Encyclopedia in Algebra and Applications

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Contents

Chapter 1

Derived categories

1.1 Introduction

Derived categories were conceived as a 'formalism for hyperhomology' [VER 96] in the early 1960s. At that time, they were only used by the circle around Grothendieck but by the 1990s, they had become widespread and had found their way into graduate textbooks [IVE 86, KAS 94, GEL 96, WEI 94, POS 11, ZIM 14].

According to Illusie [ILL 90], derived categories were invented by Grothendieck in the early 1960s. He needed them to formulate the duality theory for schemes which he had announced [GRO 58] at the International Congress in 1958. Grothendieck's student J.–L. Verdier worked out the essential constructions and, in the course of the year 1963, wrote down a summary of the principal results [VER 77]. Having at his disposal the required foundations Grothendieck exposed the duality theory he had conceived of in a huge manuscript, which served as a basis for the seminar [HAR 66] that Hartshorne conducted at Harvard in the autumn of the same year.

Derived categories found their first applications in duality theory in the coherent setting [HAR 66] and then also in the etale [VER 67, DEL 73] and in the locally compact setting [VER 63, VER 66, VER 69, GRI 85].

Chapter written by Bernhard KELLER.

At the beginning of the seventies, M. Sato [SAT 69] and M. Kashiwara [KAS 70] adapted Grothendieck–Verdier's methods to the study of systems of partial differential equation. Nowadays, derived categories have become the standard language of microlocal analysis (cf. [KAS 94, MEB 89, SAI 86, BOR 87]). Thanks to Brylinski–Kashiwara's proof [BRY 81] of the Kazhdan–Lusztig conjecture, they have penetrated the representation theory of Lie groups [BER 94] and finite Chevalley groups [SCO 87]. In this theory, a central rôle is played by certain abelian subcategories of derived categories which are modeled on the category of perverse sheaves [BEI 82], which originated in the sheaf-theoretic interpretation [DEL] of intersection cohomology [GOR 80, GOR 83].

In two ground-breaking papers [BE$\tilde{7}$8] [BER 78], Beilinson and Bernstein–Gelfand–Gelfand used derived categories to establish a beautiful relation between coherent sheaves on projective space and representations of certain non commutative finite-dimensional algebras. Their constructions had numerous generalizations [GEI 87, KAP 83, KAP 86, KAP 88]. They also lead D. Happel to a systematic investigation of the derived category of a finite-dimensional algebra [HAP 87, HAP 88]. He realized that derived categories provide the proper setting for tilting theory [BRE 80, HAP 82, BON 81, ANG 07]. This theory is the origin of J. Rickard's Morita theory for derived categories [RIC 89, RIC 91], cf. also [KEL 91, KEL 94]. Morita theory has further widened the range of applications of derived categories. Thus, Broué's conjectures in the modular representation theory of finite groups [BRO 88] are typical of the synthesis of precision with generality that can be achieved by the systematic use of this language.

In this chapter, we will present Grothendieck's quick definition of the derived category followed by Verdier's more elaborate construction. We will then describe the triangulated structure on the derived category and construct derived functors. These will be applied in derived Morita theory. Finally, we will outline the generalization from rings to differential graded (=dg) categories and conclude by discussing invariants under derived equivalences between dg categories.

1.2 Grothendieck's definition

Let $\mathscr{C}$ be a category and S a set of arrows of $\mathscr{C}$. Then there is a category $\mathscr{C}[S^{-1}]$ and a functor

$$Q : \mathscr{C} \longrightarrow \mathscr{C}[S^{-1}]$$

such that Qs is invertible for each $s \in S$ and each functor F such that Fs is invertible for all $s \in S$ factors uniquely through Q, cf. [GAB 67]. The category $\mathscr{C}[S^{-1}]$ is called the *localization of* $\mathscr{C}$ *at* S and Q is called the *localization functor*. A right or left adjoint to Q is automatically fully faithful.

Now let $\mathscr{A}$ be an abelian category [GRO 57], for example the category Mod$R$ of all right modules over a ring R. A *complex* over $\mathscr{A}$ is a diagram M of the form

$$\ldots \longrightarrow M^p \xrightarrow{d^p} M^{p+1} \longrightarrow \ldots$$

where $p \in \mathbb{Z}$ and $d^p d^{p-1} = 0$ for all $p \in \mathbb{Z}$. Thus, M is given by a $\mathbb{Z}$-graded object $(M^p)_{p\in\mathbb{Z}}$ together with a homogeneous endomorphism d of degree 1 such that

$$d^2 = 0.$$

The *homology* of a complex M is the $\mathbb{Z}$-graded object H^*M with components

$$H^p M = (\ker d^p)/(\operatorname{im} d^{p-1}).$$

A *morphism of complexes* $f : L \to M$ is a graded morphism homogeneous of degree 0 and which commutes with the differential. Clearly, the class of complexes and their morphisms form a category $\mathbf{C}(\mathscr{A})$. A morphism of complexes $s : M \to M'$ is a *quasi-isomorphism* if $H^p(s)$ is an isomorphism for all $p \in \mathbb{Z}$. Grothendieck defined the *derived category* $\mathbf{D}(\mathscr{A})$ to be the localization of the category of complexes $\mathbf{C}(\mathscr{A})$ at the class of all quasi-isomorphisms. This definition has the advantage of being quick and elegant but it does not give a useable description of the morphisms in the derived category.

1.3 Verdier's definition

As above, let $\mathscr{A}$ be an abelian category. Recall that a morphism of complexes $f : L \to M$ is *null-homotopic* if there is a graded morphism $h : L \to M$ homogeneous of degree -1 such that $f = d \circ h + h \circ d$. Clearly, sums of null-homotopic morphisms are null-homotopic. Moreover, if f is null-homotopic, so are $g \circ f$ and $f \circ k$ for arbitrary morphisms g and k composable with f. The *category up to homotopy* $\mathbf{H}(\mathscr{A})$ is defined as the category whose objects are the complexes and whose morphisms $L \to M$ are classes of morphisms of complexes modulo null-homotopic morphisms. Notice that the image of a null-homotopic morphism under homology is zero so that homology induces a well-defined functor on the category up to homotopy. A morphism s of the category up to homotopy is a quasi-isomorphism if its image $H^*(s)$ is invertible.

LEMMA.– *The following hold in the category up to homotopy* $\mathbf{H}(\mathscr{A})$*:*

a) All identities are quasi-isomorphisms.

b) If two among s, t, st are quasi-isomorphisms, so is the third.

c) If f is a morphism and s a quasi-isomorphism such that $fs = 0$, then there is a quasi-isomorphism t such that $tf = 0$.

d) Each diagram

$$\begin{array}{ccc} L & \xrightarrow{s} & L' \\ {\scriptstyle f}\big\downarrow & & \\ M & & \end{array}$$

where s is a quasi-isomorphism, can be completed to a commutative square

$$\begin{array}{ccc} L & \xrightarrow{s} & L' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ M & \xrightarrow[s']{} & M' \end{array}$$

where s' is a quasi-isomorphism.

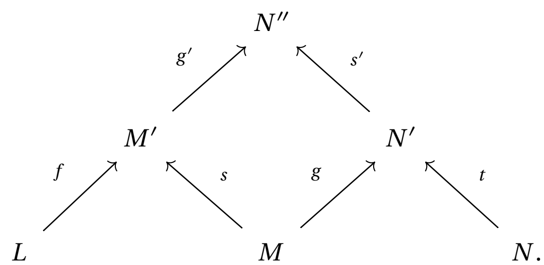
The properties in the lemma are summed up by saying that the class of quasi-isomorphisms in $\mathbf{H}(\mathscr{A})$ admits a *calculus of left fractions.* For two complexes L and M, define a *left fraction $s^{-1}f$* to be an equivalence class of diagrams

$$L \xrightarrow{f} M' \xleftarrow{s} M$$

where s is a quasi-isomorphism and two diagrams (f, s) and (g, t) are equivalent if there is a quasi-isomorphism u fitting into a commutative diagram

$$\begin{array}{ccccc} & & M' & & \\ & {\scriptstyle f}\nearrow & \big\downarrow & \nwarrow{\scriptstyle s} & \\ L & \longrightarrow & M''' & \xleftarrow{u} & M. \\ & {\scriptstyle g}\searrow & \big\uparrow & \swarrow{\scriptstyle t} & \\ & & M'' & & \end{array}$$

Verdier defines the *derived category* $\mathbf{D}(\mathscr{A})$ to have as objects all complexes and as morphisms $L \to M$ all left fractions from L to M. The *composition* of two left fractions $t^{-1}g$ and $s^{-1}f$ is defined as $(s't)^{-1}(g'f)$ using part d) of the above Lemma to complete the following commutative diagram:



It is not hard to check that Verdier's definition is equivalent to Grothendieck's. The following lemma allows us to compute morphisms in the derived category. For a category $\mathscr{C}$, we write $\mathscr{C}(X,Y)$ for the set of morphisms from X to Y.

LEMMA.–

a) If I is a left bounded complex with injective components, the canonical map

$$\mathbf{H}(\mathscr{A})(?,I) \to \mathbf{D}(\mathscr{A})(?,I)$$

is bijective.

b) if P is a right bounded complex with projective components, the canonical map

$$\mathbf{H}(\mathscr{A})(P,?) \to \mathbf{D}(\mathscr{A})(P,?)$$

is bijective.

Let $\Sigma\colon \mathbf{C}(\mathscr{A}) \to \mathbf{C}(\mathscr{A})$ be the *suspension functor*, i.e. for a complex X, we have $(\Sigma X)^p = X^{p+1}$ and $d_{\Sigma X} = -d_X$ and for a morphism of complexes f, we have $(\Sigma f)^p = f^{p+1}$. We identify $\mathscr{A}$ with the full subcategory of $\mathbf{C}(\mathscr{A})$ formed by the complexes concentrated in degree 0. Let M be an object of $\mathscr{A}$ and $M \to I$ an injective resolution, i.e. a quasi-isomorphism where I is concentrated in degrees ≥ 0 and has injective components. Then, from the lemma, we find for each complex N and each $n \in \mathbb{Z}$

$$\mathbf{D}(\mathscr{A})(N,\Sigma^n M) \xrightarrow{\sim} \mathbf{D}(\mathscr{A})(N,\Sigma^n I) \xleftarrow{\sim} \mathbf{H}(\mathscr{A})(N,\Sigma^n I)$$

If N is concentrated in degree 0, the last group is easily seen to be isomorphic to the extension group

$$\mathrm{Ext}^n_{\mathscr{A}}(N,M).$$

Here, by convention, the Ext-groups vanish in strictly negative degrees. This result also holds if we do not assume the existence of an injective resolution:

LEMMA.– *For objects N and M in $\mathscr{A}$ and each $n \in \mathbb{Z}$, we have a canonical isomorphism*

$$\mathrm{Ext}^n_{\mathscr{A}}(N,M) \xrightarrow{\sim} \mathbf{D}(\mathscr{A})(N,\Sigma^n M).$$

EXAMPLE.– If $\mathscr{A}$ is the category of vector spaces over a field k, then each object X of $\mathbf{D}(\mathscr{A})$ is canonically isomorphic to $\bigoplus_p \Sigma^{-p} H^p X$ and all extension groups vanish. So $\mathbf{D}(\mathscr{A})$ is equivalent to the category of $\mathbb{Z}$-graded vector spaces.

EXAMPLE.– If $\mathscr{A}$ is *hereditary*, i.e. we have $\mathrm{Ext}^2_{\mathscr{A}} = 0$, then each object X of $\mathbf{D}(\mathscr{A})$ is non canonically isomorphic to $\bigoplus_p \Sigma^{-p} H^p X$. The space of morphisms between two objects X and Y is isomorphic to the product over $p \in \mathbb{Z}$ of the groups

$$\mathrm{Hom}_{\mathscr{A}}(H^p X, H^p Y) \oplus \mathrm{Ext}^1_{\mathscr{A}}(H^p X, H^{p-1} Y).$$

1.4 Triangulated structure

As above, let $\mathscr{A}$ be an abelian category. The categories $\mathbf{H}(\mathscr{A})$ and $\mathbf{D}(\mathscr{A})$ are almost never abelian (they are if and only if all short exact sequences of $\mathscr{A}$ split). However, they do carry a structure induced by the short exact sequence of complexes.

A Σ-*sequence* of $\mathbf{H}(\mathscr{A})$ is a sequence of the form

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X.$$

A *morphism of Σ-sequences* is a commutative diagram of the form

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle a}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma a} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X'. \end{array}$$

For a morphism $f : L \to M$ of $\mathbf{C}(\mathscr{A})$ the *standard triangle* associated with f is the image in $\mathbf{H}(\mathscr{A})$ of the Σ-sequence

$$X \xrightarrow{f} Y \xrightarrow{i} C(f) \xrightarrow{p} \Sigma X\,,$$

where $C(f)$ is the *mapping cone* of f, i.e. the graded object $Y \oplus \Sigma X$ endowed with the differential

$$\begin{bmatrix} d_Y & f \\ 0 & d_{\Sigma X} \end{bmatrix},$$

where i and p are the canonical injection and projection. A *triangle* of $\mathbf{H}(\mathscr{A})$ is a Σ-sequence isomorphic to a standard triangle.

THEOREM.– *The following hold*

(T0) The triangles are stable under isomorphism of Σ-sequences and for each object X, the following Σ-sequence is a triangle

$$X \xrightarrow{\mathbf{1}_X} X \longrightarrow 0 \longrightarrow \Sigma X.$$

(T1) For each morphism $f : X \to Y$, there is a triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X.$$

(T2) A Σ-sequence (u, v, w) is a triangle if and only if so is $(v, w, -\Sigma u)$.

(T3) Given two triangles

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X \quad \textit{and} \quad X' \longrightarrow Y' \longrightarrow Z' \longrightarrow \Sigma X'$$

and a commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\ X' & \longrightarrow & Y' \end{array}$$

there is a (non unique) morphism c yielding a morphism of Σ-sequences

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle \Sigma a}\downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

(T4) Given two composable morphisms

$$X \xrightarrow{u} Y \xrightarrow{v} Z$$

there is a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{x} & Z' & \longrightarrow & \Sigma X \\
{\scriptstyle \mathbf{1}_X}\downarrow & & {\scriptstyle v}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathbf{1}_{\Sigma X}} \\
X & \longrightarrow & Z & \longrightarrow & Y' & \longrightarrow & \Sigma X \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Sigma u} \\
 & & X' & \xrightarrow{\mathbf{1}_{X'}} & X' & \xrightarrow{r} & \Sigma Y \\
 & & {\scriptstyle r}\downarrow & & \downarrow & & \\
 & & \Sigma Y & \xrightarrow{\Sigma x} & \Sigma Z' & &
\end{array}$$

where the first two rows and the two central columns are triangles.

A *triangulated category* is an additive category endowed with an autoequivalence Σ and a class of distinguished Σ-sequences called *triangles* such that the properties T0–T4 of the Theorem hold. Thus, the category up to homotopy $\mathbf{H}(\mathscr{A})$ is a triangulated category.

The most important consequence of the axioms T0–T3 is that, for each triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

of a triangulated category $\mathscr{T}$, the induced sequences

$$\ldots \longrightarrow \mathscr{T}(?, X) \longrightarrow \mathscr{T}(?, Y) \longrightarrow \mathscr{T}(?, Z) \longrightarrow \ldots$$

and

$$\ldots \longrightarrow \mathscr{T}(Z, ?) \longrightarrow \mathscr{T}(Y, ?) \longrightarrow \mathscr{T}(X, ?) \longrightarrow \ldots$$

are exact. Via the 5-lemma, this implies that if in a morphism of triangles, two components are invertible, then so is the third. It follows that in a triangle

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X \quad ,$$

the third term Z is unique up to (non unique) isomorphism. One also shows that the direct sum of two Σ-sequences is a triangle if and only if both summands are and that in a triangle (u,v,w), the sequence (u,v) is split exact if and only if $w=0$. The theory of triangulated categories admitting infinite sums is developed in [NEE 01]. A *triangulated subcategory* of a triangulated category is a full subcategory stable under Σ and Σ^{-1} such that with two terms of a triangle, it also contains the third term. A *thick subcategory* is a triangulated subcategory stable under taking direct summands. An object G is a *generator* of a triangulated category $\mathscr{T}$ if $\mathscr{T}$ coincides with its smallest thick subcategory containing G. Important existence theorems for generators in derived categories appearing in algebraic geometry are given in [BON 03] and [ROU 08].

If $\mathscr{S}$ and $\mathscr{T}$ are triangulated categories, a *triangle functor* $\mathscr{S}\to\mathscr{T}$ is a pair (F,ϕ) formed by an additive functor $F:\mathscr{S}\to\mathscr{T}$ and an isomorphism of functors $\phi: F\Sigma\to\Sigma F$ such that for each triangle (u,v,w) of $\mathscr{S}$ the Σ-sequence

$$FX \xrightarrow{Fu} FY \xrightarrow{Fv} FZ \xrightarrow{(\phi X)(Fw)} \Sigma FX$$

is a triangle of $\mathscr{T}$. Let $Q:\mathbf{H}(\mathscr{A})\to\mathbf{D}(\mathscr{A})$ be the canonical localization functor. We have a canonical isomorphism $\phi: Q\Sigma\to\Sigma Q$.

LEMMA.– $\mathbf{D}(\mathscr{A})$ *admits a unique structure of triangulated category such that* (Q,ϕ) *becomes a triangle functor.*

The construction of the derived category from the category up to homotopy is a special case of the *localization of triangulated categories*: Let $\mathscr{T}$ be a triangulated category and $\mathscr{N}\subset\mathscr{T}$ a thick subcategory. Define S to be the class of morphisms s such that in a triangle

$$X \xrightarrow{s} Y \longrightarrow N \longrightarrow \Sigma X$$

the cone N belongs to $\mathscr{N}$. Then it is easy to see that a triangle functor $F:\mathscr{T}\to\mathscr{S}$ vanishes on the objects of $\mathscr{N}$ if and only if it makes the morphisms of S invertible. One defines the *Verdier quotient* $\mathscr{T}/\mathscr{N}$ as the localization $\mathscr{T}[S^{-1}]$, which is constructed using a calculus of fractions in complete analogy with Verdier's definition of the derived category. In particular, it inherits a structure of triangulated category from $\mathscr{T}$. By definition, the sequence of triangulated categories

$$0 \longrightarrow \mathscr{N} \longrightarrow \mathscr{T} \longrightarrow \mathscr{T}/\mathscr{N} \longrightarrow 0$$

is *exact*. For example, we obtain the derived category by localizing the category up to homotopy $\mathscr{T}=\mathbf{H}(\mathscr{A})$ at the thick subcategory $\mathscr{N}$ formed by the acyclic complexes, i.e. the complexes with vanishing homology.

If (F,ϕ) and (G,ψ) are triangle functors $\mathscr{S}\to\mathscr{T}$, a *morphism* $(F,\phi)\to(G,\psi)$ is a morphism of functors $\alpha : F\to G$ such that the square

$$\begin{array}{ccc} F\Sigma & \xrightarrow{\phi} & \Sigma F \\ {\scriptstyle \alpha\Sigma}\downarrow & & \downarrow{\scriptstyle \Sigma\alpha} \\ G\Sigma & \xrightarrow[\psi]{} & \Sigma G \end{array}$$

commutes. The *composition* of two triangle functors $(F,\phi):\mathscr{S}\to\mathscr{T}$ and $(G,\psi):\mathscr{R}\to\mathscr{S}$ is $(FG,(\phi G)(F\psi))$. Two triangle functors $(F,\phi):\mathscr{S}\to\mathscr{T}$ and $(G,\psi):\mathscr{T}\to\mathscr{S}$ are *adjoint*, if there are morphisms $\alpha:(F,\phi)(G,\psi)\to\mathbf{1}_{\mathscr{S}}$ and $\beta:\mathbf{1}_{\mathscr{T}}\to(G,\psi)(F,\phi)$ such that $(G\alpha)(\beta G)=\mathbf{1}_G$ and $(\alpha F)(F\beta)=\mathbf{1}_F$.

LEMMA.– *A triangle functor $(F,\phi):\mathscr{S}\to\mathscr{T}$ admits a triangle right adjoint if and only if the additive functor $F:\mathscr{S}\to\mathscr{T}$ admits a right adjoint.*

THEOREM.– *Let R be a ring and* Mod$R$ *the category of right R-modules. The localization functor* $\mathbf{H}(\mathrm{Mod}R)\to\mathbf{D}(\mathrm{Mod}R)$ *admits a (fully faithful) left adjoint $M\mapsto\mathbf{p}M$ and a (fully faithful) right adjoint $M\mapsto\mathbf{i}M$.*

If M is an R-module, then $\mathbf{p}M$ is given by a projective resolution

$$\ldots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

of M and dually, $\mathbf{i}M$ is given by an injective resolution. We call $\mathbf{p}$ and $\mathbf{i}$ the *resolution functors.*

1.5 Derived functors

We follow Deligne's approach [DEL 73] to derived functors. Let $F:\mathscr{A}\to\mathscr{B}$ be an additive functor between abelian categories. It induces functors $\mathbf{C}(\mathscr{A})\to\mathbf{C}(\mathscr{B})$ and $\mathbf{H}(\mathscr{A})\to\mathbf{H}(\mathscr{B})$ which we still denote by F. Since we do not assume that F is exact, it does not, in general, induce a functor between the derived categories. Nevertheless, we may look for a functor $\mathbf{R}F:\mathbf{D}(\mathscr{A})\to\mathbf{D}(\mathscr{B})$ which comes close to making the following square commute

$$\begin{array}{ccc} \mathbf{H}(\mathscr{A}) & \xrightarrow{F} & \mathbf{H}(\mathscr{B}) \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathbf{D}(\mathscr{A}) & \xrightarrow[\mathbf{R}F]{} & \mathbf{D}(\mathscr{B}). \end{array}$$

For an object Y of $\mathbf{D}(\mathscr{A})$, to define $\mathbf{R}F(Y)$, we define the functor $\mathbf{r}F(Y)$ represented by $\mathbf{R}F(Y)$. Namely, its value at an object $X \in \mathbf{D}(\mathscr{B})$ is formed by the equivalence classes $(f|s)$ of pairs

$$X \xrightarrow{f} FY' \qquad Y' \xleftarrow{s} Y$$

consisting of a quasi-isomorphism s of $\mathbf{H}(\mathscr{A})$ and a morphism f of $\mathbf{D}(\mathscr{B})$. Two pairs $(f|s)$ and $(f'|s')$ are equivalent if there are commutative diagrams

$$\begin{array}{ccccccc} & & FY' & & Y' & & \\ & \nearrow^{f} & \downarrow{\scriptstyle Fv} & & {\scriptstyle v}\downarrow & \nwarrow^{s} & \\ X & \xrightarrow{h} & FY''' & & Y''' & \xleftarrow{u} & Y \\ & \searrow_{f'} & \uparrow{\scriptstyle Fw} & & {\scriptstyle w}\uparrow & \swarrow_{s'} & \\ & & FY'' & & Y'' & & \end{array}$$

in $\mathbf{D}(\mathscr{B})$ respectively $\mathbf{H}(\mathscr{A})$. The functor $\mathbf{R}F$ is *defined* at Y, if the functor $\mathbf{r}F(Y)$ is representable and in this case, the value $\mathbf{R}F(Y)$ is defined by the isomorphism

$$\mathrm{Hom}(?, \mathbf{R}F(Y)) = \mathbf{r}F(Y).$$

The *left derived functor* $\mathbf{L}F$ is defined dually.

LEMMA.– *The domain of definition of* $\mathbf{R}F$ *is a triangulated subcategory* $\mathscr{S}$ *of* $\mathbf{D}(\mathscr{A})$ *and* $\mathbf{R}F : \mathscr{S} \to \mathbf{D}(\mathscr{B})$ *admits a canonical structure of triangle functor.*

LEMMA.– *Suppose that* $\mathscr{A} = \mathrm{Mod}R$ *for a ring* R*. Then the left and right derived functors of* F *are defined on all of* $\mathbf{D}(\mathscr{A})$ *and we have* $\mathbf{R}F(M) = F\mathbf{i}M$ *and* $\mathbf{L}F(M) = F\mathbf{p}M$ *for all* $M \in \mathbf{D}(\mathscr{A})$*, where* $\mathbf{i}$ *and* $\mathbf{p}$ *are the resolution functors defined in the preceding section.*

1.6 Derived Morita theory

Let B be a ring and T a (right) B-module. Let A be the endomorphism ring of T. Then T becomes an A-B-bimodule and yields the adjoint pair

$$? \otimes_A T : \mathrm{Mod}A \to \mathrm{Mod}B \quad \text{and} \quad \mathrm{Hom}_B(T, ?) : \mathrm{Mod}B \to \mathrm{Mod}A.$$

The following is the main theorem of tilting theory [ANG 07]. The module T is called a *tilting module* if it satisfies the properties of ii). We put $\mathbf{D}(A) = \mathbf{D}(\mathrm{Mod}A)$.

THEOREM.– *The following are equivalent:*

i) The derived functor $\mathbf{L}(? \otimes_A T) : \mathbf{D}(A) \to \mathbf{D}(B)$ *is an equivalence.*

ii) The module T *has the following properties:*

a) The module T *has a finite resolution by finitely generated projective* B*-modules.*

b) We have $\mathrm{Ext}^p_B(T,T) = 0$ *for all* $p > 0$.

c) There is an exact sequence $0 \to A \to T^0 \to \ldots \to T^N \to 0$ *of left* A*-modules where the* T^i *are direct summands of finite direct sums of copies of* T.

Now let A and B be rings and X a complex of A-B-bimodules. For a complex M of right A-modules, define the complex $M \otimes_A X$ of right B-modules to have the components

$$\bigoplus_{p+q=n} M^p \otimes_A X^q$$

and the differential given by

$$d(m \otimes x) = (dm) \otimes x + (-1)^p m \otimes dx ,$$

where $m \in M^p$. For a complex N of right B-modules, define the complex $\mathrm{Hom}_B(X, N)$ of right A-modules to have the components

$$\prod_{-p+q=n} \mathrm{Hom}_B(X^p, N^q)$$

and the differential given by $d(f^p) = (d \circ f^p - (-1)^n f^{p+1} \circ d)$. Then the functors $? \otimes_A X$ and $\mathrm{Hom}_B(X, ?)$ form an adjoint pair between $\mathbf{C}(\mathrm{Mod}A)$ and $\mathbf{C}(\mathrm{Mod}B)$. The following theorem is due to J. Rickard [RIC 89, RIC 91]. A direct proof is given in [KEL 94, KEL 98b].

THEOREM.– *Assume that* A *and* B *are algebras over a commutative ring* k *and that* A *is* k*-flat. The following are equivalent:*

i) There is a triangle equivalence $F : \mathbf{D}(A) \to \mathbf{D}(B)$.

ii) There is a complex of B*-modules* T *such that*

a) T *is quasi-isomorphic to a bounded complex of finitely generated projective* B*-modules.*

b) We have $\mathrm{Hom}(T, \Sigma^n T) = 0$ *for all* $n \neq 0$ *and* $\mathrm{Hom}(T,T) \cong A$.

c) B *belongs to the smallest triangulated subcategory of* $\mathbf{D}(B)$ *containing* T *and closed under forming direct summands.*

(iii) There is a complex X *of* A*-*B*-bimodules such that* $\mathbf{L}(? \otimes_A X)$ *is an equivalence* $\mathbf{D}(A) \to \mathbf{D}(B)$.

The algebras A and B are *derived equivalent* if the conditions of the theorem hold. A complex T as in ii) is called a *(one-sided) tilting complex* and a bimodule complex X as in iii) is called a *two-sided tilting complex*. A direct construction of a two-sided tilting complex from a one-sided one when k is a field is given in [KEL 00].

EXAMPLE.– Let k be a field of characteristic 0 and V a k-vector space of finite dimension $n+1$. For $p \geq 0$, let S^p be the pth symmetric power of V and Λ^p the pth exterior power of the dual of V. Let A be the algebra of upper triangular $(n+1)\times(n+1)$-matrices whose (i,j)-entry lies in S^{j-i} and B the algebra of lower triangular $(n+1)\times(n+1)$-matrices whose (i,j)-entry lies in Λ^{i-j}. Let S_i be the B-module k, where B acts through the projection onto the ith diagonal entry. Then $T = \Sigma^n S_1 \oplus \Sigma^{n-1} S_2 \oplus \cdots \oplus S_{n+1}$ is a one-sided tilting complex over B with endomorphism algebra A and thus A and B have equivalent derived categories. This is an example of *Koszul duality* [BEI 96, KEL 94]. In fact, both derived categories are equivalent to the derived category of quasi-coherent sheaves on the projectivization of V, as shown by Beilinson [BE7̃8]. Notice that for $n \geq 3$, the module categories over A and B are not equivalent.

1.7 Dg categories

Triangulated categories were invented by Grothendieck–Verdier in order to axiomatize the properties of derived categories. While they do capture some key features, they suffer from serious defects. Most importantly, tensor products and functor categories formed from triangulated categories are no longer triangulated. The theory of differential graded (=dg) categories [KEL 06] [TO1̃1] was developed to overcome these limitations.

1.7.1 Dg categories and functors

Let k be a commutative ring. A *dg k-module* is a complex of k-modules. Equivalently, it is a $\mathbb{Z}$-graded k-module

$$M = \bigoplus_{n\in\mathbb{Z}} M^n$$

endowed with a differential, i.e. a k-linear endomorphism d homogeneous of degree 1 such that $d^2 = 0$. The *tensor product* $L \otimes M$ of two dg k-modules is the dg k-module with components

$$\bigoplus_{p+q=n} L^p \otimes_k M^q$$

and differential $d_L \otimes \mathbf{1}_M + \mathbf{1}_L \otimes d_M$.

A *dg k-category* is a category $\mathcal{A}$ whose morphism sets $\mathcal{A}(X,Y)$ are dg k-modules and whose compositions are morphisms of dg k-modules

$$\mathcal{A}(Y,Z)\otimes\mathcal{A}(X,Y)\to\mathcal{A}(X,Z).$$

For example, the datum of a dg k-category $\mathcal{A}$ with a single object $*$ is equivalent to that of the dg k-algebra $\mathcal{A}(*,*)$. A typical example with several objects is obtained as follows: Let B be a k-algebra. A right dg B-module is a complex of right B-modules. For two dg B-modules L and M define $\mathrm{Hom}(L,M)^n$ to be the k-module of B-linear maps $f : L\to M$ homogeneous of degree n and make the graded space $\mathrm{Hom}(L,M)$ into a dg k-module by defining

$$d(f)=d_M\circ f-(-1)^n f\circ d_L\,,$$

where f is of degree n. The dg k-category $\mathbf{C}_{dg}(B)$ has as objects all dg B-modules and as morphism spaces the dg k-modules $\mathrm{Hom}(L,M)$ with the natural composition.

Let $\mathcal{A}$ be a dg category. The *opposite dg category* $\mathcal{A}^{op}$ has the same objects as $\mathcal{A}$ and its morphisms are defined by

$$\mathcal{A}(X,Y)=\mathcal{A}(Y,X)\,;$$

the composition of $f\in\mathcal{A}^{op}(Y,X)^p$ with $g\in\mathcal{A}^{op}(Z,Y)^q$ is given by $(-1)^{pq}gf$. The *category* $Z^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and its morphisms are defined by

$$(Z^0\mathcal{A})(X,Y)=Z^0(\mathcal{A}(X,Y))\,,$$

where Z^0 is the kernel of $d:\mathcal{A}(X,Y)^0\to\mathcal{A}(X,Y)^1$. The *category* $H^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and its morphisms are defined by

$$(H^0(\mathcal{A}))(X,Y)=H^0(\mathcal{A}(X,Y))\,,$$

where H^0 denotes the 0th homology of the complex $\mathcal{A}(X,Y)$. For example, if B is a k-algebra, we have isomorphisms of categories

$$Z^0(\mathbf{C}_{dg}(B))=\mathbf{C}(\mathrm{Mod}B)\quad\text{and}\quad H^0(\mathbf{C}_{dg}(B))=\mathbf{H}(\mathrm{Mod}B).$$

Let $\mathcal{A}$ and $\mathcal{A}'$ be dg categories. A *dg functor* $F:\mathcal{A}\to\mathcal{A}'$ is given by a map F from the class of objects of $\mathcal{A}$ to the class of objects of $\mathcal{A}'$ and by morphisms of dg k-modules, for all objects X, Y of $\mathcal{A}$,

$$F(X,Y):\mathcal{A}(X,Y)\to\mathcal{A}'(FX,FY)$$

compatible with the composition and the identities. It is a *quasi-equivalence* if it induces isomorphisms, for X, Y in $\mathcal{A}$,

$$H^*(\mathcal{A}(X,Y))\to H^*(\mathcal{A}'(FX,FY))\,,$$

and the induced functor $H^0(\mathscr{A}) \to H^0(\mathscr{A}')$ is an equivalence. The *category of small dg k-categories* dgcat has the small dg k-categories as objects and the dg functors as morphisms. Note that it has an initial object, the empty dg category $\emptyset$, and a final object, the dg category with one object whose endomorphism ring is the zero ring. The *tensor product* $\mathscr{A} \otimes \mathscr{B}$ of two dg categories has as class of objects the product of the class of objects of $\mathscr{A}$ and that of $\mathscr{B}$ and the morphism spaces

$$(\mathscr{A} \otimes \mathscr{B})((X, Y), (X', Y')) = \mathscr{A}(X, X') \otimes \mathscr{B}(Y, Y')$$

with the natural compositions and units.

For two dg functors $F, G : \mathscr{A} \to \mathscr{B}$, the *dg k-module of graded morphisms* $\mathrm{Hom}(F, G)$ has as its nth component the module formed by the families of morphisms

$$\phi_X \in \mathscr{B}(FX, GX)^n$$

such that $(Gf)(\phi_X) = (\phi_Y)(Ff)$ for all $f \in (X, Y)$, $X, Y \in \mathscr{A}$. The differential is induced by that of $\mathscr{B}(FX, GX)$. The set of *morphisms $F \to G$* is by definition in bijection with $Z^0\mathrm{Hom}(F, G)$. The *dg functor category* $\mathrm{Hom}(\mathscr{A}, \mathscr{B})$ has as objects the dg functors $\mathscr{A} \to \mathscr{B}$ and as morphism complexes the dg k-modules $\mathrm{Hom}(F, G)$.

1.7.2 The derived category

Let $\mathscr{A}$ be a dg category. The category of *(right) dg $\mathscr{A}$-modules* is defined as

$$\mathbf{C}(\mathscr{A}) = Z^0\mathrm{Hom}(\mathscr{A}^{op}, \mathbf{C}_{dg}(k)).$$

Thus, a dg $\mathscr{A}$-module is a dg functor $M : \mathscr{A}^{op} \to \mathbf{C}_{dg}(k)$. With each object X of $\mathscr{A}$, it associates a dg k-module $M(X)$ functorial in $X \in \mathscr{A}^{op}$. Its *homology* is the functor $X \mapsto H^*(M(X))$ from $H^0(\mathscr{A})$ to the category of graded k-modules. A morphism of dg modules $s : M \to M'$ is a *quasi-isommorphism* if it induces an isomorphism in homology. The *category up to homotopy of dg modules* is defined as

$$\mathbf{H}(\mathscr{A}) = H^0\mathrm{Hom}(\mathscr{A}^{op}, \mathbf{C}_{dg}(k)).$$

The *derived category* $\mathbf{D}(\mathscr{A})$ is by definition the localization of $\mathbf{H}(\mathscr{A})$ at the class of quasi-isomorphisms. It is not hard to show that the category up to homotopy and the derived category are canonically triangulated. If $\mathscr{A}$ is the dg category with one object whose endomorphism dg algebra is a k-algebra B (concentrated in degree 0 and endowed with the zero differential), then $\mathbf{C}(\mathscr{A}) = \mathbf{C}(\mathrm{Mod}B)$, $\mathbf{H}(\mathscr{A}) = \mathbf{H}(\mathrm{Mod}B)$ and $\mathbf{D}(\mathscr{A}) = \mathbf{D}(\mathrm{Mod}B)$. For general $\mathscr{A}$, for each object X of $\mathscr{A}$, we have the right module *represented by X*

$$X^\wedge = \mathscr{A}(?, X).$$

For a dg module M and $X \in \mathscr{A}$, we have the Yoneda isomorphism

$$\mathrm{Hom}(X^\wedge, M) \xrightarrow{\sim} M(X)$$

which induces an isomorphism

$$\mathbf{D}(\mathscr{A})(X^\wedge, M) \xrightarrow{\sim} H^0(M(X)).$$

THEOREM.– *Let $\mathscr{A}$ be a dg category. The localization functor $\mathbf{H}(\mathscr{A}) \to \mathbf{D}(\mathscr{A})$ admits a (fully faithful) left adjoint $M \mapsto \mathbf{p}M$ and a (fully faithful) right adjoint $M \mapsto \mathbf{i}M$.*

For example, if M is a representable functor $\mathscr{A}(?, X)$, then $\mathbf{p}M = M$. In general, the dg module $\mathbf{p}M$ is constructed via a 'resolution' of M by representables, cf. [KEL 94].

If $F : \mathscr{A} \to \mathscr{B}$ is a dg functor, the composition with F yields a restriction functor $F^* : \mathbf{D}(\mathscr{B}) \to \mathbf{D}(\mathscr{A})$. The functor F is a *Morita functor* if F^* is an equivalence. It follows from the theorem that all quasi-equivalences are Morita functors.

1.7.3 Derived functors

Let $\mathscr{A}$ and $\mathscr{B}$ be small dg categories. Let X be an $\mathscr{A}$-$\mathscr{B}$-*bimodule*, i.e. a dg $\mathscr{A}^{op} \otimes \mathscr{B}$-module. Thus X is given by complexes $X(B, A)$, for all A in $\mathscr{A}$ and B in $\mathscr{B}$, and morphisms of complexes

$$\mathscr{A}(A, A') \otimes X(B, A) \otimes \mathscr{B}(B', B) \to X(B', A').$$

For each dg $\mathscr{B}$-module M, we obtain a dg $\mathscr{A}$-module

$$GM = \mathrm{Hom}(X, M) : A \mapsto \mathrm{Hom}(X(?, A), M).$$

The functor $G : \mathbf{C}(\mathscr{B}) \to \mathbf{C}(\mathscr{A})$ admits a left adjoint $F : L \mapsto L \otimes_{\mathscr{A}} X$. These functors do not respect quasi-isomorphisms in general, but their derived functors

$$\mathbf{L}F : L \mapsto F(\mathbf{p}L) \text{ and } \mathbf{R}G : M \mapsto G(\mathbf{i}M)$$

form an adjoint pair of functors between $\mathbf{D}(\mathscr{A})$ and $\mathbf{D}(\mathscr{B})$. The following lemma is proved in [KEL 94]. A dg $\mathscr{B}$-module is *perfect* if it belongs to the smallest thick subcategory of $\mathbf{D}(\mathscr{B})$ containing the representable $\mathscr{B}$-modules $\mathscr{B}(?, X)$, $X \in \mathscr{B}$. A set of objects $\mathscr{X}$ *generates* $\mathbf{D}(\mathscr{B})$ if $\mathbf{D}(\mathscr{B})$ coincides with its smallest triangulated subcategory stable under forming infinite sums and containing $\mathscr{X}$.

LEMMA.– *The functor $\mathbf{L}F : \mathbf{D}(\mathscr{A}) \to \mathbf{D}(\mathscr{B})$ is an equivalence if and only if* *a) the dg $\mathscr{B}$-module $X(?, A)$ is perfect for all A in $\mathscr{A}$,*

b) the morphism

$$\mathscr{A}(A, A') \to \mathrm{Hom}(X(?, A), X(?, A'))$$

is a quasi-isomorphism for all A, A' in $\mathscr{A}$ and

c) the dg $\mathscr{B}$-modules $X(?, A)$, $A \in \mathscr{A}$, form a generating set for $\mathbf{D}(\mathscr{B})$.

If the conditions of the lema hold, the dg categories $\mathscr{A}$ and $\mathscr{B}$ are *derived equivalent*. If $\mathscr{A}$ is a dg category, its *perfect derived category* $\mathrm{per}(\mathscr{A})$ is defined as the full subcategory of the derived category formed by the perfect objects. One can show [NEE 92] that an object X is perfect in the derived category if and only if it is *compact*, i.e. the functor $\mathrm{Hom}(X, ?)$ commutes with infinite sums. This shows that an equivalence between derived categories induces an equivalence between their perfect subcategories. The *perfect dg category* $\mathrm{per}_{dg}(\mathscr{A})$ is the full dg subcategory of

$$\mathrm{Hom}(\mathscr{A}^{op}, \mathbf{C}_{dg}(k))$$

whose objects are the resolutions $\mathbf{p}P$ of perfect dg modules P.

For two dg categories $\mathscr{A}$ and $\mathscr{B}$, the *category* $\mathrm{rep}(\mathscr{A}, \mathscr{B})$ is defined as the full triangulated subcategory of the derived category $\mathbf{D}(\mathscr{A}^{op} \otimes \mathscr{B})$ formed by the bimodules X such that $X(?, A)$ is perfect in $\mathbf{D}(\mathscr{B})$ for each A in $\mathscr{A}$. These are precisely the bimodules whose associated tensor functor $\mathbf{D}(\mathscr{A}) \to \mathbf{D}(\mathscr{B})$ takes perfect $\mathscr{A}$-modules to perfect $\mathscr{B}$-modules. By the lemma, this always holds when the tensor functor is an equivalence.

1.7.4 Dg quotients

Let Hqe denote the category obtained from the category of small dg categories dgcat by localizing at the class of all quasi-equivalences. One can show that dgcat admits a Quillen model structure whose weak equivalences are the quasi-equivalences [TAB 05a]. In particular, the morphism spaces of the localized category Hqe are sets and not classes. We need the category Hqe to lift the construction of the Verdier quotient of triangulated categories to the world of dg categories.

Let $\mathscr{A}$ be a small dg category and let $\mathscr{N}$ be a set of objects of $\mathscr{A}$. Let us say that a morphism $Q : \mathscr{A} \to \mathscr{B}$ of Hqe *annihilates* $\mathscr{N}$ if the induced functor

$$H^0(\mathscr{A}) \to H^0(\mathscr{B})$$

takes all objects of $\mathscr{N}$ to zero objects (i.e. objects whose identity morphism vanishes in $H^0(\mathscr{B})$). The following theorem is implicit in [KEL 99] and explicit in [DRI 04].

THEOREM.– *There is a morphism $Q : \mathscr{A} \to \mathscr{A}/\mathscr{N}$ of* Hqe *which annihilates $\mathscr{N}$ and is universal among the morphisms annihilating $\mathscr{N}$.*

We call $\mathscr{A}/\mathscr{N}$ the *dg quotient of $\mathscr{A}$ by $\mathscr{N}$*. If $\mathscr{A}$ is k-flat (i.e. $\mathscr{A}(X,Y)\otimes N$ is acyclic for each acyclic dg k-module N), then $\mathscr{A}/\mathscr{N}$ admits a beautiful simple construction [DRI 04]: One adjoins to $\mathscr{A}$ a contracting homotopy for each object of $\mathscr{N}$. The general case can be reduced to this one or treated using orthogonal subcategories [KEL 99]. The following theorem shows the compatibility between dg quotients and Verdier localizations. A sequence of small dg categories in Hqe

$$0 \longrightarrow \mathscr{U} \longrightarrow \mathscr{V} \longrightarrow \mathscr{W} \longrightarrow 0$$

is *exact* if the induced sequence of triangulated categories

$$0 \longrightarrow \mathbf{D}(\mathscr{U}) \longrightarrow \mathbf{D}(\mathscr{V}) \longrightarrow \mathbf{D}(\mathscr{W}) \longrightarrow 0$$

is exact as a sequence of triangulated categories, i.e. the third term identifies with the Verdier quotient of the second term by the first term.

THEOREM.– *Under the hypotheses of the above theorem, the sequence*

$$0 \longrightarrow \mathscr{N} \longrightarrow \mathscr{A} \longrightarrow \mathscr{A}/\mathscr{N} \longrightarrow 0$$

is exact.

Using dg quotients, we can construct *dg enhancements* of derived categories. For example, if $\mathscr{E}$ is a small abelian (or, more generally, exact) category, we can take for $\mathscr{A}$ the dg category of bounded complexes $\mathbf{C}^b_{dg}(\mathscr{E})$ over $\mathscr{E}$ and for $\mathscr{N}$ the dg subcategory of acyclic bounded complexes $\mathbf{Ac}^b_{dg}(\mathscr{E})$. Then we obtain the *dg-derived category*

$$\mathbf{D}^b_{dg}(\mathscr{E}) = \mathbf{C}^b_{dg}(\mathscr{E})/\mathbf{Ac}^b_{dg}(\mathscr{E})$$

so that we have

$$\mathscr{D}^b(\mathscr{E}) = H^0(\mathbf{D}^b_{dg}(\mathscr{E})).$$

1.7.5 Invariants

K-theory. If $\mathscr{T}$ is a small triangulated category, its *Grothendieck group* $K_0(\mathscr{T})$ is the free abelian group on the set of isomorphism classes of $\mathscr{T}$ modulo the subgroup generated by the elements $[X]-[Y]+[Z]$ associated with the triangles

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

of $\mathscr{T}$. If $\mathscr{A}$ is a small dg category, one defines

$$K_0(\mathscr{A}) = K_0(\mathrm{per}(\mathscr{A})).$$

By section 1.7.3, this is an invariant under derived equivalence. One defines the category Hmo_0 to have as objects all small dg categories and as morphisms the Grothendieck groups

$$K_0(\mathrm{rep}(\mathscr{A}, \mathscr{B}))$$

with the composition induced by the derived tensor product. Then the functor $\mathscr{A} \mapsto K_0(\mathscr{A})$ induces an additive functor defined on Hmo_0 with values in the category of abelian groups. By definition, an *additive invariant* of dg categories is an additive functor defined on Hmo_0. This means that the functor $\mathrm{dgcat} \to \mathrm{Hmo}_0$ is the universal additive invariant [TAB 05b]. Additive invariants do not distinguish between rather different dg categories. For example, if k is an algebraically closed field, each finite-dimensional algebra of finite global dimension becomes isomorphic in Hmo_0 to a product of copies of k [KEL 98a] but it is derived equivalent to such a product only if it is semisimple.

One defines the higher K-theory $K(\mathscr{A})$ by applying Waldhausen's construction [WAL 85] to a suitable category with cofibrations and weak equivalences: here, the category is that of perfect $\mathscr{A}$-modules, the cofibrations are the morphisms $i : L \to M$ of $\mathscr{A}$-modules which admit retractions as morphisms of graded $\mathscr{A}$-modules and the weak equivalences are the quasi-isomorphisms. This construction can be improved so as to yield a functor K from dgcat to the homotopy category of spectra. As in [THO 90], from Waldhausen's results [WAL 85], one then obtains the following

THEOREM.–

a) [DUG 04] The map $\mathscr{A} \mapsto K(\mathscr{A})$ *yields a well-defined additive functor on* Hmo_0.

b) Applied to the bounded dg-derived category $\mathbf{D}^b_{dg}(\mathscr{E})$ *of an exact category* $\mathscr{E}$, *the* K*-theory defined above agrees with Quillen* K*-theory.*

c) The functor $\mathscr{A} \mapsto K(\mathscr{A})$ *is an additive invariant. Moreover, each short exact sequence*

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0$$

of Hqe *yields a long exact sequence*

$$\ldots \to K_i(\mathscr{A}) \to K_i(\mathscr{B}) \to K_i(\mathscr{C}) \to \ldots \to K_0(\mathscr{B}) \to K_0(\mathscr{C}).$$

Hochschild homology. Let $\mathscr{A}$ be a small k-flat k-category. Following [MIT 72] the *Hochschild chain complex* of $\mathscr{A}$ is the complex $C(\mathscr{A})$ concentrated in homological degrees $p \geq 0$ whose pth component is the sum of the

$$\mathscr{A}(X_p, X_0) \otimes \mathscr{A}(X_p, X_{p-1}) \otimes \mathscr{A}(X_{p-1}, X_{p-2}) \otimes \cdots \otimes \mathscr{A}(X_0, X_1)\,,$$

where $X_0, \ldots, X_p$ range through the objects of $\mathscr{A}$, endowed with the differential

$$d(f_p \otimes \ldots \otimes f_0) = f_{p-1} \otimes \cdots \otimes f_0 f_p + \sum_{i=1}^{p} (-1)^i f_p \otimes \cdots \otimes f_i f_{i-1} \otimes \cdots \otimes f_0.$$

If $\mathscr{A}$ is a k-flat differential graded category, its Hochschild chain complex $C(\mathscr{A})$ is the sum-total complex of the bicomplex obtained as the natural re-interpretation of the above complex. The following theorem is stated for Hochschild homology but analogous theorems hold for all variants of cyclic homology [KEL 99].

THEOREM.–

a) The map $\mathscr{A} \mapsto C(\mathscr{A})$ *yields an additive functor* $\mathrm{Hmo}_0 \to \mathbf{D}(k)$. *Moreover, each exact sequence of* Hqe *yields a canonical triangle of* $\mathbf{D}(k)$.

b) If A is a k-algebra, there is a natural isomorphism $C(A) \to C(\mathrm{per}_{dg}(A))$ *in* $\mathbf{D}(k)$.

The second statement in a) may be viewed as an excision theorem analogous to [WOD 89].

Hochschild cohomology. Let $\mathscr{A}$ be a small dg category over a field k. Its cohomological Hochschild complex $C(\mathscr{A}, \mathscr{A})$ is defined as the product-total complex of the bicomplex whose 0th column is

$$\prod \mathscr{A}(X_0, X_0)\,,$$

where X_0 ranges over the objects of $\mathscr{A}$, and whose pth column, for $p \geq 1$, is

$$\prod \mathrm{Hom}_k(\mathscr{A}(X_{p-1}, X_p) \otimes \mathscr{A}(X_{p-2}, X_{p-1}) \otimes \cdots \otimes \mathscr{A}(X_0, X_1), \mathscr{A}(X_0, X_p))$$

where $X_0, \ldots, X_p$ range over the objects of $\mathscr{A}$. The horizontal differential is given by the Hochschild differential. This complex carries rich additional structure: As shown in [GET], it is a B_∞*-algebra*, i.e. its bar construction carries, in addition to its canonical differential and comultiplication, a natural *multiplication* which makes it into a dg bialgebra. The B_∞-structure contains in particular the cup product and the Gerstenhaber bracket, which both descend to the Hochschild cohomology

$$\mathrm{HH}^*(\mathscr{A}, \mathscr{A}) = H^* C(\mathscr{A}, \mathscr{A}).$$

Note that $C(\mathscr{A},\mathscr{A})$ is not functorial with respect to dg functors. However, if $F:\mathscr{A}\to\mathscr{B}$ is a fully faithful dg functor, it clearly induces a restriction map

$$F^*: C(\mathscr{B},\mathscr{B}) \to C(\mathscr{A},\mathscr{A})$$

and this map is compatible with the B_∞-structure. This can be used to construct [KEL] a morphism

$$\phi_X : C(\mathscr{B},\mathscr{B}) \to C(\mathscr{A},\mathscr{A})$$

in the homotopy category of B_∞-algebras associated with each dg $\mathscr{A}$-$\mathscr{B}$-bimodule X such that the functor

$$\mathbf{L}(?\otimes_{\mathscr{A}} X) : \mathrm{per}(\mathscr{A}) \to \mathbf{D}(\mathscr{B})$$

is fully faithful. If moreover the functor $\mathbf{L}(X\otimes_{\mathscr{B}}?) : \mathrm{per}(\mathscr{B}^{op}) \to \mathbf{D}(\mathscr{A}^{op})$ is fully faithful, then ϕ_X is an isomorphism. We refer to [LOW 05] for the closely related study of the Hochschild complex of an abelian category.

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