A characterization of cluster categories

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Outline

1. From quivers to derived categories and back
2. The cluster category, and the main theorem
3. Applications
4. Appendix: On the proof of the main theorem

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A characterization of cluster categories
A quiver is an oriented graph

Definition

A quiver $Q$ is an oriented graph: It is given by

- a set $Q_0$ (the set of vertices)
- a set $Q_1$ (the set of arrows)
- two maps
  - $s: Q_1 \to Q_0$ (taking an arrow to its source)
  - $t: Q_1 \to Q_0$ (taking an arrow to its target).

Remark

A quiver is a ‘category without composition’.
A quiver can have loops, cycles, several components.

Example

The quiver $\tilde{A}_3 : 1 \overset{\alpha}{\longleftarrow} 2 \overset{\beta}{\longleftarrow} 3$ is an orientation of the Dynkin diagram $A_3 : 1 \rightarrow 2 \rightarrow 3$.

Example

$$Q : \begin{array}{c} 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \\ \scriptstyle{\lambda} \quad \scriptstyle{\mu} \quad \scriptstyle{\alpha} \quad \scriptstyle{\beta} \quad \scriptstyle{\gamma} \quad \end{array}$$

We have $Q_0 = \{1, 2, 3, 4, 5, 6\}$, $Q_1 = \{\alpha, \beta, \ldots\}$. $\alpha$ is a loop, $(\beta, \gamma)$ is a 2-cycle, $(\lambda, \mu, \nu)$ is a 3-cycle.
Let $k$ be an algebraically closed field. Let $Q$ be a finite quiver (the sets $Q_0$ and $Q_1$ are finite).

**Definition**

A *representation* of $Q$ is a diagram of finite-dimensional vector spaces of the shape given by $Q$.

**Example**

A representation of $\tilde{A}_2 : 1 \xrightarrow{\alpha} 2$ is a diagram of two finite-dimensional vector spaces linked by one linear map

$$V : V_1 \xrightarrow{V_\alpha} V_2.$$
The category of representations of $Q$ is abelian.

**Definition**

A *morphism of representations* of $Q$ is a morphism of diagrams. 

\[ \text{rep}(Q) = \text{category of representations of } Q. \]

**Remarks**

- Direct sums, kernels and cokernels are computed componentwise.
- The category of representations is a $k$-linear abelian category with enough projectives (it is even a module category).
Definition of the derived category $\mathcal{D}_Q$

**Definition**

$\mathcal{D}_Q = \text{bounded derived category of } \text{rep}(Q)$

- objects: bounded complexes $V : \ldots \rightarrow V^p \rightarrow V^{p+1} \rightarrow \ldots$ of representations
- morphisms: obtained from morphisms of complexes by formally inverting all quasi-isomorphisms
- suspension functor: $\Sigma : \mathcal{D}_Q \rightarrow \mathcal{D}_Q$, $V \mapsto \Sigma V = V[1]
- triangles: $U' \rightarrow V' \rightarrow W' \rightarrow \Sigma U'$ obtained from short exact sequences of complexes $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$.

**Remark**

$\mathcal{D}_Q$ is $k$-linear. It is abelian iff $Q$ has no arrows.
Objects of $\mathcal{D}_Q$ decompose into indecomposables.

**Definition**

An object $V$ of $\mathcal{D}_Q$ is *indecomposable* if $V \neq 0$ and in each decomposition $V \cong V' \oplus V''$, we have $V' = 0$ or $V'' = 0$.

**Decomposition theorem**

(Azumaya-Fitting-Krull-Remak-Schmidt- . . . )

a) An object of $\mathcal{D}_Q$ is indecomposable iff its endomorphism ring is local.

b) Each object of $\mathcal{D}_Q$ decomposes into a finite sum of indecomposables, unique up to isomorphism and permutation.
Let \( \mathcal{A} \) be any \( k \)-linear category where the decomposition theorem holds. We will assign a quiver \( \Gamma_{\mathcal{A}} \) to \( \mathcal{A} \).

- The vertices of \( \Gamma_{\mathcal{A}} \) will be in bijection with the isomorphism classes of the indecomposables of \( \mathcal{A} \).
- To define the arrows, let

\[
\mathcal{R}(X, Y) = \{ \text{non invertible morphisms } f : X \to Y \},
\]

where \( X, Y \) are indecomposable in \( \mathcal{A} \). Then \( \mathcal{R} \) is an ideal (namely, the radical) of the category \( \text{ind} \mathcal{A} \) of indecomposables of \( \mathcal{A} \).
The quiver of a category with decomposition

**Definition**

The *quiver of* $\mathcal{A}$ is the quiver $\Gamma_{\mathcal{A}}$ with

- vertices: representatives $X$ of the isoclasses of indecomposables of $\mathcal{A}$
- arrows: the number of arrows from $X$ to $Y$ equals the dimension of the space

$$\text{irr}(X, Y) = R(X, Y)/R^2(X, Y) = \{\text{irreducible morphisms}\}$$

of ‘morphisms without non trivial factorization’.
The quiver of the derived category

Theorem

Suppose that $Q$ does not have oriented cycles.

a) The quiver of the category $\mathcal{P}_Q$ of projectives of $\text{rep}(Q)$ is the opposite quiver $Q^{\text{op}}$.

b) (Happel, 1986) If the underlying graph of $Q$ is a Dynkin diagram of type $A_n$, $D_n$ or $E_n$, the (Auslander-Reiten) quiver of $\mathcal{D}_Q$ is the repetition $\mathbb{Z}Q^{\text{op}}$ of the opposite quiver: It has

- vertices: $(p, x)$, for $p \in \mathbb{Z}$, $x \in Q_0$,
- arrows: for each arrow $\alpha: x \to y$ of $Q^{\text{op}}$, we have arrows
  - $(p, \alpha): (p, x) \to (p, y)$, $p \in \mathbb{Z}$, and
  - $\sigma(p, \alpha): (p - 1, y) \to (p, x)$, $p \in \mathbb{Z}$.
The example $\tilde{A}_3$

**Example**

$Q = \tilde{A}_3 : 1 \leftarrow 2 \leftarrow 3$

$\Gamma_{\mathcal{P}_Q} = Q^{op} :$

$\Gamma_{\mathcal{D}_Q} = \mathbb{Z}Q^{op} :$

\[
\begin{array}{c}
\cdots \\
P_1 \\
P_2 \\
P_3 \\
\end{array}
\begin{array}{c}
P_1 \\
P_2 \\
P_3 \\
\sum P_1 \\
\sum P_2 \\
\sum P_3 \\
\end{array}
\]
The Serre functor

Blanket assumptions

Q is a finite quiver without oriented cycles. All categories and functors are $k$-linear.

Theorem (Happel, 1986)

$\mathcal{D}_Q$ admits a **Serre functor** (=Nakayama functor), i.e. an autoequivalence $S : \mathcal{D}_Q \xrightarrow{\sim} \mathcal{D}_Q$ such that

$$D \text{Hom}(X, ?) \xrightarrow{\sim} \text{Hom}(?, SX)$$

for all $X \in \mathcal{D}_Q$, where $D = \text{Hom}_k(?, k)$. 

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A characterization of cluster categories
Calabi-Yau categories

Let $d$ be an integer and $\mathcal{T}$ a triangulated category with finite-dimensional Hom-spaces.

**Definition (Kontsevich)**

$\mathcal{T}$ is $d$-Calabi-Yau if it has a Serre functor $S$ and $S \sim \Sigma^d$ as triangle functors.

**Example**

$X$ a smooth projective variety of dimension $d$. $\mathcal{T}$ the bounded derived category of coherent sheaves on $X$. Then $S = ? \otimes \omega[d]$ and

\[ X \text{ is Calabi-Yau} \iff \omega \sim \mathcal{O} \iff \mathcal{T} \text{ is } d\text{-Calabi-Yau} \]
The cluster category

**Definition**

The **cluster category** \( \mathcal{C}_Q \) is the universal 2-Calabi-Yau category under the derived category \( \mathcal{D}_Q \):

\[
\begin{align*}
\mathcal{D}_Q & \xrightarrow{\mathcal{C}_Q} \mathcal{T} \\
(P, \pi) & \xrightarrow{(F, \phi)} \\
\mathcal{C}_Q & \xrightarrow{\geq} \mathcal{T}
\end{align*}
\]

\( \mathcal{C}_Q, \mathcal{T} \) 2-Calabi-Yau

\( P, F \) triangle functors

\( \pi : P \circ S \xrightarrow{\sim} S \circ P \)

\( \phi : F \circ S \xrightarrow{\sim} S \circ F \)

**Remark**

\( \mathcal{C}_Q \) is due to Buan-Marsh-Reineke-Reiten-Todorov for \( Q \), to Caldero-Chapoton-Schiffler for \( \tilde{A}_n \).
### Remarks

1) Strictly speaking the definition should be formulated in the homotopy category of enhanced triangulated categories.

2) Explicitly, $\mathcal{C}_Q$ is the **orbit category** of $\mathcal{D}_Q$ under the action of the automorphism $S^{-1} \circ \Sigma^2$. It has

- **objects**: same as those of $\mathcal{D}_Q$
- **morphisms**:

\[
\mathcal{C}_Q(X, Y) = \bigoplus_{p \in \mathbb{Z}} \mathcal{D}_Q(X, (S^{-1} \circ \Sigma^2)^p Y).
\]
The quiver of the cluster category

**Theorem (BMRRT)**

The decomposition theorem holds in $\mathcal{C}_Q$ and its quiver is isomorphic to the quotient of the quiver of $\mathcal{D}_Q$ under the action of the automorphism induced by $S^{-1} \circ \Sigma^2$.

**Example**

For $Q = \tilde{A}_n = (1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow n)$ the quiver of $\mathcal{C}_Q$ is a Moebius strip of width $n$ with $n(n + 3)/2$ vertices. Similarly for $Q = \tilde{E}_6$. 
$F = S^{-1} \circ \Sigma^2$

A characterization of cluster categories
The canonical cluster-tilting subcategory

Recall that we have functors \( \text{rep}(Q) \to \mathcal{D}_Q \to \mathcal{C}_Q \). Let \( \mathcal{T}_Q \) be the image of the category \( \mathcal{P}_Q \) of projectives of \( \text{rep}(Q) \).

**Theorem (BMRRT)**

a) *The quiver of* \( \mathcal{T}_Q \) *is isomorphic to* \( Q^{op} \).

b) \( \mathcal{T}_Q \subset \mathcal{C}_Q \) *is a cluster-tilting subcategory*, i.e.

1) *for all* \( T, T' \) *in* \( \mathcal{T}_Q \), *we have*

\[
0 = \text{Ext}^1(T, T') := \text{Hom}_{\mathcal{C}_Q}(T, \Sigma T'),
\]

2) *if* \( X \in \mathcal{C}_Q \) *satisfies* \( \text{Ext}^1(T, X) = 0 \) *for all* \( T \) *in* \( \mathcal{T}_Q \), *then* \( X \) *belongs to* \( \mathcal{T}_Q \).

**Definition**

\( \mathcal{T}_Q \) *is the canonical cluster-tilting subcategory.*
The main theorem

Main theorem

Let

- \( \mathcal{C} \) be a 2-Calabi-Yau triang. category (of ‘algebraic origin’),
- \( \mathcal{T} \subset \mathcal{C} \) a cluster-tilting subcategory,
- \( Q \) the opposite quiver of \( \mathcal{T} \).

If \( Q \) does not have oriented cycles, then \( \mathcal{C}_Q \xrightarrow{\sim} \mathcal{C} \).

Remarks

We only need to know \( Q \), not \( \mathcal{T} \)! The objects of \( \mathcal{T}_Q \) generate \( \mathcal{C}_Q \) as a triangulated category but for general \( T, T' \) in \( \mathcal{T} \), we have \( \text{Ext}^i(T, T') \neq 0 \) for infinitely many \( i \).
Application: Cohen-Macaulay modules

Let $k = \mathbb{C}$, $\zeta$ a primitive third root of 1. Let $G = \mathbb{Z}/3\mathbb{Z}$ act on $S = k[[X, Y, Z]]$ by multiplying the generators by $\zeta$. Then

- $R = S^G$ is an isolated singularity of dimension 3 and Gorenstein.
- The category $\text{CM}(R)$ of maximal Cohen-Macaulay modules is Frobenius.
- The stable category $\text{CM}(R)$ is 2-Calabi-Yau (Auslander).
- Decompose $S = S^G \oplus T_1 \oplus T_2$ over $R$. Then the direct sums of copies of the $T_i$ form a cluster-tilting subcategory $\mathcal{T}$ of $\mathcal{C}$ (Iyama).
- The quiver of $\mathcal{T}$ is the generalized Kronecker quiver

$$Q : 1 \rightarrow 2.$$
Cluster categories occur in nature

Conclusion

The stable category of Cohen-Macaulay modules over $S^G$ is triangle equivalent to the cluster category $C_Q$.

Consequence

New proof of Iyama-Yoshino’s classification of the rigid Cohen-Macaulay modules over $S^G$: sums of projectives and copies of $\Omega^a T_1$, $\Omega^b T_2$, $a, b \in \mathbb{Z}$.

Moral

Cluster categories occur in nature.
Let $Q$ be a quiver without loops or 2-cycles and $u$ a vertex of $Q$. The \textbf{mutation of $Q$ at $u$} (Fomin-Zelevinsky) is the quiver $Q'$ obtained from $Q$ as follows (where $v \xrightarrow{r} w = \text{arrow of multiplicity } r \geq 0$)

1) reverse all arrows incident with $u$;
2) modify the other arrows as follows:
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Summary
Appendix: On the proof of the main theorem

$Q''$ is mutation equivalent to $Q$ if $Q''$ is isomorphic to a quiver obtained from $Q$ by a finite sequence of mutations.

![Quiver Diagram]

**Theorem**

This quiver $Q$ is not mutation equivalent to a quiver $Q'$ without oriented cycles.

Proof: Use brute force (Google ‘quiver mutation’!) or use the main theorem!
Let \( \mathcal{C} \) be the stable category of finite-dimensional modules over the preprojective algebra of type \( A_5 \).

Crawley-Boevey: \( \mathcal{C} \) is 2-Calabi-Yau.

Geiss-Leclerc-Schröer’s work implies:
- \( \mathcal{C} \) contains a cluster-tilting subcategory \( \mathcal{T} \) with quiver \( Q \),
- if \( Q \sim_{\text{mut}} Q' \), then \( \mathcal{C} \) contains a cluster-tilting subcategory with quiver \( Q' \).

By the main theorem: If \( Q \sim_{\text{mut}} Q' \) and \( Q' \) does not have oriented cycles, then \( \mathcal{C} \sim_{\text{mut}} \mathcal{C}_{Q'} \).

Contradiction: The suspension functor \( \Sigma \) is of order 6 in \( \mathcal{C} \) but of order \( \infty \) in \( \mathcal{C}_{Q'} \) (\( Q' \) is not a Dynkin quiver).
Summary

- Cluster categories occur in nature.
- Google ‘quiver mutation’!
The (best) proof of the main theorem uses the universal property.

Michel Van den Bergh: We use the universal property!

\[ \begin{array}{ccc}
\mathcal{D}_Q & \rightarrow & \mathcal{C}_Q \\
\Downarrow & & \Downarrow \\
(F, \phi) & \rightarrow & \bar{F} \\
\end{array} \]

where \( \phi : F \circ S \xrightarrow{\sim} S \circ F \).

1) Construct \( \bar{F} \) via \((F, \phi)\). Subtle: Construct \( \phi \)!

2) \( \bar{F} \) is an equivalence by the
Beilinson's lemma has a cluster analogue

\[ \begin{array}{ccc} 
\mathcal{C}' & \xrightarrow{G} & \mathcal{C} \\
\uparrow & & \uparrow \\
\mathcal{T}' & \xrightarrow{G|\mathcal{T}'} & \mathcal{T} 
\end{array} \]

Lemma (cluster-Beilinson)

- \( G : \mathcal{C}' \to \mathcal{C} \) a triangle functor between 2-CY categories
- \( \mathcal{T}' \subset \mathcal{C}' \) a cluster tilting subcategory.

Then \( G \) is an equivalence iff \( \mathcal{T} = G(\mathcal{T}') \) is a cluster-tilting subcategory and the restriction of \( G \) to \( \mathcal{T}' \) is fully faithful.