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QUIVER GRASSMANNIANS ASSOCIATED WITH THE KRONECKER QUIVER

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Abstract. These notes have been the subject of a talk at "Advanced School and Conference on Homological and Geometrical Methods in Representation Theory" which took place in Trieste (Italy) the 22nd of January 2010. We study quiver Grassmannians associated with the Kronecker quiver and we obtain a geometric realization of the "canonical basis" found by Sherman and Zelevinsky in [12].

This is a joint work with Francesco Esposito

1. Introduction

Cluster algebras are commutative $\mathbb{Z}$–subalgebras of the field of rational functions in a finite number of indeterminates which have been introduced and studied by S. Fomin and A.Zelevinsky in a series of papers [6], [7], [1] and [8]. The aim of these notes is to give a geometric realization of a particular $\mathbb{Z}$–basis of a cluster algebra of type $A_1^{(1)}$ found in [12] and called canonical basis. Our approach follows the additive categorification of cluster algebras via cluster categories and via a map, called the Caldero–Chapoton map, between the representations of an acyclic quiver Q and the corresping cluster algebra $\mathcal{A}_Q$ (we address the reader to the survey [10] for more information about cluster algebras and related topics). Such map is defined in terms of Euler–Poincaré characteristic of some complex projective varieties attached to every representation M of Q and called quiver Grassmannians. By definition the quiver Grassmannian $Gr_{\mathbf{e}}(M)$ consists of all sub–representations of M of dimension vector $\mathbf{e}$. These varieties are considered in several places, e.g. [3], [2], [4], [11], and in this paper we try to add some more geometric informations about them, at least in the case of the Kronecker quiver. In [5] authors compute the Euler–Poincaré characteristic of quiver Grassmannians associated with the Kronecker quiver and they conjecture the existence of a cellular decomposition which we find here. In [9] a torus action on some quiver Grassmannians has been found and this allows to produce a cellular decomposition of them in the case they are smooth.

We study quiver Grassmannians associated with the Kronecker quiver. A representation of the Kronecker quiver (in the sequel we will briefly say a Q–representation) is a quadruple $M = (M_1, M_2, m_a, m_b)$ where M_1 and M_2 are finite dimensional complex vector spaces and $m_a, m_b : M_1 \to M_2$ are two linear maps between them. Given two non–negative integers e_1 and e_2, the variety $Gr_{(e_1,e_2)}(M)$ is defined as the set

$$\{(N_1, N_2) \in Gr_{e_1}(M_1) \times Gr_{e_2}(M_2) :\ m_a(N_1) \subset N_2,\ m_b(N_1) \subset N_2\}$$

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where $Gr_e(V)$ denotes the Grassmannians of e–dimensional vector spaces in a vector space V. This is a projective variety which is in general not smooth. When $M_1 = M_2 = \mathbf{C}^n$ and $m_a = Id$ is the identity and $m_b = J_n(\lambda)$ is an indecomposable Jordan block of eigenvalue λ, the representation M is regular indecomposable and we denote it by $R_n(\lambda)$. The corresponding quiver Grassmannians $X = X_{\mathbf{e}}(n) = Gr_{(e_1,e_2)}(R_n)$ are the main subjects of this paper. We briefly write X for one of these quiver Grassmannians. We find that a one–dimensional torus T acts on X. We provide a stratification of X

$$X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_s \tag{1}$$

for $s = \min(e_1, n - e_2)$ into closed subvarieties $X_k \simeq Gr_{(e_1-k,e_2-k)}(R_{n-2k})$. Moreover X_{k+1} is the singular locus of X_k and the difference $X_k \setminus X_{k+1}$ is a smooth quasi–projective variety which is not complete. However we are able to prove that Białynicki-Birula's theorem on cellular decomposition of smooth projective varieties applies to $X_k \setminus X_{k+1}$ and we can hence prove that X decomposes

$$X = \cup_{L \in X^T} X_L$$

into attracting sets of T–fixed points

$$X_L := \{N \in X | \lim_{\lambda \to 0} t_\lambda N = L\}$$

and theses sets are affine varieties. As a consequence of the stratification (1) we get that X is smooth if and only if $s = 0$, i.e. either $e_1 = 0$ or $e_2 = n$, in which cases the quiver grassmannian reduces to an usual Grassmannian of vector subpaces.

We consider the (coefficient–free) cluster algebra $\mathcal{A}$ of type $A_1^{(1)}$: by definition $\mathcal{A}$ is the $\mathbb{Z}$–subalgebra of $\mathcal{F} = \mathbb{Q}(x_1, x_2)$, the field of rational functions in two indeterminates with rational coefficients, generated by rational functions (called cluster variables) $\{x_k | \ k \in \mathbb{Z}\}$ recursively generated by the following relation:

$$x_k x_{k+2} = x_{k+1}^2 + 1.$$

It is not hard to see that every couple $\{x_k, x_{k+1}\}$ of consecutive cluster variables is a free–generating system for the field $\mathcal{F}$ and hence $\mathcal{F} = \mathbb{Q}(x_k, x_{k+1})$ and every cluster variable can be expressed as a rational function in every such couple (which are called the clusters of $\mathcal{A}$). Moreover every cluster variable is a Laurent polynomial in every cluster of $\mathcal{A}$ (this is the Laurent phenomenon proved in [6]). Following [2] we consider the so called Caldero–Chapoton map $M \mapsto X_M$ which associates to a Q–representation M the Laurent polynomial:

$$X_M := \frac{\sum_{\mathbf{e}} \chi(Gr_{\mathbf{e}}(M)) x_1^{2(d_2-e_2)} x_2^{2e_1}}{x_1^{d_1} x_2^{d_2}} \tag{2}$$

where $\chi(Gr_{\mathbf{e}}(M))$ denotes the Euler–Poincaré characteristic of the quiver Grassmannian $Gr_{\mathbf{e}}(M)$. In [3] it is proved that the map $M \mapsto X_M$ restricts to a bijection between the indecomposable rigid Q–representations M (i.e. $Ext^1(M, M) = 0$) and the cluster variables of $\mathcal{A}$ different from x_1 and x_2. Moreover it has the following multiplicative property:

$$X_{M \oplus N} = X_M X_N.$$

under which cluster monomials not divisible by x_1 or x_2, i.e. monomials of the form $x_k^a x_{k+1}^b$ for $k \in \mathbb{Z} \setminus \{1,2\}$ and $a, b \geq 0$, are in bijection with rigid Q–representations.

Following [12] we say that an element of $\mathcal{A}$ is *positive* if its Laurent expansion in every cluster of $\mathcal{A}$ has positive coefficients. Positive elements form a cone and the next result describes it. In [12] the authors introduce distinguished elements $\{z_n | \, n \geq 1\}$, recursively defined by:

$$\begin{aligned} z_0 &= 2 \\ z_1 &= x_0 x_3 - x_1 x_2 \\ z_{n+1} &= z_1 z_n - z_{n-1} \quad n \geq 1 \end{aligned}$$

and they prove that the set

$$\mathbf{B} := \{\text{cluster monomials}\} \cup \{z_n : n \geq 1\}$$

is a $\mathbb{Z}$–basis of $\mathcal{A}$ such that positive linear combinations of its elements coincide with the set of all positive elements of $\mathcal{A}$. They call this basis the canonical basis of $\mathcal{A}$. We give a geometric realization of $\mathbf{B}$ by using the Caldero–Chapoton map. We already know that cluster monomials are images of rigid representations. Moreover the quiver Grasmannians associated with a rigid representation is smooth [4]. Having this in mind we prove the following result:

Theorem 1.1. *For every $n \geq 1$:*

$$z_n = \frac{\sum_{\mathbf{e}} \chi(X_{\mathbf{e}}(n)^{sm}) x_1^{2(n-e_2)} x_2^{2e_1}}{x_1^n x_2^n}$$

where $X_{\mathbf{e}}(n) = Gr_{\mathbf{e}}(R_n)$ is the variety considered above and $X_{\mathbf{e}}(n)^{sm} := X_0 \setminus X_1$ denotes its smooth part.

In other words the canonical basis can be obtained by computing the CC–map in the smooth part of the quiver Grassmannians associated with all the rigid representations and of all regular indecomposable representations.

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