Representations of pointed Hopf algebras over $\mathbb{S}_3$

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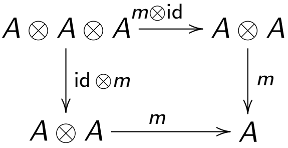
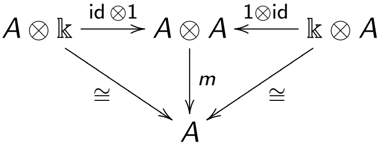
Definition and examples

Let $(A, m, 1)$ be an associative algebra with unit over a field $\mathbb{k}$. That is,

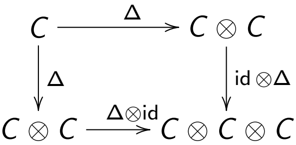
$$(ab)c = a(bc) \qquad \forall\, a, b, c, \in A \qquad \text{and}$$
$$a1 = a = 1a, \qquad \forall\, a \in A.$$

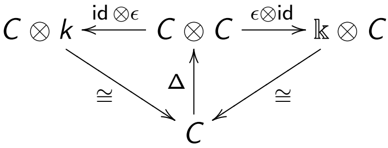
- These axioms can be codified in the commutativity of the following diagrams:





- A coassociative counital $\mathbb{k}$-*coalgebra* (C, Δ, ϵ) is a $\mathbb{k}$-vector space C together with maps $\Delta : C \to C \otimes C$ (the comultiplication) and $\epsilon : C \to \mathbb{k}$ (the counit) such the following diagrams commute:





That is,

$$(\Delta \otimes \mathrm{id})\Delta(c) = (\mathrm{id} \otimes \Delta)\Delta(c), \qquad \text{and}$$
$$(\epsilon \otimes \mathrm{id})\Delta(c) = (\mathrm{id} \otimes \epsilon)\Delta(c) = c,$$

for every $c \in C$

- A *bialgebra* B is an algebra $(B, m, 1)$ and a coalgebra (B, Δ, ϵ) such that the maps

$$\Delta : B \to B \otimes B,$$
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are algebra maps.

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- A *Hopf algebra* H is a bialgebra (H, m, Δ) together with a map $\mathcal{S} \in \mathrm{End}(H)$ (the antipode) such that the following axioms are satisfied:

$$m(\mathcal{S} \otimes \mathrm{id})\Delta(h) = \epsilon(h)1,$$
$$m(\mathrm{id} \otimes \mathcal{S})\Delta(h) = \epsilon(h)1.$$

for every $h \in H$.

Examples

- Let G be a group and $\Bbbk G$ the group algebra, that is the vector space with basis $\{e_g : g \in G\}$ and multiplication rule $e_g e_h = e_{gh}$, $g, h \in G$. Then $\Bbbk G$ is a Hopf algebra with:

$$\begin{aligned} \Delta(e_g) &= e_g \otimes e_g, \\ \epsilon(e_g) &= 1, \\ \mathcal{S}(e_g) &= e_{g^{-1}} \end{aligned}$$

for every $g \in G$.

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$$\Delta(e_g) = e_g \otimes e_g, \quad \epsilon(e_g) = 1, \quad \mathcal{S}(e_g) = e_{g^{-1}}, \quad g \in G.$$

- If $\mathfrak{g}$ is a Lie algebra, then the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra via

$$\begin{aligned} \Delta(x) &= x \otimes 1 + 1 \otimes x, \\ \epsilon(x) &= 0, \\ \mathcal{S}(x) &= -x, \end{aligned}$$

for every $x \in \mathfrak{g}$.

Some invariants

Let H be a Hopf algebra

- The *coradical* H_0 of H is the sum of all simple sub-coalgebras of H.
- If $0 \neq h \in H$ satisfies

  $$\Delta(h) = h \otimes h,$$

  then h is said to be a *grouplike element*. The set of grouplike elements of H, $G(H)$, forms a group under the multiplication in H.

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- Let Γ be a group and assume $G(H) \cong \Gamma$.

  H is called *pointed* if H_0 is the group algebra of Γ.

Technical ingredients

- A *rack* $X = (X, \triangleright)$ is a pair $(X, \triangleright)$, where X is a non-empty set and $\triangleright : X \times X \to X$ is a function, such that

$$\phi_i = i \triangleright (\cdot) : X \to X \quad \text{is a bijection} \quad \forall\, i \in X, \quad \text{and}$$

$$i \triangleright (j \triangleright k) = (i \triangleright j) \triangleright (i \triangleright k),\ \forall i, j, k \in X.$$

- A *2-cocycle* q is a function $q : X \times X \to \mathbb{k}^*$, $(i, j) \mapsto q_{ij}$ such that

$$q_{i,j \triangleright k}\, q_{j,k} = q_{i \triangleright j, i \triangleright k}\, q_{i,k},\ \forall\, i, j, k \in X.$$

- Given (X, q), let $\mathcal{R}$ be the set of equivalence classes in $X \times X$ for the relation generated by $(i, j) \sim (i \triangleright j, i)$.
  Let $C \in \mathcal{R}$, $(i, j) \in C$. Take $i_1 = j$, $i_2 = i$, and recursively, $i_{h+2} = i_{h+1} \triangleright i_h$. Set $n(C) = \# C$ and

$$\mathcal{R}' = \Big\{ C \in \mathcal{R} \mid \prod_{h=1}^{n(C)} q_{i_{h+1}, i_h} = (-1)^{n(C)} \Big\}.$$

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- Let F be the free associative algebra in the variables $\{T_l\}_{l \in X}$.
  If $C \in \mathcal{R}'$, consider the quadratic polynomial

$$\phi_C = \sum_{h=1}^{n(C)} \eta_h(C)\, T_{i_{h+1}} T_{i_h} \in F,$$

  where $\eta_1(C) = 1$ and $\eta_h(C) = (-1)^{h+1} q_{i_2 i_1} q_{i_3 i_2} \dots q_{i_h i_{h-1}}$, $h \geq 2$.

The algebra $\mathcal{H}(\mathcal{Q})$

A *quadratic lifting datum*, or ql-datum, $\mathcal{Q}$ consists of

- a rack X,
- a 2-cocycle q,
- a finite group G,
- an action $\cdot : G \times X \to X$,
- a function $g : X \to G$,
- a family of 1-cocyles $(\chi_i)_{i \in X} : G \to \Bbbk$ (*i. e.* $\chi_i(ht) = \chi_i(t)\chi_{t \cdot i}(h)$, for all $i \in X$, $h, t \in G$),
- a collection $(\lambda_C)_{C \in \mathcal{R}'} \in \Bbbk$, $(\mathcal{R}' \subset X \times X)$

subject to a (non-trivial!) set of compatibilty axioms.

Given a ql-datum $\mathcal{Q}$, we define the algebra $\mathcal{H}(\mathcal{Q})$ by generators $\{a_i, H_t : i \in X,\ t \in G\}$ and relations:

$$H_e = 1, \quad H_t H_s = H_{ts},\ t, s \in G;$$
$$H_t a_i = \chi_i(t) a_{t \cdot i} H_t,\ t \in G,\ i \in X;$$
$$\phi_C(\{a_i\}_{i \in X}) = \lambda_C(1 - H_{g_i g_j}),\ C \in \mathcal{R}',\ (i,j) \in C.$$

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Recall that:

$$\phi_C(\{a_i\}_{i \in X}) = \sum_{h=1}^{n(C)} \eta_h(C)\, a_{i_{h+1}} a_{i_h}.$$

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- $\mathcal{H}(\mathcal{Q})$ is a pointed Hopf algebra if we define the elements H_t to be group-likes and the elements a_i to be $(H_{g_i}, 1)$-primitives.
- $G(\mathcal{H}(\mathcal{Q}))$ is a quotient of the group G. And thus any $\mathcal{H}(\mathcal{Q})$-module W is G-module $W_{|G}$, by restriction.

Example Let $\mathcal{Q}_\lambda$ be the ql-datum:

- $X = \mathcal{O}_2^3$ the rack over the conjugacy class of transpositions,
- $q \equiv -1$, that is $q_{ij} = -1 \, \forall \, i \in X$,
- $G = \mathbb{S}_3$,
- $\cdot : G \times X \to X$ the conjugation,
- $g : X \hookrightarrow G$ the inclusion,
- $\chi_i(t) = \operatorname{sgn}(t)$, $\forall \, i \in X, t \in G$,
- $\{\lambda_C\}_{C \in \mathcal{R}'} = \{0, \lambda\}$.

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Then $\mathcal{A}_\lambda = \mathcal{H}(\mathcal{Q}_\lambda)$ is the algebra presented by generators $\{a_i, H_r : i \in \mathcal{O}_2^3, r \in \mathbb{S}_3\}$ and relations:

$$H_e = 1, \quad H_r H_s = H_{rs}, \;\; r, s \in \mathbb{S}_3;$$
$$H_j a_i = -a_{jij} H_j, \;\; i, j \in \mathcal{O}_2^3;$$
$$a_{(12)}^2 = 0;$$
$$a_{(12)} a_{(23)} + a_{(23)} a_{(13)} + a_{(13)} a_{(12)} = \lambda(1 - H_{(12)} H_{(23)}).$$

- The algebras $\mathcal{A}_\lambda$ were introduced in (AG).
- $\mathcal{A}_\lambda$ is a Hopf algebra of dimension 72. If H is a finite-dimensional pointed Hopf algebra with $G(H) \cong \mathbb{S}_3$, then either $H \cong \Bbbk\mathbb{S}_3$, $H \cong \mathcal{A}_0$ or $H \cong \mathcal{A}_1$. This is Thm. 4.5 in (AHS) (together with (MS,AG,AZ)).
- The algebras $\mathcal{H}(\mathcal{Q})$ were introduced in (GG). They generalize the algebras $\mathcal{A}_\lambda$ and were used to classify pointed Hopf algebras over $\mathbb{S}_4$.

(AG) Andruskiewitsch, N. and Graña, M., *From racks to pointed Hopf algebras*, Adv. in Math. **178** (2), 177–243 (2003).

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(GG) García, G. A. and García Iglesias, A., *Pointed Hopf algebras over $\mathbb{S}_4$*. Israel Journal of Math. Accepted. Also available at arXiv:0904.2558v1 [math.QA]

$\mathcal{H}(\mathcal{Q})$**-modules over** G**-characters.**

- Let $\widehat{G}$ the set of irreducible representations of G.
- Let $G_{\mathrm{ab}} = G/[G,G]$, $\widehat{G_{\mathrm{ab}}} = \mathrm{Hom}(G, \Bbbk^*) \subseteq \widehat{G}$.
- If $\chi \in \widehat{G}$, and W is a G-module, we denote by $W[\chi]$ the isotypic component of type χ, and by W_χ the corresponding simple G-module.

Isotypical modules

Let $\rho \in \widehat{G_{ab}}$.

- There exists $\bar{\rho} \in \hom_{alg}(\mathcal{H}(\mathcal{Q}), \mathbb{k})$ such that $\bar{\rho}_{|G} = \rho$ if and only if

$$0 = \lambda_C(1 - \rho(g_i g_j)) \text{ if } (i,j) \in C \text{ and } 2|n(C), \qquad (1)$$

  and there exists a family $\{\gamma_i\}_{i \in X}$ of scalars such that

$$\gamma_j = \chi_j(t)\gamma_{t \cdot j} \qquad \forall\, t \in G, j \in X, \qquad (2)$$
$$\gamma_i\gamma_j = \lambda_C(1 - \rho(g_i g_j)) \quad \text{if } (i,j) \in C \text{ and } 2|n(C)+1. \qquad (3)$$

Assume X is indecomposable and let W be an $\mathcal{H}(\mathcal{Q})$-module such that $W = W[\rho]$ for a unique $\rho \in \widehat{G_{ab}}$, $\dim W = n$.

- W is simple if and only if $n = 1$. If, in addition,

$$\chi_i(g_i) \neq 1, \quad \forall\, i \in X,$$

  then $W \cong S_\rho^{\oplus n}$.

Extensions

Let V be the space of solutions $\{f_k\}_{k\in X} \in \Bbbk^X$ of the following system, $i \in X$, $t \in G$, $C \in \mathcal{R}'$, $(i,j) \in C$,

$$\begin{cases} f_i\mu(t) = \chi_i(t) f_{t\cdot i}\rho(t), \\ (\alpha_j(C)\delta_j - \beta_j(C)\gamma_j) f_i = -\chi_i(g_i)(\alpha_i(C)\delta_i - \beta_i(C)\gamma_i) f_j \end{cases}$$

- Then $\mathrm{Ext}^1_{\mathcal{H}(\mathcal{Q})}(S^\gamma_\rho, S^\delta_\mu) \cong V$ and the set of isomorphism classes of indecomposable $\mathcal{H}(\mathcal{Q})$-modules such that

$$0 \longrightarrow S^\delta_\mu \longrightarrow W \longrightarrow S^\gamma_\rho \longrightarrow 0 \text{ is exact}$$

is in bijective correspondence with $\mathbb{P}_k(V)$.

Sums of two isotypical components

Let $\rho \neq \mu \in \widehat{G_{ab}}$.

Assume X is indecomposable and $\chi_i(g_i) = -1$, $i \in X$.
Assume further that $\exists\, C \in \mathcal{R}'$ with $n(C) > 1$.

Let $W = W[\rho] \oplus W[\mu]$ be an $\mathcal{H}(\mathcal{Q})$-module.

- Then W is a direct sum of modules of the form S_{ρ}^{γ}, S_{μ}^{δ}, $W_{\rho,\mu}^{\gamma',\delta'}$ and $W_{\mu,\rho,}^{\delta'',\gamma''}$ for various γ, δ, γ', δ', γ'', δ''.

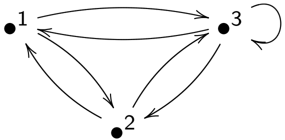
Simple $\mathbb{k}\mathbb{S}_3$-modules

- There are 3 simple $\mathbb{k}\mathbb{S}_3$-modules, namely
  1. $W_\epsilon = \mathbb{k}u$, the trivial representation, $t \cdot u = u$, $t \in \mathbb{S}_3$;
  2. $W_{\mathrm{sgn}} = \mathbb{k}z$, the sign representation, $t \cdot z = \mathrm{sgn}(t)z$, $t \in \mathbb{S}_3$;
  3. $W_{\mathrm{st}} = \mathbb{k}\{v, w\}$, the standard representation, given by

$$[(12)] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad [(23)] = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Representations of $\mathcal{A}_0$

- There are exactly three simple $\mathcal{A}_0$-modules, namely the extensions S_ϵ, S_{sgn} and S_{st} of the simple $\Bbbk\mathbb{S}_3$-modules, where a_i acts trivially, for $i \in \mathcal{O}_2^3$.
- The fusion rules for these modules coincide with those of the underlying $\Bbbk\mathbb{S}_3$-modules.
- $\mathcal{A}_0$ is of wild representation type. Its Ext-Quiver is



where we have ordered the simple modules as $\{S_\epsilon, S_{\mathrm{sgn}}, S_{\mathrm{st}}\} = \{1, 2, 3\}$.

- The projective covers of the modules S_ϵ, S_{sgn} and S_{st} have dimensions $12, 12$ and 24, respectively.

Representations of $\mathcal{A}_1$

- There are exactly six simple $\mathcal{A}_1$-modules, namely the extensions S_ϵ, S_{sgn}, and $S_{\mathrm{st}}(\mathrm{i}), S_{\mathrm{st}}(-\mathrm{i}), S_{\mathrm{st}}(\frac{\mathrm{i}}{3}), S_{\mathrm{st}}(-\frac{\mathrm{i}}{3})$.
  - These last four modules are supported on $W_{\mathrm{st}} = \Bbbk\{v, w\}$ and defined, respectively, by

$$
\begin{aligned}
a_{12}v &= \mathrm{i}(v - w), & a_{12}w &= \mathrm{i}(v - w);\\
a_{12}v &= -\mathrm{i}(v - w), & a_{12}w &= -\mathrm{i}(v - w);\\
a_{12}v &= \frac{\mathrm{i}}{3}(v + w), & a_{12}w &= -\frac{\mathrm{i}}{3}(v + w);\\
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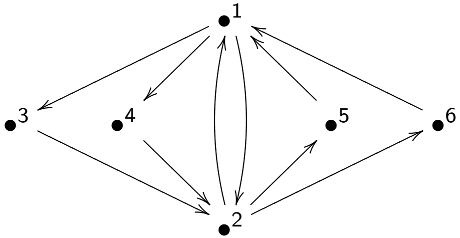
\end{aligned}
$$

- $\mathcal{A}_1$ is not of finite representation type. The Ext-Quiver of $\mathcal{A}_1$ is



  for $\{S_\epsilon, S_{\mathsf{sgn}}, S_{\mathsf{st}}(\mathrm{i}), S_{\mathsf{st}}(-\mathrm{i}), S_{\mathsf{st}}(\frac{\mathrm{i}}{3}), S_{\mathsf{st}}(-\frac{\mathrm{i}}{3})\} = \{1, 2, 3, 4, 5, 6\}$.
- $\mathcal{A}_1$ is not quasitriangular.
- The projective covers of the modules S_ϵ, S_{sgn} and $S_{\mathsf{st}}(\theta)$, $\theta \in \{\pm\mathrm{i}, \pm\frac{\mathrm{i}}{3}\}$ have dimensions $12, 12$ and 6, respectively.

(More) References

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