ON THE COMBINATORICS OF RIGID OBJECTS IN 2-CALABI-YAU CATEGORIES

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ABSTRACT. Given a triangulated 2-Calabi-Yau category $\mathcal{C}$ and a cluster-tilting subcategory $\mathcal{T}$, the index of an object X of $\mathcal{C}$ is a certain element of the Grothendieck group of the additive category $\mathcal{T}$. In this note, we show that a rigid object of $\mathcal{C}$ is determined by its index, that the indices of the indecomposables of a cluster-tilting subcategory $\mathcal{T}'$ form a basis of the Grothendieck group of $\mathcal{T}$ and that, if $\mathcal{T}$ and $\mathcal{T}'$ are related by a mutation, then the indices with respect to $\mathcal{T}$ and $\mathcal{T}'$ are related by a certain piecewise linear transformation introduced by Fomin and Zelevinsky in their study of cluster algebras with coefficients. This allows us to give a combinatorial construction of the indices of all rigid objects reachable from the given cluster-tilting subcategory $\mathcal{T}$. Conjecturally, these indices coincide with Fomin-Zelevinsky's $\mathbf{g}$-vectors.

1. Introduction

This note is motivated by the representation-theoretic approach to Fomin-Zelevinsky's cluster algebras [6] [7] [4] [8] developed by Marsh-Reineke-Zelevinsky [18], Buan-Marsh-Reineke-Reiten-Todorov [3], Geiss-Leclerc-Schröer [11] [12] and many others, *cf.* [2] for a survey. In this approach, a central rôle is played by certain triangulated 2-Calabi-Yau categories and by combinatorial invariants associated with their rigid objects (we refer to [14] [5] for different approaches). Here, our object of study is the index, which is a certain 'dimension vector' associated with each object of the given Calabi-Yau category.

More precisely, we fix a Hom-finite 2-Calabi-Yau triangulated category $\mathcal{C}$ with split idempotents which admits a cluster-tilting subcategory $\mathcal{T}$. It is known from [16] that for each object X of $\mathcal{C}$, there is a triangle

$$T_1 \to T_0 \to X \to \Sigma T_1$$

of $\mathcal{C}$, where T_1 and T_0 belong to $\mathcal{T}$. Following [19], we define the index of X to be the difference $[T_0] - [T_1]$ in the split Grothendieck group $K_0(\mathcal{T})$ of the additive category $\mathcal{T}$. We show that

- if X is rigid (*i.e.* $\mathcal{C}(X, \Sigma X) = 0$), then it is determined by its index up to isomorphism;
- the indices of the direct factors of a rigid object all lie in the same hyperquadrant of $K_0(\mathcal{T})$ with respect to the basis given by a system of representatives of the isomorphism classes of the indecomposables of $\mathcal{T}$;
- the indices of the direct factors of a rigid object are linearly independent;
- the indices of a system of representatives of the indecomposable objects of any cluster-tilting subcategory $\mathcal{T}'$ form a basis of $K_0(\mathcal{T})$. In particular, all cluster-tilting subcategories have the same (finite or infinite) number of pairwise non isomorphic indecomposable objects.

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Note that the last point was shown in Theorem I.1.8 of [1] under the additional assumption that $\mathcal{C}$ is a stable category. We then study how the index of an object transforms when we mutate the given cluster-tilting subcategory. We find that this transformation is given by the right hand side of Conjecture 7.12 of [8], *cf.* section 4. This motivates the definition of $\mathbf{g}^\dagger$-vectors as the combinatorial counterpart to indices. If, as we expect, Conjecture 7.12 of [loc. cit.] holds, then our $\mathbf{g}^\dagger$-vectors are identical with the $\mathbf{g}$-vectors of [loc. cit.], whose definition we briefly recall below. We finally show that if $\mathcal{C}$ has a cluster-structure in the sense of [1], then we have a bijection between $\mathbf{g}^\dagger$-vectors and indecomposable rigid objects reachable from $\mathcal{T}$ and between $\mathbf{g}^\dagger$-clusters and cluster-tilting subcategories reachable from T.

Our results are inspired by and closely related to the conjectures of [8] and the results of section 15 in [10]. As a help to the reader not familiar with [8], we give a short summary of the notions introduced there which are most relevant for us: Let $n \geq 1$ be an integer and B a skew-symmetric integer matrix. Let $\mathcal{F}$ be the field of rational functions $\mathbb{Q}(x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $2n$ indeterminates. Let $\mathcal{A} \subset \mathcal{F}$ be the cluster algebra with principal coefficients associated with the initial seed $(\mathbf{x}, \mathbf{y}, B)$, where $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$, *cf.* sections 1 and 2 of [8]. As shown in Proposition 3.6 of [8], each cluster variable of $\mathcal{A}$ lies in the ring $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n]$. Moreover, by Proposition 6.1 of [8], each cluster variable of $\mathcal{A}$ is homogeneous with respect to the $\mathbb{Z}^n$-grading on $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, \ldots, y_n]$ given by

$$\deg(x_i) = e_i \, , \quad \deg(y_j) = -\sum_{i=1}^{n} b_{ij} e_i \, ,$$

where the e_i form the standard basis of $\mathbb{Z}^n$. The g-vector associated with a cluster variable X is by definition the vector $\deg(X)$ of $\mathbb{Z}^n$. More generally, the g-vector of a cluster monomial M is $\deg(M)$. Now we can state the conjectures of [8] which motivated the above statements on the combinatorics of rigid objects:

- different cluster monomials have different g-vectors (part (1) of Conjecture 7.10 of [8]);
- the g-vectors of the variables in a fixed cluster all lie in the same hyperquadrant of $\mathbb{Z}^n$ (Conjecture 6.13 of [8]);
- the g-vectors of the variables in a fixed cluster form a basis of $\mathbb{Z}^n$ (part (2) of Conjecture 7.10 of [8]);
- under a mutation of the initial cluster, the g-vector of a given cluster variable transforms according to a certain piecewise linear transformation, *cf.* section 4 (Conjecture 7.12 of [8]).

In [9], the results of this paper have been used to prove these conjectures for certain classes of cluster algebras.

2. A rigid object is determined by its index

Let k be an algebraically closed field and $\mathcal{C}$ a Hom-finite k-linear triangulated category with split idempotents. In particular, the decomposition theorem holds for $\mathcal{C}$: Each object decomposes into finite sum of indecomposable objects, unique up to isomorphism, and indecomposable objects have local endomorphism rings. We write Σ for the suspension functor of $\mathcal{C}$. We suppose that $\mathcal{C}$ is 2-Calabi-Yau, *i.e.* that the square of the suspension functor (with its canonical structure of triangle functor) is a Serre functor for $\mathcal{C}$. This implies that we have bifunctorial isomorphisms

$$D\mathcal{C}(X,Y) \xrightarrow{\sim} \mathcal{C}\,(Y, \Sigma^2 X) \, ,$$

where X and Y vary in $\mathcal{C}$ and D denotes the duality functor $\mathsf{Hom}_k(?,k)$ over the ground field. Moreover, we suppose that $\mathcal{C}$ admits a cluster-tilting subcategory $\mathcal{T}$ (called a maximal 1-orthogonal subcategory in [13]). Recall from [16] that this means that $\mathcal{T}$ is a full additive subcategory such that

- $\mathcal{T}$ is functorially finite in $\mathcal{C}$, *i.e.* for all objects X of $\mathcal{C}$, the restrictions of the functors $\mathcal{C}(X,?)$ and $\mathcal{C}(?,X)$ to $\mathcal{T}$ are finitely generated, and
- an object X of $\mathcal{C}$ belongs to $\mathcal{T}$ iff we have $\mathcal{C}(T,\Sigma X)=0$ for all objects T of $\mathcal{T}$.

We call an object X of $\mathcal{C}$ *rigid* if the space $\mathcal{C}(X,\Sigma X)$ vanishes.

2.1. **Rigid objects yield open orbits.** Let X be a rigid object of $\mathcal{C}$. From [17], we know that there is a triangle

$$T_1 \xrightarrow{f} T_0 \xrightarrow{h} X \longrightarrow \Sigma X \ ,$$

where T_0 and T_1 belong to $\mathcal{T}$. The algebraic group $G=\mathsf{Aut}(T_0)\times\mathsf{Aut}(T_1)$ acts on $\mathcal{C}(T_1,T_0)$ via

$$(g_0,g_1)f' = g_0 f' g_1^{-1}.$$

Lemma. *The orbit of f under the action of G is open in $\mathcal{C}(T_1,T_0)$.*

Proof. It suffices to prove that the differential of the map $g\mapsto gf$ is a surjection from $\mathsf{Lie}(G)$ to $\mathcal{C}(T_1,T_0)$. This differential is given by

$$(\gamma_0,\gamma_1)f = \gamma_0 f - f\gamma_1.$$

Let f' be an element of $\mathcal{C}(T_1,T_0)$. Consider the following diagram

$$\begin{array}{ccccccc}
\Sigma^{-1}X & \xrightarrow{e} & T_1 & \xrightarrow{f} & T_0 & \xrightarrow{h} & X \\
 & \swarrow{\scriptstyle\gamma_1} & \downarrow{\scriptstyle f'} & \swarrow{\scriptstyle\gamma_0} & \downarrow{\scriptstyle\beta_0} & & \\
T_1 & \xrightarrow[f]{} & T_0 & \xrightarrow[h]{} & X & \longrightarrow & \Sigma T_1.
\end{array}$$

Since X is rigid, the composition $hf'e$ vanishes. So there is a β_0 such that $\beta_0 f = hf'$. Now h is a right $\mathcal{T}$-approximation. So there is a γ_0 such that $h\gamma_0=\beta_0$. It follows that we have

$$h(\gamma_0 f - f') = 0.$$

So there is a γ_1 such that

$$\gamma_0 f - f' = f\gamma_1.$$

This shows that the differential of the map $g\mapsto gf$ is indeed surjective. □

2.2. **Rigid objects have disjoint terms in their minimal presentations.** Let

$$F:\mathcal{C}\to\mathsf{mod}\,\mathcal{T}$$

be the functor taking an object Y of $\mathcal{C}$ to the restriction of $\mathcal{C}(?,Y)$ to $\mathcal{T}$. Let X be a rigid object of $\mathcal{C}$. Let

$$T_1 \longrightarrow T_0 \xrightarrow{h} X \xrightarrow{\varepsilon} \Sigma T_1$$

be a triangle such that T_0 and T_1 belong to $\mathcal{T}$ and h is a minimal right $\mathcal{T}$-approximation.

Proposition. *T_0 and T_1 do not have an indecomposable direct factor in common.*

We give two proofs of the proposition. Here is the first one:

Proof. We know that

$$FT_1 \to FT_0 \to FX \to 0$$

is a minimal projective presentation of FX. Since F induces an equivalence from $\mathcal{T}$ onto the category of projectives of $\mathsf{mod}\,\mathcal{T}$, it is enough to show that FT_1 and FT_0 do not have an indecomposable factor in common. For this, it suffices to show that no simple module S occuring in the head of FT_0 also occurs in the head of FT_1. Equivalently, we have to show that if a simple S satisfies $\mathsf{Hom}(FX, S) \neq 0$, then we have $\mathsf{Ext}^1(FX, S) = 0$. So let S be a simple admitting a surjective morphism

$$p : FX \to S.$$

Let $f : FT_1 \to S$ be a map representing an element in $\mathsf{Ext}^1(FX, S)$. Since FT_1 is projective, there is a morphism $f_1 : FT_1 \to FX$ such that $p \circ f_1 = f$. Now using the fact that F is essentially surjective and full, we choose a preimage up to isomorphism $\widetilde{S}$ of S and preimages $\widetilde{f}$, $\widetilde{p}$ and $\widetilde{f_1}$ of f, p and f_1 in $\mathcal{C}$ as in the following diagram

$$\begin{array}{ccccccc} \Sigma^{-1}X & \xrightarrow{\Sigma^{-1}\varepsilon} & T_1 & \longrightarrow & T_0 & \longrightarrow & X \\ & {\scriptstyle \widetilde{f_1}}\swarrow & \downarrow{\scriptstyle \widetilde{f}} & & & & \\ X & \xrightarrow[\widetilde{p}]{} & \widetilde{S} & & & & \end{array}$$

Denote by $\mathsf{mod}\,\mathcal{T}$ the category of finitely presented k-linear functors from $\mathcal{T}^{op}$ to the category of k-vector spaces. Since F induces a bijection

$$\mathcal{C}(T, Y) \to (\mathsf{mod}\,\mathcal{T})(FT, FY)$$

for all Y in $\mathcal{C}$, we still have $\widetilde{p} \circ \widetilde{f_1} = \widetilde{f}$. The composition $\widetilde{f_1} \circ (\Sigma^{-1}\varepsilon)$ vanishes since we have $\mathcal{C}(\Sigma^{-1}X, X) = 0$. Therefore, the composition

$$\widetilde{f} \circ (\Sigma^{-1}\varepsilon) = \widetilde{p} \circ \widetilde{f_1} \circ (\Sigma^{-1}\varepsilon)$$

vanishes. This implies that $\widetilde{f}$ factors through the morphism $T_1 \to T_0$. But then f factors through the morphism $FT_1 \to FT_0$ and f represents 0 in $\mathsf{Ext}^1(FX, S)$. □

Let us now give a second, more geometric, proof of the proposition:

Proof. Suppose that T_0 and T_1 have an indecomposable direct factor T_2 so that we have decompositions

$$T_0 = T_0' \oplus T_2 \text{ and } T_1 = T_1' \oplus T_2.$$

For a morphism $f : T_1 \to T_0$, let

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

be the matrix corresponding to f with respect to the given decompositions. Of course, up to isomorphism, the cone on f only depends on the orbit of f under the group $\mathsf{Aut}(T_0) \times \mathsf{Aut}(T_1)$. Suppose that the cone on f is isomorphic to X, which is rigid. Then we know that the orbit of f in $\mathcal{C}(T_1, T_0)$ is open. Hence there is some f' in the orbit such that the component f_{22}' is invertible. But then, using elementary operations on the rows and columns of the matrix of f', we see that the orbit of f contains a morphism f'' whose matrix is diagonal with invertible component f_{22}''. Clearly, the triangle on f'' is not minimal. This shows that T_1 and T_0 do not have a common indecomposable factor if they are the terms of a minimal triangle whose third term is the rigid object X. □

2.3. **A rigid object is determined by its index.** The (split) Grothendieck group $K_0(\mathcal{T})$ of the additive category $\mathcal{T}$ is the quotient of the free group on the isomorphism classes $[T]$ of objects T of $\mathcal{T}$ by the subgroup generated by the elements of the form

$$[T_1 \oplus T_2] - [T_1] - [T_2].$$

It is canonically isomorphic to the free abelian group on the isomorphism classes of the indecomposable objects of $\mathcal{T}$. It contains a canonical positive cone formed by the classes of objects of $\mathcal{T}$. Each element c of $K_0(\mathcal{T})$ can be uniquely written as

$$c = [T_0] - [T_1]$$

where T_0 and T_1 are objects of $\mathcal{T}$ without common indecomposable factors. Let X be an object of $\mathcal{C}$. Recall that its index [19] is the element

$$\mathsf{ind}(X) = [T_0] - [T_1]$$

of $K_0(\mathcal{T})$ where T_0 and T_1 are objects of $\mathcal{T}$ which occur in an arbitrary triangle

$$T_1 \to T_0 \to X \to \Sigma T_1.$$

Now suppose that X is rigid. We know that if we choose the above triangle minimal, then T_0 and T_1 do not have common indecomposable factors. Thus they are determined by $\mathsf{ind}(X)$. Moreover, since the $\mathcal{C}(T_1, T_0)$ is an irreducible variety (like any finite-dimensional vector space), each morphism $f : T_1 \to T_0$ whose orbit under the group $\mathsf{Aut}(T_0) \times \mathsf{Aut}(T_1)$ is open yields a cone isomorphic to X. Thus up to isomorphism, X is determined by $\mathsf{ind}(X)$. In fact, X is isomorphic to the cone on a general morphism $f : T_1 \to T_0$ between the objects T_0 and T_1 without a common indecomposable factor such that $\mathsf{ind}(X) = [T_0] - [T_1]$. We have proved the

Theorem. *The map $X \mapsto \mathsf{ind}(X)$ induces an injection from the set of isomorphism classes of rigid objects of $\mathcal{C}$ into the set $K_0(\mathcal{T})$.*

This theorem was inspired by part (1) of conjecture 7.10 in [8].

2.4. **Direct factors of rigid objects have sign-coherent indices.** Let A be a free abelian group endowed with a basis e_i, $i \in I$. A subset $X \subset A$ is *sign-coherent* if, for all elements $x, y \in X$ and for all $i \in I$, the sign of the component x_i in the decomposition

$$x = \sum x_i e_i$$

agrees with the sign of y_i, *cf.* Definition 6.12 of [8]. This means that the set X is entirely contained in a hyperquadrant of A with respect to the given basis e_i, $i \in I$. Now consider the free abelian group $K_0(\mathcal{T})$ endowed with the basis formed by the classes of indecomposable objects of $\mathcal{T}$. Suppose that X is a rigid object of $\mathcal{C}$. We claim that the set of indices of the direct factors of X is sign-coherent. Indeed, let U and V be direct factors of X. Choose minimal triangles

$$T_1^U \to T_0^U \to U \to \Sigma T_1^U \text{ and } T_1^V \to T_0^V \to V \to \Sigma T_1^V ,$$

where the T_i^U and T_i^V belong to $\mathcal{T}$. Then the triangle

$$T_1^U \oplus T_1^V \to T_0^U \oplus T_0^V \to U \oplus V \to \Sigma(T_1^U \oplus T_1^V)$$

is minimal. Since $U \oplus V$ is rigid, the two terms $T_1^U \oplus T_1^V$ and $T_0^U \oplus T_0^V$ do not have indecomposable direct factors in common. In particular, whenever an indecomposable object occurs in T_0^U (resp. T_1^U), it does not occur in T_1^V (resp. T_0^V). This shows that $\mathsf{ind}(U)$ and $\mathsf{ind}(V)$ are sign-coherent. This property is to be compared with conjecture 6.13 of [8].

2.5. **Indices of factors of rigid objects are linearly independent.** Let X be a rigid object of $\mathcal{C}$ and let X_i, $i \in I$, be a finite family of indecomposable direct factors of X which are pairwise non isomorphic. We claim that the elements $\mathsf{ind}(X_i)$, $i \in I$, are linearly independent in $K_0(\mathcal{T})$. Indeed, suppose that we have a relation

$$\sum_{i \in I_1} c_i \,\mathsf{ind}(X_i) = \sum_{j \in I_2} c_j \,\mathsf{ind}(X_j)$$

for two disjoint subsets I_1 and I_2 of I and positive integers c_i and c_j. Then the rigid objects

$$\bigoplus_{i \in I_1} X_i^{c_i} \text{ and } \bigoplus_{j \in I_2} X_j^{c_j}$$

have equal indices. So they are isomorphic. Since I_1 and I_2 are disjoint, all the c_i and c_j have to vanish.

2.6. **The indices of the indecomposables of a cluster tilting subcategory form a basis.** The following theorem was inspired by part (2) of conjecture 7.10 of [8].

Theorem. *Let $\mathcal{T}'$ be another tilting subcategory of $\mathcal{C}$. Then the elements $\mathsf{ind}(T')$, where T' runs through a system of representatives of the isomorphism classes of indecomposables of $\mathcal{T}'$, form a basis of the free abelian group $K_0(\mathcal{T})$.*

Proof. Indeed, we already know that the $\mathsf{ind}(T')$ are linearly independent. So it is enough to show that the subgroup they generate contains $\mathsf{ind}(T)$ for each indecomposable T of $\mathcal{T}$. Indeed, let T be an indecomposable of $\mathcal{T}$ and let

$$T \to T'_1 \to T'_0 \to \Sigma T$$

be a triangle with T'_i in $\mathcal{T}'$ (this triangle allows to compute the index of ΣT with respect to $\mathcal{T}'$). Then the map $FT'_1 \to FT'_0$ is surjective and therefore, we have

$$\mathsf{ind}(T) - \mathsf{ind}(T'_1) + \mathsf{ind}(T'_0) = 0$$

by Proposition 6 of [19]. Thus, $\mathsf{ind}(T)$ is in the subgroup of $K_0(\mathcal{T})$ generated by the $\mathsf{ind}(T')$, where T' runs through the indecomposables of $\mathcal{T}'$. □

3. How the index transforms under change of cluster-tilting subcategory

Let $\mathcal{T}'$ be another cluster-tilting subcategory. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are *related by a mutation*, *i.e.* there is an indecomposable S of $\mathcal{T}$ and an indecomposable S^* of $\mathcal{T}'$ such that, if indec denotes the set of isomorphism classes of indecomposables, we have

$$\mathsf{indec}(\mathcal{T}') = \mathsf{indec}(\mathcal{T}) \setminus \{S\} \cup \{S^*\},$$

and that there exist triangles

$$S^* \to B \to S \to \Sigma S^* \text{ and } S \to B' \to S^* \to \Sigma S$$

with B and B' belonging to $\mathcal{T} \cap \mathcal{T}'$, *cf. e.g.* [3] [12] [15]. We define two linear maps

$$\phi_+ : K_0(\mathcal{T}) \to K_0(\mathcal{T}') \text{ and } \phi_- : K_0(\mathcal{T}) \to K_0(\mathcal{T}')$$

which both send each indecomposable T'' belonging to both $\mathcal{T}$ and $\mathcal{T}'$ to itself and such that

$$\phi_+(S) = [B] - [S^*] \text{ and } \phi_-(S) = [B'] - [S^*].$$

For an object X of $\mathcal{C}$, we denote by $\mathsf{ind}_{\mathcal{T}}(X)$ the index of X with respect to $\mathcal{T}$ and by $[\mathsf{ind}_{\mathcal{T}}(X) : S]$ the coefficient of S in the decomposition of $\mathsf{ind}_{\mathcal{T}}(X)$ with respect to the basis given by the indecomposables of $\mathcal{T}$. The following theorem is inspired by Conjecture 7.12 of [8].

Theorem. *Let X be a rigid object of $\mathcal{C}$. We have*

$$\mathsf{ind}_{\mathcal{T}'}(X) = \begin{cases} \phi_+(\mathsf{ind}_{\mathcal{T}}(X)) & \textit{if} \quad [\mathsf{ind}_{\mathcal{T}}(X) : S] \geq 0 \ ; \\ \phi_-(\mathsf{ind}_{\mathcal{T}}(X)) & \textit{if} \quad [\mathsf{ind}_{\mathcal{T}}(X) : S] \leq 0. \end{cases}$$

Proof. Let

$$T_1 \to T_0 \to X \to \Sigma T_1$$

be a triangle with T_0 and T_1 in $\mathcal{T}$. Suppose first that S occurs neither as a direct factor of T_1 nor of T_0. Then clearly the triangle yields both the index of X with respect to $\mathcal{T}$ and with respect to $\mathcal{T}'$ and we have

$$\phi_+(\mathsf{ind}_{\mathcal{T}}(X)) = \phi_-(\mathsf{ind}_{\mathcal{T}}(X)) = \mathsf{ind}_{\mathcal{T}'}(X).$$

Now suppose that the multiplicity $[\mathsf{ind}_{\mathcal{T}}(X) : S]$ equals a positive integer $i \geq 1$. This means that S occurs with multiplicity i in T_0 but does not occur as a direct factor of T_1. Choose a decomposition $T_0 = T_0'' \oplus S^i$. From the octahedron constructed over the composition

$$T_0'' \oplus B^i \to T_0'' \oplus S^i \to X \ ,$$

we extract the following commutative diagram, whose rows and columns are triangles

$$\begin{array}{ccccccc}
\Sigma S^{*i} & \xrightarrow{\mathbf{1}} & \Sigma S^{*i} & & & & \\
\uparrow & & \uparrow & & & & \\
T_1 & \longrightarrow & T_0'' \oplus S^i & \longrightarrow & X & \longrightarrow & \Sigma T_1 \\
\uparrow & & \uparrow & & \uparrow{\scriptstyle \mathbf{1}} & & \uparrow \\
T_1' & \longrightarrow & T_0'' \oplus B^i & \longrightarrow & X & \longrightarrow & \Sigma T_1' \\
\uparrow & & \uparrow & & & & \\
S^{*i} & \xrightarrow{\mathbf{1}} & S^{*i}. & & & &
\end{array}$$

Since there are no non zero morphisms from T_1 to ΣS^{*i} (T_1 and S^* belong to $\mathcal{T}'$), the leftmost column is a split triangle and T_1' is isomorphic to $S^{*i} \oplus T_1$. Thus, the third line yields the index of X with respect to $\mathcal{T}'$, which equals

$$\mathsf{ind}_{\mathcal{T}'}(X) = [T_0'' \oplus B^i] - [T_1'] = [T_0''] - [T_1] + i([B] - [S^*]) = \phi_+(\mathsf{ind}_{\mathcal{T}}(X)).$$

Finally, suppose that the multiplicity $[\mathsf{ind}_{\mathcal{T}}(X) : S]$ is equals a negative integer $-i \leq -1$. This means that S occurs with multiplicity i in T_1 but does not occur in T_0. Choose a decomposition $T_1 = T_1'' \oplus S^i$. From the octahedron over the composition

$$\Sigma^{-1} X \to T_1'' \oplus S^i \to T_1'' \oplus B'^i \ ,$$

we extract the following diagram, whose rows and columns are triangles

$$
\begin{array}{ccccccc}
 & & \Sigma^{-1}S^{*i} & \xrightarrow{\mathbf{1}} & \Sigma^{-1}S^{*i} & & \\
 & & \downarrow & & \downarrow & & \\
\Sigma^{-1}X & \longrightarrow & T_1'' \oplus S^i & \longrightarrow & T_0 & \longrightarrow & X \\
\mathbf{1}\downarrow & & \downarrow & & \downarrow & & \downarrow\mathbf{1} \\
\Sigma^{-1}X & \longrightarrow & T_1'' \oplus B'^i & \longrightarrow & T_0' & \longrightarrow & X \\
 & & \downarrow & & \downarrow & & \\
 & & S^{*i} & \xrightarrow[\mathbf{1}]{} & S^{*i} & &
\end{array}
$$

Since there are no non zero morphisms from $\Sigma^{-1}S^{*i}$ to T_0 (S^* and T_0 belong to $\mathcal{T}'$), the object T_0' is isomorphic to $T_0 \oplus S^i$ and we can read $\mathsf{ind}_{\mathcal{T}'}(X)$ off the third line of the diagram:

$$\mathsf{ind}_{\mathcal{T}'}(X) = [T_0'] - [T_1'' \oplus B'^i] = [T_0 \oplus S^{*i}] - [T_1''] - i[B'] = [T_0] - [T_1''] - i([B'] - [S^*]) = \phi_-(\mathsf{ind}_{\mathcal{T}}(X)).$$

□

4. $\mathbf{g}^\dagger$-vectors and $\mathbf{g}^\dagger$-clusters

In this section, we recall fundamental constructions from [8] in a language adapted to our applications. We will define $\mathbf{g}^\dagger$-vectors using the right hand side of Conjecture 7.12 of [loc. cit.]. If, as we expect, this conjecture holds, then our $\mathbf{g}^\dagger$-vectors are identical with the $\mathbf{g}$-vectors of [loc. cit.].

Let Q be a quiver. Thus Q is given by a set of vertices $I = Q_0$, a set of arrows Q_1 and two maps s and t from Q_1 to $I = Q_0$ taking an arrow to its source, respectively its target. We assume that Q is *locally finite*, *i.e.* for each given vertex i of Q there are only finitely many arrows α such that $s(\alpha) = i$ or $t(\alpha) = i$. Moreover, we assume that Q has no loops (*i.e.* arrows α such that $s(\alpha) = t(\alpha)$) and no 2-cycles (*i.e.* pairs of distinct arrows $\alpha \neq \beta$ such that $s(\alpha) = t(\beta)$ and $t(\beta) = s(\alpha)$). The quiver Q is thus determined by the set I and the skew-symmetric integer matrix $B = (b_{ij})_{I\times I}$ such that, whenever the coefficient b_{ij} is positive, it equals the number of arrows from i to j in Q. Notice that if, for an integer x, we write $[x]_+ = \max(x, 0)$, then the number of arrows from i to j in Q is $[b_{ij}]_+$. The *mutation* $\mu_k(Q)$ of Q at a vertex k is by definition the quiver with vertex set I whose numbers of arrows are given by the mutated matrix $B' = \mu_k(B)$ as defined, for example, in definition 2.4 of [8]:

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \mathrm{sgn}(b_{ik})[b_{ij}b_{kj}]_+ & \text{otherwise.} \end{cases}$$

As in definition 2.8 of [8], we let $\mathbb{T} = \mathbb{T}_I$ be the regular tree whose edges are labeled by the elements of I such that for each vertex t and each element k of I, there is precisely one edge incident with t and labeled by k. We fix a vertex t_0 of $\mathbb{T}$ and define $Q_{t_0} = Q$. Clearly, there is a unique map assigning a quiver Q_t to each vertex t such that if t and t' are linked by an edge labeled by k, we have $Q_{t'} = \mu_k(Q_t)$. In analogy with the terminology of [8], we call the map $t \mapsto Q_t$ the *quiver pattern* associated with t_0 and Q.

Now for each vertex t of $\mathbb{T}$, we define K_t to be the free abelian group on the symbols e_i^t, $i \in I$. For two vertices t and t' linked by an edge labeled k, we let

$$\phi^+_{t',t} : K_t \to K_{t'} \text{ respectively } \phi^-_{t',t} : K_t \to K_{t'}$$

be the linear map sending e_j^t to $e_j^{t'}$ for each $j \neq k$ and sending e_k to

$$-e_k^{t'} + \Sigma_j [b_{jk}^t]_+ \, e_j^{t'} \text{ respectively } - e_k^{t'} + \Sigma_j [b_{kj}^t]_+ \, e_j^{t'} \, ,$$

where (b_{ij}^t) is the skew-symmetric matrix associated with the quiver Q_t. We define the piecewise linear transformation

$$\phi_{t',t} : K_t \to K_{t'}$$

to be the map whose restriction to the halfspace of elements with positive e_k^t-coordinate is $\phi_{t',t}^+$ and whose restriction to the opposite halfspace is $\phi_{t',t}^-$. Thus, the image of an element g with coordinates g_j, $j \in I$, is the element g' with coordinates

$$g'_j = \begin{cases} -g_j & \text{if } j = k; \\ g_j + [b_{jk}^t]_+ \, g_k & \text{if } j \neq k \text{ and } g_k \geq 0; \\ g_j + [b_{kj}^t]_+ \, g_k & \text{if } j \neq k \text{ and } g_k \leq 0. \end{cases}$$

It is easy to check that this rule agrees with formula (7.18) in Conjecture 7.12 of [8].

If t and t' are two arbitrary vertices of $\mathbb{T}$, there is a unique path

$$t = t_1 \text{ ---- } t_2 \text{ ---- } \cdots \text{ ---- } t_N = t'$$

of edges leading from t to t' and we define $\phi_{t',t}$ to be the composition

$$\phi_{t_N, t_{N-1}} \circ \cdots \circ \phi_{t_2, t_1} .$$

For a vertex t of $\mathbb{T}$ and a vertex l of Q, the $\mathbf{g}^\dagger$-*vector* $\mathbf{g}_{l,t}^\dagger$ is the element of the abelian group K_{t_0} defined by

$$\mathbf{g}_{l,t}^\dagger = \phi_{t_0,t}(e_l^t).$$

The $\mathbf{g}^\dagger$-*cluster* associated with a vertex t of $\mathbb{T}$ is the set of $\mathbf{g}^\dagger$-vectors $\mathbf{g}_{l,t}^\dagger$, $l \in I$. If Conjecture 7.12 of [8] holds for the cluster algebra with principal coefficients associated with the matrix B, then it is clear that in the notations of formula (6.4) of [8], we have

$$\mathbf{g}_{l,t}^\dagger = \mathbf{g}_{l,t}$$

for all vertices t of $\mathbb{T}$ and all $l \in I$, , *i.e.* the $\mathbf{g}^\dagger$-vectors equal the $\mathbf{g}$-vectors for the cluster algebra with principal coefficients associated with the skew-symmetric matrix B.

5. Rigid objects in 2-Calabi-Yau categories with cluster structure

Let $\mathcal{C}$ be a Hom-finite 2-Calabi-Yau category with a cluster-tilting subcategory $\mathcal{T}$. Let $Q = Q(\mathcal{T})$ be the *quiver of* $\mathcal{T}$. Recall that this means that the vertices of Q are the isomorphism classes of indecomposable objects of $\mathcal{T}$ and that the number of arrows from the isoclass of T_1 to that of T_2 equals the dimension of the space of irreducible morphisms

$$\mathsf{irr}(T_1, T_2) = \mathsf{rad}(T_1, T_2)/\mathsf{rad}^2(T_1, T_2) \, ,$$

where rad denotes the radical of $\mathcal{T}$, *i.e.* the ideal such that $\mathsf{rad}(T_1, T_2)$ is formed by all non isomorphisms from T_1 to T_2.

We make the following *assumption on* $\mathcal{C}$: For each cluster-tilting subcategory $\mathcal{T}'$ of $\mathcal{C}$, the quiver $Q(\mathcal{T}')$ does not have loops or 2-cycles. We refer to section 1, page 11 of [1] for a list of classes of examples where this assumption holds. By theorem 1.6 of [1], the assumption implies that the cluster-tilting subcategories of $\mathcal{C}$ determine a cluster structure for $\mathcal{C}$. Let us recall what this means:

1) For each cluster-tilting subcategory $\mathcal{T}'$ of $\mathcal{C}$ and each indecomposable S of $\mathcal{T}'$, there is a unique (up to isomorphism) indecomposable S^* not isomorphic to M and such that the additive subcategory $\mathcal{T}'' = \mu_S(\mathcal{T}')$ of $\mathcal{C}$ with
$$\mathsf{indec}(\mathcal{T}'') = \mathsf{indec}(\mathcal{T}') \setminus \{S\} \cup \{S^*\}$$
is a cluster-tilting subcategory;
2) the space of morphisms from S to ΣS^* is one-dimensional and in the non-split triangles
$$S^* \to B \to S \to \Sigma S^* \text{ and } S \to B' \to S^* \to \Sigma S$$
the objects B and B' belong to $\mathcal{T}' \cap \mathcal{T}''$;
3) the multiplicity of an indecomposable L of $\mathcal{T}' \cap \mathcal{T}''$ in B equals the number of arrows from L to S in $Q(\mathcal{T}')$ and that from S^* to L in $Q(\mathcal{T}'')$; the multiplicity of L in B' equals the number of arrows from S to L in $Q(\mathcal{T}')$ and that from L to S^* in $Q(\mathcal{T}'')$;
4) finally, we have $Q(\mathcal{T}'') = \mu_S(Q(\mathcal{T}'))$.

Let $Q = Q(\mathcal{T})$ be the quiver of $\mathcal{T}$. Notice that its set of vertices is the set $Q_0 = I$ of isomorphism classes of indecomposables of $\mathcal{T}$. Let $\mathbb{T}$ be the regular tree associated with Q as in section 4. We fix a vertex t_0 of $\mathbb{T}$ and put $\mathcal{T}_{t_0} = \mathcal{T}$. For two cluster tilting subcategories $\mathcal{T}'$ and $\mathcal{T}''$ as above, let $\psi_{\mathcal{T}'',\mathcal{T}'} : \mathsf{indec}(\mathcal{T}') \to \mathsf{indec}(\mathcal{T}'')$ be the bijection taking S to S^* and fixing all other indecomposables.

Thanks to point 1), with each vertex t of $\mathbb{T}$, we can associate

a) a unique cluster-tilting subcatgory $\mathcal{T}_t$ and
b) a unique bijection
$$\psi_{t,t_0} : \mathsf{indec}(\mathcal{T}_{t_0}) \to \mathsf{indec}(\mathcal{T}_t)$$

such that $\mathcal{T}_{t_0} = \mathcal{T}$ and that, whenever two vertices t and t' are linked by an edge labeled by an indecomposable S of $\mathcal{T} = \mathcal{T}_{t_0}$, we have

a) $\mathcal{T}_{t'} = \mu_{S'}(\mathcal{T}_t)$, where $S' = \psi_{t,t_0}(S)$, and
b) $\psi_{t',t_0} = \psi_{t',t} \circ \psi_{t,t_0}$.

Moreover, thanks to point 4), the map $t \mapsto Q(\mathcal{T}_t)$ is the quiver-pattern associated with Q and t_0 in section 4. Notice that the group $K_0(\mathcal{T})$ with the basis formed by the isomorphism classes of indecomposables canonically identifies with the free abelian group K_{t_0} of section 4. We define a cluster-tilting subcategory $\mathcal{T}'$ to be *reachable from* $\mathcal{T}$ if we have $\mathcal{T}' = \mathcal{T}_t$ for some vertex t of the tree $\mathbb{T}$. We define a rigid indecomposable M to be *reachable from* $\mathcal{T}$ if it belongs to a cluster-tilting subcategory which is reachable from $\mathcal{T}$.

Theorem. a) *The index* $\mathsf{ind}(M)$ *of a rigid indecomposable reachable from* $\mathcal{T}$ *is a* $\mathbf{g}^\dagger$*-vector and the map* $M \mapsto \mathsf{ind}(M)$ *induces a bijection from the set of isomorphism classes of rigid indecomposables reachable from* $\mathcal{T}$ *onto the set of* $\mathbf{g}^\dagger$*-vectors.*
b) *Under the bijection* $M \mapsto \mathsf{ind}(M)$ *of a), the cluster-tilting subcategories reachable from* $\mathcal{T}$ *are mapped bijectively to the* $\mathbf{g}^\dagger$*-clusters.*

Proof. a) By assumption, there is a vertex t of $\mathbb{T}$ such that M belongs to $\mathcal{T}_t$. Now we use theorem 3 and induction on the length of the path joining t_0 to t in the tree $\mathbb{T}$ to conclude that
$$\mathsf{ind}(M) = \mathbf{g}^\dagger_{M',t}\,, \quad \text{where } M = \psi_{t,t_0}(M').$$
This formula shows that the map $M \mapsto \mathsf{ind}(M)$ is a well-defined surjection onto the set of $\mathbf{g}^\dagger$-vectors. By theorem 2.3, the map $M \mapsto \mathsf{ind}(M)$ is also injective. b) By assumption, a reachable cluster-tilting subcategory $\mathcal{T}'$ is of the form $\mathcal{T}_t$ for some vertex t of the tree $\mathbb{T}$. Thus its image is the $\mathbf{g}^\dagger$-cluster associated with t. This shows that the map is well-defined and surjective. It follows from a) that it is also injective. □

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