition (T3) can be replaced by

$$[(T3')] \ 0 \to R \to T_0 \to T_1 \to \ldots \to T_n \to 0$$

where the $T_i$’s are in $\text{add } T$.

Each classical $n$-tilting module is good.

**Proposition**

Every $n$-tilting module admits an equivalent good $n$-tilting module.
**Proposition**
Let $T_R$ be a good $n$-tilting module, $S = \text{End}_R(T)$. Then,

(T1) there exists $0 \to Q_n \to \ldots \to Q_0 \to S \cdot T \to 0$

$Q_i$ finitely generated projective left $S$-modules,

(T2) $\text{Ext}^i_S(T, T) = 0$ for each $i \geq 0$, and $R \cong \text{End}(S \cdot T)$.

Thus, $S \cdot T$ is a partial classical $n$-tilting $S$-module.

**Lemma Miyashita**
Let $T_R$ be a good $n$-tilting module with endomorphism ring $S$.

Then, for each injective module $I_R$

- $\text{Hom}_R(T, I) \otimes_S T \cong I$;

- $\text{Hom}_R(T, I)$ is an $(- \otimes_S T)$-acyclic right $S$-module;

For each projective right $S$-module $P_S$

- $P \otimes_S T$ is an $\text{Hom}_R(T, -)$-acyclic right $R$-module.
Generalization of the derived equivalence

- $T_R$ $R$-module, $\text{End}(T) = S$.
- $\mathcal{D}(R)$, $\mathcal{D}(S)$ derived categories of $\text{Mod-}R$ and $\text{Mod-}S$.
- The adjoint pair

$$H = \text{Hom}_R(T, -): \text{Mod-}R \leftrightarrow \text{Mod-}S: G = - \otimes_S T$$

induces an adjoint pair of total derived functors

$$\mathbb{R}H = \mathbb{R}\text{Hom}_R(T, -): \mathcal{D}(R) \leftrightarrow \mathcal{D}(S): \mathbb{L}G = - \otimes_S T$$
**Theorem**

$T_R$ a good $n$-tilting module, $\text{End}(T) = S$.

$\mathcal{R}H = \mathcal{R}\text{Hom}_R(T, -)$, \hspace{0.5cm} $\mathcal{L}G = - \otimes_S T$

The following hold:

1. The counit of the adjunction $\psi: \mathcal{L}G \circ \mathcal{R}H \to \text{Id}_{\mathcal{D}(R)}$ is invertible.

2. There is a triangle equivalence $\Theta: \mathcal{D}(S)/\text{Ker}(\mathcal{L}G) \to \mathcal{D}(R)$

3. $\Sigma$: system of morphisms $u \in \mathcal{D}(S)$ such that $\mathcal{L}G(u)$ is invertible in $\mathcal{D}(R)$. $\Sigma$ admits a calculus of left fractions and

\[ \mathcal{D}(S)[\Sigma^{-1}] \cong \mathcal{D}(S)/\text{Ker}(\mathcal{L}G) \]
The key fact is that the counit of the adjunction

\[ \psi : \mathbb{L}G \circ \mathbb{R}H \to \text{Id}_{D(R)} \]

is invertible.

Obtained by using:

the functors \( \text{Hom}_R(\, T, -) \) and \( - \otimes_S T \) have finite homological dimension
and their total derived functors can be computed on complexes with acyclic components.

\[ \mathbb{R}\text{Hom}(\, T, I^\bullet) \mathbb{L} \otimes_S T = \text{Hom}(\, T, I^\bullet) \otimes_S T, \]

\( I^\bullet \) complex whose terms are injective right \( R \)-modules.

\[ \mathbb{R}\text{Hom}(\, T, P^\bullet \mathbb{L} \otimes_S T) = \text{Hom}(\, T, P^\bullet \otimes_S T), \]

\( P^\bullet \) complex whose terms are projective right \( S \)-modules.

The rest follows by Proposition 1.3 in Gabriel-Zisman’s book.
The perpendicular subcategory

\[ \phi : 1_{\mathcal{D}(S)} \to \mathbb{R}H \circ \mathbb{L}G \] the unit of the adjunction

\[ \psi : \mathbb{L}G \circ \mathbb{R}H \to 1_{\mathcal{D}(R)} \] the counit of the adjunction is invertible.

**Proposition**

The functor \( L := \mathbb{R}H \circ \mathbb{L}G : \mathcal{D}(S) \to \mathcal{D}(S) \) is a Bousfield localization.

So the kernel \( L \), i.e. \( \mathcal{E} = \text{Ker}\mathbb{L}G \) is a localizing subcategory, and if \( \mathcal{E}_\perp \) is the perpendicular category

\[ \mathcal{E}_\perp := \{ X \in \mathcal{D}(S) : \text{Hom}_{\mathcal{D}(S)}(\mathcal{E}, X) = 0 \} \]

\( L \) factorizes as

\[
\mathcal{D}(S) \xrightarrow{q} \mathcal{D}(S)/\text{Ker}\mathbb{L}G \xrightarrow{\rho} \mathcal{E}_\perp \xleftarrow{j} \mathcal{D}(S)
\]

where \( q \) is the canonical quotient functor and \( \rho \) is an equivalence.
**Theorem**

Let $T_R$ be a good $n$-tilting $R$-module and $S = \text{End}(T)$. Let $\mathcal{E}$ be the kernel of $\mathbb{L}G$ we have triangle equivalence:

$$
\mathcal{D}(R) \xrightarrow{\mathbb{R}H} \mathbb{L}G \xrightarrow{\mathbb{L}G} \mathcal{E}_\perp
$$

($\mathbb{R}H$ and $\mathbb{L}G$ corestriction and restriction) and we have a commutative diagram:

$$
\begin{array}{ccc}
\mathcal{D}(R) & \xrightarrow{\mathbb{R}H \cong} & \mathcal{E}_\perp \\
\downarrow \cong & & \downarrow \rho \\
\mathcal{D}(S) & \xrightarrow{\Theta} & \mathcal{D}(S) / \text{Ker}\mathbb{L}G
\end{array}
$$

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**Derived equivalence induced by $n$-tilting modules**
**Proposition**

The following are equivalent.

- $T_R$ is a classical $n$-tilting module;
- $\mathcal{E} = 0$ or equivalently $\mathcal{E}_\perp = \mathcal{D}(S)$;
- the class $\mathcal{E}$ is **smashing**, i.e. $\mathcal{E}_\perp$ is closed under direct sums.
Using the canonical embeddings
\[ \text{Mod-} R \to \mathcal{D}(R) \quad \text{Mod-} S \to \mathcal{D}(S) \]

we have a generalization to \textit{infinitely generated} \( n \)-tilting modules of Brenner-Blutler, Colby-Fuller and Miyashita equivalences:

\[
\begin{array}{ccc}
\text{KE}_i & \leftrightarrow & \text{KT}_i \cap \mathcal{E}_\perp \\
\text{Ext}_R^i(T, -) & \leftrightarrow & \text{Tor}_S^i(-, T)
\end{array}
\]