A characterization of cluster categories

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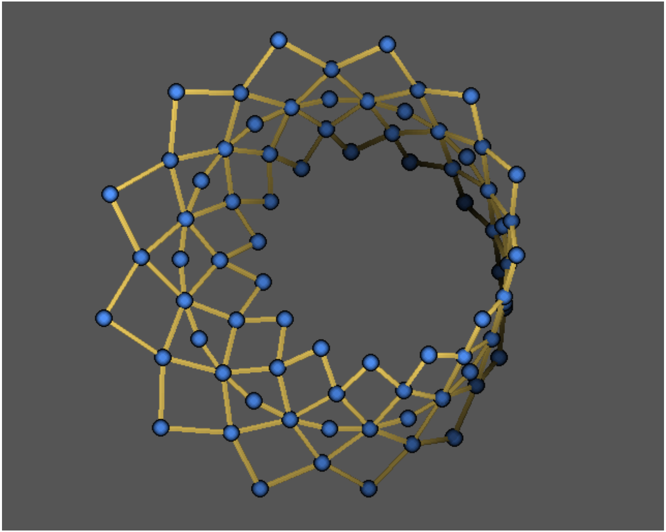


Outline

A quiver is an oriented graph

Definition

A *quiver* Q is an oriented graph: It is given by

- a set Q_0 (the set of vertices)
- a set Q_1 (the set of arrows)
- two maps
  - $s : Q_1 \to Q_0$ (taking an arrow to its source)
  - $t : Q_1 \to Q_0$ (taking an arrow to its target).

Remark

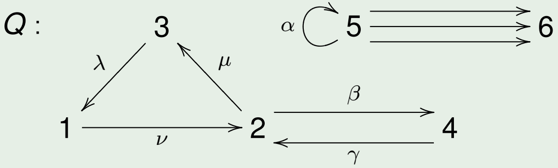
A quiver is a ‘category without composition’.

A quiver can have loops, cycles, several components.

Example

The quiver $\vec{A}_3 : \; 1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3$ is an orientation of the Dynkin diagram $A_3 : \; 1 \text{ --- } 2 \text{ --- } 3$.

Example



We have $Q_0 = \{1, 2, 3, 4, 5, 6\}$, $Q_1 = \{\alpha, \beta, \ldots\}$.
α is a *loop*, (β, γ) is a 2-*cycle*, (λ, μ, ν) is a 3-*cycle*.

representation of a quiver = diagram of vector spaces

Let k be an algebraically closed field.
Let Q be a finite quiver (the sets Q_0 and Q_1 are finite).

Definition

A *representation* of Q is a diagram of finite-dimensional vector spaces of the shape given by Q.

Example

A representation of $\vec{A}_2 : \ 1 \xrightarrow{\alpha} 2$ is a diagram of two finite-dimensional vector spaces linked by one linear map

$$V : \ V_1 \xrightarrow{V_\alpha} V_2 \, .$$

The category of representations of Q is abelian.

Definition

A *morphism of representations* of Q is a morphism of diagrams.
$\mathrm{rep}(Q)$ = category of representations of Q.

Remarks

- Direct sums, kernels and cokernels are computed componentwise.
- The category of representations is a k-linear abelian category with enough projectives (it is even a module category).

Definition of the derived category $\mathcal{D}_Q$

Definition

$\mathcal{D}_Q$ = bounded derived category of rep(Q)

- objects: bounded complexes $V : \ldots \to V^p \to V^{p+1} \to \ldots$ of representations
- morphisms: obtained from morphisms of complexes by formally inverting all quasi-isomorphisms
- suspension functor: $\Sigma : \mathcal{D}_Q \to \mathcal{D}_Q$, $V \mapsto \Sigma V = V[1]$
- triangles: $U' \to V' \to W' \to \Sigma U'$ obtained from short exact sequences of complexes $0 \to U \to V \to W \to 0$.

Remark

$\mathcal{D}_Q$ is k-linear. It is abelian iff Q has no arrows.

Objects of $\mathcal{D}_Q$ decompose into indecomposables.

Definition

An object V of $\mathcal{D}_Q$ is *indecomposable* if $V \neq 0$ and in each decomposition $V \cong V' \oplus V''$, we have $V' = 0$ or $V'' = 0$.

Decomposition theorem (Azumaya-Fitting-Krull-Remak-Schmidt- . . .)

a) An object of $\mathcal{D}_Q$ is indecomposable iff its endomorphism ring is local.

b) Each object of $\mathcal{D}_Q$ decomposes into a finite sum of indecomposables, unique up to isomorphism and permutation.

Back from categories to quivers ...

Let $\mathcal{A}$ be any k-linear category where the decomposition theorem holds. We will assign a quiver $\Gamma_{\mathcal{A}}$ to $\mathcal{A}$.

- The vertices of $\Gamma_{\mathcal{A}}$ will be in bijection with the isomorphism classes of the indecomposables of $\mathcal{A}$.
- To define the arrows, let

$$\mathcal{R}(X, Y) = \{\text{non invertible morphisms } f : X \to Y\} ,$$

  where X, Y are indecomposable in $\mathcal{A}$. Then $\mathcal{R}$ is an ideal (namely, the radical) of the category $\operatorname{ind} \mathcal{A}$ of indecomposables of $\mathcal{A}$.

The quiver of a category with decomposition

Definition

The *quiver of* $\mathcal{A}$ is the quiver $\Gamma_{\mathcal{A}}$ with

- vertices: representatives X of the isoclasses of indecomposables of $\mathcal{A}$
- arrows: the number of arrows from X to Y equals the dimension of the space

$$\mathrm{irr}(X, Y) = \mathcal{R}(X, Y)/\mathcal{R}^2(X, Y) = \{\text{irreducible morphisms}\}$$

of 'morphisms without non trivial factorization'.

The quiver of the derived category

Theorem

Suppose that Q does not have oriented cycles.

a) *The quiver of the category $\mathcal{P}_Q$ of projectives of* $\mathrm{rep}(Q)$ *is the opposite quiver Q^{op}.*

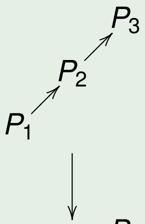
b) *(Happel, 1986) If the underlying graph of Q is a Dynkin diagram of type A_n, D_n or E_n, the (Auslander-Reiten) quiver of $\mathcal{D}_Q$ is the* repetition $\mathbb{Z}Q^{op}$ *of the opposite quiver: It has*
   - *vertices: (p, x), for $p \in \mathbb{Z}$, $x \in Q_0$,*
   - *arrows: for each arrow $\alpha : x \to y$ of Q^{op}, we have arrows*
     - $(p, \alpha) : (p, x) \to (p, y)$, $p \in \mathbb{Z}$, *and*
     - $\sigma(p, \alpha) : (p - 1, y) \to (p, x)$, $p \in \mathbb{Z}$.

The example $\vec{A}_3$

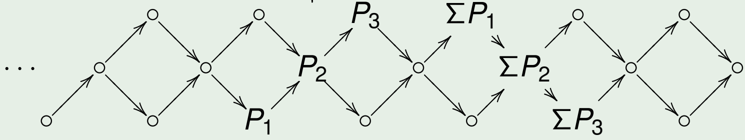
Example

$$Q = \vec{A}_3 : \; 1 \longleftarrow 2 \longleftarrow 3$$

$\Gamma_{\mathcal{P}_Q} = Q^{op}$:



$\Gamma_{\mathcal{D}_Q} = \mathbb{Z}Q^{op}$:



The Serre functor

Blanket assumptions

Q is a finite quiver without oriented cycles.
All categories and functors are k-linear.

Theorem (Happel, 1986)

$\mathcal{D}_Q$ admits a Serre functor (=Nakayama functor), i.e. *an autoequivalence $S : \mathcal{D}_Q \xrightarrow{\sim} \mathcal{D}_Q$ such that*

$$D\,\mathrm{Hom}(X, ?) \xrightarrow{\sim} \mathrm{Hom}(?, SX)$$

for all $X \in \mathcal{D}_Q$, where $D = \mathrm{Hom}_k(?, k)$.

Calabi-Yau categories

Let d be an integer and $\mathcal{T}$ a triangulated category with finite-dimensional Hom-spaces.

Definition (Kontsevich)

$\mathcal{T}$ is *d-Calabi-Yau* if it has a Serre functor S and $S \xrightarrow{\sim} \Sigma^d$ as triangle functors.

Example

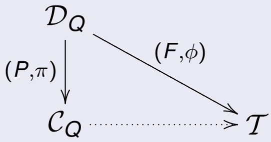
X a smooth projective variety of dimension d.
$\mathcal{T}$ the bounded derived category of coherent sheaves on X.
Then $S = ? \otimes \omega[d]$ and

$$X \text{ is Calabi-Yau} \Leftrightarrow \omega \xrightarrow{\sim} \mathcal{O} \Leftrightarrow \mathcal{T} \text{ is } d\text{-Calabi-Yau}$$

The cluster category

Definition

The cluster category $\mathcal{C}_Q$ is the universal 2-Calabi-Yau category under the derived category $\mathcal{D}_Q$:



$\mathcal{C}_Q, \mathcal{T}$ 2-Calabi-Yau
P, F triangle functors
$\pi : P \circ S \xrightarrow{\sim} S \circ P$
$\phi : F \circ S \xrightarrow{\sim} S \circ F$

Remark

$\mathcal{C}_Q$ is due to Buan-Marsh-Reineke-Reiten-Todorov for Q, to Caldero-Chapoton-Schiffler for $\vec{A}_n$.

Explicit construction of the cluster category

Remarks

1) Strictly speaking the definition should be formulated in the homotopy category of enhanced triangulated categories.
2) Explicitly, $\mathcal{C}_Q$ is the orbit category of $\mathcal{D}_Q$ under the action of the automorphism $S^{-1} \circ \Sigma^2$. It has

   objects: same as those of $\mathcal{D}_Q$

   morphisms:

$$\mathcal{C}_Q(X, Y) = \bigoplus_{p \in \mathbb{Z}} \mathcal{D}_Q(X, (S^{-1} \circ \Sigma^2)^p Y).$$

The quiver of the cluster category

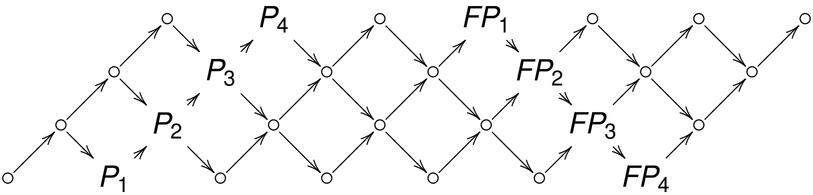
Theorem (BMRRT)

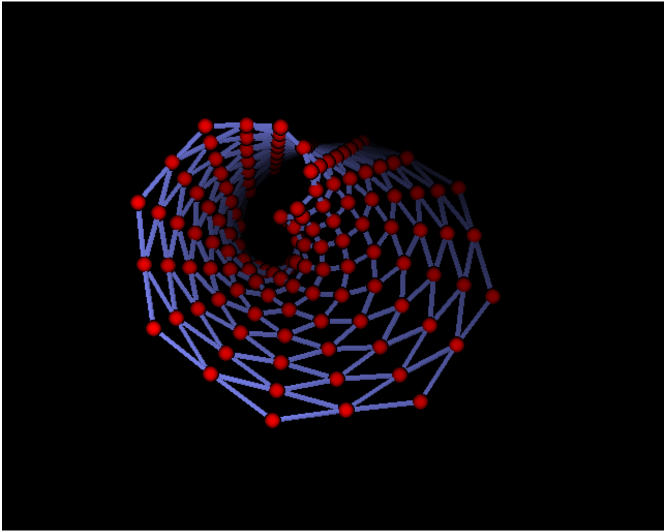
The decomposition theorem holds in $\mathcal{C}_Q$ and its quiver is isomorphic to the quotient of the quiver of $\mathcal{D}_Q$ under the action of the automorphism induced by $S^{-1} \circ \Sigma^2$.

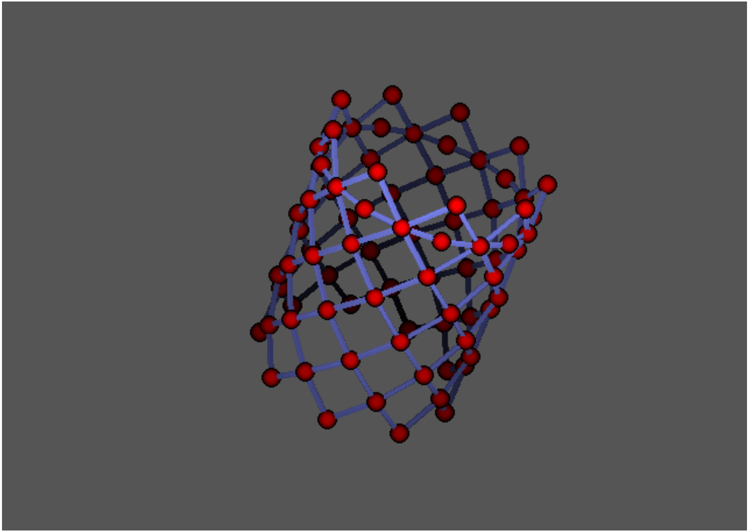
Example

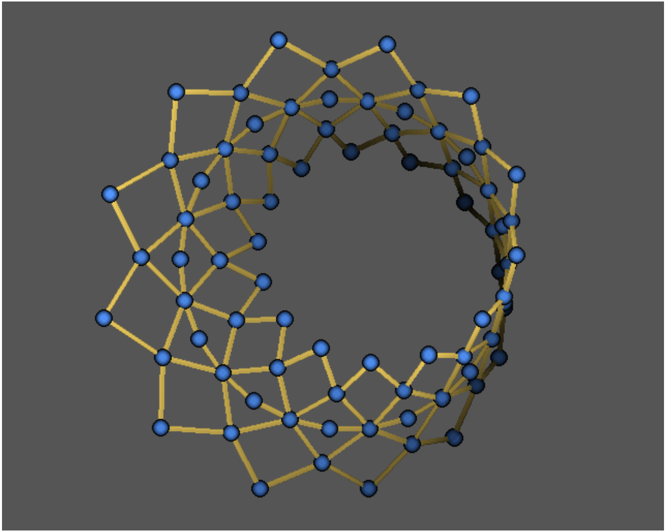
For $Q = \vec{A}_n = (1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow n)$ the quiver of $\mathcal{C}_Q$ is a Moebius strip of width n with $n(n+3)/2$ vertices. Similarly for $Q = \vec{E}_6$.

$$F = S^{-1} \circ \Sigma^2$$









The canonical cluster-tilting subcategory

Recall that we have functors $\mathrm{rep}(Q) \longrightarrow \mathcal{D}_Q \longrightarrow \mathcal{C}_Q$. Let $\mathcal{T}_Q$ be the image of the category $\mathcal{P}_Q$ of projectives of $\mathrm{rep}(Q)$.

Theorem (BMRRT)

a) *The quiver of $\mathcal{T}_Q$ is isomorphic to Q^{op}.*

b) $\mathcal{T}_Q \subset \mathcal{C}_Q$ *is a cluster-tilting subcategory,* i.e.

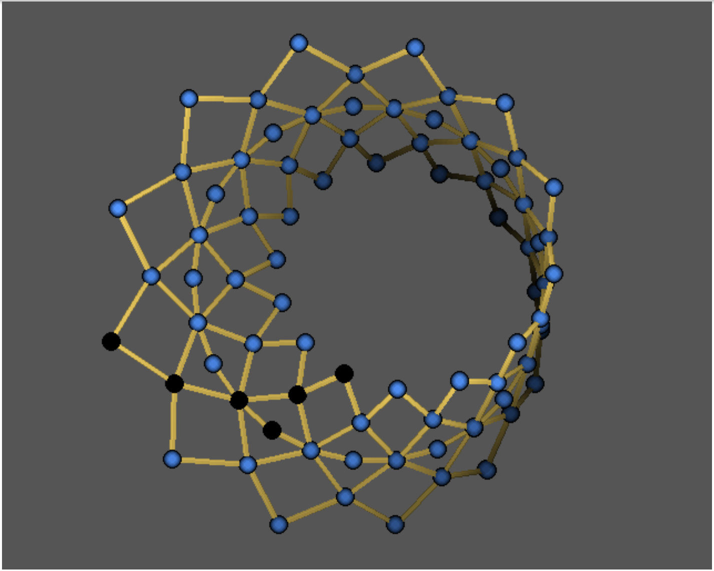
1) *for all T, T' in $\mathcal{T}_Q$, we have*

$$0 = \mathrm{Ext}^1(T, T') := \mathrm{Hom}_{\mathcal{C}_Q}(T, \Sigma T'),$$

2) *if $X \in \mathcal{C}_Q$ satisfies* $\mathrm{Ext}^1(T, X) = 0$ *for all T in $\mathcal{T}_Q$, then X belongs to $\mathcal{T}_Q$.*

Definition

$\mathcal{T}_Q$ is the *canonical cluster-tilting subcategory*.



The main theorem

Main theorem

Let

- $\mathcal{C}$ be a 2-Calabi-Yau triang. category (of 'algebraic origin'),
- $\mathcal{T} \subset \mathcal{C}$ a cluster-tilting subcategory,
- Q the opposite quiver of $\mathcal{T}$.

If Q does not have oriented cycles, then $\mathcal{C}_Q \xrightarrow{\sim} \mathcal{C}$.

Remarks

We only need to know Q, not $\mathcal{T}$! The objects of $\mathcal{T}_Q$ generate $\mathcal{C}_Q$ as a triangulated category but for general T, T' in $\mathcal{T}$, we have $\mathrm{Ext}^i(T, T') \neq 0$ for infinitely many i.

Application: Cohen-Macaulay modules

Let $k = \mathbb{C}$, ζ a primitive third root of 1. Let $G = \mathbb{Z}/3\mathbb{Z}$ act on $S = k[[X, Y, Z]]$ by multiplying the generators by ζ. Then

- $R = S^G$ is an isolated singularity of dimension 3 and Gorenstein.
- The category CM(R) of maximal Cohen-Macaulay modules is Frobenius.
- The stable category $\underline{\mathrm{CM}}(R)$ is 2-Calabi-Yau (Auslander).
- Decompose $S = S^G \oplus T_1 \oplus T_2$ over R. Then the direct sums of copies of the T_i form a cluster-tilting subcategory $\mathcal{T}$ of $\mathcal{C}$ (Iyama).
- The quiver of $\mathcal{T}$ is the generalized Kronecker quiver

$$Q: \; 1 \overset{\textstyle\longrightarrow}{\underset{\textstyle\longrightarrow}{\longrightarrow}} 2 \, .$$

Cluster categories occur in nature

Conclusion

The stable category of Cohen-Macaulay modules over S^G is triangle equivalent to the cluster category $\mathcal{C}_Q$.

Consequence

New proof of Iyama-Yoshino's classification of the rigid Cohen-Macaulay modules over S^G: sums of projectives and copies of $\Omega^a T_1, \Omega^b T_2, a, b \in \mathbb{Z}$.

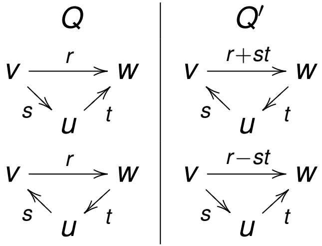
Moral

Cluster categories occur in nature.

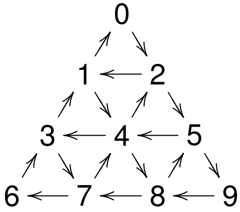
Application: Quiver mutation, I

Let Q be a quiver without loops or 2-cycles and u a vertex of Q. The mutation of Q at u (Fomin-Zelevinsky) is the quiver Q' obtained from Q as follows (where $v \xrightarrow{r} w$ = arrow of multiplicity $r \geq 0$)

1) reverse all arrows incident with u;
2) modify the other arrows as follows:



Q'' is mutation equivalent to Q if Q'' is isomorphic to a quiver obtained from Q by a finite sequence of mutations.



Theorem

This quiver Q is not mutation equivalent to a quiver Q' without oriented cycles.

Proof: Use brute force (Google 'quiver mutation'!) or use the main theorem!

Sketch of the proof via the main theorem

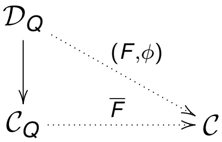
- Let $\mathcal{C}$ be the stable category of finite-dimensional modules over the preprojective algebra of type A_5.
- Crawley-Boevey: $\mathcal{C}$ is 2-Calabi-Yau.
- Geiss-Leclerc-Schröer's work implies:
  - $\mathcal{C}$ contains a cluster-tilting subcategory $\mathcal{T}$ with quiver Q,
  - if $Q \sim_{mut} Q'$, then $\mathcal{C}$ contains a cluster-tilting subcategory with quiver Q'.
- By the main theorem: If $Q \sim_{mut} Q'$ and Q' does not have oriented cycles, then $\mathcal{C} \xrightarrow{\sim} \mathcal{C}_{Q'}$.
- Contradiction: The suspension functor Σ is of order 6 in $\mathcal{C}$ but of order ∞ in $\mathcal{C}_{Q'}$ (Q' is not a Dynkin quiver).

Summary

- Cluster categories occur in nature.
- Google ‘quiver mutation’!

The (best) proof of the main theorem uses the universal property

Michel Van den Bergh: We use the universal property!



where $\phi : F \circ S \xrightarrow{\sim} S \circ F$.

1) Construct $\overline{F}$ via (F, ϕ). Subtle: Construct ϕ !
2) $\overline{F}$ is an equivalence by the

Beilinson's lemma has a cluster analogue

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\quad G \quad} & \mathcal{C} \\ \uparrow & & \uparrow \\ \mathcal{T}' & \xrightarrow{\quad G|\mathcal{T}' \quad} & \mathcal{T} \end{array}$$

Lemma (cluster-Beilinson)

- *$G : \mathcal{C}' \to \mathcal{C}$ a triangle functor between 2-CY categories*
- *$\mathcal{T}' \subset \mathcal{C}'$ a cluster tilting subcategory.*

Then G is an equivalence iff $\mathcal{T} = G(\mathcal{T}')$ is a cluster-tilting subcategory and the restriction of G to $\mathcal{T}'$ is fully faithful.