THE HALL ALGEBRA OF A SPHERICAL OBJECT

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Abstract. We determine the Hall algebra, in the sense of B. Toën, of the algebraic triangulated category generated by a spherical object.

2000 Mathematics Subject Classifications: 17B37, 18E30, 16E45.

1. Introduction

This note is motivated by recent developments in the categorification of cluster algebras and cluster varieties. Let us recall the context: To a finite quiver Q without loops and without 2-cycles, one can associate the cluster algebra $\mathcal{A}_Q$ and the cluster variety $\mathcal{X}_Q$ (endowed with a Poisson structure), *cf.* [5] and [4]. If Q does not have oriented cycles, we have at our disposal a very good categorical model for the combinatorics of the cluster algebra $\mathcal{A}_Q$, *cf.* the surveys [1] [19] [20] [12]. In contrast, for the moment, there is no corresponding theory for the cluster variety $\mathcal{X}_Q$. Ongoing work by Kontsevich-Soibelman [16], T. Bridgeland [3] and others shows that there is a close link between the quantized version [4] of $\mathcal{X}_Q$ and the Hall algebra [22] of a certain triangulated 3-Calabi-Yau category $\mathcal{T}_Q$ associated with Q. The category $\mathcal{T}_Q$ can be described as the algebraic triangulated category generated by the objects in a 'generic' collection of 3-spherical objects whose extension spaces have dimensions encoded by the quiver Q. Alternatively, it may be described as the derived category of dg (=differential graded) modules with finite-dimensional total homology over the Ginzburg dg algebra [6] associated with Q and a generic potential. In this note, we consider the case where Q is reduced to a single vertex without any arrows. This amounts to considering the (algebraic) triangulated category $\mathcal{T}_Q$ generated by a single spherical object. We first show that this category is indeed well-determined up to a triangle equivalence (Theorem 2.1). Then we classify the objects of $\mathcal{T}_Q$ (Theorem 4.1, due to P. Jørgensen [9]), compute the Hall algebra of $\mathcal{T}_Q$ (Theorem 5.1) and establish the link with the cluster variety, which in this case is just a one-dimensional torus (Section 6). The Hall algebra of the algebraic triangulated category generated by a spherical object of arbitrary dimension can be determined similarly. We give the result in Section 7. For the classification theorem, we establish more generally the classification of the indecomposable objects in a triangulated category admitting a generator whose graded endomorphism algebra is hereditary, a result which may be useful in other contexts as well.

Date: October 5, 2008. Last modified on October 17, 2009.

Acknowledgments

The second author gratefully acknowledges a postdoctoral fellowship granted by the Université Pierre et Marie Curie, during which this work was carried out. The third author benefited from financial support via postdoctoral fellowships from the Ecole Polytechnique and from the network 'Representation theory of algebras and algebraic Lie theory'. Both of them would like to thank the first author and Professor Steffen König for their support. All three authors thank Peter Jørgensen for pointing out references [9] and [10] and an anonymous referee for helpful comments on a previous version of this article.

2. The triangulated category generated by a spherical object

Let k be a field and $\mathcal{T}$ a k-linear algebraic triangulated category (*cf.* Section 3.6 of [14] for this terminology). We write Σ for the suspension functor of $\mathcal{T}$. We assume that $\mathcal{T}$ is idempotent complete, i.e. each idempotent endomorphism of an object of $\mathcal{T}$ comes from a direct sum decomposition.

Let d be an integer and G a d-spherical object of $\mathcal{T}$. This means that the graded endomorphism algebra

$$B = \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p G)$$

is isomorphic to $k\langle s\rangle/(s^2)$, where s is of degree d. We also view B as a dg algebra whose differential vanishes. We refer to Section 3 of [14] for the definition of the derived category $\mathcal{D}(B)$. The perfect derived category $\mathrm{per}\,(B)$ is defined as the smallest thick subcategory of $\mathcal{D}(B)$ containing B. We say that G *classically generates* $\mathcal{T}$ if $\mathcal{T}$ coincides with its smallest triangulated subcategory stable under taking direct factors and containing G.

Theorem 2.1. *If G classically generates $\mathcal{T}$, there is a triangle equivalence from $\mathcal{T}$ to the perfect derived category of B.*

Proof. According to Theorem 7.6.0.6 of [17], there is a triangle equivalence between $\mathcal{T}$ and the perfect derived category of a minimal strictly unital A_∞-algebra whose underlying graded algebra is B. This A_∞-structure is given by linear maps

$$m_p : (ks)^{\otimes p} \to B$$

defined for $p \geq 3$ and homogeneous of degree $2-p$. For degree reasons, these maps vanish. The claim follows because the perfect derived category of B considered as a dg algebra is equivalent to the perfect derived category of B considered as an A_∞-algebra by Lemma 4.1.3.8 of [17]. $\surd$

Thus, it makes sense to speak about 'the' algebraic triangulated category generated by a spherical object of dimension d. Notice that Koszul duality provides us with another realization of this category: If $d \neq 1$, we have a triangle equivalence

$$\mathrm{per}\,(B) \xrightarrow{\sim} \mathcal{D}_{fd}(A) \ ,$$

where the dg algebra A is the free dg algebra on a closed generator t of degree $-d+1$. Here the category $\mathcal{D}_{fd}(A)$ is the full subcategory of the derived category $\mathcal{D}A$ formed by the dg modules whose homology is of finite total dimension. If d equals 3, then A is the Ginzburg algebra [6] associated with the quiver A_1. If d equals 1, then $\mathrm{per}\,(B)$ is triangle equivalent to the full subcategory of $\mathcal{D}(k[t])$, where t is of degree 0, formed by the dg modules whose homology is of finite total dimension and annihilated by some power of t.

3. Classification

In this section, we present a general classification theorem for indecomposable objects in a triangulated category admitting a 'generator' G whose graded endomorphism algebra is hereditary. We first consider the case where G compactly generates a triangulated category with arbitrary direct sums. Then we consider the case where G is a classical generator. We apply it to the perfect and the finite dimensional derived categories of the Ginzburg algebra of type A_1.

3.1. **Compactly generated case.** Let k be a commutative ring, and $\mathcal{T}$ a k-linear triangulated category with suspension functor Σ. Assume $\mathcal{T}$ has arbitrary direct sums. Let G be a *compact generator* for $\mathcal{T}$, i.e. the functor $\mathrm{Hom}_{\mathcal{T}}(G, ?)$ commutes with arbitrary direct sums, and given an object X of $\mathcal{T}$, if $\mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p X)$ vanishes for all integers p, then X vanishes. Let

$$A = \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p G)$$

be the graded endomorphism algebra of G. Then for any object X of $\mathcal{T}$, the graded vector space

$$\bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p X)$$

has a natural graded (right) module structure over A. We define a functor

$$F : \mathcal{T} \to \mathrm{Grmod}(A), X \mapsto \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p X)\ ,$$

where $\mathrm{Grmod}(A)$ denotes the category of all graded right A-modules. Notice that since G is a compact generator, a morphism of $\mathcal{T}$ is invertible if and only if its image under F is invertible.

We say that A is *graded hereditary*, if the category $\mathrm{Grmod}(A)$ of graded A-modules is hereditary, or in other terms, each subobject of a projective object of $\mathrm{Grmod}(A)$ is projective.

Theorem 3.1. *With the notations above, suppose that A is graded hereditary. The functor $F : \mathcal{T} \to \mathrm{Grmod}(A)$ is full, essentially surjective, and its kernel has square zero. In particular, it induces a bijection from the class of isoclasses of objects (respectively, of indecomposable objects) of $\mathcal{T}$ to that of* $\mathrm{Grmod}(A)$.

Remarks. *a) Notice that we have an isomorphism of functors $F\circ\Sigma \simeq [1]\circ F$, where $[1]$ denotes the shift functor in* $\mathrm{Grmod}(A)$.

b) The functor F is obviously a homological functor. We will use this fact implicitly.

The theorem is a consequence of the following lemmas.

For a class $\mathcal{S}$ of objects of an additive category $\mathcal{A}$ with arbitrary direct sums, we denote by $\mathrm{Add}(\mathcal{S})$ the closure of $\mathcal{S}$ under taking all direct sums and direct summands.

Lemma 3.2. *a) The functor $F : \mathcal{T} \to \mathrm{Grmod}(A)$ induces an equivalence between $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$ and $\mathrm{Add}(A[p] | p \in \mathbb{Z})$.*

b) An object X belongs to $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$ if and only if FX belongs to $\mathrm{Add}(A[p] | p \in \mathbb{Z})$.

Notice that the sufficiency of the condition in b) is not immediate from a). For example, the functor F might 'retract' the whole category $\mathcal{T}$ onto $\mathrm{Add}(A[p] | p \in \mathbb{Z})$.

Proof. a) By definition, we have

$$\mathrm{Hom}_{\mathcal{T}}(G, X) = (FX)_0 \cong \mathrm{Hom}_{\mathrm{Grmod}(A)}(A, FX)$$

for any object X in $\mathcal{T}$, and so the map

$$F(G, X) : \mathrm{Hom}_{\mathcal{T}}(G, X) \to \mathrm{Hom}_{\mathrm{Grmod}(A)}(FG, FX)$$

is bijective. Therefore the map

$$F(G_0, X) : \mathrm{Hom}_{\mathcal{T}}(G_0, X) \to \mathrm{Hom}_{\mathrm{Grmod}(A)}(FG_0, FX)$$

is an isomorphism for any G_0 in $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$ and any X in $\mathcal{T}$. Taking X in $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$, this proves that, considered as a functor from $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$ to $\mathrm{Add}(A[p] | p \in \mathbb{Z})$, the functor F is fully faithful. Moreover, since G is compact, F commutes with arbitrary coproducts. The proof of essential surjectivity is therefore easy.

b) The necessity of the condition follows from a). Let us prove the sufficiency. Let X be an object of $\mathcal{T}$ such that there is an isomorphism $f : FG_0 \to FX$ in $\mathrm{Add}(A[p] | p \in \mathbb{Z})$ for some $G_0 \in \mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$. We can lift f to a morphism $\tilde{f} : G_0 \to X$ in $\mathcal{T}$. As we have observed above, since G is a compact generator, a morphism of $\mathcal{T}$ is invertible iff its image under F is invertible. Since we have $F\tilde{f} = f$, it follows that $\tilde{f}$ is invertible and X is isomorphic to G_0. $\surd$

It is well known that the class of projective objects of $\mathrm{Grmod}(A)$ is exactly $\mathrm{Add}(A[p] | p \in \mathbb{Z})$.

Lemma 3.3. *The functor F is essentially surjective.*

Proof. Let M be a graded A-module. Since A is graded hereditary, there exists a short exact sequence of graded A-modules

$$0 \to P_1 \xrightarrow{u} P_0 \to M \to 0$$

with P_0 and P_1 in $\mathrm{Add}(A[p] | p \in \mathbb{Z})$. By Lemma 3.2 a), we can lift u to a morphism v in $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$. Letting X be a cone of v, one checks that FX is isomorphic to M. $\surd$

Lemma 3.4. *The functor F is full.*

Proof. We prove this in three steps.

Step 1: By the first paragraph of the proof of Lemma 3.2, the map

$$F(G_0, X) : \mathrm{Hom}_{\mathcal{T}}(G_0, X) \to \mathrm{Hom}_{\mathrm{Grmod}(A)}(FG_0, FX)$$

is an isomorphism for any G_0 in $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$ and any X in $\mathcal{T}$.

Step 2: Let X be an object of $\mathcal{T}$. We will show that there exists a triangle

$$G_1 \to G_0 \to X \to \Sigma G_1$$

in $\mathcal{T}$ such that G_0, G_1 belong to $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$. We choose $w : G_0 \to X$ such that Fw is surjective. We form the triangle

$$Y \to G_0 \xrightarrow{w} X \to \Sigma Y.$$

We apply F and obtain an exact sequence

$$F(\Sigma^{-1} G_0) \xrightarrow{F(\Sigma^{-1} w)} F(\Sigma^{-1} X) \longrightarrow FY \longrightarrow FG_0 \xrightarrow{Fw} FX \longrightarrow F(\Sigma Y).$$

Both Fw and $F(\Sigma^{-1} w) = (Fw)[-1]$ are surjective, so we obtain a short exact sequence

$$0 \to FY \to FG_0 \to FX \to 0.$$

Thus FY belongs to $\mathrm{Add}(A[p] | p \in \mathbb{Z})$ since $\mathrm{Grmod}(A)$ is hereditary. By Lemma 3.2 b), the object Y belongs to $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$. Now it suffices to take $G_1 = Y$.

Step 3: Let X, Y be objects in $\mathcal{T}$. By Step 2, there is a triangle in $\mathcal{T}$

$$G_1 \to G_0 \to X \to \Sigma G_1,$$

where G_0, G_1 belong to $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$, whose image under F is a short exact sequence in $\mathrm{Grmod}(A)$

$$0 \to FG_1 \to FG_0 \to FX \to 0.$$

If we apply $\mathrm{Hom}_{\mathcal{T}}(?, Y)$ to the triangle and $\mathrm{Hom}_{\mathrm{Grmod}(A)}(?, FY)$ to the short exact sequence, we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccc} \mathrm{Hom}_{\mathcal{T}}(\Sigma G_1, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(X, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(G_0, Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{T}}(G_1, Y) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (FX, FY) & \longrightarrow & (FG_0, FY) & \longrightarrow & (FG_1, FY), \end{array}$$

where the parentheses $(,)$ in the second row denote the groups of homogeneous A-linear maps. By Step 1, the rightmost two vertical maps are isomorphisms. Therefore, the leftmost vertical map is surjective. Since X and Y are arbitrary, we have proved that F is full. $\surd$

Lemma 3.5. *Let* $J = \{f \in \mathrm{Mor}(\mathcal{T}) | Ff = 0\}$*. Then* $J^2 = 0$*.*

Proof. Let $f : X \to Y$ be a morphism in J, that is, for any $p \in \mathbb{Z}$ and for any morphism $u : G \to \Sigma^p X$, we have $\Sigma^p f \circ u = 0$.

Let $G_1 \xrightarrow{u} G_0 \xrightarrow{v} X \xrightarrow{w} \Sigma G_1$ be a triangle in $\mathcal{T}$ such that G_0, G_1 belong to $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$. Since f belongs to J, we have $f \circ v = 0$. Therefore, the morphism f factors through w, that is, there is $f' \in \mathrm{Hom}_{\mathcal{T}}(\Sigma G_1, Y)$ such that $f = f' \circ w$.

Let $G_1' \xrightarrow{u'} G_0' \xrightarrow{v'} Y \xrightarrow{w'} \Sigma G_1'$ be a triangle in $\mathcal{T}$ such that G_0', G_1' belong to $\mathrm{Add}(\Sigma^p G | p \in \mathbb{Z})$, Fu' is injective and Fv' is surjective. Then the induced homomorphism

$$\mathrm{Hom}_{\mathcal{T}}(\Sigma G_1, G_0') \to \mathrm{Hom}_{\mathcal{T}}(\Sigma G_1, Y)$$

is surjective. Therefore, there is $h \in \mathrm{Hom}_{\mathcal{T}}(\Sigma G_1, G_0')$ such that $f' = v' \circ h$.

Now let $g : Y \to Z$ be another morphism in J. By the arguments in the second paragraph there is $g' : \Sigma G_1' \to Z$ such that $g = g' \circ w'$. Thus we have $g \circ f = g' \circ w' \circ v' \circ h \circ w = 0$, and we are done. $\surd$

3.2. **Classically generated case.** Let k be a commutative ring and let $\mathcal{T}$ be a k-linear triangulated category with suspension functor Σ. Let G be a *classical generator* for $\mathcal{T}$, i.e. $\mathcal{T}$ is the closure of G under taking shifts, extensions and direct summands. Let

$$A = \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p G)$$

be the graded endomorphism algebra of G. We assume that the category $\mathrm{grmod}(A)$ of finitely presented graded A-modules is abelian (i.e. A is graded right coherent) and hereditary.

Theorem 3.6. *The functor*

$$F : \mathcal{T} \to \mathrm{grmod}(A), X \mapsto \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(G, \Sigma^p X)$$

is well-defined, full, essentially surjective, and its kernel has square zero. In particular, it induces a bijection from the set of isoclasses of objects (respectively, of indecomposable objects) of $\mathcal{T}$ to that of $\mathrm{grmod}(A)$.

Proof. Lemma 3.2, 3.3, 3.4 and 3.5 and their proofs are still valid, mutatis mutandis. For example, we need to replace Add by add in the statement of Lemma 3.2, where for a class $\mathcal{S}$ of objects of an additive category $\mathcal{A}$, we denote by $\mathrm{add}(\mathcal{S})$ the closure of $\mathcal{S}$ under taking direct summands and finite direct sums. It remains to prove that F is well-defined, that is, for any object X of $\mathcal{T}$, the graded A-module FX is indeed finitely presented.

Let $\mathcal{T}'$ be the full additive subcategory of $\mathcal{T}$ consisting of those objects X such that $F(X)$ is a finitely presented A-module. Evidently G belongs to $\mathcal{T}'$. Thus, in order to conclude that $\mathcal{T}'$ equals $\mathcal{T}$, it suffices to show that $\mathcal{T}'$ is stable under shifts, direct summands and extensions. The first two points are clear.

Suppose that we have a triangle

$$Y \xrightarrow{u} Z \xrightarrow{v} X \xrightarrow{w} \Sigma Y$$

in $\mathcal{T}$ such that FY and FX are finitely presented. Then the objects

$$F(\Sigma^{-1} X) = (FX)[-1] \text{ and } F(\Sigma Y) = (FY)[1]$$

are also finitely presented. We apply F to the above triangle to obtain an exact sequence

$$F(\Sigma^{-1}X) \xrightarrow{F(\Sigma^{-1}w)} F(Y) \xrightarrow{Fu} FZ \xrightarrow{Fv} FX \xrightarrow{Fw} F(\Sigma Y).$$

Note that all components except possibly FZ are finitely presented. Since the category $\mathrm{grmod}(A)$ of finitely presented graded A-modules is abelian, the kernel $\ker Fv = \mathrm{coker} F(\Sigma^{-1}w)$ of Fv and the image $\mathrm{im} Fv = \ker Fw$ of Fv are also finitely presented. Consequently FZ is finitely presented and $\mathcal{T}'$ is stable under extensions. Therefore, the functor F is well-defined. $\surd$

Examples 3.7. *a) Let B be a finite dimensional hereditary algebra over a field k. Let $\mathcal{T} = \mathcal{D}^b(\mathrm{mod}B)$ be the bounded derived category of finite dimensional B-modules, and let G be the free B-module of rank* 1. *Then $A = B$ and the functor $F : \mathcal{D}^b(\mathrm{mod}B) \to \mathrm{grmod}(B)$ takes X to its total homology H^*X.*

b) Let R be a discrete valuation ring with a uniformizing parameter π. Denote $B = R/(\pi^2)$ and $k = R/(\pi)$. Let $\mathcal{T} = \mathcal{D}^b(\mathrm{mod}B)$ be the bounded derived category of finitely generated B-modules, and let G be the simple module k. Then the graded endomorphism algebra A of G in $\mathcal{T}$ is isomorphic to the graded algebra $k[u]$ with $\deg(u) = 1$. Now Theorem 3.6 gives the classification of the indecomposable objects of $\mathcal{T}$ which was previously obtained by I. Burban in his thesis ([2]*) and also by M. Künzer in* [15, Lemma 3.1]. *In fact, using the notations of M. Künzer, up to isomorphism, the indecomposables are the complexes $X^{[a,b]}$ and $X^{]-\infty,b]}$, where, for given integers $a \leq b$, we denote by $X^{[a,b]}$ the complex*

$$\cdots \to 0 \to \underbrace{B}_{a} \xrightarrow{\pi} B \xrightarrow{\pi} \cdots \xrightarrow{\pi} B \xrightarrow{\pi} \underbrace{B}_{b} \to 0 \to \cdots$$

and by $X^{]-\infty,b]}$ the complex

$$\cdots \xrightarrow{\pi} B \xrightarrow{\pi} B \xrightarrow{\pi} \underbrace{B}_{b} \to 0 \to \cdots .$$

c) Let $\tilde{A}$ be a dg algebra such that the category of finitely presented graded modules over the graded algebra $A = H^*(\tilde{A})$ is abelian and hereditary. Let $\mathcal{T} = \mathrm{per}(\tilde{A})$ be the perfect derived category and let G be the free dg $\tilde{A}$-module of rank 1. *Then the functor F takes X to its total homology viewed as a graded A-module.*

4. Application of the classification

Let k be a field. Let Γ denote the Ginzburg dg algebra of type A_1 over k, i.e. Γ is the dg algebra $k[t]$ with $\deg(t) = -2$ and trivial differential.

Recall that the perfect derived category $\mathrm{per}(\Gamma)$ is the smallest thick subcategory of the derived category $\mathcal{D}(\Gamma)$ containing Γ. We denote by $\mathcal{D}_{fd}(\Gamma)$ the finite dimensional derived category, i.e. the full triangulated subcategory consisting of the dg Γ-modules whose homology is of finite total dimension (*cf.* [13]). The triangulated category $\mathcal{D}_{fd}(\Gamma)$ is Hom-finite and 3-Calabi-Yau (*cf.* [9] or [11]), classically generated by the simple dg Γ-module $S = \Gamma/(t\Gamma)$ concentrated in degree 0, which is a spherical object of dimension 3.

Let [1] denote the shift functor of the category $\mathrm{grmod}(\Gamma)$ of finitely presented graded Γ-modules. For an integer p and a strictly positive integer n, the finite dimensional graded Γ-module $\Gamma/(t^n\Gamma)[p]$, viewed as an object in $\mathcal{D}_{fd}(\Gamma)$, is indecomposable.

Theorem 4.1 (Jørgensen [9])**.** *a) Each indecomposable object in* $\mathrm{per}(\Gamma)$ *is isomorphic to either* $\Gamma/(t^n\Gamma)[p]$ *for some integer* p *and some strictly positive integer* n *or* $\Gamma[p]$ *for some integer* p*.*

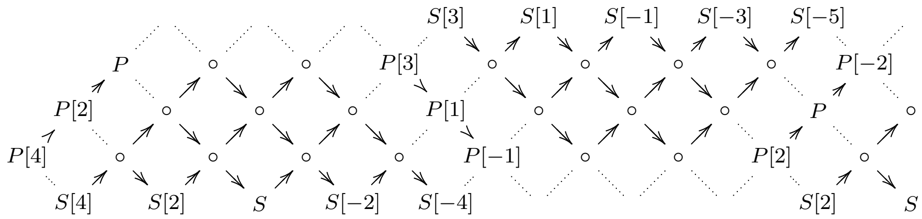
b) Each indecomposable object in $\mathcal{D}_{fd}(\Gamma)$ *is isomorphic to* $\Gamma/(t^n\Gamma)[p]$ *for some integer* p *and some strictly positive integer* n*.*

Proof. It is readily seen that the category $\mathrm{grmod}(A)$ for $A = H^*(\Gamma)(= \Gamma$ as graded algebras) is abelian and hereditary. We are therefore in a particular case of Example 3.7 c). The functor

$$F = H^* : \mathrm{per}(\Gamma) \to \mathrm{grmod}(\Gamma)$$

induces a bijection between the set of isoclasses of indecomposable objects of $\mathrm{per}(\Gamma)$ and that of $\mathrm{grmod}(A)$. Moreover, the full subcategory $\mathcal{D}_{fd}(\Gamma)$ of $\mathrm{per}(\Gamma)$ is sent by F to the full subcategory of $\mathrm{grmod}(\Gamma)$ consisting of finite dimensional graded Γ-modules. Now the theorem follows from the classification of indecomposable objects for the latter category, which is well-known. $\surd$

Remark. *It is not hard to check that the Auslander-Reiten quiver of the perfect derived category has the following shape*



where the picture is periodic as indicated by the labels. The Auslander-Reiten quiver of $\mathcal{D}_{fd}(\Gamma)$ *is the subquiver consisting of the components containing the simples* S *and* $S[1]$*. This latter quiver was first determined by P. Jørgensen in* [9]*; he considerably generalized the result in* [10]*.*

5. The Hall algebra

In this section, we prove the structure theorem (Theorem 5.1) for the (derived) Hall algebra of the the Ginzburg dg algebra of type A_1. We begin with some reminders on Hall algebras of triangulated categories. We refer to [21] for an excellent introduction to non derived Hall algebras.

5.1. **The Hall algebra.** We follow [22] and [23]. Let $\mathbb{Q}$ be the field of rational numbers, q be a prime power and $\mathbb{F}_q$ be the finite field with q elements. Let $\mathcal{C}$ be a Hom-finite triangulated $\mathbb{F}_q$-category with suspension functor Σ, such that for all objects X and Y of $\mathcal{C}$, the space of morphisms from X to $\Sigma^{-i}Y$ vanishes for all but finitely many positive integers i.

Let X, Y and Z be three objects of $\mathcal{C}$. We denote by $\mathrm{Aut}(Y)$ the group of automorphisms of Y and by $[Y, Z]_X$ the set of morphisms from Y to Z with cone isomorphic to X. Following [22], we define the Hall number by

$$F_{XY}^Z = \frac{|[Y,Z]_X|}{|\mathrm{Aut}(Y)|} \cdot \frac{\prod_{i>0} |\mathrm{Hom}(Y, \Sigma^{-i}Z)|^{(-1)^i}}{\prod_{i>0} |\mathrm{Hom}(Y, \Sigma^{-i}Y)|^{(-1)^i}},$$

where $|\cdot|$ denotes the cardinality. The *Hall algebra* of $\mathcal{C}$ over $\mathbb{Q}$, denoted by $\mathcal{H}(\mathcal{C})$, is the $\mathbb{Q}$-vector space with basis the isoclasses $[X]$ of objects X of $\mathcal{C}$ whose multiplication is given by

$$[X][Y] = \sum_{[Z]} F_{XY}^{Z}[Z].$$

It is shown in [22] [23] that it is an associative algebra with unit [0]. Notice however that the algebra we define here is opposite to that in [22] [23].

5.2. **The structure theorem.** Let R be the $\mathbb{Q}$-algebra with generators x_i and y_i, $i \in \mathbb{Z}$, subject to the following relations:

$$x_i^2 x_{i-1} - (1+q^{-1}) x_i x_{i-1} x_i + q^{-1} x_{i-1} x_i^2 \tag{1}$$
$$x_i x_{i-1}^2 - (1+q^{-1}) x_{i-1} x_i x_{i-1} + q^{-1} x_{i-1}^2 x_i \tag{2}$$
$$x_i x_j - x_j x_i \qquad \text{if } |i-j|>1 \tag{3}$$
$$y_i x_i - q x_i y_i + \tfrac{q}{q-1} \tag{4}$$
$$y_i x_{i+1} - q^{-1} x_{i+1} y_i - \tfrac{1}{q-1} \tag{5}$$
$$y_i x_j - x_j y_i \qquad \text{if } j \neq i, i+1 \tag{6}$$
$$y_i^2 y_{i-1} - (1+q^{-1}) y_i y_{i-1} y_i + q^{-1} y_{i-1} y_i^2 \tag{7}$$
$$y_i y_{i-1}^2 - (1+q^{-1}) y_{i-1} y_i y_{i-1} + q^{-1} y_{i-1}^2 y_i \tag{8}$$
$$y_i y_j - y_j y_i \qquad \text{if } |i-j|>1. \tag{9}$$

Let Γ be the Ginzburg dg algebra of type A_1 over the finite field $\mathbb{F}_q$, and $\mathcal{D}_{fd}(\Gamma)$ the finite dimensional derived category with suspension functor Σ. Let $\mathcal{H} = \mathcal{H}(\mathcal{D}_{fd}(\Gamma))$ be the Hall algebra.

Theorem 5.1. *We have a $\mathbb{Q}$-algebra isomorphism*

$$\phi : R \longrightarrow \mathcal{H}, \qquad x_i \mapsto [\Sigma^{-2i} S], y_i \mapsto [\Sigma^{-2i-1} S],$$

where we recall that $S = \Gamma/(t\Gamma)$ is the simple dg Γ-module concentrated in degree 0.

One checks by a direct computation that ϕ is indeed an algebra homomorphism, i.e. the relations (1)–(9) are satisfied if we replace x_i and y_i by $[\Sigma^{-2i} S]$ and $[\Sigma^{-2i-1} S]$ respectively. It remains to prove the surjectivity and the injectivity.

5.3. **Surjectivity of ϕ.**

Proposition 5.2. *The $\mathbb{Q}$-algebra $\mathcal{H}$ is generated by the $[\Sigma^p S], p \in \mathbb{Z}$.*

Proof. Let M be an object of $\mathcal{D}_{fd}(\Gamma)$. Thanks to Theorem 4.1, we may assume that the differential of M is trivial. We have a short exact sequence in the category of graded Γ-modules

$$0 \longrightarrow \mathrm{rad} M \longrightarrow M \longrightarrow \mathrm{top} M \longrightarrow 0,$$

where $\mathrm{rad}M$ is the radical of M, and $\mathrm{top}M = M/\mathrm{rad}M$ is the maximal semisimple quotient of M. This sequence can be viewed as a sequence in the category of dg Γ-modules, and yields a triangle in $\mathcal{D}_{fd}(\Gamma)$

$$\mathrm{rad} M \longrightarrow M \longrightarrow \mathrm{top} M \longrightarrow \Sigma \mathrm{rad} M.$$

Now let E be an extension of top$M$ by rad$M$ in $\mathcal{D}_{fd}(\Gamma)$, i.e. there is a triangle in $\mathcal{D}_{fd}(\Gamma)$

$$\mathrm{rad}M \longrightarrow E \longrightarrow \mathrm{top}M \xrightarrow{f} \Sigma\mathrm{rad}M.$$

Applying the cohomological functor H^*, we obtain a long exact sequence in the category of graded Γ-modules

$$\mathrm{top}M[-1] \xrightarrow{H^*f[-1]} \mathrm{rad}M \longrightarrow H^*E \longrightarrow \mathrm{top}M \xrightarrow{H^*f} \mathrm{rad}M[1].$$

If $H^*f \neq 0$, then the dimension of H^*E is strictly smaller than that of H^*M. If $H^*f = 0$, then the dimensions are equal but the number of indecomposable direct summands of H^*E is greater than or equal to that of H^*M, and these two numbers equal if and only if H^*E is isomorphic to H^*M. Hence by Theorem 3.6, the number of indecomposable direct summands of E is greater than or equal to that of M, and these two numbers equal if and only if E is isomorphic to M in $\mathcal{D}_{fd}(\Gamma)$. Thus

$$[\mathrm{top}M][\mathrm{rad}M] = F^M_{\mathrm{top}M,\mathrm{rad}M}[M] + \sum_{[E]:E\ncong M} F^E_{\mathrm{top}M,\mathrm{rad}M}[E]$$

where $F^M_{\mathrm{top}M,\mathrm{rad}M}$ is nonzero and $F^E_{\mathrm{top}M,\mathrm{rad}M}$ $(E \ncong M)$ is zero unless E has strictly more indecomposable direct summands than M or the dimension of H^*E is strictly smaller than that of H^*M. By induction the proof reduces to the case where M is semisimple, namely, the case where M is isomorphic to $\bigoplus_{p\in\mathbb{Z}}(\Sigma^p S)^{\oplus m_p}$, where the m_p's are nonnegative integers and only finitely many of them are nonzero. Applying a suitable shift, we may assume that m_p is zero for all positive p and m_0 is nonzero. Let E be an extension of S by $M' = \bigoplus_{p\leq 0}(\Sigma^p S)^{\oplus m'_p}$, where $m'_p = m_p$ if p is negative and $m'_0 = m_0 - 1$. Recall that S is a 3-spherical object. It follows that any morphism from S to $\Sigma M'$ either is 0 or induces an isomorphism between S and a direct summand of $\Sigma M'$. In both cases the above triangle splits. So E is isomorphic either to M or to $M'' = \bigoplus_{p\leq 0}(\Sigma^p S)^{\oplus m''_p}$ (when it is well-defined), where $m''_p = m'_p$ if $p \neq -1$ and $m''_{-1} = m'_{-1} - 1$. Thus we have

$$[S][M'] = F^M_{S,M'}[M] + F^{M''}_{S,M'}[M''],$$

where $F^M_{S,M'}$ is nonzero. Now the proposition follows by induction on $\sum_{p\in\mathbb{Z}} m_p$. $\surd$

As a consequence, we have

Corollary 5.3. *The algebra homomorphism $\phi : R \to \mathcal{H}$ is surjective.*

5.4. **Injectivity of ϕ.** Let R_x (respectively, R_y) be the subalgebra of R generated by $\{x_i|i \in \mathbb{Z}\}$ (respectively, by $\{y_i|i \in \mathbb{Z}\}$). The image of R_x under ϕ is the subalgebra of $\mathcal{H}$ generated by $\{[\Sigma^{2i}S]|i \in \mathbb{Z}\}$, denoted by $\mathcal{H}_x$, which has a $\mathbb{Q}$-basis $\{[M]|M \in \mathcal{D}_{fd}(\Gamma), H^{odd}M = 0\}$, where $H^{odd}M$ is the direct sum of homology spaces of M in odd degrees. Similarly, the image $\mathcal{H}_y$ of R_y under ϕ is the subalgebra of $\mathcal{H}$ generated by $\{[\Sigma^{2i+1}S]|i \in \mathbb{Z}\}$, and has a $\mathbb{Q}$-basis $\{[M]|M \in \mathcal{D}_{fd}(\Gamma), H^{even}M = 0\}$, where $H^{even}M$ is the direct sum of homology spaces of M in even degrees.

Thanks to (4)(5)(6), we have an isomorphism of $\mathbb{Q}$-vector spaces

$$\psi : R_x \otimes R_y \to R, \qquad f(x) \otimes g(y) \mapsto f(x)g(y).$$

In particular, the product of a basis of R_x and a basis of R_y is a basis of R. Now the injectivity is implied by the following two lemmas.

Lemma 5.4. *The restriction* $\phi|_{R_x} : R_x \to \mathcal{H}_x$ *(respectively,* $\phi|_{R_y} : R_y \to \mathcal{H}_y$*) is an isomorphism.*

Lemma 5.5. *The set* $\{[M][N] | M, N \in \mathcal{D}_{fd}(\Gamma), H^{odd}M = H^{even}N = 0\}$ *is a* $\mathbb{Q}$*-basis of* $\mathcal{H}$*.*

Proof of Lemma 5.4: On one hand, the algebra R_x is the Hall algebra of the quiver $\vec{A}_\infty^\infty$ of type A_∞^∞ (infinite to both sides) with linear (any) orientation. It is $\mathbb{N}I$-graded with $I = \mathbb{Z}$. For each $\underline{d} \in \mathbb{N}I$, the dimension of the degree $\underline{d}$ component of R_x equals the number of ways of expressing $\underline{d}$ as sum of dimension vectors of indecomposable representations over $\vec{A}_\infty^\infty$.

On the other hand, the algebra $\mathcal{H}_x$ is also $\mathbb{N}I$-graded with $\deg([M]) = (\dim H^{2i}(M))_{i \in I}$. Indeed, if E is an extension of M by M', where $H^{odd}M = H^{odd}M' = 0$, then we have exact sequences

$$0 = H^{2i-1}(M') \to H^{2i}(M) \to H^{2i}(E) \to H^{2i}(M') \to H^{2i+1}(M) = 0.$$

As a result, we have the equality

$$\deg([E]) = \deg([M]) + \deg([M']).$$

For $\underline{d} \in \mathbb{N}I$, the dimension of the degree $\underline{d}$ component of $\mathcal{H}_x$ equals the number of ways of expressing $\underline{d}$ as sum of $\deg([M])$ with $M \in \mathcal{D}_{fd}(\Gamma)$ indecomposable and $H^{odd}M = 0$.

Now the homomorphism $\phi|_{R_x}$ is $\mathbb{N}I$-graded and restricts to an isomorphism in each degree $\underline{d} \in \mathbb{N}I$ because the map deg defines a bijection from the set of isoclasses of indecomposable objects of $\mathcal{D}_{fd}(\Gamma)$ whose homology is concentrated in even degrees to the set of dimension vectors of indecomposable representations over the quiver $\vec{A}_\infty^\infty$. Hence it is an isomorphism. $\surd$

Proof of Lemma 5.5: By the surjectivity of ϕ, the set of products

$$\{[M][N] | M, N \in \mathcal{D}_{fd}(\Gamma), H^{odd}M = H^{even}N = 0\}$$

generates the $\mathbb{Q}$-vector space $\mathcal{H}$. It remains to prove that these products are linearly independent.

Following [8], cf. also [7], we define a partial order $\leq_\Delta$ on the set of isoclasses of objects in $\mathcal{D}_{fd}(\Gamma)$ as follows: if X and Y are two objects of $\mathcal{D}_{fd}(\Gamma)$, then $[Y] \leq_\Delta [X]$ if there exists an object Z of $\mathcal{D}_{fd}(\Gamma)$ and a triangle in $\mathcal{D}_{fd}(\Gamma)$:

$$X \to Y \oplus Z \to Z \to \Sigma X.$$

We extend the partial order $\leq_\Delta$ to a total order $\preceq$.

Now suppose $(M_1, N_1), \ldots, (M_r, N_r)$ are pairwise distinct pairs of objects of $\mathcal{D}_{fd}(\Gamma)$ such that

$$H^{odd}M_1 = \ldots = H^{odd}M_r = H^{even}N_1 = \ldots = H^{even}N_r = 0.$$

Suppose that $\lambda_1, \ldots, \lambda_r$ are rational numbers such that

$$\lambda_1[M_1][N_1] + \ldots + \lambda_r[M_r][N_r] = 0.$$

By the assumption on the M_i's and N_i's, there is a unique maximal element among all $[M_i \oplus N_i]$'s, say $[M_1 \oplus N_1]$. Then we have

$$\lambda_1[M_1][N_1] + \ldots + \lambda_r[M_r][N_r] = \lambda_1 F_{M_1 N_1}^{M_1\oplus N_1}[M_1 \oplus N_1] + \text{smaller terms},$$

since a nontrivial extension of two objects is always smaller than the direct sum of them. The (derived) Hall number $F_{M_1N_1}^{M_1\oplus N_1}$ is a nonzero rational number. Therefore λ_1 has to be zero. An induction on r shows that $\lambda_1 = \ldots = \lambda_r = 0$. $\surd$

6. From the Hall algebra to the torus

Let $v = \sqrt{q}$. We tensor R with $\mathbb{Q}(v)$ over $\mathbb{Q}$, and still denote the resulting algebra by R. Let I be the ideal of R generated by the space $[R, R]$ of commutators of R.

Lemma 6.1. *The assignment $\varphi : x_i \mapsto \frac{v}{v^2-1}x, y_i \mapsto \frac{v}{v^2-1}x^{-1}$ defines an algebra homomorphism from R to $\mathbb{Q}(v)[x, x^{-1}]$ with kernel I.*

Proof. We have

$$R/I \cong \mathbb{Q}(v)[x_i, y_i]_{i\in\mathbb{Z}}/(x_i y_i = x_{i+1} y_i = \frac{q}{(q-1)^2}).$$

Now it is clear that $x_i \mapsto \frac{v}{v^2-1}x, y_i \mapsto \frac{v}{v^2-1}x^{-1}$ defines an algebra isomorphism from R/I to $\mathbb{Q}(v)[x, x^{-1}]$. $\surd$

7. General case

The general case can be treated similarly. Here we only give the final result and the key points of the proof.

Theorem 7.1. *Let d be an integer, and $d' = d - 1$. Let $\mathcal{D}$ be the algebraic triangulated category classically generated by a d-spherical object, and let $\mathcal{H}(\mathcal{D})$ be the Hall algebra of $\mathcal{D}$ over $\mathbb{Q}$.*

(i) When $d \geq 3$ (i.e. $d' \geq 2$), the algebra $\mathcal{H}(\mathcal{D})$ is generated by z_i, $i \in \mathbb{Z}$, subject to the following relations:

$$z_i^2 z_{i-d'} - (q+1)q^{(-1)^d} z_i z_{i-d'} z_i + q^{1+2(-1)^d} z_{i-d'} z_i^2 \tag{10}$$

$$z_i z_{i-d'}^2 - (q+1)q^{(-1)^d} z_{i-d'} z_i z_{i-d'} + q^{1+2(-1)^d} z_{i-d'}^2 z_i \tag{11}$$

$$z_i z_{i+1} - q^{-1} z_{i+1} z_i - \tfrac{1}{q-1} \tag{12}$$

$$z_i z_j - q^{(-1)^{j-i}(1+(-1)^{-d})} z_j z_i \qquad \textit{if } i - j < -d' \tag{13}$$

$$z_i z_j - q^{(-1)^{j-i}} z_j z_i \qquad \textit{if } -d' < i - j < -1. \tag{14}$$

(ii) When $d = 2$ (i.e. $d' = 1$), the algebra $\mathcal{H}(\mathcal{D})$ is generated by z_i, $i \in \mathbb{Z}$, subject to the following relations:

$$z_i^2 z_{i-1} - (q+1)q z_i z_{i-1} z_i + q^3 z_{i-1} z_i^2 - q(q+1) z_i \tag{15}$$

$$z_i z_{i-1}^2 - (q+1)q z_{i-1} z_i z_{i-1} + q^3 z_{i-1}^2 z_i - q(q+1) z_{i-1} \tag{16}$$

$$z_i z_j - q^{2(-1)^{j-i}} z_j z_i \qquad \textit{if } i - j < -1. \tag{17}$$

(iii) When $d = 1$ (i.e. $d' = 0$), the algebra $\mathcal{H}(\mathcal{D})$ is generated by $z_{i,j}$, $i \in \mathbb{Z}$, $j \in \mathbb{N}$, subject to the following relations:

$$z_{i,j} z_{i',j'} - z_{i',j'} z_{i,j}, \qquad \text{if } i - i' \neq \pm 1 \tag{18}$$

$$z_{i,j} z_{i+1,j'} - \sum\nolimits_{0 \leq l \leq \min\{j,j'\}} F^l_{j,j'} z_{i+1,j'-l} z_{i,j-l} \tag{19}$$

where

$$F^l_{j,j'} = \begin{cases} 1, & \text{if } l = 0 \\ \frac{q-1}{q^{l+1}}, & \text{if } 0 < l < \min\{j,j'\} \\ q^{-j'}, & \text{if } l = j' < j \\ q^{-j}, & \text{if } l = j < j' \\ \frac{1}{q^{j-1}(q-1)}, & \text{if } l = j = j'. \end{cases}$$

(iv) When $d = 0$ (i.e. $d' = -1$), the algebra $\mathcal{H}(\mathcal{D})$ is generated by z_i, $i \in \mathbb{Z}$, subject to the following relations

$$z_i^2 z_{i+1} - (q+1) q^{-2} z_i z_{i+1} z_i + q^{-3} z_{i+1} z_i^2 - (q+1) q^{-3} z_i \tag{20}$$

$$z_i z_{i+1}^2 - (q+1) q^{-2} z_{i+1} z_i z_{i+1} + q^{-3} z_{i+1}^2 z_i - (q+1) q^{-3} z_{i-1} \tag{21}$$

$$z_i z_j - q^{2(-1)^{j-i}} z_j z_i \qquad \text{if } i - j < -1. \tag{22}$$

(v) When $d \leq -1$ (i.e. $d' \leq -2$), the algebra $\mathcal{H}(\mathcal{D})$ is generated by z_i, $i \in \mathbb{Z}$, subject to the following relations:

$$z_i^2 z_{i-d'} - (q+1) q^{-1-(-1)^{-d}} z_i z_{i-d'} z_i + q^{-1-2(-1)^{-d}} z_{i-d'} z_i^2 \tag{23}$$

$$z_i z_{i-d'}^2 - (q+1) q^{-1-(-1)^{-d}} z_{i-d'} z_i z_{i-d'} + q^{-1-2(-1)^{-d}} z_{i-d'}^2 z_i \tag{24}$$

$$z_i z_{i+1} - q^{-1} z_{i+1} z_i - \frac{1}{q^{(-1)^{-d}}(q-1)} \tag{25}$$

$$z_i z_j - q^{(-1)^{j-i}(1+(-1)^{-d})} z_j z_i \qquad \text{if } i - j < d' \tag{26}$$

$$z_i z_j - q^{(-1)^{j-i}} z_j z_i \qquad \text{if } d' < i - j < -1. \tag{27}$$

Proof. Let S be the d-spherical object, and Σ be the suspension functor.

(i) and (v): Similar to Theorem 5.1, with z_i representing $\Sigma^{-i} S$.

(ii) and (iv): Notice that both the Hall algebra $\mathcal{H}(\mathcal{D})$ and the desired algebra are filtered, and the algebra homomorphism from the desired algebra to the Hall algebra $\mathcal{H}(\mathcal{D})$ is a morphism of filtered algebras, and the associated graded algebra homomorphism is an isomorphism, which has a similar proof to that for Theorem 5.1, with z_i representing for $\Sigma^{-i} S$.

(iii) In this case, the triangulated category $\mathcal{D}$ is equivalent to the bounded derived category of the hereditary abelian category of finite dimensional representations over the Jordan quiver. Then the desired result follows from [22, Proposition 7.1] and the classical result on the Hall algebra of the above hereditary abelian category (*cf.* for example [18]), with $z_{i,j}$ representing $\Sigma^{-i} M_j$, where M_j is the indecomposable nilpotent representation of the Jordan quiver of dimension j. $\surd$

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