ON LECLERC'S CONJECTURAL CLUSTER STRUCTURES FOR OPEN RICHARDSON VARIETIES

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Abstract. In 2016, Leclerc constructed conjectural cluster structures on open Richardson varieties using representations of preprojective algebras. A variant with more explicit seeds was obtained by Ménard in his thesis. We show that Ménard's seeds do yield *upper* cluster algebra structures on open Richardson varieties and discuss the problems that remain in order to prove that they are cluster algebra structures.

1. Introduction

Open Richardson varieties were introduced by Kazhdan–Lusztig [20]. They are relevant for Kazhdan–Lusztig polynomials [20, 7], the study of total positivity in the Grassmannian [24, 34, 26, 35], the Poisson geometry of the flag variety [17] and many other subjects.

It is natural to ask whether open Richardson varieties carry cluster structures compatible with total positivity and Poisson geometry. In 2016, Leclerc [23] gave a conjectural positive answer using representations of preprojective algebras. His conjecture was slightly modified and made more explicit by Ménard [27, 28]. In this note, based on [21], we show that Ménard's seed does provide an *upper* cluster algebra structure on each open Richardson variety. We also discuss the problems that remain in order to prove that it is a cluster algebra structure.

In type A, a (possibly different) upper cluster algebra structure was obtained using completely different methods by Gracie Ingermanson in her thesis [19] under the supervision of David Speyer. Open Richardson varieties are special cases of braid varieties. In this more general framework, much stronger results will be contained in the forthcoming work of two groups of mathematicians:

- Roger Casals, Eugene Gorsky, Mikhail Gorsky, Ian Le, Linhui Shen, and José Simental in [4] and
- Pavel Galashin, Thomas Lam and Melissa Sherman-Bennett in [11, 12], cf. also [10].

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2. Open Richardson varieties

Let Δ be a simply laced Dynkin diagram, for example the diagram A_4 given by a chain of length 4. Let G be the associated simple, simply connected complex algebraic group, for example $Sl_5(\mathbb{C})$. Let $B, B^- \subseteq G$ be opposite Borel subgroups, for example the subgroups of upper/lower triangular matrices in $Sl_5(\mathbb{C})$. Let $H = B \cap B^-$ be the associated maximal torus and $W = N_B(H)/H$ the Weyl group, for example the subgroup $H \subseteq Sl_5(\mathbb{C})$ of

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diagonal matrices and the symmetric group S_5. Let $X = B^- \setminus G$ be the flag variety and $\pi : G \to X$ the canonical projection, for example the variety of complete flags

$$0 = F_0 \subset F_1 \subset \cdots \subset F_5 = \mathbb{C}^5$$

in the space of rows $\mathbb{C}^5$ with its natural right action by $Sl_5(\mathbb{C})$. We have the Schubert decomposition

$$X = \coprod_{w \in W} C_w$$

into the Schubert cells $C_w = \pi(wB^-)$, which are affine spaces of dimension equal to the length $l(w)$. Dually, we have the opposite Schubert decomposition

$$X = \coprod_{v \in W} C^v$$

into the opposite Schubert cells $C^v = \pi(vB)$, which are affine spaces of dimension equal to $l(w_0) - l(v)$, where w_0 is the longest element of W. For a pair of Weyl group elements (v, w), the *open Richardson variety* $\mathcal{R}_{v,w}$ is defined as the intersection $C^v \cap C_w$. It is non-empty if and only if $v \leq w$ for the Bruhat order and in this case, it is a smooth, irreducible, locally closed subvariety of C_w of dimension $l(w) - l(v)$ which is affine but not an affine space, in general. In the minimal example where $G = Sl_2(\mathbb{C})$, the flag variety $X = B^- \setminus G$ identifies with the projective line $\mathbb{P}^1(\mathbb{C})$ by the map sending a 2×2-matrix to the line generated by its first row. Under this identification, if s generates the Weyl group S_2, we have

$$\begin{array}{ccc} C_e = \{0\} & C_s = \mathbb{P}^1(\mathbb{C}) \setminus \{0\} & \\ C^e = \mathbb{A}^1(\mathbb{C}) & C^s = \{\infty\} & \\ \mathcal{R}_{e,e} = \{0\} & \mathcal{R}_{e,s} = \mathbb{P}^1(\mathbb{C}) \setminus \{0, \infty\} & \mathcal{R}_{s,s} = \infty. \end{array}$$

We refer to section 1 for the most relevant references on open Richardson varieties. It is natural to ask whether they carry cluster structures compatible with total positivity and Poisson geometry. In 2016, Leclerc [23] gave a conjectural positive answer using representations of preprojective algebras. There is a direct link between such representations and the coordinate algebra $\mathbb{C}[N]$ of the unipotent radical N of B. We will recall it in the next section. In turn, the coordinate algebra $\mathbb{C}[N]$ is linked to the coordinate algebra $\mathbb{C}[\mathcal{R}_{v,w}]$ of the affine variety $\mathcal{R}_{v,w}$ as follows: Put $N(v) = N \cap v^{-1}N^- v$, where N^- is the unipotent radical of B^-, and $N'(w) = N \cap w^{-1}Nw$. Let $\mathbb{C}[N]^{v,w} \subseteq \mathbb{C}[N]$ be the subalgebra of double invariants

$$\mathbb{C}[N]^{v,w} = {}^{N(v)}\mathbb{C}[N]^{N'(w)}.$$

Let $M_{v,w}$ be the multiplicative subset of $\mathbb{C}[N]^{v,w}$ generated by the irreducible factors of

$$D_{v,w} = \prod_{i \in I} \Delta_{v^{-1}(\varpi_i), w^{-1}(\varpi_i)} ,$$

where I is the set of vertices of the Dynkin diagram and the factors of the product are generalized minors, *cf.* section 2.2 of [23]. In section 2.8 of [23], Leclerc constructs an algebra isomorphism

$$\mathbb{C}[N]^{v,w}[M_{v,w}^{-1}] \xrightarrow{\sim} \mathbb{C}[\mathcal{R}_{v,w}].$$

We thus obtain the following diagram summing up the relations between the coordinate algebras $\mathbb{C}[N]$ and $\mathbb{C}[\mathcal{R}_{v,w}]$

$$\begin{array}{ccc} \mathbb{C}[N]^{v,w} & \hookrightarrow & \mathbb{C}[N] \\ \big\downarrow \text{can} & & \\ \mathbb{C}[N]^{v,w}[M_{v,w}^{-1}] & \xrightarrow{\sim} & \mathbb{C}[\mathcal{R}_{v,w}]. \end{array}$$

3. Additive categorification and Leclerc's conjecture

3.1. The case of $N \xrightarrow{\sim} C_{w_0}$. We keep the notations and assumptions of section 2. Let Λ be the preprojective algebra of Δ over $k = \mathbb{C}$. For example, if Δ is the diagram A_4

$$1 \text{ --- } 2 \text{ --- } 3 \text{ --- } 4 \ ,$$

then, up to isomorphism, Λ is the k-algebra presented by the quiver

$$1 \underset{\alpha^*}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta^*}{\overset{\beta}{\rightleftarrows}} 3 \underset{\gamma^*}{\overset{\gamma}{\rightleftarrows}} 4$$

with relations

$$-\alpha^*\alpha = 0\ ,\ \alpha\alpha^* - \beta^*\beta = 0\ ,\ \beta\beta^* - \gamma^*\gamma = 0\ ,\ \gamma\gamma^* = 0.$$

Let us recall some important properties of Λ and the category $\operatorname{mod}\Lambda$ of k-finite-dimensional (right) Λ-modules:

a) The algebra Λ is finite-dimensional and selfinjective so that the category $\operatorname{mod}\Lambda$ becomes a Frobenius category.
b) As shown by Crawley-Boevey [5], for finite-dimensional Λ-modules L and M, we have a bifunctorial isomorphism
$$D\operatorname{Ext}^1_\Lambda(L, M) \xrightarrow{\sim} \operatorname{Ext}_\Lambda(M, L)\ ,$$
where $D = \operatorname{Hom}_k(?, k)$ is the duality over the ground field. This means that the Frobenius category $\operatorname{mod}\Lambda$ is *stably 2-Calabi–Yau*.
c) The category $\operatorname{mod}\Lambda$ contains (basic) *cluster-tilting objects*
$$T = T_1 \oplus \cdots \oplus T_m\ ,$$
where the T_i are indecomposable (and pairwise non isomorphic) and m is the length of the longest element w_0. These can be *mutated* at each non projective summand T_i.
d) Each reduced expression $\overline{w_0}$ of the longest element w_0 yields a canonical cluster-tilting object $T_{\overline{w_0}}$ which, up to mutation, is independent of the choice of $\overline{w_0}$.
e) We have a canonical *cluster-character*
$$\varphi : \operatorname{mod}\Lambda \to \mathbb{C}[N]$$
constructed by Geiss–Leclerc–Schröer [13] using work of Lusztig [25].

For an ice quiver Q, let us write $\mathcal{A}^+(Q)$ for the cluster algebra with *non invertible coefficients* associated with Q and $\mathcal{A}(Q)$ for the cluster algebra with *invertible coefficients* associated with Q. We always denote the initial cluster variables by x_i, $i \in Q_0$.

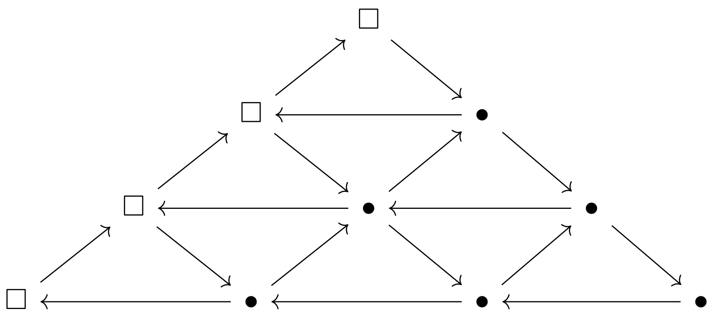
Theorem 3.2 ([13]). *If $T = T_1 \oplus \cdots \oplus T_m$ is a basic cluster-tilting object mutation-equivalent to $T_{\overline{w_0}}$ and $Q(T)$ is the ice quiver of its endomorphism algebra* $\operatorname{End}_\Lambda(T)$ *with frozen vertices corresponding to the projective-injective indecomposable summands of T,* *then* $(Q(T),(\varphi(T_i)))$ *is an initial seed for a cluster structure on* $\mathbb{C}[N]$. *Moreover, the algebra isomorphism morphism taking* x_i *to* $\varphi(T_i)$ *fits into a commutative diagram*

$$\begin{array}{ccc} \mathcal{A}^+(Q(T)) & \xrightarrow{\sim} & \mathbb{C}[N] \\ \downarrow & & \downarrow \\ \mathcal{A}(Q(T)) & \xrightarrow{\sim} & \mathbb{C}[\mathcal{R}_{e,w_0}]. \end{array}$$

For example, suppose that $\Delta = A_4$ and

$$\overline{w_0} = s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_1 s_2 s_1.$$

We refer to [14] for the construction of the canonical cluster-tilting object $T = T_{\overline{w_0}}$ and the computation of the corresponding ice quiver $Q(T)$ given by



where the squares denote frozen vertices. In this case, the subgroup N is formed by the upper unitriangular 4×4-matrices and the isomorphism

$$\mathcal{A}^+(Q(T)) \xrightarrow{\sim} \mathbb{C}[N]$$

sends the x_i to certain maximal minors.

3.3. **Case of C_w.** Recall that a *torsion pair* in $\operatorname{mod}\Lambda$ is a pair $(\mathcal{T},\mathcal{F})$ of strictly full subcategories such that we have

a) $\operatorname{Hom}(\mathcal{T},\mathcal{F}) = 0$ and
b) for each $M \in \operatorname{mod}\Lambda$, there is a short exact sequence

$$0 \longrightarrow M_{\mathcal{T}} \longrightarrow M \longrightarrow M^{\mathcal{F}} \longrightarrow 0\,,$$

where $M_{\mathcal{T}}$ belongs to $\mathcal{T}$ and $M^{\mathcal{F}}$ to $\mathcal{F}$.

Here the submodule $M_{\mathcal{T}}$ is called the *torsion part* and $M^{\mathcal{F}}$ the *torsion-free part* of M. The subcategory $\mathcal{T}$ is called a *torsion class* and $\mathcal{F}$ a *torsion-free class*. Torsion classes ordered by inclusion form a poset.

For two elements v and w of the Weyl group W, we write $v \leq_R w$ if w admits a reduced expression $\overline{w}$ equal to the concatenation $\overline{v}\overline{x}$ of a reduced expression $\overline{v}$ for v with a reduced word $\overline{x}$. The relation $\leq_R$ is called the *weak right order* on W. Clearly, the relation $v \leq_R w$ implies that $v \leq w$ in the Bruhat order but the converse does not hold in general.

Recall the a subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is *functorially finite* if, for each $M \in \operatorname{mod}\Lambda$, there are morphisms $C_0 \to M \to C^0$ with $C_0, C^0 \in \mathcal{C}$ such that for each $C \in \mathcal{C}$, each morphism $C \to M$ factors through $C_0 \to M$ and each morphism $M \to C$ factors through $M \to C^0$.

Theorem 3.4 ([29])**.** *We have a canonical isomorphism of posets* $w \mapsto \mathcal{C}_w$ *from* $(W, \leq_R)$ *to the poset of functorially finite torsion classes of* $\operatorname{mod}\Lambda$.

Theorem 3.5 ([6])**.** *If* A *is a finite-dimensional* k*-algebra such that* $\operatorname{mod} A$ *admits only finitely many functorially finite torsion classes, then each torsion class is functorially finite.*

By combining the two theorems, we see that all torsion classes of $\operatorname{mod}\Lambda$ are functorially finite and that they are canonically parametrized by the elements of W via the bijection $w \mapsto \mathcal{C}_w$.

Theorem 3.6 ([2]). *If $\mathcal{C} \subseteq \operatorname{mod}\Lambda$ is an extension-closed, functorially finite full subcategory, it becomes a Frobenius category (for the exact structure inherited from $\operatorname{mod}\Lambda$) which is stably 2-Calabi–Yau and has a cluster structure. In particular, the category $\mathcal{C}$ contains a cluster-tilting object.*

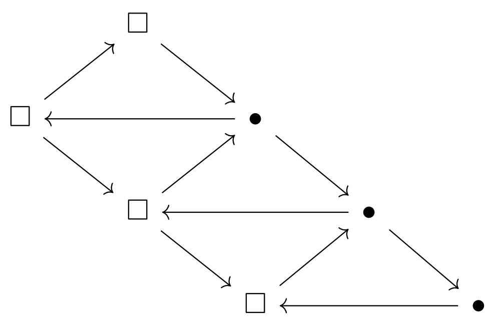
For an ice quiver Q, let we write $\mathcal{U}^+(Q)$ for the upper cluster algebra with *non invertible* coefficients associated with Q.

We say that an ice quiver Q has a *reddening sequence* if the non frozen part of Q has a reddening sequence in the sense of [22].

Theorem 3.7 ([14, 2]). *Fix $w \in W$ and let $\overline{w}$ be a reduced expression for w.*

a) *There is a canonical cluster-tilting object $T_{\overline{w}}$ of $\mathcal{C}_w$ which, up to mutation, only depends on w.*
b) *The ice quiver $Q_{\overline{w}}$ of the endomorphism algebra of $T_{\overline{w}}$ has an explicit description (up to the determination of the frozen subquiver).*
c) *The ice quiver $Q_{\overline{w}}$ has a reddening sequence [22] and we have $\mathcal{A}^+(Q_{\overline{w}}) = \mathcal{U}^+(Q_{\overline{w}})$.*

As an example, consider $\Delta = A_4$ and $\overline{w} = s_1 s_2 s_3 s_1 s_2 s_4 s_3$. Then the quiver $Q_{\overline{w}}$ is given by



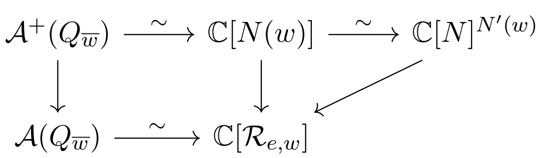
Let $w \in W$ and let $\overline{w}$ be a reduced expression for w. Recall that we have defined $N(w) = N \cap w^{-1} N^- w$, where N^- is the unipotent radical of B^-, and $N'(w) = N \cap w^{-1} N w$. We have a canonical isomorphism

$$\mathbb{C}[N(w)] \xrightarrow{\sim} \mathbb{C}[N]^{N'(w)}.$$

Theorem 3.8 ([14, 15]). *Choose a decomposition into indecomposables*

$$T_{\overline{w}} = T_1 \oplus \cdots \oplus T_{l(w)}.$$

Then the pair $(Q_{\overline{w}}, (\varphi(T_i)))$ is an initial seed for a cluster structure on $\mathbb{C}[N(w)]$, i.e. the algebra morphism taking x_i to $\varphi(T_i)$ is an isomorphism. Moreover, it makes the following square commutative



3.9. **The case of $\mathcal{R}_{v,w}$.** We follow [23]. Let $v \leq w$ be elements of W. Let $(\mathcal{C}_v, \mathcal{C}^v)$ be the torsion pair associated with v. Define

$$\mathcal{C}_{v,w} = \mathcal{C}^v \cap \mathcal{C}_w \subseteq \operatorname{mod} \Lambda.$$

Clearly, this subcategory is extension-closed. By the results of Auslander–Smalø [1], it is also functorially finite in $\operatorname{mod}\Lambda$. By Theorem 3.6, the category $\mathcal{C}_{v,w}$ has a cluster structure. If T is a cluster-tilting object of $\mathcal{C}_w$, then its $\mathcal{C}^v$-torsion-free part $T^{\mathcal{C}^v}$ is a cluster-tilting object of $\mathcal{C}_{v,w}$, by Proposition 3.12 of [23]. However, in general, the module $T^{\mathcal{C}^v}$ is not basic. Let $T_{v,\overline{w}}$ be a maximal basic summand of the cluster-tilting object $T_{\overline{w}}^{\mathcal{C}^v}$ and let $Q_{v,\overline{w}}$ be the quiver of the endomorphism algebra of $T_{v,\overline{w}}$.

Theorem 3.10 (Leclerc [23])**.** a) *The $\mathbb{C}$-span of $\varphi(\mathcal{C}_{v,w}) \subseteq \mathbb{C}[N]$ equals*

$$\mathbb{C}[N]^{v,w} = {}^{N(v)}\mathbb{C}[N]^{N'(w)}.$$

b) *The map $\varphi : \mathcal{C}_{v,w} \to \mathbb{C}[N]$ induces injective algebra morphisms*

$$\begin{array}{ccc} \mathcal{A}^+(Q_{v,\overline{w}}) & \overset{\widetilde{\varphi}}{\hookrightarrow} & \mathbb{C}[N]^{v,w} \\ \downarrow & & \downarrow \\ \mathcal{A}(Q_{v,\overline{w}}) & \underset{\widetilde{\varphi}_{loc}}{\hookrightarrow} & \mathbb{C}[\mathcal{R}_{v,w}] \end{array}$$

and $\dim \mathcal{R}_{v,w}$ equals the number of vertices of $Q_{v,\overline{w}}$.

c) *The algebra embedding $\widetilde{\varphi}_{loc}$ is an isomorphism if $v \leq_R w$ or if $\mathcal{C}_{v,w}$ has only finitely many indecomposables (up to isomorphism).*

Conjecture 3.11 (Leclerc [23])**.** *The algebra embedding $\widetilde{\varphi}_{loc}$ is always an isomorphism.*

One difficulty arises from the fact that Leclerc's seed $(Q_{v,\overline{w}}, (\varphi(T_i)))$ is not known explicitly. The following theorem is the first to have made it explicit in certain cases.

Theorem 3.12 (Serhiyenko–Sherman-Bennett–Williams [36])**.** *Leclerc's seed equals the canonical seed defined by a plabic graph if $\mathcal{R}_{v,w}$ is an open Schubert variety (in type A).*

This theorem implies Leclerc's conjecture for these cases since we have $v \leq_R w$ if $\mathcal{R}_{v,w}$ is an open Schubert variety.

Theorem 3.13 (Galashin–Lam [9])**.** *Leclerc's seed seed equals the canonical seed defined by a plabic graph if $\mathcal{R}_{v,w}$ is an open positroid variety (i.e. a type A open Richardson variety in the Grassmannian). Moreover, the conjecture holds in this case.*

Notice that in the situation of the theorem, we may have $v \not\leq_R w$.

4. Ménard's results

We keep the notations and assumptions of the preceding section.

Theorem 4.1 (Ménard [28])**.** *There is an explicit sequence of mutations transforming $T_{\overline{w}}$ into a cluster-tilting object $T'_{\overline{w}}$ such that any maximal direct summand $M_{v,\overline{w}}$ of $T'_{\overline{w}}$ lying in $\mathcal{C}_{v,w}$ is a cluster-tilting object of $\mathcal{C}_{v,w}$.*

The sequence of mutations in the theorem was conjectured by Jan Schröer. The cluster-tilting object $M_{v,\overline{w}}$ is expected to be isomorphic to $T_{v,\overline{w}}$. It yields a (possibly new) candidate seed for $\mathbb{C}[N]^{v,w}$ and $\mathbb{C}[\mathcal{R}_{v,w}]$. In his thesis [27, 28], Ménard has developed an algorithm allowing to explicitly compute the seed associated with $M_{v,\overline{w}}$. It follows from his theorem above that the quiver $Q(M_{v,\overline{w}})$ is a *cluster reduction* of $Q(T_{\overline{w}})$, i.e. it is obtained from $Q(T_{\overline{w}})$ by mutating, freezing vertices and deleting certain frozen vertices (in this order).

By a theorem of Muller [31], the existence of reddening sequences is preserved under cluster reduction. Since the existence of a reddening sequence for the ice quiver $Q(T_{\overline{w}})$ is known [15], it follows that the ice quiver $Q(M_{v,\overline{w}})$ has a reddening sequence. Moreover, the exchange matrix associated with $Q(M_{v,\overline{w}})$ has full rank (this follows essentially from [2]). Thus, the upper cluster algebra $\mathcal{U}(Q(M_{v,\overline{w}}))$ (with invertible coefficients) admits a theta basis in the sense of Gross–Hacking–Keel–Kontsevich [18] and also a generic basis, as shown by Qin [33]. In particular, the image of the Caldero–Chapoton map spans the upper cluster algebra $\mathcal{U}(Q(M_{v,\overline{w}}))$ over the algebra of Laurent polynomials in the coefficients.

5. Upper cluster structure

We keep the notations and assumptions of the preceding section. Let T be Ménard's cluster-tilting object $M_{v,\overline{w}}$. Let $\varphi : \mathcal{A}(Q(T)) \to \mathbb{C}[N]^{v,w}$ be the algebra morphism associated with the seed $(Q(T), (\varphi(T_i)))$.

Theorem 5.1. *The map φ yields a commutative square*

$$\begin{array}{ccc} \mathcal{U}^+(Q(T)) & \xrightarrow[\widetilde{\varphi}]{} & \mathbb{C}[N]^{v,w} \\ \downarrow & & \downarrow \\ \mathcal{U}(Q(T)) & \xrightarrow[\widetilde{\varphi}_{loc}]{\sim} & \mathbb{C}[\mathcal{R}_{v,w}]. \end{array}$$

where the bottom map $\widetilde{\varphi}_{loc}$ is an isomorphism.

Remark 5.2. *We do not know whether the statement of the theorem also holds if T is Leclerc's cluster tilting object $T_{v,\overline{w}}$.*

Proof. Since the $\varphi(T_i)$ are algebraically independent, the map $x_i \mapsto \varphi(T_i)$ defines a field embedding

$$\mathbb{C}(x_i) \xhookrightarrow{\widetilde{\varphi}} \mathbb{C}(N).$$

Let $CC : \mathcal{C}_{v,w} \to \mathbb{C}(x_i)$ denote the cluster character associated with the cluster-tilting object $T \in \mathcal{C}_{v,w}$ in [8]. By Theorem 4 of [16], the triangle

$$\begin{array}{ccccc} \mathbb{C}(x_i) & & \xhookrightarrow{\widetilde{\varphi}} & & \mathbb{C}(N) \\ & \nwarrow_{CC} & & \nearrow_{\varphi} & \\ & & \mathcal{C}_{v,w} & & \end{array}$$

commutes. Now by definition, the map $CC : \mathcal{C}_{v,w} \to \mathbb{C}(x_i)$ actually takes its values in $\mathbb{C}[x_i^{\pm}]$ and by Theorem 1.1 of [32], it even takes its values in the upper cluster algebra $\mathcal{U}^+ = \mathcal{U}^+(Q(T))$ with non invertible coefficients. Clearly, the field embedding

$$\widetilde{\varphi} : \mathbb{C}(x_i) \to \mathbb{C}(N)$$

induces an isomorphism

$$\mathcal{U}^+ \xrightarrow{\sim} \widetilde{\varphi}(\mathcal{U}^+) \subset \mathbb{C}(N)$$

and we have the commutative square

$$\begin{array}{ccc} \mathcal{C}_{v,w} & \xrightarrow{\varphi} & \mathbb{C}(N) \\ {\scriptstyle CC}\downarrow & & \uparrow \\ \mathcal{U}^+ & \xrightarrow{\sim} & \widetilde{\varphi}(\mathcal{U}^+). \end{array}$$

By Theorem 3.10 (a), the $\mathbb{C}$-span of $\varphi(\mathcal{C}_{v,w})$ equals $\mathbb{C}[N]^{v,w}$. This implies the inclusion

$$\mathbb{C}[N]^{v,w} \subseteq \widetilde{\varphi}(\mathcal{U}^+).$$

Since $Q(T)$ is of full rank, the upper cluster algebra with non invertible coefficients is a *finite* intersection of Laurent polynomial rings (by Cor. 1.9 of [3], the 'starfish lemma'). Therefore, the upper cluster algebra with invertible coefficients $\mathcal{U}$ is the localization of $\mathcal{U}^+$ at the coefficients. Therefore, from the above inclusion, we deduce that we have

$$\mathbb{C}[N]^{v,w}[M_{v,w}^{-1}] \subseteq \widetilde{\varphi}(\mathcal{U}).$$

Here, the symbol $M_{v,w}$ denotes the multiplicative set in $\mathbb{C}[N]^{v,w}$ introduced at the end of section 2.

Since T is Ménard's cluster-tilting object, we know that the non frozen part of the ice quiver $Q(T)$ has a reddening sequence. By [33, Theorem 1.2.3], the upper cluster algebra $\mathcal{U}$ has a generic basis. This implies that $CC(\mathcal{C}_{v,w})$ contains a generating set for the $\mathbb{C}[M_{v,w}^{\pm 1}]$-algebra $\mathcal{U}$. Since the $\mathbb{C}$-span of $\varphi(\mathcal{C}_{v,w})$ equals $\mathbb{C}[N]^{v,w}$, we have the reverse inclusion

$$\widetilde{\varphi}(\mathcal{U}) \subseteq \mathbb{C}[N]^{v,w}[M_{v,w}^{-1}].$$

Thus, we obtain the equality

$$\widetilde{\varphi}(\mathcal{U}) = \mathbb{C}[N]^{v,w}[M_{v,w}^{-1}] = \mathbb{C}[\mathcal{R}_{v,w}].$$

This is what we had to prove. $\surd$

6. Towards a cluster structure

Our hope is that for Ménard's cluster-tilting object $M_{v,\overline{w}}$, we have $\mathcal{A} = \mathcal{U}$ for the corresponding cluster and upper cluster algebra with invertible coefficients. Recall that by Ménard's theorem, the ice quiver $Q(M_{v,\overline{w}})$ is obtained from $Q(T_{\overline{w}})$ by cluster reduction, i.e. by mutation, freezing and deletion of frozen vertices (in this order).

Theorem 6.1 (Geiss–Leclerc–Schröer [15]). a) *We have $\mathcal{A} = \mathcal{U}$ for $Q(T_{\overline{w}})$.*
b) *The ice quiver $Q(T_{\overline{w}})$ admits a reddening sequence.*

The second property is preserved under cluster reduction by Muller's theorem [31]. However, it is not clear under which conditions this holds for the first property.

6.2. **Preservation of $\mathcal{U} = \mathcal{A}$ under freezing?** Let Q be an ice quiver and Q' the quiver obtained from Q by freezing the cluster variable x associated with a non frozen vertex. We then have the algebra inclusions

$$\mathcal{A}' \subseteq \mathcal{A}[x^{-1}] \subseteq \mathcal{U}[x^{-1}] \subseteq \mathcal{U}' ,$$

where $\mathcal{A} = \mathcal{A}(Q)$, ... Following Muller [30], we define $\mathcal{A}'$ to be a *cluster localization* of $\mathcal{A}$ at x if $\mathcal{A}' = \mathcal{A}[x^{-1}]$. Unfortunately, it is not clear whether the freezing occuring in the passage from Geiss–Leclerc–Schröer's seed for C_w to Ménard's for $\mathcal{R}_{v,w}$ is a composition of cluster localizations. The following theorem may nevertheless be useful.

Theorem 6.3. *Suppose the exchange matrix associated with the ice quiver Q is of full rank. Let $\mathcal{A}$ and $\mathcal{U}$ be the associated cluster and upper cluster algebra. If we have $\mathcal{A} = \mathcal{U}$ and $\mathcal{A}'$ is a cluster localization of $\mathcal{A}$ at x, then we have $\mathcal{A}' = \mathcal{U}'$.*

Proof. Since the exchange matrix B associated with the ice quiver Q is of full rank and Q' is obtained from Q by freezing the non frozen vertex of Q labeled by x, we know that the exchange matrix B' associated with the ice quiver Q' is also of full rank. Hence, the starfish lemma [3, Corollary 1.9] holds for $\mathcal{U}$ and $\mathcal{U}'$.

Denote by t_0 the initial seed of $\mathcal{U}$ and I_{uf} the set of non frozen vertices of the ice quiver Q. Let k be the non frozen vertex of Q such that $x = x_{k;t_0}$. By the starfish lemma [3, Corollary 1.9], we have

$$\mathcal{U} = \mathcal{L}(t_0) \bigcap \left(\bigcap_{i \in I_{\mathrm{uf}}} \mathcal{L}(\mu_i(t_0)) \right), \tag{6.1}$$

$$\mathcal{U}' = \mathcal{L}(t_0) \bigcap \left(\bigcap_{k \neq i \in I_{\mathrm{uf}}} \mathcal{L}(\mu_i(t_0)) \right), \tag{6.2}$$

where $\mathcal{L}(\mu_i(t_0))$ is the Laurent polynomial ring associated with the seed $\mu_i(t_0)$.

Since $\mathcal{A} = \mathcal{U}$ and $\mathcal{A}'$ is a cluster localization of $\mathcal{A}$ at x, we have that

$$\mathcal{A}' = \mathcal{A}[x^{-1}] = \mathcal{U}[x^{-1}] \subseteq \mathcal{U}'.$$

It remains to show the converse inclusion $\mathcal{U}' \subseteq \mathcal{U}[x^{-1}]$.

By the equality (6.2), we know that for any $v \in \mathcal{U}'$, there exists a positive integer d such that the exponents of $x = x_{k;t_0}$ in the Laurent expansion of vx^d with respect to the seed $\mu_i(t_0)$ are positive for any $k \neq i \in I_{\mathrm{uf}}$. In this case, we have $vx^d \in \mathcal{L}(\mu_k(t_0))$. Then by the equality (6.1), we get $vx^d \in \mathcal{U}$ and thus $v \in \mathcal{U}[x^{-1}]$. So we have $\mathcal{U}' \subseteq \mathcal{U}[x^{-1}]$ and

$$\mathcal{A}' = \mathcal{A}[x^{-1}] = \mathcal{U}[x^{-1}] = \mathcal{U}'.$$

$\surd$

6.4. **Preservation of $\mathcal{A} = \mathcal{U}$ under deletion?** In general, the property $\mathcal{A} = \mathcal{U}$ is not preserved under deletion of frozen vertices. The hypotheses of the following proposition do hold for the deletion occuring in the passage from Geiss–Leclerc–Schröer's seed for C_w to Ménard's for $\mathcal{R}_{v,w}$.

Proposition 6.5. *Suppose Q' is obtained from Q by deleting a frozen vertex, that Q and Q' are of full rank and that the ice quiver Q admits a reddening sequence. Denote by $\mathcal{A}$, $\mathcal{U}$, $\mathcal{A}'$ and $\mathcal{U}'$ the cluster algebras and the upper cluster algebras associated with Q and Q'. If $\mathcal{A} = \mathcal{U}$, then $\mathcal{A}' = \mathcal{U}'$.*

Proof. Let $\mathbb{P}$ and $\mathbb{P}'$ denote the groups of Laurent monomials in the coefficients of $\mathcal{A}$ and $\mathcal{A}'$. Consider the diagram

$$\begin{array}{ccccc} \mathcal{A} & = & \mathcal{U} & \hookrightarrow & \mathbb{C}[x_i^{\pm 1}][\mathbb{P}] \\ \downarrow & & \downarrow \pi & & \downarrow \\ \mathcal{A}' & \hookrightarrow & \mathcal{U}' & \hookrightarrow & \mathbb{C}[x_i^{\pm 1}][\mathbb{P}'] \end{array}$$

Let CC and CC' be the Caldero–Chapoton maps associated with Q and Q' and let $\pi : \mathbb{C}[x_i^{\pm 1}][\mathbb{P}] \to \mathbb{C}[x_i^{\pm 1}][\mathbb{P}']$ be the specialization map. We have $\pi \circ CC = CC'$. By Qin's work [33], we know that the image of CC generates $\mathcal{U}$ as a $\mathbb{C}[\mathbb{P}]$-module. Thus, the image of $\pi \circ CC$ generates $\pi(\mathcal{U})$ as a $\mathbb{C}[\mathbb{P}']$-module. Now the image of $\pi \circ CC$ equals that of CC' and the image of CC' generates $\mathcal{U}'$ as a $\mathbb{C}[\mathbb{P}']$-module since the ice quiver Q' also has a reddening sequence. It follows that $\pi(\mathcal{U})$ equals $\mathcal{U}'$ and this implies $\mathcal{A}' = \mathcal{U}'$. $\surd$

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