Minimal Prime Ideals of Ore Extensions over Commutative Dedekind Domains

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Background

- Various linear systems can be defined by means of matrices with entries in non commutative algebras of functional operators. An important class of such algebras is Ore extensions.

- Irving and Leroy-Matczuk consider primes of Ore extensions over commutative Noetherian rings.

- Chin, Ferrero-Matczuk, Passman consider prime ideals of Ore extensions of derivation type.

- Amir-Marubayashi-Wang consider minimal prime ideals minimal prime rings of Ore extensions of derivation type.

Aim: To extend the result of Amir-Marubayashi-Wang to general Ore extensions of automorphism type, in order to study the structure of the corresponding factor rings.

Definitions

A **(left) skew derivation** on a ring D is a pair (σ, δ) where σ is a ring endomorphism of D and δ is a **(left) σ-derivation** on D; that is, an additive map from D to itself such that

$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b \text{ for all } a, b \in D.$$

Let D be a ring with identity 1 and (σ, δ) be a (left) skew derivation on the ring D.

The Ore Extension $D[x; \sigma, \delta]$ over D with respect to the skew derivation (σ, δ) is the ring consisting of all polynomials over D with an indeterminate x, $D[x; \sigma, \delta] = \{f(x) = a_n x^n + \cdots + a_0 : a_i \in D\}$ satisfying the following equation:

$$xa = \sigma(a)x + \delta(a) \text{ for all } a \in D.$$

Example

Let k be the real or complex numbers $\mathbb{R}$ or $\mathbb{C}$. The *Weyl Algebra* $A(k)$ consists of all differential operators in x with polynomial coefficients

$$f_n(x)\partial_x^n + \cdots + f_1(x)\partial_x + f_0(x).$$

Let's write $y = d/dx$. What should xy-yx be?

Apply this operator to x^n.

$$xy\ (x^n) = x.\ d/dx(x^n) = nx^n.$$

$$yx\ (x^n) = d/dx(x^{n+1}) = (n+1)x^n.$$

So $xy - yx(x^n) = x^n$ again. That is $xy - yx$ is the identity operator or $xy - yx = 1$.

Definition Let Σ be a set of map from the ring D to itself (e.g. $\Sigma = \{\sigma\}, \Sigma = \{\delta\}$ or $\Sigma = \{\sigma, \delta\}$). A **$\Sigma$-ideal** of D is any ideal I of D such that $\alpha(I) \subseteq I$ for all $\alpha \in \Sigma$.
A **Σ-prime ideal** is any proper Σ-ideal I such that whenever J, K are Σ-ideals satisfying $JK \subseteq I$, then either $J \subseteq I$ or $K \subseteq I$.

Teorema 1 (Amir-Marubayashi-Wang) *Let $R = D[x, \sigma]$ be a skew polynomial ring over a commutative Dedekind domain D, where σ is an automorphism of D and let P be a prime ideal of R. Then*

1. P is a minimal prime ideal of R if and only if either $P = \mathfrak{p}[x; \sigma]$, where $\mathfrak{p}$ is either a non-zero σ-prime ideal of D or $P \in Spec_0(R)$ with $P \neq (0)$.

2. If $P = \mathfrak{p}[x; \sigma]$, where $\mathfrak{p}$ is a non-zero σ-prime ideal of D, then R/P is a hereditary prime ring. In particular, R/P is a Dedekind prime ring if and only if $\mathfrak{p} \in Spec(D)$.

3. If $P \in Spec_0(R)$ with $P = xR$, then R/P is a Dedekind prime ring. If the order of σ is infinite, then $P = xR$ is the only minimal prime ideal belonging to $Spec_0(R)$.

4. If $P \in Spec_0(R)$ with $P \neq xR$ and $P \neq (0)$, then R/P is a hereditary prime ring if and only if P is not a subset of M^2 for any maximal ideal M of R.

Setting let D be a commutative Dedekind domain and $R = D[x;\sigma,\delta]$ be the Ore extension over D, for (σ,δ) is a skew derivation, $\sigma \neq 1$ is an automorphism of D and $\delta \neq 0$.

Teorema 2 (Goodearl) *If $\mathfrak{p}$ is any ideal of D which is (σ,δ)-prime, then $\mathfrak{p} = P \cap R$ for some prime ideal P of R and more specially $\mathfrak{p}R \in$ Spec(R) where $Spec(R)$ denotes the set of all prime ideal in R.*

Lema 3 *If $P = \mathfrak{p}[x;\sigma,\delta]$ is a minimal prime ideal of R where $\mathfrak{p}$ is a (σ,δ)-prime ideal of D, then $\mathfrak{p}$ is a minimal (σ,δ)-prime ideal of D.*

Result

Teorema 4 *Let P be a prime ideal of R and $P \cap D = \mathfrak{p} \neq (0)$. Then P is a minimal prime ideal of R if and only if either $P = \mathfrak{p}[x;\sigma,\delta]$ where $\mathfrak{p}$ is a minimal (σ,δ)-prime ideal of D or* (0) *is the largest (σ,δ)-ideal of D in $\mathfrak{p}$.*

Proof

$\Rightarrow$ By [Goodearl, Theorem 3.1], there are two cases:

Case 1: $\mathfrak{p}$ is a (σ, δ)-prime ideal of D. Then $\mathfrak{p}R \in \text{Spec}(R)$ ([Goodearl, Theorem 3.1]). So, $\mathfrak{p}R = P$ because $\mathfrak{p}R \subseteq P$ and P is a minimal prime ideal. Since $\mathfrak{p}R = \mathfrak{p}[x; \sigma, \delta]$, then $P = \mathfrak{p}[x; \sigma, \delta]$ and $\mathfrak{p}$ is a minimal (σ, δ)-prime ideal of D, by Lemma 3.

Case 2: $\mathfrak{p}$ is a prime ideal of D and $\sigma(\mathfrak{p}) \neq \mathfrak{p}$. Let $\mathfrak{m}$ be the largest (σ, δ)-ideal contained in $\mathfrak{p}$ and assume that $\mathfrak{m} \neq (0)$. Then by primeness of $\mathfrak{p}$ it can be shown that $\mathfrak{m}$ is a (σ, δ)-prime ideal of D. So, $\mathfrak{m}R$ is a prime ideal of R ([Goodearl, Proposition 3.3]). On the other hand, since $\sigma(\mathfrak{p}) \neq \mathfrak{p}$, we have $\mathfrak{m} \subsetneq \mathfrak{p}$. So, $\mathfrak{m}R \subsetneq \mathfrak{p}R \subseteq P$, i.e, P is not a minimal prime. This is a contradiction. So, (0) is the largest (σ, δ)-ideal of D in $\mathfrak{p}$.

$\Leftarrow$

For the case $P = \mathfrak{p}[x;\sigma,\delta]$, where $\mathfrak{p}$ is a minimal (σ,δ)-prime ideal of D, by [Goodearl, Theorem 3.3], $P = \mathfrak{p}[x;\sigma,\delta]$ is a prime ideal of R. Let Q be a prime ideal of R where $Q \subseteq P$. Set $\mathfrak{q} = Q \cap D$, then $\mathfrak{q} = Q \cap D \subseteq P \cap D = \mathfrak{p}$. By [Goodearl, Theorem 3.1] we have two cases

Case 1: $\mathfrak{q}$ is a (σ,δ)-prime ideal of D. Suppose $\mathfrak{q}$ is a (σ,δ)-prime ideal of D. Then $\mathfrak{q} = \mathfrak{p}$ because $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is a minimal (σ,δ)-prime ideal of D. So, $P = \mathfrak{p}[x;\sigma,\delta] = \mathfrak{q}[x;\sigma,\delta] \subseteq Q$. This implies $P = Q$.

Case 2: $\mathfrak{q}$ is a prime ideal of D. Then $\mathfrak{q} = \mathfrak{p}$ because D is a Dedekind domain. So, $P = \mathfrak{p}[x;\sigma,\delta] = \mathfrak{q}[x;\sigma,\delta] \subseteq Q$. This implies $P = Q$.

For the case (0) is the largest (σ, δ)-ideal of D in $\mathfrak{p}$, let Q be a prime nonzero ideal of R satisfying $Q \subseteq P$. Set $\mathfrak{q} = Q \cap D$, then $\mathfrak{q} = Q \cap D \subseteq P \cap D = \mathfrak{p}$. We have two cases:

1. $\mathfrak{q}$ is a (σ, δ)-prime ideal of D. But if this happens, because of (0) being the largest (σ, δ)-ideal of D in $\mathfrak{p}$, $\mathfrak{q} = (0)$ implying a contradiction $Q \cap D = 0$. (see [Goodearl-Warfield, Lemma 2.19]

2. $\mathfrak{q}$ is a prime ideal of D with $\sigma(\mathfrak{q}) \neq \mathfrak{q}$. Then $\mathfrak{q} = \mathfrak{p}$. So $Q \cap D = P \cap D$, which, according to [Goodearl-Warfield, Proposition 3.5], implies $Q = P$. Thus P is the minimal prime ideal of R. QED

Reseach on going: Structure of factor rings (generalization of Theorem 1).

References

W. Chin, *Prime ideals in differential operator rings and crossed product of infinite groups*, J. Algebra 106 (1987), 78-104.

M. Ferrero, J. Matczuk, *Prime ideals in skew polinomial rings of derivation type*, Comm. Algebra 18 (3) (1990), 689-710.

K.R. Goodearl, *Prime ideals in skew polinomial ring and quantized Weyl algebras*, J. of Algebra 150, (1992), 324-377

K.R. Goodearl, R.B. Warfield, JR *An Introduction to Noncommutative Noetherian rings*, London Mathematical Society Student Text, 16 (1989).

R.S. Irving, *Prime ideals of Ore extension over commutative rings* , J. Algebra 56 (1979), 315-342

A. Leroy, J. Matczuk, *The extended centeroid and X-inner automorphism of Ore extensions*, J. Algebra 145 (1992), 143-177.

A.K. Amir,H. Marubayashi, Y. Wang, *Prime factor rings of skew polynomial rings over a commutative Dedekind domain*, (submitted).

D.S. Passman, *Prime ideals in enveloping rings*, Trans. Amer. Math. Soc. 302(2) (1987), 535-560.