DERIVED INVARIANCE OF THE TAMARKIN–TSYGAN CALCULUS OF AN ALGEBRA

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Abstract. We prove that derived equivalent algebras have isomorphic differential calculi in the sense of Tamarkin–Tsygan.

1. Introduction

Let k be a commutative ring and A an associative k-algebra projective as a module over k. We write $\otimes$ for the tensor product over k. We point out that all the constructions and proofs of this paper extend to small dg categories cofibrant over k. The Hochschild homology $HH_\bullet(A)$ and cohomology $HH^\bullet(A)$ are derived invariants of A, see [3, 4, 9, 10, 12]. Moreover, these k-modules come with operations, namely the cup product

$$\cup : HH^n(A) \otimes HH^m(A) \to HH^{n+m}(A),$$

the Gerstenhaber bracket

$$[-,-] : HH^n(A) \otimes HH^m(A) \to HH^{n+m-1}(A),$$

the cap product

$$\cap : HH_n(A) \otimes HH^m(A) \to HH_{n-m}(A)$$

and Connes' differential

$$B : HH_n(A) \to HH_{n+1}(A),$$

such that $B^2 = 0$ and

$$[Bi_\alpha - (-1)^{|\alpha|} i_\alpha B, i_\beta] = i_{[\alpha,\beta]}, \tag{1}$$

where $i_\alpha(z) = (-1)^{|\alpha||z|} z \cap \alpha$. This is the first example [2, 11] of a *differential calculus* or a *Tamarkin-Tsygan calculus*, which is by definition a collection

$$(\mathcal{H}^\bullet, \cup, [-,-], \mathcal{H}_\bullet, \cap, B),$$

such that $(\mathcal{H}^\bullet, \cup, [-,-])$ is a Gerstenhaber algebra, the cap product $\cap$ endows $\mathcal{H}_\bullet$ with the structure of a graded Lie module over the Lie algebra $(\mathcal{H}^\bullet[1], \cup, [-,-])$ and the map $B : \mathcal{H}_n \to \mathcal{H}_{n+1}$ squares to zero and satisfies the equation (1). The Gerstenhaber algebra $(HH^\bullet(A), \cup, [-,-])$ has been proved to be a derived invariant [8, 6]. The cap product is also a derived invariant [1]. In this work, we use an isomorphism induced from the cyclic

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functor [7] to prove derived invariance of Connes' differential and of the ISB-sequence. To obtain derived invariance of the differential calculus, we need to prove that this isomorphism equals the isomorphism between Hochschild homologies used in [1] to prove derived invariance of the cap product.

2. The cyclic functor

Let **Alg** be the category whose objects are the associative dg (=differential graded) k-algebras cofibrant over k (i.e. 'closed' in the sense of section 7.5 of [7]) and whose morphisms are morphisms of dg k-algebras which do not necessarily preserve the unit. Let $\mathrm{rep}(A,B)$ be the full subcategory of the derived category $D(A^{op}\otimes B)$ whose objects are the dg bimodules X such that the restriction X_B is compact in $D(B)$, i.e. lies in the thick subcategory generated by the free module B_B. Define **ALG** to be the category whose objects are those of **Alg** and whose morphisms from A to B are the isomorphism classes in $\mathrm{rep}(A,B)$. The composition of morphisms in **ALG** is given by the total derived tensor product [7]. The identity of A is the isomorphism class of the bimodule ${}_AA_A$. There is a canonical functor $\mathbf{Alg}\to\mathbf{ALG}$ that associates to a morphism $f:A\to B$ the bimodule ${}_fB_B$ with underlying space $f(1)B$ and A-B-action given by $a.f(1)b.b' = f(a)bb'$.

Let Λ be the dg algebra $k[\epsilon]/(\epsilon^2)$ where $|\epsilon| = -1$ and the differential vanishes. As in [5, 7], we will identify the category of dg Λ-modules with the category of mixed complexes. Denote by DMix the derived category of dg Λ-modules. Let $C:\mathbf{Alg}\to\mathrm{DMix}$ be the *cyclic functor* [7], that is, the underlying dg k-module of $C(A)$ is the mapping cone over $(1-t)$ viewed as a morphism of complexes $(A^{\otimes *+1}, b')\to(A^{\otimes *}, b)$ and the first and second differentials of the mixed complex $C(A)$ are

$$\begin{bmatrix} b & 1-t \\ 0 & -b' \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & 0 \\ N & 0 \end{bmatrix}.$$

Clearly, a dg algebra morphism $f:A\to B$ (even if it does not preserve the unit) induces a morphism $C(f):C(A)\to C(B)$ of dg Λ-modules. Let X be an object of $\mathrm{rep}(A,B)$. We assume, as we may, that X is cofibrant (i.e. 'closed' in the sense of section 7.5 of [7]). This implies that X_B is cofibrant as a dg B-module and thus that morphism spaces in the derived category with source X_B are isomorphic to the corresponding morphism spaces in the homotopy category. Consider the morphisms

$$A \xrightarrow{\alpha_X} \mathrm{End}_B(B\oplus X) \xleftarrow{\beta_X} B$$

where $\mathrm{End}_B(B\oplus X)$ is the differential graded endomorphism algebra of $B\oplus X$, the morphism α_X be given by the left action of A on X and β_X is induced by the left action of B on B. Note that these morphisms do not preserve the units. The second author proved in [7] that $C(\beta_X)$ is invertible in DMix and defined $C(X) = C(\beta_X)^{-1} \circ C(\alpha_X)$. We recall that C is well defined on **ALG** and that this extension of C from **Alg** to **ALG** is unique by Theorem 2.4 of [7].

Let $X : A \to B$ be a morphism of **ALG** where X is cofibrant. Put $X^\vee = \mathrm{Hom}_B(X, B)$. We can choose morphisms $u_X : A \to X \overset{\mathbf{L}}{\otimes}_B X^\vee$ and $v_X : X^\vee \overset{\mathbf{L}}{\otimes}_B X \to B$ such that the following triangles commute

$$\begin{array}{ccc} X & \xrightarrow{u_X \otimes 1} & X \overset{\mathbf{L}}{\otimes}_B X^\vee \overset{\mathbf{L}}{\otimes}_A X \\ & \searrow^{=} & \downarrow{\scriptstyle 1\otimes v_X} \\ & & X \end{array} \qquad \begin{array}{ccc} X^\vee & \xrightarrow{1 \otimes u_X} & X^\vee \overset{\mathbf{L}}{\otimes}_A X \overset{\mathbf{L}}{\otimes}_B X^\vee \\ & \searrow^{=} & \downarrow{\scriptstyle v_X\otimes 1} \\ & & X^\vee. \end{array}$$

Then the functors

$$? \overset{\mathbf{L}}{\otimes}_{A^e} (X \otimes X^\vee) : D(A^e) \to D(B^e)$$

and

$$? \overset{\mathbf{L}}{\otimes}_{B^e} (X^\vee \otimes X) : D(B^e) \to D(A^e)$$

form an adjoint pair. We will identify $X \overset{\mathbf{L}}{\otimes}_B X^\vee \xrightarrow{\sim} (X \otimes X^\vee) \overset{\mathbf{L}}{\otimes}_{B^e} B$ and $X^\vee \overset{\mathbf{L}}{\otimes}_A X \xrightarrow{\sim} (X^\vee \otimes X) \overset{\mathbf{L}}{\otimes}_{A^e} A$, and still call u_X and v_X the same morphisms when composed with this identification. Since k is a commutative ring, the tensor product over k is symmetric. We will denote the symmetry isomorphism by τ. Let $D(k)$ denote the derived category of k-modules. We define a functor $\psi : \mathbf{Alg} \to D(k)$ by putting $\psi(A) = A \overset{\mathbf{L}}{\otimes}_{A^e} A$, and $\psi(f) = f \otimes f$ for a morphism $f : A \to B$. There is a canonical quasi-isomorphism $\psi(A) \to \varphi(A)$ for any algebra A, where $\varphi(A)$ is the underlying complex of $C(A)$. Therefore, the functors φ and ψ take isomorphic values on objects. We now define ψ on morphisms of **ALG** as follows: Let X be a cofibrant object of $\mathrm{rep}(A, B)$. Define $\psi(X)$ to be the composition

$$\begin{array}{rcl} A \overset{\mathbf{L}}{\otimes}_{A^e} A & \to & A \overset{\mathbf{L}}{\otimes}_{A^e} X \otimes X^\vee \overset{\mathbf{L}}{\otimes}_{B^e} B \\ & \xrightarrow{\sim} & B \overset{\mathbf{L}}{\otimes}_{B^e} X^\vee \otimes X \overset{\mathbf{L}}{\otimes}_{A^e} A \\ & \to & B \overset{\mathbf{L}}{\otimes}_{B^e} B. \end{array}$$

That is, we put $\psi(X) = (1 \otimes v_X) \circ \tau \circ (1 \otimes u_X)$.

Theorem 2.1. *The assignments $A \mapsto \psi(A)$, $X \mapsto \psi(X)$ define a functor on* $\boldsymbol{ALG}$ *that extends the functor $\varphi : \boldsymbol{Alg} \to D(k)$.*

Corollary 2.2. *The functors φ and $\psi : \boldsymbol{ALG} \to D(k)$ are isomorphic.*

Proof of the Corollary. This is immediate from Theorem 2.4 of [7] and the remark following it. □

Proof of the Theorem. For ease of notation, we write $\otimes$ and Hom instead of $\overset{\mathbf{L}}{\otimes}$ and RHom. Let $f : A \to B$ be a morphism of **Alg**. The associated morphism in **ALG** is $X = {}_fB_B$. Note that $X^\vee = {}_BB_f$. The diagrams

$$\begin{array}{ccc} A \otimes_{A^e} ({}_fB \otimes_B B_f) & & \\ \simeq \downarrow & \searrow^{\simeq} & \\ A \otimes_{A^e} ({}_fB \otimes B_f) \otimes_{B^e} B & \xrightarrow[\simeq]{} & A \otimes_{A^e} {}_fB_f \end{array}$$

and

$$\begin{array}{ccc} A \otimes_{A^e} ({}_fB \otimes B_f) \otimes_{B^e} B & \xrightarrow[\simeq]{} & A \otimes_{A^e} {}_fB_f \\ \tau \downarrow & & \downarrow \tau \\ B \otimes_{B^e} B_f \otimes {}_fB \otimes_{A^e} A & \xrightarrow{\simeq} & {}_fB_f \otimes_{A^e} A \end{array}$$

are commutative. Since

$$\begin{array}{ccc} {}_fB_f \otimes_{A^e} A & \xrightarrow{\tau} & A \otimes_{A^e} {}_fB_f \\ 1\otimes f \downarrow & & \downarrow f\otimes 1 \\ B \otimes_{B^e} B & \xrightarrow{\tau} & B \otimes_{B^e} B \end{array}$$

is also commutative and the bottom morphism equals the identity, we get that $\psi({}_fB_B)$ is the morphism induced by f from $A \otimes_{A^e} A$ to $B \otimes_{B^e} B$. Therefore $\psi({}_fB_B) = \varphi({}_fB_B)$. Let $X : A \to B$ and $Y : B \to C$ be morphisms in **ALG**. We have canonical isomorphisms

$$\begin{aligned} \mathrm{Hom}_C(Y, C) \otimes_B \mathrm{Hom}_B(X, B) &\xrightarrow{\sim} \mathrm{Hom}_B(X, \mathrm{Hom}_C(Y, C)) \\ &\xrightarrow{\sim} \mathrm{Hom}_C(X \otimes_B Y, C). \end{aligned}$$

Whence the identification

$$(X \otimes_B Y)^\vee = Y^\vee \otimes_B X^\vee.$$

Put $Z = X \otimes_B Y$. For u_Z, we choose the composition

$$A \xrightarrow{u_X} X \otimes_B X^\vee \xrightarrow{1\otimes u_Y \otimes 1} X \otimes_B Y \otimes_C \otimes Y^\vee \otimes_B X^\vee$$

and for v_Z the composition

$$(Y^\vee \otimes_B X^\vee) \otimes_A (X \otimes_B Y) \xrightarrow{1\otimes v_X \otimes 1} Y^\vee \otimes_B Y \xrightarrow{v_Y} C\ .$$

By definition, the composition $\psi(Y) \circ \psi(X)$ is the composition of $(1 \otimes v_Y) \circ \tau \circ (1 \otimes u_Y)$ with $(1 \otimes v_X) \circ \tau \circ (1 \otimes u_X)$. We first examine the composition $(1 \otimes u_Y) \circ (1 \otimes v_X)$:

$$B \otimes_{B^e} (X^\vee \otimes X) \otimes_{A^e} A \xrightarrow{1\otimes v_X} B \otimes_{B^e} B \xrightarrow{1\otimes u_Y} B \otimes_{B^e} (Y \otimes Y^\vee) \otimes_{C^e} C$$

Clearly, the following square is commutative

$$\begin{array}{ccc} B \otimes_{B^e} (X^\vee \otimes X) \otimes_{A^e} A & \xrightarrow{c} & ((X^\vee \otimes X) \otimes_{A^e} A) \otimes_{B^e} B \\ \downarrow{\scriptstyle 1\otimes v_X} & & \downarrow{\scriptstyle v_X \otimes 1} \\ B \otimes_{B^e} B & \xrightarrow{\tau} & B \otimes_{B^e} B \end{array},$$

where c is the obvious cyclic permutation. Notice that

$$\tau : B \otimes_{B^e} B \to B \otimes_{B^e} B$$

equals the identity. Thus, we have $1 \otimes u_Y = (1 \otimes u_Y) \circ \tau$ and

$$(1 \otimes u_Y) \circ (1 \otimes v_X) = (1 \otimes u_Y) \circ \tau \circ (1 \otimes v_X) = (1 \otimes u_Y) \circ (v_X \otimes 1) \circ c.$$

Let σ

$$((X^\vee \otimes X) \otimes_{A^e} A) \otimes_{B^e} (Y \otimes Y^\vee) \otimes_{C^e} C \xrightarrow{\sim} A \otimes_{A^e} (X \otimes_B Y) \otimes (Y^\vee \otimes_B X^\vee) \otimes_{C^e} C$$

be the natural isomorphism given by reordering the factors. Then we have $\psi(Y) \circ \psi(X) = f \circ g$, where $f = \sigma \circ (1 \otimes u_Y) \circ c \circ \tau \circ (1 \otimes u_X)$ and $g = (v_Y \otimes 1) \circ \tau \circ (v_X \otimes 1) \circ \sigma^{-1}$. It is not hard to see that f equals $1 \otimes u_Z$ and g equals $(1 \otimes v_Z) \circ \tau$. Intuitively, the reason is that given the available data, there is only one way to go from $A \otimes_{A^e} A$ to

$$A \otimes_{A^e} (X \otimes_B Y) \otimes (Y^\vee \otimes_B X^\vee) \otimes_{C^e} C$$

and only one way to go from here to $C \otimes_{C^e} C$. It follows that $\psi(Y) \circ \psi(X) = \psi(Z)$. □

3. Derived invariance

Let A and B be derived equivalent algebras and X a cofibrant object of $\mathrm{rep}(A, B)$ such that $? \overset{\mathbf{L}}{\otimes}_A X : D(A) \to D(B)$ is an equivalence. Then $C(X)$ is an isomorphism of DMix and $\varphi(X)$ an isomorphism of $D(k)$. There is a canonical short exact sequence of dg Λ-modules

$$0 \to k[1] \to \Lambda \to k \to 0$$

giving rise to a triangle

$$k[1] \xrightarrow{B'} \Lambda \xrightarrow{I} k \xrightarrow{S} k[1] .$$

We apply the isomorphism of functors $? \overset{\mathbf{L}}{\otimes}_\Lambda C(A) \xrightarrow{\sim} ? \overset{\mathbf{L}}{\otimes}_\Lambda C(B)$ to this triangle to get an isomorphism of triangles in $D(k)$, where we recall that $\varphi(A)$ is the underlying complex of $C(A)$

$$\begin{array}{ccccccc} k[1] \overset{\mathbf{L}}{\otimes}_\Lambda C(A) & \xrightarrow{B'} & \varphi(A) & \xrightarrow{I} & k \overset{\mathbf{L}}{\otimes}_\Lambda C(A) & \xrightarrow{S} & k[2] \overset{\mathbf{L}}{\otimes}_\Lambda C(A) \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\varphi(X)} & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ k[1] \overset{\mathbf{L}}{\otimes}_\Lambda C(B) & \xrightarrow{B'} & \varphi(B) & \xrightarrow{I} & k \overset{\mathbf{L}}{\otimes}_\Lambda C(B) & \xrightarrow{S} & k[2] \overset{\mathbf{L}}{\otimes}_\Lambda C(B) \end{array}$$

Taking homology and identifying $H_j(k \overset{\mathbf{L}}{\otimes}_\Lambda C(A)) = HC_j(A)$ as in [5], gives an isomorphism of the ISB-sequences of A and B,

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & HC_{n-1}(A) & \xrightarrow{B'_{n-1}} & HH_n(A) & \xrightarrow{I_n} & HC_n(A) & \xrightarrow{S_n} & HC_{n-2}(A) & \longrightarrow \cdots \\
& \cong\downarrow & & \downarrow HH_n(X) & & \cong\downarrow & & \cong\downarrow & \\
\cdots \longrightarrow & HC_{n-1}(B) & \xrightarrow{B'_{n-1}} & HH_n(B) & \xrightarrow{I_n} & HC_n(B) & \xrightarrow{S_n} & HC_{n-2}(B) & \longrightarrow \cdots
\end{array},$$

where $HH_n(X)$ is the map induced by $\varphi(X)$. In terms of the differential calculus, Connes' differential is the map

$$B_n : HH_n(A) \to HH_{n+1}(A),$$

given by $B_n = B'_n I_n$. This shows that B_n is derived invariant via $HH_n(X)$. By Theorem 2.1, the map $HH_n(X)$ is equal to the map induced by $\psi(X)$ used in the proof of the derived invariance of the cap product [1]. Therefore, we get the following

Theorem 3.1. *The differential calculus of an algebra is a derived invariant.*

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