

Elsevier Editorial System(tm) for Journal of Algebra
Manuscript Draft

Manuscript Number: JALGEBRA-D-07-00076R1

Title: Bounded derived categories and repetitive algebras

Article Type: Research Paper

Section/Category: General

Keywords: derived categories, repetitive algebras, almost split triangles

Corresponding Author: idun reiten, ph.d

Corresponding Author's Institution: NTNU

First Author: idun reiten, ph.d

Order of Authors: idun reiten, ph.d; dieter happel; bernhard keller

Manuscript Region of Origin:

BOUNDED DERIVED CATEGORIES AND REPETITIVE ALGEBRAS

DIETER HAPPEL, BERNHARD KELLER AND IDUN REITEN

INTRODUCTION

Let Λ be a finite dimensional algebra over a field k . It was proved in [H1] that there is a full and faithful embedding of the bounded derived category $\mathbf{D}^b(\Lambda)$ into the stable category $\underline{\text{mod}}\widehat{\Lambda}$ of finite dimensional modules over the repetitive algebra $\widehat{\Lambda}$. This embedding is an equivalence if and only if Λ has finite global dimension [H1]. The category $\mathbf{D}^b(\Lambda)$ is a triangulated category which does not have almost split triangles when Λ has infinite global dimension [H2], whereas $\underline{\text{mod}}\widehat{\Lambda}$ is triangulated and always has almost split triangles [H1] [H2].

The purpose of this paper is to investigate the relationship between $\mathbf{D}^b(\Lambda)$ and $\underline{\text{mod}}\widehat{\Lambda}$ from various points of view, which is of course meaningful only for algebras Λ of infinite global dimension. The most satisfactory results are obtained for Gorenstein algebras, especially for selfinjective algebras.

We investigate the embedding $\mathbf{D}^b(\Lambda) \subset \underline{\text{mod}}\widehat{\Lambda}$ from the point of view of universal properties with respect to triangle functors to triangulated categories with almost split triangles, and also to which extent $\underline{\text{mod}}\widehat{\Lambda}$ is the smallest category containing $\mathbf{D}^b(\Lambda)$ with these properties. The first question has a positive answer for Gorenstein algebras, and is not true in general. The second question has a negative answer even for selfinjective algebras.

We also investigate the behavior of almost split triangles and irreducible maps under the embedding functor, and show that both are actually preserved. While it is known from [H2] what the end terms of almost split triangles in $\mathbf{D}^b(\Lambda)$ look like, and hence the left and right end terms of certain irreducible maps, we do not know in general so much about irreducible maps in $\mathbf{D}^b(\Lambda)$. However in the selfinjective case we show that there are no irreducible maps not associated with almost split triangles when $(\text{rad } \Lambda)^2 \neq 0$, and we describe them all when $(\text{rad } \Lambda)^2 = 0$. We believe that also for arbitrary Λ there should be very few irreducible maps not associated with almost split triangles. As an application of our results we show that when Λ is selfinjective, all the components of the AR-quiver of the category $\mathbf{K}^b(\mathcal{P})$ of bounded complexes of projective modules are of the form $\mathbb{Z}A_\infty$.

The paper is organised as follows. In section 1 we give some background material from [H1] on the categories $\mathbf{D}^b(\Lambda)$ and $\underline{\text{mod}}\widehat{\Lambda}$, including properties of almost split triangles. In section 2 we give an example showing that in general the embedding $\mathbf{D}^b(\Lambda) \subset \underline{\text{mod}}\widehat{\Lambda}$ is not universal among triangle functors from $\mathbf{D}^b(\Lambda)$ to triangulated categories with almost split triangles. We also show that the embedding $\text{mod } \Lambda \subset \underline{\text{mod}}\widehat{\Lambda}$ has a weak universal property with respect to triangle functors from $\mathbf{D}^b(\Lambda)$ to triangulated categories where the Nakayama functor becomes an equivalence. We

deduce that if Λ is Gorenstein, there is a natural triangle functor from $\underline{\text{mod}}\widehat{\Lambda}$ to $\mathbf{D}^b(\Lambda)$. In section 3 we show that even when Λ is selfinjective, there is an infinite strictly descending chain of triangulated subcategories of $\underline{\text{mod}}\widehat{\Lambda}$ with almost split triangles and containing $\mathbf{D}^b(\Lambda)$. In section 4 we show that irreducible maps in $\mathbf{D}^b(\Lambda)$ stay irreducible in $\underline{\text{mod}}\widehat{\Lambda}$, and give sufficient conditions for the existence of irreducible maps in $\mathbf{D}^b(\Lambda)$ of the form $S[-1] \rightarrow T$ where S and T are simple Λ -modules. In section 5 we show that almost split triangles in $\mathbf{D}^b(\Lambda)$ stay almost split in $\underline{\text{mod}}\widehat{\Lambda}$, and give the shape of the components of the AR-quiver of $\mathbf{K}^b(\mathcal{P})$ for selfinjective algebras. We also give necessary conditions for having irreducible maps in $\mathbf{D}^b(\Lambda)$ not coming from almost split triangles for Gorenstein algebras, and deduce the result on irreducible maps in $\mathbf{D}^b(\Lambda)$ when Λ is selfinjective. In section 6 we deal with arbitrary finite dimensional algebras, and give some results supporting the suspicion that there are very few irreducible maps not associated with almost split triangles. We also show that the natural questions of a connection between irreducible maps between infinite complexes of projective modules and their finite parts have negative answers.

1. PRELIMINARIES

In this section we will fix the notation and recall some of the results frequently used in the subsequent sections. For the proofs of the stated propositions we refer to [H1]. Let Λ be a finite dimensional algebra over a field k .

We denote by $\text{mod } \Lambda$ the category of finitely generated left Λ -modules and by ${}_{\Lambda}\mathcal{P}$ (resp. ${}_{\Lambda}\mathcal{I}$) the full subcategory of projective (resp. injective) Λ -modules. For a simple Λ -module S we denote by $P(S)$ (resp. $I(S)$) the projective cover (resp. injective envelope) of S . We denote by $\nu_{\Lambda}: {}_{\Lambda}\mathcal{P} \rightarrow {}_{\Lambda}\mathcal{I}$ the Nakayama functor defined by $\nu_{\Lambda} = D \text{Hom}(-, {}_{\Lambda}\Lambda)$, where D is the duality with respect to k , and by $\nu_{\Lambda}^{-}: {}_{\Lambda}\mathcal{I} \rightarrow {}_{\Lambda}\mathcal{P}$ the inverse Nakayama functor which is defined by $\nu_{\Lambda}^{-} = \text{Hom}(D\Lambda, -)$. We denote by $\mathbf{D}^b(\Lambda)$ the bounded derived category of $\text{mod } \Lambda$. The Nakayama functors ν_{Λ} and ν_{Λ}^{-} induce inverse equivalences of triangulated categories still denoted by $\nu_{\Lambda}: \mathbf{K}^b({}_{\Lambda}\mathcal{P}) \rightarrow \mathbf{K}^b({}_{\Lambda}\mathcal{I})$ and $\nu_{\Lambda}^{-}: \mathbf{K}^b({}_{\Lambda}\mathcal{I}) \rightarrow \mathbf{K}^b({}_{\Lambda}\mathcal{P})$. We denote by $\mathbf{K}^{-,b}({}_{\Lambda}\mathcal{P})$ the homotopy category of complexes over ${}_{\Lambda}\mathcal{P}$ bounded above with bounded cohomology groups. Note that $\mathbf{K}^{-,b}({}_{\Lambda}\mathcal{P}) \simeq \mathbf{D}^b(\Lambda)$. For a complex $Z = (Z^i, d^i)$ in $\mathbf{D}^b(\Lambda)$ and $n \in \mathbb{Z}$ we always have a triangle $Z_{\geq n} \rightarrow Z \rightarrow Z_{< n} \rightarrow Z_{\geq n}[1]$ in $\mathbf{D}^b(\Lambda)$, where $Z_{\geq n}^i = Z^i$ for $i \geq n$, $Z_{< n}^i = Z^i$ for $i < n$, and zero otherwise, with the induced differentials.

To Λ we may associate the repetitive algebra $\widehat{\Lambda}$ and its category $\text{mod } \widehat{\Lambda}$ of finitely generated modules. The $\widehat{\Lambda}$ -modules X are given by $X = (X_i, f_i)$ where $X_i \in \text{mod } \Lambda$ and $X_i = 0$ for almost all i and $f_i: X_i \rightarrow \nu_{\Lambda}^{-} X_{i+1}$ such that $f_i \nu_{\Lambda}^{-}(f_{i+1}) = 0$ for all i . A morphism of $\widehat{\Lambda}$ -modules is defined in an obvious way. There is an automorphism $\nu_{\widehat{\Lambda}}: \text{mod } \widehat{\Lambda} \rightarrow \text{mod } \widehat{\Lambda}$ defined by $(\nu_{\widehat{\Lambda}} X)_i = X_{i+1}$. The inverse is denoted by $\nu_{\widehat{\Lambda}}^{-}$. The category $\text{mod } \widehat{\Lambda}$ is a Frobenius category in the sense of [H1]. The indecomposable projective-injective $\widehat{\Lambda}$ -modules are given by $I = P = (X_i, f_i)$ with $X_i = P(S)$, $X_{i+1} = I(S)$, $f_i = \text{id}_{P(S)}$ and zero otherwise. Note that $\text{top } P = \nu_{\widehat{\Lambda}} \text{soc } P$. Clearly there are enough projectives. So for each X we obtain exact sequences $0 \rightarrow X \rightarrow I(X) \rightarrow \Omega_{\widehat{\Lambda}}^{-} X \rightarrow 0$ and $0 \rightarrow \Omega_{\widehat{\Lambda}} X \rightarrow P(X) \rightarrow X \rightarrow 0$. We denote by $\underline{\text{mod}}\widehat{\Lambda}$ the stable

category. This is a triangulated category where $\Omega_{\widehat{\Lambda}}^-$ serves as a translation functor. If $X \in \underline{\text{mod}}\widehat{\Lambda}$, we may choose a representative, again denoted by $X \in \text{mod}\widehat{\Lambda}$, without indecomposable projective direct summands. This fact will be used frequently later on.

There is a triangle functor $\mu: \mathbf{D}^b(\Lambda) \rightarrow \underline{\text{mod}}\widehat{\Lambda}$ which is full and faithful such that μ extends the identity functor on $\text{mod}\Lambda$, where $\text{mod}\Lambda$ is embedded in $\mathbf{D}^b(\Lambda)$ (resp. $\underline{\text{mod}}\widehat{\Lambda}$) as complexes (resp. modules) concentrated in degree zero. It is known [H2] that μ is an equivalence if and only if $\text{gl. dim}\Lambda < \infty$.

In general we recall from [GK] the following description of $\text{Im}\mu$. $Z = (Z_i, g_i) \in \text{Im}\mu$ if and only if there is some $n \geq 0$ such that $(\Omega_{\widehat{\Lambda}}^{-n}Z)_j = 0$ for $j > 0$ and $(\Omega_{\widehat{\Lambda}}^n Z)_j = 0$ for $j < 0$. Also note that for a $\widehat{\Lambda}$ -module $Z = (Z_i, g_i)$ with $Z_i = 0$ for $i < 0$ also $(\Omega_{\widehat{\Lambda}}^r Z)_j = 0$ for $j < 0$ and all $r \geq 0$.

This has the following immediate consequence for Gorenstein algebras (see also [CZ]).

Corollary 1.1. *Let Λ be a Gorenstein algebra. Then $\text{Im}\mu = \{Z = (Z_i, g_i) \in \text{mod}\widehat{\Lambda} \mid \text{pd}_{\Lambda} Z_i < \infty \text{ for } i \neq 0\}$*

Proof. If Λ is a Gorenstein algebra then the modules of finite projective dimension coincide with the modules of finite injective dimension. Moreover this dimension is bounded by the projective dimension of $D\Lambda_{\Lambda}$, which coincides with the injective dimension of ${}_{\Lambda}\Lambda$. Suppose that $Z = (Z_i, g_i) \in \text{mod}\widehat{\Lambda}$ satisfies $\text{pd}_{\Lambda} Z_i < \infty$ for $i \neq 0$, then it follows immediately from the criterion mentioned above from [GK] that $Z \in \text{Im}\mu$. Conversely let $Z = (Z_i, g_i) \in \text{Im}\mu$ and assume that $Z = (Z_i, g_i) \in \text{mod}\widehat{\Lambda}$ satisfies $\text{pd}_{\Lambda} Z_i = \infty$ for some $i \neq 0$. We may assume that $i > 0$. Choose i maximal with this property. So $\text{pd}_{\Lambda} Z_j < \infty$ for $j > i$. By the first part of the proof and the fact that $\text{Im}\mu$ is a triangulated category the factor module $Z' = (Z'_j, g_j)$ with $Z'_j = Z_j$ for $j \leq i$ and $Z'_j = 0$ for $j > i$ is contained in $\text{Im}\mu$. But then $(\Omega_{\widehat{\Lambda}}^{-n} Z')_i \neq 0$ for all $n \geq 0$, in contrast to the result recalled from [GK]. \square

Let \mathcal{C} be a Hom-finite triangulated category, that is, the homomorphism spaces are finite dimensional over k . Assume that \mathcal{C} is also Krull-Schmidt, that is, the indecomposable objects have local endomorphism rings. We say that there is an almost split triangle ending at Z provided there is a triangle in \mathcal{C} of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

where (i) X is indecomposable, (ii) for all $f: W \rightarrow Z$ not split epi there is some $g: W \rightarrow Y$ with $f = gv$ and (iii) $w \neq 0$.

We refer to [H1] for equivalent formulations and the connection to irreducible maps.

In case there is an almost split triangle ending at Z , the starting term X is uniquely determined up to isomorphism. We then define $\tau_{\mathcal{C}}Z = X$.

Almost split sequences exist in $\text{mod}\widehat{\Lambda}$ and the translation τ is DTr , where Tr denotes the transpose [H2]. It follows easily from this that $\underline{\text{mod}}\widehat{\Lambda}$ has almost split triangles with $\tau = \text{DTr}$. It is well known and can be shown using this description of τ that for $Z \in \underline{\text{mod}}\widehat{\Lambda}$ indecomposable, we have $\tau_{\widehat{\Lambda}}Z = \nu_{\widehat{\Lambda}}\Omega_{\widehat{\Lambda}}^2 X$.

In the case of $\mathbf{D}^b(\Lambda)$ the following is known [H2]. Let $Z \in \mathbf{D}^b(\Lambda)$ be indecomposable. Then there is an almost split triangle $X \rightarrow Y \rightarrow Z \xrightarrow{w} X[1]$ if and only if $Z \in \mathbf{K}^b(\Lambda\mathcal{P})$. In this case $\tau_{\mathbf{D}^b(\Lambda)}Z = \nu_\Lambda Z[-1]$. Thus $\mathbf{D}^b(\Lambda)$ has almost split triangles, that is, for any indecomposable object Z in $\mathbf{D}^b(\Lambda)$ there is an almost split triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ in $\mathbf{D}^b(\Lambda)$ if and only if $\text{gl. dim } \Lambda < \infty$.

2. A COUNTEREXAMPLE AND A WEAK UNIVERSAL PROPERTY

Problem: Let Λ be an artin algebra and

$$\mu : \mathbf{D}^b(\Lambda) \rightarrow \underline{\text{mod}} \widehat{\Lambda}$$

the embedding of [H1]. Let \mathcal{C} be a triangulated category with almost split triangles and $F : \mathbf{D}^b(\Lambda) \rightarrow \mathcal{C}$ a triangle functor. Does there exist a triangle functor

$$G : \underline{\text{mod}} \widehat{\Lambda} \rightarrow \mathcal{C}$$

such that $F \xrightarrow{\sim} \mu G$?

The following example shows that the answer is no, in general.

Example: Let Λ be given as a factor algebra of a path algebra of a field k by an ideal:

$$\beta \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} T \xleftarrow{\alpha} S, \langle \beta^2, \alpha\beta \rangle.$$

Let S, T be the two simple Λ -modules. Then

$$P(S) = \begin{pmatrix} S \\ T \end{pmatrix} \text{ and } P(T) = \begin{pmatrix} T \\ T \end{pmatrix}$$

are the indecomposable projective Λ -modules and $I(S) = S$ and

$$I(T) = \begin{pmatrix} S & T \\ & T \end{pmatrix}$$

the indecomposable injective Λ -modules. It is easy to see that Λ is not Gorenstein ($I(T)$ is of infinite projective dimension). Let $\Gamma = \text{End}_\Lambda(P(T))$, so $\Gamma = k[x]/(x^2)$ and let $F = \text{Hom}(P(T), -)$ be a functor from $\text{mod } \Lambda$ to $\text{mod } \Gamma$. Now F is exact, so F induces a functor $\mathbf{D}^b(\Lambda) \rightarrow \mathbf{D}^b(\Gamma)$. Since Γ is selfinjective, there is a functor [Ric]

$$\pi : \mathbf{D}^b(\Gamma) \rightarrow \underline{\text{mod}} \Gamma,$$

so there is a triangle functor $\phi : \mathbf{D}^b(\Lambda) \rightarrow \underline{\text{mod}} \Gamma$ and $\underline{\text{mod}} \Gamma$ has almost split triangles. We are now going to show that there is no triangle functor $G : \underline{\text{mod}} \widehat{\Lambda} \rightarrow \underline{\text{mod}} \Gamma$ such that $\phi = \mu G$.

Suppose there exists a triangle functor $G : \underline{\text{mod}} \widehat{\Lambda} \rightarrow \underline{\text{mod}} \Gamma$ such that $\phi = \mu G$. Let $X = (X_i, f_i)$ be an object of $\underline{\text{mod}} \widehat{\Lambda}$ with $X_1 = S$ and $X_i = 0$ for $i \neq 1$. Then $\Omega_{\widehat{\Lambda}}^- X = P(S)$, the stalk module concentrated in degree zero. So $G\Omega_{\widehat{\Lambda}}^- X = \phi(P(S)) = T$ and

$$G\Omega_{\widehat{\Lambda}}^- X \cong \Omega_{\Gamma}^- G(X) = G(X),$$

so $G(X) \cong T$. Also $G(S) = \phi(S) = 0$ and $G(T) = \phi(T) = T$. But then also $G(\Omega_{\widehat{\Lambda}}T) = T$. Now $\Omega_{\widehat{\Lambda}}T = (Y_i, f_i)$ where $Y_0 = T$, $Y_1 = I(T)$ and $f_0 : T \rightarrow P(T)$ the canonical map, and $Y_i = 0$ for $i \neq 0, 1$. Consider the exact sequence in $\text{mod } \Lambda$:

$$0 \rightarrow T \rightarrow P(T) \rightarrow T \rightarrow 0.$$

It gives rise to a triangle

$$T[-1] \xrightarrow{f} T \rightarrow P(T) \rightarrow T \quad (*)$$

in $\mathbf{D}^b(\Lambda)$. Since $\phi(P(T)) = 0$, the map $\phi(f)$ is invertible. Now we also have the exact sequence in $\text{mod } \widehat{\Lambda}$:

$$0 \rightarrow Z \rightarrow \Omega_{\widehat{\Lambda}}T \rightarrow T \rightarrow 0.$$

This gives rise to a triangle

$$Z \rightarrow \Omega_{\widehat{\Lambda}}T \xrightarrow{\mu(f)} T \rightarrow Z[1],$$

which identifies with the image of the triangle $(*)$ under μ . Applying G then shows that $G(Z) = 0$ because $\mu(G(f)) = \phi(f)$ is invertible. Now $Z = (Z_i, f_i)$, where $Z_1 = I(T)$ and $Z_i = 0$ for $i \neq 1$. Let $U = (U_i, f_i)$ with $U_1 = T$ and $U_i = 0$ for $i \neq 1$. Then we obtain an exact sequence in $\text{mod } \widehat{\Lambda}$

$$0 \rightarrow U \rightarrow Z \rightarrow U \oplus X \rightarrow 0,$$

which gives rise to a triangle

$$U \rightarrow Z \rightarrow U \oplus X \rightarrow U[1]$$

in $\underline{\text{mod}} \widehat{\Lambda}$, and so

$$G(U) \rightarrow G(Z) \rightarrow G(U) \oplus G(X) \rightarrow G(U)[1]$$

is a triangle in $\underline{\text{mod}} \Gamma$. Now $G(Z) = 0$ by the computation above and $G(X) = T$, so the triangle is of the form

$$G(U) \rightarrow 0 \rightarrow G(U) \oplus T \rightarrow G(U)[1],$$

a contradiction.

A weak universal property. As the above counterexample shows, the repetitive category is not the ‘universal triangulated category with Auslander-Reiten triangles containing the derived category’. However, we will see that if we take into account additional structure, we do get a weak universal property for the embedding

$$\text{mod } \Lambda \rightarrow \underline{\text{mod}} \widehat{\Lambda}.$$

Roughly speaking this embedding is the ‘universal functor to a triangulated category where the Nakayama functor becomes an equivalence’. In the case where Λ is Gorenstein, we will use this property to construct a natural triangle functor from the stable category of the repetitive category to the bounded derived category.

Let us now construct the additional structure we need: For short, let us write \mathcal{M} for $\text{mod } \Lambda$ and \mathcal{R} for $\text{mod } \widehat{\Lambda}$. We write $\Sigma : \mathcal{M} \rightarrow \mathcal{M}$ for the right exact extension of the Nakayama functor defined in section 1: Thus, we have $\Sigma(M) = (D\Lambda) \otimes_{\Lambda} M$ for

all M in \mathcal{M} . We now define an exact functor $\mathcal{R} \rightarrow \mathcal{R}$, which we will also denote by Σ . Namely, we put

$$\Sigma(X) = \nu_{\Lambda}(\Omega X),$$

where Ω is the syzygy functor $\mathcal{R} \rightarrow \mathcal{R}$ constructed as follows: If X is an object of \mathcal{R} with structure maps f_i , $i \in \mathbb{Z}$, we define the object $P(X)$ to have the i th component

$$(\Lambda \otimes_k X_i) \oplus (D\Lambda \otimes_k X_{i-1})$$

and the structure maps

$$\begin{bmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{bmatrix} : \nu P(X)_i \rightarrow P(X)_{i-1}.$$

Thus, the object $P(X)$ is projective-injective. We define the canonical map $P(X) \rightarrow X$ to have the components

$$[\text{can}, g_{i-1}] : P(X)_i \rightarrow X_i$$

where can is the canonical map from $\Lambda \otimes_k X_i$ to X_i and g_{i-1} is the map $D\Lambda \otimes_k X_{i-1} \rightarrow X_i$ induced by $\nu(f_{i-1})$. Thus, the map $P(X) \rightarrow X$ is a functorial projective right approximation of X . We define ΩX to be the kernel of $P(X) \rightarrow X$.

The functor $\Sigma : \mathcal{R} \rightarrow \mathcal{R}$ is exact, preserves projective-injectives and induces an equivalence in the stable category (namely, the Serre functor). Moreover, if $F_0 : \mathcal{M} \rightarrow \mathcal{R}$ denotes the canonical embedding, we have a morphism of functors

$$\phi_0 : F_0 \Sigma \rightarrow \Sigma F_0.$$

Namely, for an object M of \mathcal{M} , the only non vanishing component of the morphism $F_0 \Sigma(M) \rightarrow \Sigma F_0(M)$ is induced by the canonical map $D\Lambda \otimes_k M \rightarrow D\Lambda \otimes_{\Lambda} M$. It is easy to check that if P is a projective Λ -module, then $\phi_0(P)$ becomes an isomorphism in the stable category of \mathcal{R} . To summarize, we have

- a k -linear Frobenius category \mathcal{R} endowed with an exact functor $\Sigma : \mathcal{R} \rightarrow \mathcal{R}$ preserving projective-injectives and inducing an equivalence in the stable category,
- an exact functor $F_0 : \mathcal{M} \rightarrow \mathcal{R}$ endowed with a morphism

$$\phi_0 : F_0 \Sigma \rightarrow \Sigma F_0$$

such that $\phi_0(P)$ becomes an isomorphism in the stable category for each projective module P .

Theorem 2.1. *Let \mathcal{E} be a k -linear Frobenius category endowed with an exact functor $\Sigma : \mathcal{E} \rightarrow \mathcal{E}$ preserving projective-injectives and inducing an equivalence in the stable category. Let $F : \mathcal{M} \rightarrow \mathcal{E}$ be an exact functor endowed with a morphism $\phi : \Sigma F \rightarrow F \Sigma$ such that $\phi(P)$ becomes an isomorphism in the stable category for each projective module P . Then there is a triangle functor*

$$G : \underline{\mathcal{R}} \rightarrow \underline{\mathcal{E}},$$

such that G commutes with Σ up to isomorphism and the triangle

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F_0} & \mathcal{R} \\ & \searrow F & \downarrow G \\ & & \underline{\mathcal{E}} \end{array}$$

commutes up to isomorphism.

The theorem will be proved below. Note that it does not make any claim about uniqueness. In fact, one could obtain a more intrinsic formulation and a uniqueness statement by working in a more sophisticated framework based on towers of triangulated categories [K2], or derivators [G] or the homotopy category of dg categories [Ta] [To] [K1]. However, this would go beyond the scope of this article.

Corollary 2.2. *Suppose that Λ is Gorenstein. Then there is a triangle functor*

$$G : \underline{\text{mod}} \widehat{\Lambda} \rightarrow \mathbf{D}^b(\Lambda)$$

which commutes with the inclusion of $\text{mod } \Lambda$ and such that we have a functorial isomorphism

$$\Sigma \circ G \xrightarrow{\sim} G \circ \Sigma,$$

where $\Sigma : \underline{\text{mod}} \widehat{\Lambda} \rightarrow \underline{\text{mod}} \widehat{\Lambda}$ is the Serre functor and $\Sigma : \mathbf{D}^b(\Lambda) \rightarrow \mathbf{D}^b(\Lambda)$ the functor $M \mapsto D\Lambda \overset{L}{\otimes}_{\Lambda} M$.

Proof of the corollary. Let \mathcal{E} be the category of right bounded complexes of projective Λ -modules with bounded homology. Then the stable category of \mathcal{E} is triangle equivalent to the bounded derived category. For each Λ -bimodule B , write $\mathbf{p}(B)$ for a projective bimodule resolution of B . Let $\Sigma : \mathcal{E} \rightarrow \mathcal{E}$ be the (total) tensor product over Λ by the complex of bimodules $\mathbf{p}(D\Lambda)$. Let F_0 be the functor taking a module M to $\mathbf{p}(\Lambda) \otimes_{\Lambda} M$. To construct $\phi : F_0 \Sigma \rightarrow \Sigma F_0$, it suffices to construct a quasi-isomorphism of bimodule complexes

$$\widetilde{\phi} : \mathbf{p}(D\Lambda) \otimes_{\Lambda} \mathbf{p}(\Lambda) \rightarrow \mathbf{p}(\Lambda) \otimes_{\Lambda} D\Lambda.$$

Indeed, since the morphism

$$\mathbf{p}(D\Lambda) \otimes_{\Lambda} \mathbf{p}(\Lambda) \rightarrow D\Lambda$$

is a projective resolution, it lifts (in the homotopy category) along the quasi-isomorphism

$$\mathbf{p}(\Lambda) \otimes_{\Lambda} D\Lambda \rightarrow D\Lambda$$

and we define $\widetilde{\phi}$ to be a representative of a lift. If P is a projective module, then in the square (commutative in the homotopy category),

$$\begin{array}{ccc} \mathbf{p}(D\Lambda) \otimes_{\Lambda} \mathbf{p}(\Lambda) \otimes_{\Lambda} P & \longrightarrow & D\Lambda \otimes_{\Lambda} P \\ \phi(P) \downarrow & & \downarrow \mathbf{1} \\ \mathbf{p}(\Lambda) \otimes_{\Lambda} D\Lambda \otimes_{\Lambda} P & \longrightarrow & D\Lambda \otimes_{\Lambda} P \end{array}$$

the two horizontal morphisms are quasi-isomorphisms and so the left vertical morphism is a homotopy equivalence. This means that $\phi(P)$ becomes an isomorphism

in the stable category. Thus, the hypotheses of the theorem are satisfied and we get, if Λ is Gorenstein, a natural triangle functor

$$G : \underline{\text{mod}} \widehat{\Lambda} \rightarrow \mathbf{D}^b(\Lambda)$$

which extends the inclusion of $\text{mod } \Lambda$ and commutes with Σ up to isomorphism of triangle functors.

Proof of the theorem. It is not hard to see that it suffices to define a functor with the required properties on the full subcategory of objects X of \mathcal{R} with $X_i = 0$ for $i > 0$. Let X be such an object of \mathcal{R} with structure maps $f_i : \Sigma(X_i) \rightarrow X_{i+1}$, $i \in \mathbb{Z}$. We define $G_1(X)$ to be the complex over \mathcal{E} with components $\Sigma^i(F(X_{-i}))$ and with the differential

$$\Sigma^i(F(X_{-i})) \rightarrow \Sigma^{i-1}(F(X_{-i+1}))$$

given by $(\Sigma^{i-1}\phi(X_{-i}))(\Sigma^{i-1}F(f_i))$. It is straightforward to check that the square of the differential vanishes and that with the natural definition of G_1 on morphisms, we get a k -linear functor

$$G_1 : \mathcal{R} \rightarrow \mathbf{C}^b(\mathcal{E})$$

taking exact sequences of \mathcal{R} to componentwise conflations of the category $\mathbf{C}^b(\mathcal{E})$ of bounded complexes over \mathcal{E} . Moreover, the functor G_1 takes an indecomposable projective injective object given by a projective P (put in degree -1 , for simplicity of notation) and the identity $\Sigma P \rightarrow \Sigma P$ to a complex of the shape

$$\cdots \longrightarrow 0 \longrightarrow \Sigma(F(P)) \xrightarrow{\phi(P)} F(\Sigma(P)) \longrightarrow 0 \longrightarrow \cdots$$

Now since \mathcal{E} is a Frobenius category, we have a canonical triangle functor [KV] [Ric]

$$\mathbf{D}^b(\mathcal{E}) \rightarrow \underline{\mathcal{E}}$$

extending the natural projection functor $\mathcal{E} \rightarrow \underline{\mathcal{E}}$. We define G_2 to be the composition

$$\mathbf{C}^b(\mathcal{E}) \rightarrow \mathbf{D}^b(\mathcal{E}) \rightarrow \underline{\mathcal{E}}$$

and we put $G_3 = G_1 \circ G_2 : \mathcal{R} \rightarrow \underline{\mathcal{E}}$. Then G_3 takes projective-injectives to zero-objects: Indeed, a complex of the form

$$\cdots \longrightarrow 0 \longrightarrow \Sigma(F(P)) \xrightarrow{\phi(P)} F(\Sigma(P)) \longrightarrow 0 \longrightarrow \cdots$$

is the cone over the morphism $\phi(P)$ (between complexes concentrated in degree 0). Since $\phi(P)$ becomes invertible in $\underline{\mathcal{E}}$ by assumption, the image of the cone under G_2 is a zero object. Thus, G_3 induces a k -linear functor G . It is clear from the construction that $F_0 G$ is isomorphic to F . Since G_1 takes conflations to componentwise conflations and the projection $\mathbf{C}^b(\mathcal{E}) \rightarrow \mathbf{D}^b(\mathcal{E})$ transforms each componentwise conflation into a canonical triangle, the functor G is in fact a triangle functor. Therefore, to construct a commutation isomorphism $\Sigma G \rightarrow G \Sigma$, it suffices to construct such a commutation isomorphism for $\Omega^{-1} \circ \Sigma$. Now in \mathcal{R} the composition $\Omega^{-1} \circ \Sigma$ is isomorphic to the degree shifting functor $\nu_{\widehat{\Lambda}}$. For an object X , the image $G_1(\nu_{\widehat{\Lambda}} X)$ is isomorphic to $\Sigma(G_1(X))[-1]$, where we denote by Σ the functor from $\mathbf{C}^b(\mathcal{E})$ to itself obtained by applying $\Sigma : \mathcal{E} \rightarrow \mathcal{E}$ to each component. Now the canonical triangle functor

$$\mathbf{D}^b(\mathcal{E}) \rightarrow \underline{\mathcal{E}}$$

is functorial with respect to exact functors preserving projective-injectives and thus canonically commutes with Σ . Moreover, since it is a triangle functor, it is compatible with shifts. So we get a canonical isomorphism

$$\Omega^{-1}\Sigma(G(X)) \xrightarrow{\sim} G_2(\Sigma(G_1(X))[-1]) \xrightarrow{\sim} G_1(G_2(\Sigma(\Omega^{-1}X))).$$

Note that for Gorenstein algebras we now have a positive answer to the question posed in the beginning of the section.

3. INFINITE CHAIN OF SUBCATEGORIES

In this section we construct triangulated subcategories of $\underline{\text{mod}}\widehat{\Lambda}$ containing $\mathbf{D}^b(\Lambda)$ for Λ a selfinjective algebra. Recall from Section 1 that in this case $\mathbf{D}^b(\Lambda)$ can be identified with the full subcategory of $\underline{\text{mod}}\widehat{\Lambda}$ with objects $X = (X_i, f_i)$ such that X_i is a projective Λ -module for $i \neq 0$.

Lemma 3.1. *Let Λ be a selfinjective algebra. Let $I \subseteq \mathbb{Z}$ and let $\mathcal{C}_I \subset \{X = (X_i, f_i) \in \underline{\text{mod}}\widehat{\Lambda} \mid X_i \text{ is a projective } \Lambda\text{-module if } i \notin I\}$. Then the following hold*

- i) $\mathcal{C}_I \subseteq \underline{\text{mod}}\widehat{\Lambda}$ is a triangulated subcategory.
- ii) $0 \in I$ if and only if $\mathbf{D}^b(\Lambda) \subseteq \mathcal{C}_I$.
- iii) If $I, I' \subseteq \mathbb{Z}$ with $I \subseteq I'$, then $\mathcal{C}_I \subseteq \mathcal{C}_{I'}$.

Proof. If P is a projective-injective $\widehat{\Lambda}$ -module, then $P \in \mathcal{C}_I$, since Λ is selfinjective. So $\mathcal{C}_I \subseteq \underline{\text{mod}}\widehat{\Lambda}$.

Let $X \in \mathcal{C}_I$ and consider an exact sequence $0 \rightarrow X \rightarrow I(X) \rightarrow \Omega_{\widehat{\Lambda}}^- X \rightarrow 0$ with $I(X)$ injective in $\underline{\text{mod}}\widehat{\Lambda}$. Then for each $i \in \mathbb{Z}$ we have an exact sequence $0 \rightarrow X_i \rightarrow I(X_i) \oplus \nu^- I(X_{i+1}) \rightarrow Z_i \rightarrow 0$ where $I(X_i)$ is the Λ -injective envelope of X_i . If $i \notin I$, the sequence splits, since X_i is projective, hence $\Omega_{\widehat{\Lambda}}^- X \in \mathcal{C}_I$. Thus \mathcal{C}_I is closed under the translation functor in $\underline{\text{mod}}\widehat{\Lambda}$. Finally, let $X \rightarrow Y \rightarrow Z \rightarrow C[1]$ be a triangle in $\underline{\text{mod}}\widehat{\Lambda}$ with $X, Y \in \mathcal{C}_I$. Then the triangle gives an exact sequence $0 \rightarrow X \rightarrow I(X) \oplus Y \rightarrow Z \rightarrow 0$ in $\underline{\text{mod}}\widehat{\Lambda}$. So for each $j \in \mathbb{Z}$ we obtain an exact sequence $0 \rightarrow X_j \rightarrow I(X)_j \oplus Y_j \rightarrow Z_j \rightarrow 0$ in $\text{mod } \Lambda$. If $i \notin I$ then X_i, Y_i are projective, so Z_i is projective, hence $Z \in \mathcal{C}_I$, so \mathcal{C}_I is a triangulated subcategory of $\underline{\text{mod}}\widehat{\Lambda}$.

(ii) and (iii) are obvious. □

Example 3.2. *Let $n \in \mathbb{N}$ and let $I = (n+1)\mathbb{Z}$. Let $\mathcal{C}_n = \mathcal{C}_I$. Then $\nu_{\widehat{\Lambda}}^{n+1}$ is an automorphism on \mathcal{C}_n . In fact, if $X = (X_i, f_i) \in \mathcal{C}_n$, then $(\nu_{\widehat{\Lambda}}^{n+1} X)_i = X_{i+n+1}$. Since $j \notin (n+1)\mathbb{Z}$ if and only if $j+n+1 \notin (n+1)\mathbb{Z}$, we see that $\nu_{\widehat{\Lambda}}^{n+1} X \in \mathcal{C}_n$.*

If we choose $n+1 = 2^k$ and let $\mathcal{D}_k = \mathcal{C}_n$ we obtain a descending chain of subcategories $\cdots \subseteq \mathcal{D}_2 \subseteq \mathcal{D}_1 \subseteq \underline{\text{mod}}\widehat{\Lambda}$ and clearly $\mathbf{D}^b(\Lambda) = \bigcap_{i \geq 1} \mathcal{D}_i$.

Let Λ be a symmetric algebra and let $F = \nu_{\widehat{\Lambda}} \Omega_{\widehat{\Lambda}}$ be the Serre functor on $\underline{\text{mod}}\widehat{\Lambda}$. So for all $X, Y \in \underline{\text{mod}}\widehat{\Lambda}$ we have $\eta_{X,Y} : \underline{\text{Hom}}(X, Y) \xrightarrow{\sim} D\underline{\text{Hom}}(Y, F(X))$ natural in X and Y . We will show that F^{n+1} is a Serre functor on \mathcal{C}_n . For this we will construct $\eta_X : F^{n+1}(X) \rightarrow F(X)$ such that η_X is natural in X and for all $X, Y \in \mathcal{C}_n$ we have

that $\underline{\text{Hom}}(Y, F^{n+1}(X)) \xrightarrow{\sim} \underline{\text{Hom}}(Y, F(X))$. This then implies that F^{n+1} is a Serre functor on $\underline{\mathcal{C}}_n$, hence $\underline{\mathcal{C}}_n$ has Auslander-Reiten triangles. (Compare [RV])

Lemma 3.3. *For all $X \in \text{mod } \widehat{\Lambda}$ there is an exact sequence $0 \rightarrow K_X \xrightarrow{\mu_X} F(X) \xrightarrow{\pi_X} X \rightarrow 0$ which is natural in X .*

Proof. Since Λ is finite-dimensional, there is a functor $P: \text{mod } \Lambda \rightarrow_{\Lambda} \mathcal{P}$ and an exact sequence $0 \rightarrow \Omega_{\Lambda} X \xrightarrow{\alpha_X} P(X) \xrightarrow{\beta_X} X \rightarrow 0$ natural in X for $X \in \text{mod } \Lambda$ (compare section 2). Now let $X = (X_i, f_i) \in \text{mod } \widehat{\Lambda}$. Applying $\nu_{\widehat{\Lambda}}^-$ if necessary we may assume that $X_i = 0$ for $i < 0$. Now P extends to a functor $\widetilde{P}: \text{mod } \widehat{\Lambda} \rightarrow_{\widehat{\Lambda}} \mathcal{P}$ and we have an exact sequence $0 \rightarrow \Omega_{\widehat{\Lambda}} X \rightarrow \widetilde{P}(X) \rightarrow X \rightarrow 0$. Explicitly we have for $i \geq 0$ a commutative diagram of the form

$$\begin{array}{ccc} P(X_{i-1}) \oplus \Omega_{\Lambda} X_i & \xrightarrow{\begin{pmatrix} P(f_{i-1}) & 0 \\ \alpha_{X_i} & \Omega_{\Lambda} f_i \end{pmatrix}} & P(X_i) \oplus \Omega_{\Lambda} X_{i+1} \\ \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{X_i} \end{pmatrix} \downarrow & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{X_{i+1}} \end{pmatrix} \\ P(X_{i-1}) \oplus P(X_i) & \xrightarrow{\begin{pmatrix} -P(f_{i-1}) & 0 \\ 1 & P(f_i) \end{pmatrix}} & P(X_i) \oplus P(X_{i+1}) \\ \begin{pmatrix} 0 \\ \beta_{X_i} \end{pmatrix} \downarrow & & \downarrow \begin{pmatrix} 0 \\ \beta_{X_{i+1}} \end{pmatrix} \\ X_i & \xrightarrow{f_i} & X_{i+1} \end{array}$$

The map $\pi_X: F(X) \rightarrow X$ is now defined by $(\pi_X)_i: P(X_i) \oplus \Omega X_{i+1} \xrightarrow{\begin{pmatrix} \beta_{X_i} \\ 0 \end{pmatrix}} X_i$. Clearly π_X is surjective and K_X is described by the following commutative diagram

$$\begin{array}{ccc} \Omega X_i \oplus \Omega_{\Lambda} X_{i+1} & \xrightarrow{\begin{pmatrix} \Omega f_i & 0 \\ 1 & \Omega_{\Lambda} f_{i+1} \end{pmatrix}} & \Omega X_{i+1} \oplus \Omega X_{i+2} \\ \begin{pmatrix} \alpha_{X_i} & 0 \\ 0 & 1 \end{pmatrix} \downarrow & & \downarrow \begin{pmatrix} \alpha_{X_{i+1}} & 0 \\ 0 & 1 \end{pmatrix} \\ P(X_i) \oplus \Omega X_{i+1} & \xrightarrow{\begin{pmatrix} P(f_i) & 0 \\ \alpha_{X_i} & \Omega_{\Lambda} f_{i+1} \end{pmatrix}} & P(X_{i+1}) \oplus \Omega X_{i+2} \\ \begin{pmatrix} \beta_{X_i} \\ 0 \end{pmatrix} \downarrow & & \downarrow \begin{pmatrix} \beta_{X_{i+1}} \\ 0 \end{pmatrix} \\ X_i & \xrightarrow{f_i} & X_{i+1} \end{array}$$

Since P, \widetilde{P} are functors, the exact sequence $0 \rightarrow K_X \xrightarrow{\mu_X} F(X) \xrightarrow{\pi_X} X \rightarrow 0$ is natural in X . \square

So for each $1 \leq i \leq n$ we obtain an exact sequence

$$(*)_i: 0 \rightarrow F^i(K_X) \rightarrow F^{i+1}(X) \rightarrow F^i(X) \rightarrow 0$$

Now $(*)_i$ induces $\eta_X: F^{n+1}(X) \rightarrow F(X)$ natural in X . Thus for all $X, Y \in \text{mod } \widehat{\Lambda}$ we have $\eta_{X,Y}: \text{Hom}(Y, F^{n+1}(X)) \rightarrow \text{Hom}(Y, F(X))$ natural in X and Y .

For the following lemma we need some notation. Let $X \in \text{mod } \Lambda$ and denote by $\delta^i(X) = (Z_j, \gamma_j)$ the $\widehat{\Lambda}$ -module with $Z_{i-1} = Z_i = X$, $Z_j = 0$ for $j \neq i-1, i$, $\gamma_{i-1} = 1_X$ and $\gamma_j = 0$ for $j \neq i-1$.

Lemma 3.4. *If $X \in \mathcal{C}_n$, then $K_X \simeq \bigoplus_{i \in (n+1)\mathbb{Z}} \delta^i(\Omega X_i)$ in $\underline{\text{mod}} \widehat{\Lambda}$*

Proof. If $X = (X_i, f_i) \in \mathcal{C}_n$, then by definition X_i is projective for $i \notin (n+1)\mathbb{Z}$. Moreover it follows that $\delta^i(\Omega X_i)$ is projective as a $\widehat{\Lambda}$ -module if X_i is projective. It follows from the previous lemma that for each i we have that $\delta^i(\Omega X_i)$ is a submodule of K_X . Explicitly consider the following commutative diagram

$$\begin{array}{ccc} \Omega X_{i+1} & \xrightarrow{1} & \Omega X_{i+1} \\ \downarrow (0 \ 1) & \left(\begin{array}{cc} \Omega f_i & 0 \\ 1 & \Omega f_{i+1} \end{array} \right) & \downarrow (1 \ \Omega f_{i+1}) \\ \Omega X_i \oplus \Omega X_{i+1} & \xrightarrow{\quad} & \Omega X_{i+1} \oplus \Omega X_{i+2} \end{array}$$

So if $i \notin (n+1)\mathbb{Z}$ we see that $\delta^i(\Omega X_i)$ is a direct summand of K_X . But then it follows from the description of K_X in the previous lemma that $K_X \simeq \bigoplus_{i \in (n+1)\mathbb{Z}} \delta^i(\Omega X_i)$ in $\underline{\text{mod}} \widehat{\Lambda}$. \square

Lemma 3.5. *Let $X \in \text{mod } \Lambda$ and $i \in \mathbb{Z}$. Then*

- i) $F(\delta^i(X)) \simeq \delta^{i-1}(\Omega X)$ in $\underline{\text{mod}} \widehat{\Lambda}$.
- ii) If $Y \in \mathcal{C}_n$, then $\underline{\text{Hom}}(Y, \delta^i(X)) = 0$, if $i \notin (n+1)\mathbb{Z}$

Proof. i) Clearly, if $X, Y \in \text{mod } \Lambda$ then $\Omega_{\widehat{\Lambda}} \delta^i(X) = \delta^i(\Omega_{\Lambda} X)$ in $\underline{\text{mod}} \widehat{\Lambda}$ and $\nu_{\widehat{\Lambda}} \delta^i(Y) = \delta^{i-1}(Y)$, so the assertion follows.

ii) Let $Y = (Y_i, g_i) \in \mathcal{C}_n$ and let $\varphi \in \text{Hom}(Y, \delta^i(X))$. So we have the following commutative diagram with $\varphi = (\varphi_j)$

$$\begin{array}{ccccccc} \dots & & Y_{i-2} & \xrightarrow{g_{i-2}} & Y_{i-1} & \xrightarrow{g_{i-1}} & Y_i & \xrightarrow{g_i} & Y_{i+1} \\ & & \downarrow & & \downarrow \varphi_{i-1} & & \downarrow \varphi_i & & \\ \dots & & 0 & \longrightarrow & X & \xrightarrow{1_X} & X & \longrightarrow & 0 \end{array}$$

Consider $\delta^i(P(X)) \xrightarrow{\pi} \delta^i(X)$, then $\delta^i(P(X))$ is a projective $\widehat{\Lambda}$ -module. Since $i \notin (n+1)\mathbb{Z}$, we have that Y_i is a projective Λ -module, so there is some $\alpha_i: Y_i \rightarrow P(X)$ such that $\alpha_i \pi_i = \varphi_i$, where $\pi = (\pi_j)$ and $\pi_j = 0$ for $j \neq i-1, i$ and $\pi_{i-1} = \pi = \beta_X$.

Let $\alpha_{i-1} = g_{i-1} \alpha_i$ and $\alpha_j = 0$ for $j \neq i-1, i$. Then $\alpha = (\alpha_j)$ is a map $Y \rightarrow \delta^i(P(X))$ such that $\alpha \pi = \varphi$, hence $\underline{\text{Hom}}(Y, \delta^i(X)) = 0$ \square

Proposition 3.6. *For all $X, Y \in \mathcal{C}_n$, the natural transformation $\eta_{X,Y}: \underline{\text{Hom}}(Y, F^{n+1}(X)) \rightarrow \underline{\text{Hom}}(Y, F(X))$ is an isomorphism*

Proof. By the previous considerations we have for each $X \in \mathcal{C}_n$ and each $1 \leq i \leq n$ an exact sequence $0 \rightarrow F^i(K_X) \rightarrow F^{i+1}(X) \rightarrow F^i(X) \rightarrow 0$ in $\text{mod } \widehat{\Lambda}$ which gives rise to a triangle $F^i(K_X) \rightarrow F^{i+1}(X) \rightarrow F^i(X) \rightarrow F^i(K_X)[1]$ in $\underline{\text{mod}} \widehat{\Lambda}$. Applying $\underline{\text{Hom}}(Y, -)$ to this triangle for $Y \in \mathcal{C}_n$ gives an exact sequence

$$\underline{\text{Hom}}(Y, F^i(K_X)) \rightarrow \underline{\text{Hom}}(Y, F^{i+1}(X)) \rightarrow \underline{\text{Hom}}(Y, F^i(X)) \rightarrow \underline{\text{Hom}}(Y, F^i(K_X)[1])$$

By the description of K_X and the previous lemma we see that $\underline{\text{Hom}}(Y, F^i(K_X)) = 0 = \underline{\text{Hom}}(Y, F^i(K_X)[1])$, hence $\underline{\text{Hom}}(Y, F^{i+1}(K_X)) \xrightarrow{\sim} \underline{\text{Hom}}(Y, F^i(X))$.

For each $1 \leq i \leq n$ we also have an exact sequence $0 \rightarrow K^i \rightarrow F^{i+1}(X) \rightarrow F(X) \rightarrow 0$ in $\text{mod } \widehat{\Lambda}$ which gives rise to a triangle $K^i \rightarrow F^{i+1}(X) \rightarrow F(X) \rightarrow K^i[1]$

in $\underline{\text{mod}}\widehat{\Lambda}$ where $F^{i+1}(X) \rightarrow F(X)$ is obtained from the composition $F^{i+1}(X) \rightarrow F^i(X) \rightarrow F(X)$. By the octahedral axiom we have a triangle $F^i(K_X) \rightarrow K^i \rightarrow K^{i-1} \rightarrow F^i(K_X)[1]$. By induction and the previous considerations we have that $\underline{\text{Hom}}(Y, K^i) = 0 = \underline{\text{Hom}}(Y, K^i[1])$, hence $\underline{\text{Hom}}(Y, F^{i+1}(X)) \simeq \underline{\text{Hom}}(Y, F(X))$, thus $\eta_{X,Y}$ is an isomorphism for all $X, Y \in \underline{\mathcal{C}}_n$. \square

As pointed out above this implies

Corollary 3.7. *For each $n \in \mathbb{N}$ the category $\underline{\mathcal{C}}_n$ has almost split triangles*

4. IRREDUCIBLE MAPS IN $\mathbf{D}^b(\Lambda)$

In this section we study the behavior of the embedding $\mu: \mathbf{D}^b(\Lambda) \rightarrow \underline{\text{mod}}\widehat{\Lambda}$ under irreducible maps. If $X = (X^i, f^i) \in \mathbf{D}^b(\Lambda)$ we write $\mu(X) = (X_i, f_i)$. If $X = (X^i, f^i)$ satisfies $X^i = 0$ for $i < 0$, then $X_i = 0$ for $i < 0$. Here of course we assume that $\mu(X)$ has no projective-injective indecomposable summands. If $X^i = 0$ for $i > 0$ then $X_i = 0$ for $i > 0$.

We denote by $\text{mod}^{\geq 0}\widehat{\Lambda} = \{(X_i, f_i) | X_i = 0, i < 0\}$ and $\text{mod}^{> 0}\widehat{\Lambda} = \{(X_i, f_i) | X_i = 0, i \leq 0\}$. The categories $\text{mod}^{\leq 0}\widehat{\Lambda}$ and $\text{mod}^{< 0}\widehat{\Lambda}$ are defined analogously. Clearly $\text{mod}^{\geq 0}\widehat{\Lambda}$ and $\text{mod}^{> 0}\widehat{\Lambda}$ are stable under $\Omega_{\widehat{\Lambda}}$. Moreover we clearly have $\text{Hom}(X, Y) = 0$ for $X \in \text{mod}^{\geq 0}\widehat{\Lambda}$ and $Y \in \text{mod}^{< 0}\widehat{\Lambda}$. This yields the following easy lemma.

Lemma 4.1. *Let $X \in \text{mod}^{\geq 0}\widehat{\Lambda}$ and $Y \in \text{mod}^{< 0}\widehat{\Lambda}$. Then $\text{Ext}_{\widehat{\Lambda}}^1(X, Y) = 0$.*

Proof. We have $\text{Ext}_{\widehat{\Lambda}}^1(X, Y) \simeq \underline{\text{Hom}}(X, \Omega_{\widehat{\Lambda}}^- Y) = 0$, since $\Omega_{\widehat{\Lambda}}^- Y \in \text{mod}^{< 0}\widehat{\Lambda}$. \square

Lemma 4.2. *Let $Z = (Z_i, f_i) \in \text{mod}^{\geq 0}\widehat{\Lambda}$ with $Z \in \text{Im } \mu$. Consider the exact sequence $0 \rightarrow Z^{> 0} \rightarrow Z \rightarrow Z_0 \rightarrow 0$. Then $\nu_{\widehat{\Lambda}} Z^{> 0} \in \text{Im } \mu$.*

Proof. We verify the condition mentioned in section 1 from [GK]. Since $\nu_{\widehat{\Lambda}} Z^{> 0} \in \text{mod}^{\geq 0}\widehat{\Lambda}$ we have that $\Omega_{\widehat{\Lambda}}^r \nu_{\widehat{\Lambda}} Z^{> 0} \in \text{mod}^{\geq 0}\widehat{\Lambda}$ for all $r \geq 0$. The exact sequence $0 \rightarrow Z^{> 0} \rightarrow Z \rightarrow Z_0 \rightarrow 0$ gives a triangle $\Omega_{\widehat{\Lambda}}^- Z_0 \rightarrow Z^{> 0} \rightarrow Z \rightarrow Z_0$ in $\underline{\text{mod}}\widehat{\Lambda}$. So for each $n \geq 0$ we obtain a triangle

$$\Omega_{\widehat{\Lambda}}^{-n} Z_0 \rightarrow \Omega_{\widehat{\Lambda}}^{-n-1} Z^{> 0} \rightarrow \Omega_{\widehat{\Lambda}}^{-n-1} Z \rightarrow \Omega_{\widehat{\Lambda}}^{-n-1} Z_0$$

For each $n \geq 0$ we clearly have $\Omega_{\widehat{\Lambda}}^{-n} Z_0 \in \text{mod}^{\leq 0}\widehat{\Lambda}$. Since $Z \in \text{Im } \mu$ there is n_0 such that $\Omega_{\widehat{\Lambda}}^{-n} Z \in \text{mod}^{\leq 0}\widehat{\Lambda}$ for all $n \geq n_0$ by [GK]. Let $n \geq n_0$ and assume that $\Omega_{\widehat{\Lambda}}^{-n-1} Z^{> 0} \notin \text{mod}^{\leq 0}\widehat{\Lambda}$. Then there is some $X \in \text{mod}^{> 0}\widehat{\Lambda}$ such that $\underline{\text{Hom}}(X, \Omega_{\widehat{\Lambda}}^{-n-1} Z^{> 0}) \neq 0$. Since $\underline{\text{Hom}}(X, \Omega_{\widehat{\Lambda}}^{-n} Z_0) = 0 = \underline{\text{Hom}}(X, \Omega_{\widehat{\Lambda}}^{-n-1} Z)$ we obtain a contradiction. \square

Theorem 4.3. *Let X, Y be indecomposable in $\mathbf{D}^b(\Lambda)$ and let $f: X \rightarrow Y$ be irreducible. Then $\mu(f): \mu(X) \rightarrow \mu(Y)$ is irreducible in $\underline{\text{mod}}\widehat{\Lambda}$.*

Proof. We consider the almost split triangle (*) $\tau_{\widehat{\Lambda}} \mu(Y) \xrightarrow{\alpha} E \xrightarrow{\beta} \mu(Y) \xrightarrow{\gamma} \tau_{\widehat{\Lambda}} \mu(Y)[1]$ in $\underline{\text{mod}}\widehat{\Lambda}$. Now $\mu(f)$ is not split epi, since f is irreducible, hence we get $g: \mu(X) \rightarrow E$ such that $\mu(f) = g\beta$. Let $\mu(X) = (X_i, f_i)$ and $\mu(Y) = (Y_i, g_i)$. We may assume that $\mu(X), \mu(Y) \in \text{mod}^{\geq 0}\widehat{\Lambda}$. Now $\tau_{\widehat{\Lambda}} \mu(Y) = \nu_{\widehat{\Lambda}} \Omega_{\widehat{\Lambda}}^2 \mu(Y) = (Z_i, h_i)$ satisfies $Z_i = 0$ for

$i < -1$, $Z_{-1} = \Omega_{\Lambda}^2 Y_0$. Thus $E = (E_i, u_i)$ satisfies $E_i = 0$ for $i < -1$, $E_{-1} = \Omega_{\Lambda}^2 Y_0$ and $u_{-1}: \Omega_{\Lambda}^2 Y_0 \rightarrow \nu^{-} E_0$. Let $P \xrightarrow{\pi} \Omega_{\Lambda}^2 Y_0$ be epi with P a projective Λ -module. Let $\tilde{E} = (\tilde{E}_i, v_i) \in \text{mod } \hat{\Lambda}$ defined by $\tilde{E}_i = 0$ for $i < -1$, $\tilde{E}_{-1} = P$, $\tilde{E}_i = E_i$ for $i \geq 0$, $v_{-1} = \pi u_{-1}$, $v_i = u_i$ for $i \geq 0$.

Now $F = \Omega_{\Lambda}^2 \mu(Y) \in \text{Im } \mu$, since $\text{Im } \mu$ is a triangulated category. By Lemma 4.2 we have that $\nu_{\Lambda} F^{>0} \in \text{Im } \mu$, but $\nu_{\Lambda} F^{>0} = E^{\geq 0}$. Since $\nu_{\Lambda} P \in \text{Im } \mu$ we see that $\tilde{E} \in \text{Im } \mu$. The construction of \tilde{E} clearly yields a triangle

$$\nu_{\Lambda} \mu(\text{Ker } \pi) \xrightarrow{\delta} \tilde{E} \xrightarrow{\epsilon} E \xrightarrow{\eta} \nu_{\Lambda} \mu(\text{Ker } \pi)[1]$$

The factorization $\mu(f) = g\beta$ induces another factorization as follows:

$$\begin{array}{ccccc} \mu(X) & \xlongequal{\quad} & \mu(X) & & \\ \downarrow \tilde{g} & & \downarrow g & & \\ \nu_{\Lambda} \mu(\text{Ker } \pi) & \xrightarrow{\delta} & \tilde{E} & \xrightarrow{\epsilon} & E & \xrightarrow{\eta} & \nu_{\Lambda} \mu(\text{Ker } \pi)[1] \\ \downarrow \tilde{\beta} & & \downarrow \beta & & & & \\ \mu(Y) & \xlongequal{\quad} & \mu(Y) & & \end{array}$$

where $\tilde{\beta} = \epsilon\beta$. Since $g\eta = 0$ by Lemma 3.1 we obtain \tilde{g} with $\tilde{g}\epsilon = g$. Now $\tilde{g}\tilde{\beta} = \tilde{g}\epsilon\beta = g\beta = \mu(f)$.

Since $\tilde{E} \in \text{Im } \mu$ and f is irreducible, we get that \tilde{g} is a split mono or $\tilde{\beta}$ is a split epi. If $\tilde{\beta}$ is split epi, there is $\tilde{\beta}_1: \mu(Y) \rightarrow \tilde{E}$ such that $\tilde{\beta}_1\beta = 1_{\mu(Y)}$. Let $\beta_1 = \tilde{\beta}_1 c$. Then $\beta_1\beta = \tilde{\beta}_1 c\beta = \tilde{\beta}_1 \tilde{\beta} = 1_{\mu(Y)}$, so β is a split epi, in contrast to (*) being an almost split triangle. So \tilde{g} is split mono, hence there is some $\tilde{g}_1: \tilde{E} \rightarrow \mu(X)$ such that $\tilde{g}\tilde{g}_1 = 1_{\mu(X)}$. Since $\text{Hom}(\nu_{\Lambda} \text{Ker } \pi, \mu(X)) = 0$, we have $\delta\tilde{g}_1 = 0$, so there is some $g_1: E \rightarrow \mu(X)$ such that $\epsilon g_1 = \tilde{g}_1$. Now $g g_1 = \tilde{g}\epsilon g_1 = \tilde{g}\tilde{g}_1 = 1_{\mu(X)}$ shows that g is split mono, hence X is an indecomposable direct summand of E and $\mu(f)$ is a component of β , hence $\mu(f)$ is irreducible. \square

Next we show how certain irreducible maps in $\mathbf{D}^b(\Lambda)$ arise quite naturally from extensions of simple Λ -modules. This will be of interest in section 5.

Proposition 4.4. *Let S and T be simple Λ -modules with $\text{Ext}_{\Lambda}^1(S, T) \neq 0$. If $\text{rad } P(S)$ and $I(T)/T$ are both semisimple, then there is an irreducible map $f: S[-1] \rightarrow T$ in $\mathbf{D}^b(\Lambda)$.*

Proof. We will show that there is an irreducible map $\varphi: \mu(S[-1]) \rightarrow \mu(T)$ in $\underline{\text{mod}} \hat{\Lambda}$. For simplicity let $\mu(S) = S$ and $\mu(T) = T$. Then $S[-1] = \Omega_{\Lambda} S \simeq \text{rad}_{\hat{\Lambda}} P(S)$, where $P(S)$ is the $\hat{\Lambda}$ -projective cover of S . We consider the almost split sequence in $\text{mod } \hat{\Lambda}$ starting in $S[-1] = \text{rad}_{\hat{\Lambda}} P(S)$. It is well-known [AR, Prop. 4.1] that this is of the form

$$0 \rightarrow \text{rad}_{\hat{\Lambda}} P(S) \rightarrow_{\hat{\Lambda}} P(S) \oplus \text{rad}_{\hat{\Lambda}} P(S) / \text{soc}_{\hat{\Lambda}} P(S) \rightarrow P(S) / \text{soc}_{\hat{\Lambda}} P(S) \rightarrow 0$$

Clearly, $\text{soc}_{\hat{\Lambda}} P(S) = \nu_{\Lambda}^{-} S$. Let $0 \rightarrow S \xrightarrow{\alpha} I(S) \xrightarrow{\beta} I(S)/S \rightarrow 0$ be exact in $\text{mod } \Lambda$, with $I(S)$ the Λ -injective envelope of S . Applying the Nakayama functor $\nu_{\Lambda}^{-} =$

$\text{Hom}(D\Lambda_\Lambda, -)$ yields an exact sequence

$$0 \rightarrow \text{Hom}(D\Lambda_\Lambda, S) \xrightarrow{\nu_\Lambda^-(\alpha)} P(S) = \text{Hom}(D\Lambda_\Lambda, I(S)) \xrightarrow{\nu_\Lambda^-(\beta)} \text{Hom}(D\Lambda_\Lambda, I(S)/S)$$

Let g be the composition of $\text{rad } P(S) \xrightarrow{\gamma} P(S) \xrightarrow{\nu_\Lambda^-(\beta)} \nu_\Lambda^- I(S)/S$, so $g = \gamma \nu_\Lambda^- \beta$. Then $\text{rad}_{\widehat{\Lambda}} P(S)/\text{soc}_{\widehat{\Lambda}} P(S) = (Z_i, g_i) = Z$ with $Z_0 = \text{rad}_\Lambda P(S)$, $Z_1 = I(S)/S$, $g_0 = g$, and zero otherwise. By assumption, $\text{rad } P(S)$ is semisimple and $\text{Ext}_\Lambda^1(S, T) \neq 0$, so T is an indecomposable direct summand of $\text{rad } P(S)$. Let $0 \neq \delta: I(T) \rightarrow I(S)$ be a map which is not an isomorphism. Since $I(T)/T$ is semisimple, δ factors over α , hence $g(T) = 0$, or equivalently T is an indecomposable direct summand of Z . Hence there is an irreducible map $f: \Omega_{\widehat{\Lambda}} S \rightarrow T$ in $\underline{\text{mod}} \widehat{\Lambda}$, so there is an irreducible map $f: S[-1] \rightarrow T$ in $\mathbf{D}^b(\Lambda)$ \square

5. COMPONENTS

We consider the embedding $\mu: \mathbf{D}^b(\Lambda) \rightarrow \underline{\text{mod}} \widehat{\Lambda}$. The category $\underline{\text{mod}} \widehat{\Lambda}$ has almost split triangles, where for an indecomposable $X \in \underline{\text{mod}} \widehat{\Lambda}$, the translate $\tau_{\widehat{\Lambda}} X = \nu_{\widehat{\Lambda}}[-2]X$. In general, $\mathbf{D}^b(\Lambda)$ will not have almost split triangles. However it was shown in [H2] that for each $P \in \mathbf{K}^b(\Lambda \mathcal{P})$ indecomposable there is an almost split triangle in $\mathbf{D}^b(\Lambda)$ of the form $\nu P[-1] \xrightarrow{u} E \xrightarrow{v} P \xrightarrow{w} \nu P$ where $\nu: \mathbf{K}^b(\Lambda \mathcal{P}) \rightarrow \mathbf{K}^b(\Lambda \mathcal{I})$ is the Nakayama functor. We will show first that this triangle is sent under μ to the almost split triangle in $\underline{\text{mod}} \widehat{\Lambda}$ ending at $\mu(P)$ and then apply this to determine the structure of the components of the AR-quiver of $\mathbf{K}^b(\Lambda \mathcal{P})$ in case Λ is a Gorenstein algebra.

Lemma 5.1. *If $P \in \mathbf{K}^b(\Lambda \mathcal{P})$, then $\tau_{\widehat{\Lambda}} \mu(P) \in \text{Im } \mu$.*

Proof. Let $P = (P^i, d^i) \in \mathbf{K}^b(\Lambda \mathcal{P})$. Since μ commutes with the translation functors we may assume that $P^i = 0$ for $i > 0$ and it is enough

to show that $\nu_{\widehat{\Lambda}} \mu(P) \in \text{Im } \mu$. Since $P \in \mathbf{K}^b(\Lambda \mathcal{P})$ there is m_0 such that $P^m = 0$ for $m < m_0$. We proceed by induction on m_0 . If $m_0 = 0$, then P is a stalk complex concentrated in degree 0, so $\mu(P)$ is the stalk module P^0 concentrated in degree zero. But $\nu_{\widehat{\Lambda}} P^0 \cong \nu P^0[1]$ shows that $\nu_{\widehat{\Lambda}} P^0 \in \text{Im } \mu$. If $m_0 < 0$, let $P' = (P'^i, d'^i)$ with $P'^i = P^i$ for $i < 0$ and $P'^0 = 0$, $d'^i = d^i$ for $i < -1$ and $d'^i = 0$ for $i \geq -1$ be the truncated complex.

We clearly have a map of complexes $P'[-1] \xrightarrow{u} P^0$ whose mapping cone is P . So we obtain a triangle $P'[-1] \rightarrow P^0 \rightarrow P \rightarrow P'$ in $\mathbf{K}^b(\Lambda \mathcal{P})$. This yields a triangle $\nu_{\widehat{\Lambda}} \mu(P')[-1] \rightarrow \nu_{\widehat{\Lambda}} P^0 \rightarrow \nu_{\widehat{\Lambda}} \mu(P) \rightarrow \nu_{\widehat{\Lambda}} \mu(P')$ in $\underline{\text{mod}} \widehat{\Lambda}$.

By induction the first two terms belong to $\text{Im } \mu$, hence so does the third, since μ is a triangle functor. \square

Proposition 5.2. *Let $P \in \mathbf{K}^b(\Lambda \mathcal{P})$ and let $\nu P[-1] \xrightarrow{u} E \xrightarrow{v} P \xrightarrow{w} \nu P$ be the almost split triangle in $\mathbf{D}^b(\Lambda)$ ending at P . Then $\mu(\nu P)[-1] \xrightarrow{\mu(u)} \mu(E) \xrightarrow{\mu(v)} \mu(P) \xrightarrow{\mu(w)} \mu(\nu P)$ is the almost split triangle in $\underline{\text{mod}} \widehat{\Lambda}$ ending at $\mu(P)$.*

Proof. Let

$$\tau_{\widehat{\Lambda}} \mu(P) \xrightarrow{u} F \xrightarrow{v} \mu(P) \xrightarrow{w} \tau_{\widehat{\Lambda}} \mu(P)[1] \quad (*)$$

be the almost split triangle in $\underline{\text{mod}}\widehat{\Lambda}$ ending at $\mu(P)$. By Lemma 5.1 there is $X \in \mathbf{D}^b(\Lambda)$ such that $\mu(X) = \tau_{\widehat{\Lambda}}\mu(P)[1]$. So there is some $w' : P \rightarrow X$ such that $\mu w' = w$. Let

$$X[-1] \xrightarrow{u'} E \xrightarrow{v'} P \xrightarrow{w'} X \quad (**)$$

be a triangle in $\mathbf{D}^b(\Lambda)$. By construction $\mu(X)[-1] \rightarrow \mu(E) \rightarrow \mu(P) \xrightarrow{w} \mu(X)$ is isomorphic to $(*)$. Since μ is an embedding and $(*)$ is an almost split triangle, we infer that $(**)$ is the almost split triangle in $\mathbf{D}^b(\Lambda)$ ending at P . \square

Note that 5.2 is related to [KL, sections 7,8] where an adjoint of an extension of the functor $\mu : \mathbf{D}^b(\Lambda) \rightarrow \underline{\text{mod}}\widehat{\Lambda}$ is constructed and used to compute almost split triangles.

In the following let Λ be a Gorenstein algebra. Then the Nakayama functor $\nu : \mathbf{K}^b({}_{\Lambda}\mathcal{P}) \rightarrow \mathbf{K}^b({}_{\Lambda}\mathcal{I})$ is an endofunctor, hence $\mathbf{K}^b({}_{\Lambda}\mathcal{P})$ has almost split triangles, which are almost split triangles in $\mathbf{D}^b(\Lambda)$, and therefore by Proposition 5.2 also almost split triangles in $\underline{\text{mod}}\widehat{\Lambda}$. Hence we get

Corollary 5.3. *Let \mathcal{C} be a connected component of the AR-quiver of $\mathbf{K}^b({}_{\Lambda}\mathcal{P})$. Then \mathcal{C} is a connected component of the AR-quiver of $\underline{\text{mod}}\widehat{\Lambda}$.*

We will now investigate the shape of the components of the AR-quiver of $\mathbf{K}^b({}_{\Lambda}\mathcal{P})$ for Λ a selfinjective algebra.

Theorem 5.4. *Let Λ be a connected selfinjective algebra, which is not semisimple. Let \mathcal{C} be a connected component of the AR-quiver of $\mathbf{K}^b({}_{\Lambda}\mathcal{P})$. Then \mathcal{C} is of the form $\mathbb{Z}\mathbf{A}_{\infty}$.*

Proof. Let \mathcal{C} be a connected component of the AR-quiver of $\mathbf{K}^b({}_{\Lambda}\mathcal{P})$. By Corollary 5.3, \mathcal{C} is a connected component of the AR-quiver of $\underline{\text{mod}}\widehat{\Lambda}$. In fact \mathcal{C} is a connected component of the AR-quiver of $\underline{\text{mod}}\widehat{\Lambda}$, since otherwise there would exist an indecomposable projective $\widehat{\Lambda}$ -module P such that $\text{rad } P \in \mathcal{C}$, in particular $\text{rad } P \in \text{Im } \mu|_{\mathbf{K}^b({}_{\Lambda}\mathcal{P})}$.

But it follows from the description of $\text{Im } \mu$ in Section 3 that $\text{Im } \mu|_{\mathbf{K}^b({}_{\Lambda}\mathcal{P})} = \{(X_i, f_i) \mid X_i \text{ is a projective } \Lambda\text{-module}\}$, so $\text{rad } P \notin \text{Im } \mu|_{\mathbf{K}^b({}_{\Lambda}\mathcal{P})}$, since Λ is not semisimple. Consider $l : \mathcal{C} \rightarrow \mathbb{N}$ defined by $l(X) = |X|$, the length of X as a $\widehat{\Lambda}$ -module. Then l is an additive function on \mathcal{C} , since \mathcal{C} is a component of the AR-quiver of $\underline{\text{mod}}\widehat{\Lambda}$: If $P \in \mathbf{K}^b({}_{\Lambda}\mathcal{P})$ and $\mu(P) = (X_i, f_i)$ we have that $X_i = 0$ for $|i| > m$ and some m and X_i is projective for $|i| \leq m$. Since Λ is selfinjective there exists $n \in \mathbb{N}$ such that $\nu_{\Lambda}^n P \cong P$ for each projective Λ -module P . If $\mu(P) = (X_i, f_i)$, so that $l(\mu(P)) = \sum_i |X_i|$, then $l(\Omega_{\widehat{\Lambda}}\mu(P)) = \sum_i |\nu_{\Lambda} X_i|$.

So $l(\mu(P)) = l(\Omega_{\widehat{\Lambda}}^n \mu(P))$, hence $l(\tau_{\widehat{\Lambda}}^n \mu(P)) = l(\mu(P))$, showing that l is a $\tau_{\widehat{\Lambda}}$ -periodic additive function. Let $X \in \mathcal{C}$ and let $0 \neq f : P \rightarrow X$ with P an indecomposable projective $\widehat{\Lambda}$ -module. Since \mathcal{C} does not contain any projective $\widehat{\Lambda}$ -modules we obtain for each i a chain of irreducible maps $X_i \xrightarrow{f_i} X_{i-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{f_1} X_0 = X$ such that $f_i \cdots f_1 \neq 0$ and $X_i \in \mathcal{C}$. By the lemma of Harada-Sai (see [ARS, VI Cor. 1.3]) we know that the length of the indecomposable modules in \mathcal{C} is unbounded; so l is unbounded on \mathcal{C} . In particular \mathcal{C} contains infinitely many $\tau_{\widehat{\Lambda}}$ -orbits. By [F]

the tree class of \mathcal{C} is \mathbf{A}_∞ . Trivially \mathcal{C} does not contain any $\tau_{\widehat{\Lambda}}$ -periodic vertices. So $\mathcal{C} \cong \mathbb{Z}\mathbf{A}_\infty$. \square

If Λ is a Gorenstein algebra, then in Section 2 we constructed a functor $G : \underline{\text{mod}}\widehat{\Lambda} \rightarrow \mathbf{D}^b(\Lambda)$ such that $\mu G \cong \mathbf{1}_{\mathbf{D}^b(\Lambda)}$. Let $\tilde{\nu} : \mathbf{D}^b(\Lambda) \rightarrow \mathbf{D}^b(\Lambda)$ be the equivalence induced by $\nu_\Lambda = D \text{Hom}(-, {}_\Lambda \Lambda)$, then we obtain a commutative diagram

$$\begin{array}{ccc} \underline{\text{mod}}\widehat{\Lambda} & \xrightarrow{G} & \mathbf{D}^b(\Lambda) \\ \downarrow \nu_{\widehat{\Lambda}} & & \downarrow \tilde{\nu}[1] \\ \underline{\text{mod}}\widehat{\Lambda} & \longrightarrow & \mathbf{D}^b(\Lambda) \end{array}$$

Proposition 5.5. *Let Λ be a Gorenstein algebra and $X, Y \in \mathbf{D}^b(\Lambda)$ indecomposable. If $f : X \rightarrow Y$ is irreducible and $Y \notin \mathbf{K}^b({}_\Lambda \mathcal{P})$, then $X \cong \tilde{\nu}Y[-1]$.*

Proof. We consider the almost split triangle in $\underline{\text{mod}}\widehat{\Lambda}$

$$\tau_{\widehat{\Lambda}}\mu(Y) \xrightarrow{\alpha} E \xrightarrow{\beta} \mu(Y) \xrightarrow{\gamma} \tau_{\widehat{\Lambda}}\mu(Y)[1] \quad (*)$$

Since $f : X \rightarrow Y$ is irreducible and by Theorem 4.3, also $\mu(f)$ is irreducible, we see that $E \cong \mu(X) \oplus C$ and $\beta = (\mu(f), g)^t$ for some $g : C \rightarrow \mu(Y)$. Since $\tau_{\widehat{\Lambda}} = \nu_{\widehat{\Lambda}}\Omega_{\widehat{\Lambda}}^2$ and using the diagram above, we see that G applied to $(*)$ yields a triangle in $\mathbf{D}^b(\Lambda)$

$$\tilde{\nu}Y[-1] \xrightarrow{G(\alpha)} X \oplus G(C) \xrightarrow{G(\beta)} Y \xrightarrow{G(\gamma)} \tilde{\nu}Y \quad (**)$$

We claim that $G(\gamma) = 0$. Otherwise, let $h : Z \rightarrow Y$ be a map which is not a split epi, then $\mu(h)$ is not a split epi. But then $\mu(h)\gamma = 0$, hence $0 = G(\mu(h)\gamma) = hG(\gamma)$. Since Y and $\tilde{\nu}Y[-1]$ are indecomposable, $(**)$ would be an almost split triangle in $\mathbf{D}^b(\Lambda)$. Since $Y \notin \mathbf{K}^b({}_\Lambda \mathcal{P})$, this contradicts the existence theorem in [H2]. So $G(\gamma) = 0$, hence $G(\beta)$ is a split epi. Since X is not isomorphic to Y , we get that $X \cong \tilde{\nu}Y[-1]$. \square

We will now show that for selfinjective algebras Λ irreducible maps in $\mathbf{D}^b(\Lambda)$ outside $\mathbf{K}^b({}_\Lambda \mathcal{P})$ are rare. For this we will need the following easy fact, but first we will define the relevant class of algebras Λ_n for $n \geq 1$. Let $\Lambda_1 = k[x]/(x^2)$ and let Λ_n for $n \geq 2$ be defined by the following quiver

$$\begin{array}{ccc} & 1 \xrightarrow{\alpha_1} 2 & \\ \alpha_n \nearrow & & \searrow \alpha_2 \\ n & & 3 \\ & \cdots & \end{array}$$

over k , with relations $\alpha_i\alpha_{i+1} = 0$ for $1 \leq i \leq n$ where $\alpha_{n+1} = \alpha_1$.

We collect the relevant information in the following well-known lemma (see [ARS, IV.2])

Lemma 5.6. *Let Λ be a basic selfinjective algebra over an algebraically closed field k , which is not semisimple.*

- (1) $\text{rad}^2 \Lambda = 0$ if and only if $\Lambda \cong \Lambda_n$ for some n .

- (2) If $S(i)$ is a simple Λ_n -module, then $\nu S(i) = S(i-1)$ where $S(0) = S(n)$.
(3) Λ_n is symmetric if and only if $n = 1$.

Theorem 5.7. *Let Λ be a basic selfinjective algebra which is not semisimple. Let $Y \in \mathbf{D}^b(\Lambda) \setminus \mathbf{K}^b(\Lambda\mathcal{P})$ be indecomposable. There exists an irreducible map $f: X \rightarrow Y$ in $\mathbf{D}^b(\Lambda)$ if and only if $\Lambda \cong \Lambda_n$, $Y \cong S(i-1)[j]$, $X \cong S(i)[j-1]$ for some $1 \leq i \leq n$ and $j \in \mathbb{Z}$.*

Proof. If $\Lambda = \Lambda_n$ for some n , we have seen in Proposition 4.4 that for each arrow α_i we have an irreducible map $\nu S(i)[-1] \rightarrow S(i+1)$ in $\mathbf{D}^b(\Lambda)$. Since $\text{pd}_{\Lambda_n} S(i+1) = \infty$ we have $S(i+1) \notin \mathbf{K}^b(\Lambda\mathcal{P})$.

Conversely, let $f: X \rightarrow Y$ be irreducible in $\mathbf{D}^b(\Lambda)$ and $Y \notin \mathbf{K}^b(\Lambda\mathcal{P})$. We choose $Y \in \mathbf{K}^{-b}(\Lambda\mathcal{P})$ and may assume that $Y = (P^i, d^i)$ satisfies $Y^i = 0$ for $i > 0$ and $H^0(Y) \neq 0$. By Proposition 5.5 we know that $X \cong \tilde{\nu}Y[-1]$. Since Λ is selfinjective, we have that ν_Λ is exact, hence $\tilde{\nu}Y = (\nu_\Lambda P^i, \nu_\Lambda d^i)$. Consider the triangle $\nu_\Lambda P^0[-1] \xrightarrow{\alpha} \nu Y[-1] \xrightarrow{\beta} \nu Y[-1]_{\leq 0} \xrightarrow{\gamma} \nu_\Lambda P^0$. Since $\text{Hom}(\nu_\Lambda P^0[-1], Y) = 0$, there is some $\bar{f}: \nu Y[-1]_{\leq 0} \rightarrow Y$ such that $f = \beta\bar{f}$. $H^0(Y) \neq 0$ implies that $\alpha \neq 0$. Thus β is not split mono, so \bar{f} is split epi, since f is irreducible. Hence $H^i(\bar{f}): H^i(\tilde{\nu}Y[-1]_{\leq 0}) \rightarrow H^i(Y)$ is split epi for all i . Since ν is exact, we have that $H^i(\tilde{\nu}Y) \cong \nu_\Lambda H^i(Y)$ for all i . Also we have that $H^i(\tilde{\nu}Y[-1]_{\leq 0}) = H^i(\tilde{\nu}Y[-1])$ for all $i \leq -1$ and $H^i(\tilde{\nu}Y[-1]) = H^{i-1}(\tilde{\nu}Y)$. Since $Y \in \mathbf{K}^{-b}(\Lambda\mathcal{P})$ there is some $n_0 \leq 0$ such that $H^n(Y) = 0$ for all $n \leq n_0$. Choose n_0 maximal with this property, so $H^{n_0}(Y) = 0$ and $H^{n_0+1}(Y) \neq 0$. We claim that $Y \in \text{mod } \Lambda$, or equivalently $n_0 = -1$. Otherwise $n_0 \leq -2$. But then $0 = H^{n_0}(Y) = H^{n_0}(\tilde{\nu}Y) = H^{n_0+1}(\tilde{\nu}Y[-1]) = H^{n_0+1}(\tilde{\nu}Y[-1]_{\leq 0}) \rightarrow H^{n_0+1}(Y)$, hence $H^{n_0+1}(Y) = 0$ in contrast to the choice of n_0 . Hence $n_0 = -1$, so Y is an indecomposable Λ -module. But then $Y^{\leq -1} \cong \Omega_\Lambda Y[1]$, so $Y^{\leq -1}$ is indecomposable, since Λ is selfinjective. But then $\tilde{\nu}Y[-1]_{\leq 0}$ is indecomposable, so \bar{f} is an isomorphism, so $\nu_\Lambda \Omega_\Lambda Y \cong Y$. If Y is not simple, there is a proper epi $Y \xrightarrow{\pi} S$ for some simple S . So there is $h: Y \rightarrow I(S)$, with $\text{Ker } h \neq 0$ and $\text{Coker } h \neq 0$. Since $\text{Hom}(I(S), Y[1]) = \text{Ext}_\Lambda^1(I(S), Y) = 0$ we obtain a triangle

$$C_h[-1] \xrightarrow{g} Y \xrightarrow{h} I(S) \rightarrow C_h$$

with $C_h[-1]$ indecomposable and $H^0(C_h[-1]) = \text{Ker } h$, $H^1(C_h[-1]) = \text{Coker } h$. Since $\text{Hom}(\nu_\Lambda Y[-1], I(S)) = 0$, there is some $f': \nu_\Lambda Y[-1] \rightarrow C_h[-1]$ such that $f'g = f$. Since $h \neq 0$, g is not split epi, hence f' is split mono, since f is irreducible. Since $C_h[-1]$ is indecomposable, we have that f' is an isomorphism, in contrast to $H^0(C_h[-1]) \neq 0 \neq H^1(C_h[-1])$, so Y is a simple Λ -module. Since $Y \cong \nu_\Lambda \Omega_\Lambda Y$, we see that $\Omega_\Lambda Y$ is a simple Λ -module. But then $\text{rad}^2 \Lambda = 0$, since Λ is selfinjective, and the assertion follows from Lemma 5.6. \square

6. BEHAVIOR OF IRREDUCIBLE MAPS

In this section we show that beyond the Gorenstein algebras the behavior of irreducible maps in $\mathbf{D}^b(\Lambda)$ is not so regular. In particular, we show that some natural conjectures have a negative answer.

For a non-zero map $f: P \rightarrow Q$ between indecomposable objects in $\mathbf{D}^b(\Lambda)$ but not in $\mathbf{K}^b(\Lambda\mathcal{P})$ we investigate the connection between $f: P \rightarrow Q$ being irreducible

and $f_{\geq -n} : P_{\geq -n} \rightarrow Q_{\geq -n}$ being irreducible for some n . We also give some sufficient condition for an irreducible map in $\text{mod}\Lambda$ not be irreducible in $\mathbf{D}^b(\Lambda)$.

We start with a general result on mapping cones of irreducible maps, where the analogous result in abelian categories is well known.

Proposition 6.1. *Let X and Y be indecomposable in $\mathbf{D}^b(\Lambda)$, for a finite dimensional algebra Λ , and assume that we have an irreducible map $f : X \rightarrow Y$. Then the mapping cone C_f is indecomposable.*

Proof. This can be proved in a similar way as the abelian analog. Here we give a slightly shorter proof using Theorem 4.3. Let $\mu : \mathbf{D}^b(\Lambda) \rightarrow \underline{\text{mod}}\widehat{\Lambda}$ be as usual the natural embedding. Then we know from Theorem 4.3 that $\mu(f) : \mu(X) \rightarrow \mu(Y)$ is irreducible in $\underline{\text{mod}}\widehat{\Lambda}$. This is induced by an irreducible map $f' : \mu(X) \rightarrow \mu(Y)$ in $\text{mod}\widehat{\Lambda}$. If f' is mono, we have an exact sequence $0 \rightarrow \mu(X) \rightarrow \mu(Y) \rightarrow \mu(Y)/\mu(X) \rightarrow 0$, and if f' is epi, we have an exact sequence $0 \rightarrow \text{Ker } f' \rightarrow \mu(X) \rightarrow \mu(Y) \rightarrow 0$. We know that in the first case $\mu(Y)/\mu(X)$ is indecomposable and in the second case $\text{Ker } f'$ is indecomposable, see [ARS, V Prop. 5.6]. So in any case we have a triangle $\mu(X) \xrightarrow{\mu(f)} \mu(Y) \rightarrow Z$ in $\underline{\text{mod}}\widehat{\Lambda}$, where Z is indecomposable. Since $\mu(C_f) \cong Z$, it follows that C_f is indecomposable. \square

Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be an almost split sequence in $\text{mod}\Lambda$. Then it is known that if $\text{id}_\Lambda A \leq 1$ and $\text{pd}_\Lambda C \leq 1$, then the sequence gives rise to an almost split triangle in $\mathbf{D}^b(\Lambda)$ (see [H1, 4.7]). Consequently the corresponding irreducible maps $f_i : A \rightarrow B_i$ and $g_i : B_i \rightarrow C$ stay irreducible, where $B = \bigoplus_{i=1}^t B_i$ with B_i indecomposable. But the normal behavior is that irreducible maps in $\text{mod}\Lambda$ do not stay irreducible in $\mathbf{D}^b(\Lambda)$. We illustrate this with the following result.

Proposition 6.2. *Let Λ be a finite dimensional algebra, X and Y indecomposable Λ -modules with $\text{pd}_\Lambda X < \infty$ and $\text{pd}_\Lambda Y \geq \text{pd}_\Lambda X + 2$. Then there is no irreducible map $f : X \rightarrow Y$ in $\mathbf{D}^b(\Lambda)$.*

Proof. Assume that we have an irreducible map $f : X \rightarrow Y$ in $\text{mod}\Lambda$, with X and Y indecomposable, $\text{pd } X = i < \infty$ and $\text{pd } Y \geq i + 2$. Let $P : \dots \rightarrow P^{-(i+2)} \rightarrow \dots \rightarrow P^{-1} \rightarrow P^0 \rightarrow 0$ be a minimal projective resolution of Y and $Q : 0 \rightarrow Q^{-i} \rightarrow \dots \rightarrow Q^{-1} \rightarrow Q^0 \rightarrow 0$ a minimal projective resolution of X . Let C denote the complex $0 \rightarrow P^{-(i+1)} \rightarrow \dots \rightarrow P^{-1} \rightarrow P^0 \rightarrow 0$. Then $f : Q \rightarrow P$ factors as $Q \xrightarrow{g} C \xrightarrow{h} P$, since we have the commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Q^{-i} & \longrightarrow & \dots & \longrightarrow & Q^{-1} & \longrightarrow & Q^0 & \longrightarrow & 0 \\
& & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
& & & & f^{-i} & \downarrow & & & f^{-1} & \downarrow & & f^0 \\
0 & \longrightarrow & P^{-(i+1)} & \longrightarrow & P^{-i} & \longrightarrow & \dots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 \\
& & \parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
\dots & \longrightarrow & P^{-(i+2)} & \longrightarrow & P^{-(i+1)} & \longrightarrow & P^{-i} & \longrightarrow & \dots & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0
\end{array}$$

We want to show that g is not a split monomorphism and h is not a split epimorphism. If $g : Q \rightarrow C$ was a split monomorphism, the induced map $H^0(Q) = X \rightarrow$

$H^0(C) = Y$ would be a split monomorphism. Since X and Y are indecomposable nonisomorphic modules, this is impossible.

The diagram

$$\begin{array}{ccccccc} & & \Omega^{-(i+2)}Y & & & & \\ & & \downarrow & & & & \\ & & P^{-(i+1)} & \longrightarrow & \dots & \longrightarrow & P^{-1} & \longrightarrow & P^0 \\ & & & & & & & & \downarrow \\ & & & & & & & & Y \end{array}$$

gives rise to the triangle $\Omega^{-(i+2)}Y[i+1] \rightarrow C \rightarrow Y[0] \xrightarrow{\alpha} \Omega^{-(i+2)}Y[i+2]$. Here α is given by the sequence $0 \rightarrow \Omega^{-(i+2)}Y \rightarrow P^{-(i+1)} \rightarrow \dots \rightarrow P^{-1} \rightarrow P^0 \rightarrow Y \rightarrow 0$, which does not represent the zero element since $\text{pd}_\Lambda Y \geq i+2$. Hence $h : C \rightarrow Y[0]$, or equivalently $h : C \rightarrow P$, is not a split epimorphism. It follows that $f : Q \rightarrow P$ is not irreducible. \square

Note that if Λ is hereditary, then each irreducible map in $\text{mod } \Lambda$ stays irreducible in $\mathbf{D}^b(\Lambda)$. In this case $\text{pd } X$ is 0 or 1, hence we can never have $\text{pd } Y \geq \text{pd } X + 2$.

The next natural question is to which extent we have irreducible maps $X \rightarrow Y[1]$, where X and Y are indecomposable in $\text{mod } \Lambda$, corresponding to elements of $\text{Ext}_\Lambda^1(X, Y)$. Here we have seen some sufficient conditions in Section 3. Normally we do not have such irreducible maps.

Proposition 6.3. *Let $f : X \rightarrow Y[1]$ be an irreducible map, where X and Y are indecomposable Λ -modules. Then Y must be a summand of ΩX .*

Proof. We have the factorization $X \xrightarrow{h} \Omega X[1] \xrightarrow{g[1]} Y[1]$ of $f : X \rightarrow Y[1]$, as is seen by considering the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega X & \longrightarrow & P_X & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & Y & \longrightarrow & E & \longrightarrow & X & \longrightarrow & 0 \end{array}$$

Then $h : X \rightarrow \Omega X[1]$ is not a split monomorphism since $H^0(h)$ is not a split monomorphism. Since $f : X \rightarrow Y[1]$ is irreducible, it follows that $g : \Omega X \rightarrow Y$ is a split epimorphism, so that Y is a summand of ΩX . \square

We now give another situation where there are no irreducible maps, containing the case $X \rightarrow Y[2]$, corresponding to elements of $\text{Ext}_\Lambda^2(X, Y)$, as a special case.

Proposition 6.4. *Let P and Q be indecomposable objects in $\mathbf{D}^b(\Lambda)$ for a finite dimensional algebra Λ , represented by complexes of projective Λ -modules with no split exact summands, with $P^0 \neq 0$, $P^i = 0$ for $i > 0$ and $Q^i = 0$ for $i \geq -1$. Then there is no irreducible map $f : P \rightarrow Q$ in $\mathbf{D}^b(\Lambda)$.*

Proof. Let $f : P \rightarrow Q$ be a map in $\mathbf{D}^b(\Lambda)$. Consider the factorization of f given by

$$\begin{array}{ccccccc}
\dots & \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \xrightarrow{a} & P^0 & \longrightarrow & 0 & & P \\
& & \downarrow & & \downarrow & & \downarrow & & & & \downarrow h \\
\dots & \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & 0 & \longrightarrow & 0 & & P_{\leq -1} \\
& & \downarrow & & \downarrow & & & & & & \downarrow g \\
\dots & \longrightarrow & Q^{-2} & \longrightarrow & 0 & & & & & & Q
\end{array}$$

We have $H^0(P) = P^0/\text{Im } a$, which is not zero since P has no split exact direct summands. Since $H^0(P_{\leq -1}) = 0$, $h : P \rightarrow P_{\leq -1}$ cannot be a split monomorphism.

Assume now that g is a split epimorphism, and consider the triangle $P_{\leq -1} \xrightarrow{g} Q \xrightarrow{u} C_g \rightarrow$. Then $u : Q \rightarrow C_g$ must be homotopic to 0, that is, we have the diagram

$$\begin{array}{ccccccc}
\dots & \longrightarrow & Q^{-3} & \xrightarrow{b_{-2}} & Q^{-2} & \xrightarrow{b_{-1}} & 0 \\
& & \downarrow (0,1) & \swarrow s_{-2} & \downarrow (0,1) & \swarrow s_{-1} & \\
\dots & \longrightarrow & P^{-2} \oplus Q^{-3} & \xrightarrow{c_{-2}} & P^{-1} \oplus Q^{-2} & \xrightarrow{c_{-1}} & 0
\end{array}$$

where $b_{-i}s_{-i} + s_{-(i+1)}c_{-(i+1)} = (0, 1)$ for all $i \geq 1$. Using the same maps s_i we see that in the triangle $P \xrightarrow{f} Q \xrightarrow{v} C_f \rightarrow$, the map v must be 0, so that f would also be a split epimorphism. Since P and Q are indecomposable, f would be an isomorphism, which is impossible because $H^0(P) \neq 0$ and $H^0(Q) = 0$. We conclude that g is not a split epimorphism. Since we already have that h is not a split monomorphism, it follows that $f : P \rightarrow Q$ is not irreducible. \square

The following sufficient condition for the mapping cone to be indecomposable will be useful.

Lemma 6.5. *Let $f : P \rightarrow Q$ be a map between indecomposable objects in a Hom-finite Krull-Schmidt triangulated category \mathcal{C} with shift $[1]$, and assume that f is not zero and not invertible. Complete to a triangle $P \xrightarrow{f} Q \xrightarrow{g} C \rightarrow P[1]$. If $\text{Hom}(Q, P[1]) = 0$, then C is indecomposable.*

Proof. Assume to the contrary that \mathcal{C} is not indecomposable, and write $C = \bigoplus_{i=1}^r Z_i$, where $r > 1$ and each Z_i is indecomposable. Let $g = (g_1, \dots, g_r)$ and $h = (h_1, \dots, h_r)^t$. Then we know from [R] that $g_i \neq 0$ and $h_i \neq 0$ for each $i = 1, \dots, r$.

Consider the map

$$\varphi = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \vdots \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} : C \rightarrow C,$$

where $1 = \mathbf{1}_{Z_1}$. We then have the diagram

$$\begin{array}{ccccccc} P & \xrightarrow{f} & Q & \xrightarrow{g} & C & \xrightarrow{h} & P[1] \\ & & & & \downarrow \varphi & & \\ P & \xrightarrow{f} & Q & \xrightarrow{g} & C & \xrightarrow{h} & P[1] \end{array}$$

Since by the assumption $\text{Hom}(Q, P[1]) = 0$, it follows that $g\varphi h = 0$. Hence there is a map $\varphi_Q : Q \rightarrow Q$ such that $\varphi_Q g = g\varphi$. We have $g\varphi = (g_1, 0, \dots, 0)$ and $\varphi_Q g = (\varphi_Q g_1, \dots, \varphi_Q g_r)$, so that $\varphi_Q g_1 = g_1$ and $\varphi_Q g_i = 0$ for $2 \leq i \leq r$. Since $\varphi^2 = \varphi$, we have $\varphi_Q^n g = g\varphi^n = g\varphi$, so that $\varphi_Q^n g_1 = g_1$ and $\varphi_Q^n g_i = 0$ for $2 \leq i \leq r$. Since Q is indecomposable and \mathcal{C} is Hom-finite, any map $t : Q \rightarrow Q$ is nilpotent or an isomorphism, so that we have a contradiction. It follows that C is indecomposable. \square

We now consider the following question. If we have an irreducible map $f : P \rightarrow Q$ between unbounded complexes of projective modules, not objects in $\mathbf{K}^b(\Lambda\mathcal{P})$, is then $f_{\geq -n} : P_{\geq -n} \rightarrow Q_{\geq -n}$ irreducible for all n , where $f_{\geq -n}$ is a nonzero map between indecomposable objects?

For selfinjective algebras Λ , the existence of an irreducible map $f : P \rightarrow Q$ not in $\mathbf{K}^b(\Lambda\mathcal{P})$ implies that Λ is selfinjective with $\text{rad}^2 \Lambda = 0$ and that we have $f : S \rightarrow T[1]$, where S and T are simple Λ -modules. In this case $f_{\geq -n} : P_{\geq -n} \rightarrow Q_{\geq -n}$ is irreducible for $n \geq 2$.

We now give an example which gives a negative answer to the above question. Let Λ be the path algebra of the quiver

$$\gamma \circlearrowleft 1 \xleftarrow{\alpha} 2$$

with relations $\alpha\gamma = 0$, $\gamma^2 = 0$. Denote by S the simple module at vertex 1 and by T the simple module at vertex 2. Then the indecomposable projective Λ -modules have Loewy series $\begin{smallmatrix} S \\ S \end{smallmatrix}$ and $\begin{smallmatrix} T \\ S \end{smallmatrix}$, and we have T and $\begin{smallmatrix} ST \\ S \end{smallmatrix}$ for the indecomposable injectives. We know from Proposition 4.4 that the map $f : T \rightarrow S[1]$ is irreducible, and we can write this as $f : P \rightarrow Q$ given by

$$\begin{array}{ccccccc} \dots & \longrightarrow & \begin{smallmatrix} S \\ S \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} S \\ S \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} T \\ S \end{smallmatrix} \\ & & \parallel & & \parallel & & \downarrow \\ \dots & \longrightarrow & \begin{smallmatrix} S \\ S \end{smallmatrix} & \longrightarrow & \begin{smallmatrix} S \\ S \end{smallmatrix} & \longrightarrow & 0 \end{array}$$

We then have the following.

Proposition 6.6. *Let Λ be as above. In the above notation we have that $f : P \rightarrow Q$ is irreducible, while $f_{\geq -1} : P_{\geq -1} \rightarrow Q_{\geq -1}$ is a map between indecomposable objects which is not irreducible.*

Proof. We have already seen that $f : P \rightarrow Q$ is irreducible. We now want to show that $f_{\geq -1} : P_{\geq -1} \rightarrow Q_{\geq -1}$ is not irreducible. We have $Q_{\geq -1} = \begin{smallmatrix} S \\ S \end{smallmatrix}[1]$, and hence $\nu Q_{\geq -1} = \begin{smallmatrix} ST \\ S \end{smallmatrix}$, so that we have an almost split triangle $\begin{smallmatrix} ST \\ S \end{smallmatrix} \rightarrow \begin{smallmatrix} S \\ S \end{smallmatrix} \xrightarrow{\alpha} \begin{smallmatrix} ST \\ S \end{smallmatrix} \rightarrow \begin{smallmatrix} S \\ S \end{smallmatrix}[1] \rightarrow$, where $\text{Im } \alpha = S$. We claim that $X = \begin{smallmatrix} S \\ S \end{smallmatrix} \xrightarrow{\alpha} \begin{smallmatrix} ST \\ S \end{smallmatrix}$ is indecomposable. For this, it is sufficient to show that $\text{Hom}(\begin{smallmatrix} ST \\ S \end{smallmatrix}, \begin{smallmatrix} S \\ S \end{smallmatrix}[1]) = 0$ by Lemma 6.5, that is that $\text{Ext}_{\Lambda}^1(\begin{smallmatrix} ST \\ S \end{smallmatrix}, \begin{smallmatrix} S \\ S \end{smallmatrix}) = 0$.

For it is clear that the induced map is nonzero and is in the socle of $\text{End}(Q_{\geq -2})$. Taking the mapping cone of α we obtain $P_{\geq -2}[1]$, so that $E \cong P_{\geq -2}$. This shows that $f_{\geq -2} : P_{\geq -2} \rightarrow Q_{\geq -2}$ is irreducible.

We next show that $P_{\geq -2}$ is indecomposable. We give a proof which at the same time illustrates the previous theory, rather than giving a direct computational proof. We know from Theorem 5.4 that the components of the AR-quiver of $\mathbf{K}^b({}_{\Lambda}\mathcal{P})$ are of the form $\mathbb{Z}\mathbf{A}_{\infty}$, and that the image of a component for $\mathbf{K}^b({}_{\Lambda}\mathcal{P})$ is a component of the AR-quiver for $\underline{\text{mod}}\widehat{\Lambda}$. All $\widehat{\Lambda}$ -modules in such a component \mathcal{C} are given by projective modules, the same ones as for $\mathbf{K}^b({}_{\Lambda}\mathcal{P})$. Then \mathcal{C} is also a component for $\text{mod}\widehat{\Lambda}$. This follows since any indecomposable projective object in $\text{mod}\widehat{\Lambda}$ has an irreducible map to this object modulo its socle, and this object is not given by only projective modules.

If $P_{\geq -2}$ was not indecomposable, then $\Lambda \xrightarrow{\cdot x} \Lambda$ would not be at the border of the $\mathbb{Z}\mathbf{A}_{\infty}$ -component. Hence we would have an irreducible epimorphism starting at $\Lambda \xrightarrow{\cdot x} \Lambda$, which would then have to end at Λ , since the terms must be projective. But on the other hand we have an almost split triangle $\Lambda[-1] \rightarrow (\Lambda \xrightarrow{\cdot x^2} \Lambda) \rightarrow \Lambda$ in $\mathbf{K}^b({}_{\Lambda}\mathcal{P})$, which gives a contradiction.

(2) We first show that P and Q are indecomposable. This is obvious for Q . Assume $P = P' \oplus P''$ is a nontrivial decomposition. Then we have $P_{\geq -2} = P'_{\geq -2} \oplus P''_{\geq -2}$. Since $H^0(P) = S$, $H^{-1}(P) = \begin{smallmatrix} S \\ S \end{smallmatrix}$ and $H^{-i}(P) = 0$ for $i \neq 0, 1$, we must have, say $P' \cong S$ and $P'' \cong \begin{smallmatrix} S \\ S \end{smallmatrix}[1]$. But then $P'_{\geq -2}$ and $P''_{\geq -2}$ are both nonzero, contradicting that $P_{\geq -2}$ is indecomposable.

That $f : P \rightarrow Q$ is not irreducible follows since Λ is selfinjective and $\text{rad}^2 \Lambda \neq 0$ and P and Q are not in $\mathbf{K}^b({}_{\Lambda}\mathcal{P})$. We could alternatively give a direct argument by considering the following factorization of the map $f : P \rightarrow Q$:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & \Lambda & \xrightarrow{\cdot x} & \Lambda & \xrightarrow{\cdot x^2} & \Lambda & \xrightarrow{(x,0)} & \Lambda \oplus \Lambda & \xrightarrow{\begin{smallmatrix} x^2 \\ x \end{smallmatrix}} & \Lambda & & P \\
& & \parallel & & \parallel & & \downarrow (1,0) & & \parallel & & \parallel & & \downarrow g \\
\dots & \longrightarrow & \Lambda & \xrightarrow{\cdot x} & \Lambda & \xrightarrow{(x^2,0)} & \Lambda \oplus \Lambda & \xrightarrow{\begin{smallmatrix} x & 0 \\ 0 & x^2 \end{smallmatrix}} & \Lambda \oplus \Lambda & \xrightarrow{\begin{smallmatrix} x^2 \\ x \end{smallmatrix}} & \Lambda & & U \\
& & \parallel & & \parallel & & \downarrow (\delta) & & \downarrow (\delta) & & \downarrow & & \downarrow h \\
\dots & \longrightarrow & \Lambda & \xrightarrow{\cdot x} & \Lambda & \xrightarrow{\cdot x^2} & \Lambda & \xrightarrow{\cdot x} & \Lambda & \longrightarrow & 0 & & Q
\end{array}$$

and showing that g is not a split monomorphism and h is not a split epimorphism. The first claim follows directly by considering the homology of P and U , and the second claim is also not hard to show. \square

REFERENCES

- [AR] Auslander, Maurice; Reiten, Idun, Representation theory of Artin algebras. IV. Invariants given by almost split sequences. *Comm. Algebra* 5 (1977), no. 5, 443–518.
- [ARS] Auslander, Maurice; Reiten, Idun; Smalø, Sverre O., Representation theory of Artin algebras. *Cambridge Studies in Advanced Mathematics*, 36. Cambridge University Press, Cambridge, 1995.
- [CZ] Chen Xiao-Wu; Zhang, Pu, *Quotient triangulated categories*, preprint.

- [G] Alexandre Grothendieck, *Les dérivateurs*, Manuscript, 1990, edited electronically by M. Künzer, J. Malgoire and G. Maltsiniotis.
- [GK] Geiss, Christof; Krause, Henning, On the notion of derived tameness. *J. Algebra Appl.* 1 (2002), no. 2, 133–157.
- [F] Farnsteiner, Rolf, Stable representation quivers: Subadditive functions and Webb’s theorem. Preprint 2006, <http://www.mathematik.uni-bielefeld.de/~sek/select/RF10.pdf>.
- [H1] Happel, Dieter, *Triangulated categories in the representation theory of finite-dimensional algebras*. London Mathematical Society Lecture Note Series, 119. Cambridge University Press, Cambridge, 1988.
- [H2] Happel, Dieter, Auslander-Reiten triangles in derived categories of finite-dimensional algebras. *Proc. Amer. Math. Soc.* 112 (1991), no. 3, 641–648.
- [K1] Keller, Bernhard, *On differential graded categories*, preprint, contribution to the proceedings of the ICM 2006.
- [K2] ———, *Derived categories and universal problems*, *Comm. in Algebra* **19** (1991), 699–747.
- [KV] Keller, Bernhard; Vossieck, Dieter, *Sous les catégories dérivées*, *C. R. Acad. Sci. Paris Sér. I Math.* **305** (1987), no. 6, 225–228.
- [KL] Krause, Henning; Le, Jue, The Auslander-Reiten formula for complexes of modules, *Adv. Math.* 207 (2006), no. 1, 133–148.
- [RV] Reiten, Idun; Van den Bergh, Michel, Noetherian hereditary abelian categories satisfying Serre duality. *J. Amer. Math. Soc.* 15 (2002).
- [Ric] Rickard, Jeremy, *Derived categories and stable equivalence*, *J. Pure and Appl. Algebra* **61** (1989), 303–317.
- [Rin] Ringel, Claus M., Hereditary triangulated categories. *Compositio Mathematica*. To appear.
- [Ta] Tabuada, Gonçalo, *Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories*, *C. R. Math. Acad. Sci. Paris* **340** (2005), no. 1, 15–19.
- [To] Toën, Bertrand, *The homotopy theory of dg-categories and derived Morita theory*, to appear in *Inv. Math.*, arXiv:math.AG/0408337.

We have taken all the comments of the referee into account.