HOCHSCHILD (CO)HOMOLOGY AND DERIVED CATEGORIES

BERNHARD KELLER

Abstract. These are slightly expanded notes of lectures given in April 2019 at the Isfahan school and conference on representations of algebras. We recall the formalism of derived categories and functors and survey invariance results for the Hochschild (co)homology of differential graded algebras and categories.

1. Introduction

Let k be a field. Hochschild homology and cohomology are classical invariants of an associative k-algebra or, more generally, a k-category. They may be viewed as derived versions of the trace space and the center of the algebra, which we recall in section 2. They are invariant under Morita equivalences and also under derived equivalences, as we show in section 5 after a reminder on derived categories in section 4. These invariance results suggest that Hochschild (co)homology should be defined directly using the derived category. This is not possible in general but we show in section 6 that they may be obtained from the canonical differential graded enhancement of the derived category. Hochschild homology and cohomology are endowed with higher structure: the Hochschild chain complex is a mixed complex and the Hochschild cochain complex a B_∞-algebra as we recall in section 7, where we also state the corresponding invariance results. In the final section 8, we report on recent results on Tate–Hochschild cohomology after [31] and [27]. In particular, we sketch a proof of the fact that Tate–Hochschild cohomology of an algebra is isomorphic, as an algebra, to the classical Hochschild cohomology of its differential graded singularity category.

2. Center and trace space for algebras and categories

For simplicity, we work over a field k. Let A be a k-algebra, *i.e.* an associative, possibly non commutative, unital k-algebra. The *center* $Z(A)$ *of* A is formed by the elements $a \in A$ such that

$$ab = ba$$

for all $b \in A$. The *trace space* $\mathrm{Tr}(A)$ is the quotient $A/[A, A]$, where $[A, A]$ denotes the subspace of A generated by all commutators $[a, b] = ab - ba$, $a, b \in A$. Notice that the center is naturally a commutative algebra whereas the trace space is just a vector space. For example, if $n \geq 1$ and $M_n(A)$ denotes the algebra of $n \times n$-matrices with coefficients in A, we have an isomorphism

$$Z(A) \xrightarrow{\sim} Z(M_n(A))$$

taking an element a to the diagonal matrix whose diagonal coefficients all equal a, and an isomorphism

$$\mathrm{Tr}(M_n(A)) \xrightarrow{\sim} \mathrm{Tr}(A)$$

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taking the class of a matrix $M = (m_{ij})$ to the class of its trace

$$\sum_i m_{ii},$$

cf. [38]. Observe that by definition, we have a short exact sequence

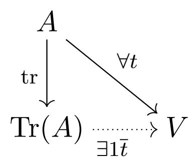
$$0 \longrightarrow Z(A) \longrightarrow A \longrightarrow \mathrm{Hom}_k(A, A)$$

where the second map is the inclusion and the third one takes $a \in A$ to the commutator $[a, ?]$. Somehow dually, we have a short exact sequence

$$0 \longleftarrow \mathrm{Tr}(A) \longleftarrow A \longleftarrow A \otimes_k A$$

where the third map takes $a \otimes b$ to the commutator $ab - ba$ and the second map is the natural projection. This parallelism suggests some similarity between the center and the trace space but we will now argue that they are actually very different (for non commutative algebras).

Let $f : A \to B$ be an algebra morphism (it need not preserve the unit). Clearly, we have an induced map $\mathrm{Tr}(f) : \mathrm{Tr}(A) \to \mathrm{Tr}(B)$ taking the class of an element a to the class of $f(a)$. Thus, the assignment $A \mapsto \mathrm{Tr}(A)$ becomes a functor. The deeper reason for this functoriality is the following universal property: The map $\mathrm{tr} : A \to \mathrm{Tr}(A)$ taking a to the class of a is a *trace map*, *i.e.* it takes values in a vector space, is k-linear and satisfies $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in A$, and it is universal among all trace maps $t : A \to V$, i.e. we have $t = \bar{t} \circ \mathrm{tr}$ for a unique linear map $\bar{t} : \mathrm{Tr}(A) \to V$:



On the other hand, for an element a of the center $Z(A)$, there is no reason for $f(a)$ to lie in $Z(B)$ and there is no induced map between the centers in general. Hence the assignment $A \mapsto Z(A)$ is *not a functor*. However, we will see below how, by passing from algebras to categories, we do gain some functoriality for the center.

Recall [17] that a *k-category* is a category $\mathcal{A}$ whose morphism sets are endowed with k-vector space structures such that the compositions are bilinear. We may (and will) identify a k-algebra A with the k-category with a single object whose endomorphism algebra is A. A general k-category should be viewed as a *k-algebra with several objects* [43]. Recall [17] that a *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is a set of 'vertices', Q_1 a set of 'arrows' and s and t are maps $Q_1 \to Q_0$ taking an arrow to its source respectively its target. In particular, each (small) k-category $\mathcal{A}$ has an underlying quiver and the functor taking the category to its quiver admits a left adjoint $Q \mapsto k\text{-cat}(Q)$, where the objects of $k\text{-cat}(Q)$ are the vertices of Q and the space of morphisms $x \to y$ is the vector space of formal linear combinations of paths (=formal compositions of arrows) of length ≥ 0 from x to y.

Let $\mathcal{A}$ be a small k-category. We define the *center* $Z(\mathcal{A})$ to be the algebra of endomorphisms of the identity functor $\mathbf{1}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$. Thus, an element of the center is a family φX, $X \in \mathcal{A}$, of endomorphisms $\varphi X : X \to X$ of objects of $\mathcal{A}$ such that for each morphism

$f : X \to Y$, the square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle \varphi X}\downarrow & & \downarrow{\scriptstyle \varphi Y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

commutes. Thus, we have the short exact sequence

$$0 \longrightarrow Z(\mathcal{A}) \longrightarrow \prod_{X\in\mathcal{A}} \mathcal{A}(X,X) \longrightarrow \prod_{X,Y\in\mathcal{A}} \mathrm{Hom}_k(\mathcal{A}(X,Y),\mathcal{A}(X,Y)) \ , \qquad (2.0.1)$$

where the third map takes a family (φX) to the family of maps taking $f : X \to Y$ to $(\varphi Y)\circ f - f \circ (\varphi X)$. 'Dually', we define the trace space $\mathrm{Tr}(\mathcal{A})$ of $\mathcal{A}$ by the sequence

$$0 \longleftarrow \mathrm{Tr}(\mathcal{A}) \longleftarrow \bigoplus_{X\in\mathcal{A}} \mathcal{A}(X,X) \longleftarrow \bigoplus_{X,Y\in\mathcal{A}} \mathcal{A}(X,Y)\otimes\mathcal{A}(Y,X) \ , \qquad (2.0.2)$$

where the third map takes $f\otimes g$ to $f\circ g - g\circ f$. Notice that the middle terms of the sequences are isomorphic if and only if $\mathcal{A}$ has only finitely many non zero objects.

We may ask whether we can reduce the computation of the center and the trace space of the k-category $\mathcal{A}$ to that of the center and trace space for some k-algebra A. The natural candidate is the *matrix algebra* defined by

$$k[\mathcal{A}] = \bigoplus_{X,Y\in\mathcal{A}} \mathcal{A}(X,Y)$$

and endowed with matrix multiplication. Notice that it is associative but unital only if $\mathcal{A}$ has only finitely many non zero objects. In general, it is still locally unital, i.e. for each finite set of elements a_i, $i\in I$, we may find an idempotent e such that $ea_i = a_i = a_i e$ for all i.

Lemma 2.1. a) *We have a canonical isomorphism* $\mathrm{Tr}(k[\mathcal{A}]) \xrightarrow{\sim} \mathrm{Tr}(A)$.
b) *We have a natural injective algebra morphism* $Z(k[\mathcal{A}]) \to Z(\mathcal{A})$ *which is an isomorphism iff* $\mathcal{A}$ *has only finitely many non zero objects.*

The proof is left to the reader as an exercise. Of course, the assignment $\mathcal{A}\mapsto\mathrm{Tr}(\mathcal{A})$ is functorial with respect to k-linear functors $F : \mathcal{A}\to\mathcal{B}$. We also have some fonctoriality for the assignment $\mathcal{A}\mapsto Z(\mathcal{A})$. Indeed, if $\mathcal{B}\subset\mathcal{A}$ is a full subcategory, then the restriction map

$$(\varphi X)_{X\in\mathcal{A}} \mapsto (\varphi X)_{X\in\mathcal{B}}$$

is an algebra morphism $Z(\mathcal{A})\to Z(\mathcal{B})$. We see that $\mathcal{A}\mapsto Z(\mathcal{A})$ is *contravariant with respect to fully faithful embeddings.*

Let A be a k-algebra and $\mathsf{Mod}\,A$ the k-category of all right A-modules (by choosing suitable universes, we can dispense with the set-theoretical problem that $\mathsf{Mod}\,A$ is not small). We have a fully faithful embedding $A\subset\mathsf{Mod}\,A$ taking the unique object to the free A-module of rank one.

Lemma 2.2. *The restriction along* $A\subset\mathsf{Mod}\,A$ *is an isomorphism* $Z(\mathsf{Mod}\,A)\xrightarrow{\sim} Z(A)$.

Proof. Let ρ denote the restriction. We define $\sigma : Z(A)\to Z(\mathsf{Mod}\,A)$ by sending an element $z\in Z(A)$ to the family $\sigma(z)$ whose component at a module M is right multiplication by z. Since z is central, the map $\sigma(z)M$ is indeed an endomorphism of M and since any $f : L\to M$ is A-linear, the family $\sigma(z)$ does lie in $Z(\mathsf{Mod}\,A)$. It is also clear that $\rho(\sigma(z)) = z$ for each $z\in Z(A)$. Thus, the map $\rho : Z(\mathsf{Mod}\,A)\to Z(A)$ is injective. Suppose that φ is in its kernel. Then $\varphi A = 0$ and in fact $\varphi F = 0$ for each free A-module. If M is an arbitrary A-module, we have an exact sequence

$$F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_1 and F_0 are free. The fact that φF_1 and φF_0 vanish then immediately implies that φM vanishes. Thus, the map ρ is also injective. $\surd$

Recall that a *generator* (resp. *cogenerator*) of $\mathsf{Mod}\, A$ is a module G such that each module is a quotient of a coproduct of copies of G (resp. a submodule of a product of copies of G). Let $\mathsf{proj}\, A$ denote the category of finitely generated projective modules, $\mathsf{Proj}\, A$ the category of all projective modules and $\mathsf{Inj}\, A$ the category of all injective modules. It is easy to adapt the above proof to obtain the

Lemma 2.3. *Let $\mathcal{B} \subset \mathsf{Mod}\, A$ be a full subcategory containing a generator or a cogenerator of $\mathsf{Mod}\, A$. Then restriction to $\mathcal{B}$ is an isomorphism $Z(\mathsf{Mod}\, A) \xrightarrow{\sim} Z(\mathcal{B})$. In particular, we have isomorphisms*

$$Z(A) \xleftarrow{\sim} Z(\mathsf{proj}\, A) \xleftarrow{\sim} Z(\mathsf{Proj}\, A) \xleftarrow{\sim} Z(\mathsf{Mod}\, A) \xrightarrow{\sim} Z(\mathsf{Inj}\, A).$$

The following lemma shows that the trace space behaves entirely differently. The proof is based on the 'Eilenberg swindle'.

Lemma 2.4. *We have* $\mathrm{Tr}(\mathsf{Mod}\, A) = 0$*. More generally, we have* $\mathrm{Tr}(\mathcal{A}) = 0$ *for any k-category $\mathcal{A}$ admitting countable coproducts.*

Proof. We sketch a proof. The details may be filled in using [33]. Let $\mathcal{A}$ and $\mathcal{B}$ be k-categories and $F, G : \mathcal{A} \to \mathcal{B}$ k-linear functors. In a first step, one shows that if F and G are isomorphic functors, we have an equality $\mathrm{Tr}(F) = \mathrm{Tr}(G)$ between the induced maps $\mathrm{Tr}(\mathcal{A}) \to \mathrm{Tr}(\mathcal{B})$. Now assume that $\mathcal{B}$ has finite direct sums. Then one shows that $\mathrm{Tr}(F \oplus G) = \mathrm{Tr}(F) + \mathrm{Tr}(G)$. Now assume that $\mathcal{A}$ has countable coproducts. For $X \in \mathcal{A}$, let $X^{(\mathbb{N})}$ be the coproduct of a countable set of copies of X. Then we can choose a functorial isomorphism

$$X \oplus X^{(\mathbb{N})} \xrightarrow{\sim} X^{(\mathbb{N})}.$$

Thus, if $F : \mathcal{A} \to \mathcal{A}$ is the functor $X \mapsto X^{(\mathbb{N})}$, then we have $\mathbf{1}_{\mathcal{A}} \oplus F \xrightarrow{\sim} F$ as functors $\mathcal{A} \to \mathcal{A}$. It follows from the two previous steps that we have

$$\mathrm{Tr}(\mathbf{1}_{\mathcal{A}}) + \mathrm{Tr}(F) = \mathrm{Tr}(\mathbf{1}_{\mathcal{A}} \oplus F) = \mathrm{Tr}(F)$$

and therefore

$$\mathbf{1}_{\mathrm{Tr}(\mathcal{A})} = \mathrm{Tr}(\mathbf{1}_{\mathcal{A}}) = 0.$$

$\surd$

On the other hand, if we restrict to finite sums (and summands), this phenomenon does not occur.

Lemma 2.5 ([33])**.** *The inclusion $A \subset \mathsf{proj}\, A$ induces an isomorphism* $\mathrm{Tr}(A) \xrightarrow{\sim} \mathrm{Tr}(\mathsf{proj}\, A)$.

3. Hochschild (co)homology of algebras

Let k be a field. We write $\otimes$ for $\otimes_k$. Let A be a k-algebra. The *Hochschild chain complex of A* is the complex $C_* A$ concentrated in homological degrees ≥ 0

$$A \longleftarrow A \otimes A \longleftarrow \ldots \longleftarrow A^{\otimes p} \longleftarrow A^{\otimes (p+1)} \longleftarrow \ldots$$

with $C_pA = A^{\otimes(p+1)}$, $p \geq 0$, and differential given by

$$d(a_0, \ldots, a_p) = \sum_{i=0}^{p-1} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_p) + (-1)^p (a_p a_0, \ldots, a_{p-1}) , \tag{3.0.1}$$

where we write $(a_0, \ldots, a_p)$ for $a_0 \otimes \cdots \otimes a_p$. Notice that the first differential takes $a \otimes b$ to the commutator $ab - ba$. The *Hochschild homology* $HH_*(A)$ is the graded space with $HH_p(A) = H_p(C_*A)$.

The *Hochschild* [26] *cochain complex of* A is the complex C^*A concentrated in cohomological degrees ≥ 0

$$A \longrightarrow \mathrm{Hom}_k(A, A) \longrightarrow \mathrm{Hom}_k(A \otimes A, A) \longrightarrow \ldots \longrightarrow \mathrm{Hom}_k(A^{\otimes p}, A) \longrightarrow \ldots$$

whose differential is given by
(3.0.2)

$$(df)(a_0, \ldots, a_p) = a_0 f(a_1, \ldots, a_p) - \sum_{i=0}^{p-1} (-1)^i f(a_0, \ldots, a_i a_{i+1}, \ldots, a_p) + (-1)^p f(a_0, \ldots, a_{p-1}) a_p.$$

Notice that the first two differentials are given by

$$a \mapsto [a, ?] \quad \text{and} \quad f \mapsto (a \otimes b \mapsto f(a)b - f(ab) + af(b)).$$

We see that that in degree 0 we recover the trace space $\mathrm{Tr}(A) = HH_0(A)$ respectively the center $Z(A) = HH^0(A)$. We also see that $HH^1(A)$ is equal to the Lie algebra of outer derivations of A (with the bracket induced by the commutator of derivations). Both structures, the commutative multiplication and the Lie bracket, extend to the whole of Hochschild cohomology, as we will see in section 7.

Let $A^e = A^{op} \otimes A$. We identify the category $\mathsf{Mod}(A^e)$ of right A^e-modules with the (isomorphic) category of A-A-bimodules via the rule

$$amb = m(b \otimes a).$$

In particular, we have the *identity bimodule* ${}_AA_A$ given by the algebra A considered as a bimodule over itself.

Theorem 3.1 (Cartan–Eilenberg [8]). *We have canonical isomorphisms*

$$\mathsf{Ext}^*_{A^e}(A, A) \xrightarrow{\sim} HH^*(A) \quad \textit{and} \quad \mathsf{Tor}^*_{A^e}(A, A) \xrightarrow{\sim} HH_*(A).$$

To prove the theorem, one computes the derived functors using the bar resolution of the first argument: Recall that the (augmented) bar resolution is the complex of bimodules

$$0 \longleftarrow {}_AA_A \longleftarrow A \otimes A \longleftarrow \ldots \longleftarrow A \otimes A^{\otimes p} \otimes A \longleftarrow \ldots$$

where the augmentation is the multiplication of A and the differential is given by

$$d(a_0, \ldots, a_{p+1}) = \sum_{i=0}^{p} (-1)^i (a_0, \ldots, a_i a_{i+1}, \ldots, a_{p+1}).$$

Corollary 3.2. *Hochschild cohomology $HH^*(A)$ carries a natural graded multiplication, the* cup product*, extending that of $Z(A) = HH^0(A)$.*

It is shown in [8] that the cup product is induced by the following associative multiplication on cochains: For $f \in C^pA$ and $g \in C^qA$, define

$$(f \cup g)(a_1, \ldots, a_p, a_{p+1}, \ldots, a_{p+q}) = (-1)^{pq} f(a_1, \ldots, a_p) g(a_{p+1}, \ldots, a_{p+q}).$$

4. Reminder on derived categories

We collect basic definitions and results on derived categories. We refer to [35] for more details and references. As before, k is a field and A a k-algebra. The category $\mathcal{C}A = \mathcal{C}\,\mathsf{Mod}\,A$ has as objects the cochain complexes

$$\ldots \longrightarrow M^p \xrightarrow{d_M} M^{p+1} \longrightarrow \ldots$$

of (right) A-modules. Notice that each such complex has an underlying $\mathbb{Z}$-graded A-module given by the sequence of the M^p, $p \in \mathbb{Z}$. It is endowed with the differential d_M, which is a homogeneous endomorphism of degree $+1$. The *morphisms* of $\mathcal{C}A$ are the morphisms $f : L \to M$ of graded A-modules which are homogeneous of degree 0 and satisfy $d_M \circ f = f \circ d_L$. The *suspension functor* $\Sigma : \mathcal{C}A \to \mathcal{C}A$ takes a complex M to ΣM with components $(\Sigma M)^p = M^{p+1}$ and differential $d_{\Sigma M} = -d_M$. Two morphisms f and $g : L \to M$ of $\mathcal{C}A$ are *homotopic* if there is a homogeneous morphism of graded A-modules $h : L \to M$ of degree -1 such that

$$f - g = d_M \circ h + h \circ d_L.$$

The *homotopy category* $\mathcal{H}A$ has the same objects as $\mathcal{C}A$; its morphisms are homotopy classes of morphisms of $\mathcal{C}A$. A morphism $s : L \to M$ of $\mathcal{C}A$ or $\mathcal{H}A$ is a *quasi-isomorphism* if the induced morphism in homology $H^*(s) : H^*(L) \to H^*(M)$ is invertible. The *derived category* is the localization of $\mathcal{C}A$ (or $\mathcal{H}A$) at the class of all quasi-isomorphisms. Thus, it has the same objects as $\mathcal{C}A$ and its morphisms are equivalence classes of formal compositions of morphisms of $\mathcal{C}A$ (or $\mathcal{H}A$) and formal inverses of quasi-isomorphisms.

The homotopy category $\mathcal{H}A$ is canonically triangulated with suspension functor Σ. Each componentwise split short exact sequence of $\mathcal{C}A$

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

yields a canonical triangle

$$L \longrightarrow M \longrightarrow N \longrightarrow \Sigma L\ .$$

The derived category $\mathcal{D}A$ is triangulated with suspension functor Σ. Each short exact sequence of $\mathcal{C}A$ yields a canonical triangle.

We identify A-modules $M \in \mathsf{Mod}\,A$ with complexes concentrated in degree 0

$$\ldots \longrightarrow 0 \longrightarrow M \longrightarrow 0 \longrightarrow \ldots\ .$$

Then, for A-modules L and M, we have canonical isomorphisms

$$\mathsf{Ext}^p_A(L, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}A}(L, \Sigma^p M)$$

for all $p \in \mathbb{Z}$ (with the convention that Ext^p vanishes for $p < 0$). Moreover, the composition in $\mathcal{D}A$ identifies with the Yoneda product on Ext.

Theorem 4.1. *The projection $\mathcal{H}A \to \mathcal{D}A$ admits a fully faithful left adjoint* $\mathbf{p}$ *and a fully faithful right adjoint* $\mathbf{i}$.

Notice that the analogous statement for the category of complexes $\mathcal{C}A$ instead of the homotopy category $\mathcal{H}A$ is wrong. This is one of the main reasons for introducing $\mathcal{H}A$. The functors $\mathbf{p}$ and $\mathbf{i}$ generalize projective respectively injective resolutions. Indeed, if M is an A-module and $P \to M$ a projective resolution (*i.e.* a quasi-isomorphism where P is right bounded with projective components), then we have $P \xrightarrow{\sim} \mathbf{p}M$ in $\mathcal{H}A$. Similarly, if $M \to I$ is an injective resolution, then $\mathbf{i}M \xrightarrow{\sim} I$ in $\mathcal{H}A$.

Now let B be another algebra and X a complex of A-B-bimodules. For $M \in \mathcal{C}A$, define the $M \otimes_A X \in \mathcal{C}B$ by

$$(M \otimes_A X)^n = \bigoplus_{p+q=n} M^p \otimes_A X^q \quad \text{and} \quad d(m \otimes a) = (dm) \otimes a + (-1)^p m \otimes (dx).$$

For $L \in \mathcal{C}B$, define $\mathrm{Hom}_B(X, L) \in \mathcal{C}A$ as the complex whose nth component is formed by the morphisms $f : X \to L$ of graded B-modules, homogeneous of degree n and whose differential is given by

$$d(f) = d_L \circ f - (-1)^n f \circ d_X.$$

Define objects of the derived categories

$$L \overset{L}{\otimes}_A X = (\mathbf{p}L) \otimes_A X \quad \text{and} \quad \mathrm{RHom}_B(X, L) = \mathrm{Hom}_B(X, ?).$$

Example 4.2. *For example, we have*

$$HH_*(A) = \mathsf{Tor}_*^{A^e}(A, A) = H_*(A \overset{L}{\otimes}_{A^e} A) \quad \textit{and} \quad HH^*(A) = \mathsf{Ext}^*_{A^e}(A, A) = H^*(\mathrm{RHom}_{A^e}(A, A)).$$

The following lemma is easy to prove.

Lemma 4.3. *We have an adjoint pair*

$$? \overset{L}{\otimes}_A X : \mathcal{D}A \rightleftarrows \mathcal{D}B : \mathrm{RHom}_B(X, ?)\ .$$

It is natural to ask when these adjoints are equivalences. To answer this question, let us observe that $\mathcal{D}A$ has arbitrary coproducts, which are given by coproducts of complexes. An object P of $\mathcal{D}A$ is *compact* if the functor $\mathrm{Hom}(P, ?) : \mathcal{D}A \to \mathsf{Mod}\, k$ commutes with arbitrary coproducts. It is *perfect* if it is quasi-isomorphic to a bounded complex of finitely generated projective modules. For example, the free A-module A_A is compact because

$$\mathrm{Hom}(A, M) \xrightarrow{\sim} H^0 M$$

and of course it is also perfect.

Proposition 4.4. *An object of $\mathcal{D}A$ is compact if and only if it is perfect.*

The *perfect derived category* $\mathsf{per}(A)$ is the full subcategory of $\mathcal{D}A$ formed by the perfect objects. Clearly it is a *thick subcategory*, *i.e.* a triangulated subcategory stable under taking direct summands.

Proposition 4.5. *The functor $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ is an equivalence if and only if*

a) *X_B is perfect in $\mathcal{D}B$ and*
b) *X_B generates $\mathcal{D}B$ as a triangulated category with arbitrary coproducts and*
c) *the natural map $A \to \mathrm{Hom}_{\mathcal{D}B}(X_B, X_B)$ given by left multiplication is an isomorphism and $\mathrm{Hom}_{\mathcal{D}B}(X_B, \Sigma^p X_B) = 0$ for all $p \neq 0$.*

By definition, the bimodule complex X is a *two-sided tilting complex* if these conditions hold. We have the following important class of examples:

Theorem 4.6 (Happel [24])**.** *If T is an A-B-bimodule, then T is a two-sided tilting complex iff T_B is a tilting module and the left action yields an isomorphism $A \xrightarrow{\sim} \mathsf{End}_B(T)$.*

For a proof of the theorem in this form, *cf.* [35].

Theorem 4.7 (Rickard [45, 46])**.** *There is a triangle equivalence $\mathcal{D}A \to \mathcal{D}B$ if and only if there is a two-sided tilting complex X.*

Notice that the theorem does not claim that a given triangle equivalence $F : \mathcal{D}A \xrightarrow{\sim} \mathcal{D}B$ is isomorphic to a derived functor $? \overset{L}{\otimes}_A X$ for a two-sided tilting complex X. It is open whether this always holds.

Define $\mathsf{rep}(A, B)$ to be the full subcategory of $\mathcal{D}(A^{op} \otimes B)$ formed by the bimodule complexes X such that X_B is perfect. We think of the objects of $\mathsf{rep}(A, B)$ as 'representations up to homotopy' of A in $\mathsf{per}(B)$. Notice that a bimodule complex X belongs to $\mathsf{rep}(A, B)$ if and only if the functor $? \overset{L}{\otimes}_A X$ takes $\mathsf{per}(A)$ to $\mathsf{per}(B)$. For $X \in \mathsf{rep}(A, B)$, put

$$X^\vee = \mathrm{RHom}_B(X, B)$$

and notice that this is naturally an object of $\mathcal{D}(B^{op} \otimes A)$.

Lemma 4.8. *We have a canonical isomorphism*

$$? \overset{L}{\otimes}_B X^\vee \xrightarrow{\sim} \mathrm{RHom}_B(X, ?).$$

Thus, the functor $? \overset{L}{\otimes}_B X^\vee$ is right adjoint to $? \overset{L}{\otimes}_A X$. The adjunction morphisms are produced by the *action morphism*

$$\alpha : A \to \mathrm{RHom}_B(X, X) \xleftarrow{\sim} X \overset{L}{\otimes}_B X^\vee \text{ in } \mathcal{D}(A^e)$$

and the *evaluation morphism*

$$\varepsilon : X^\vee \overset{L}{\otimes}_A X = \mathrm{RHom}_B(X, B) \overset{L}{\otimes}_A X \to B \text{ in } \mathcal{D}(B^e).$$

Notice that $? \overset{L}{\otimes}_A X$ is fully faithful if and only if the action morphism $A \to X \overset{L}{\otimes}_B X^\vee$ is invertible.

5. Invariance theorems

Let A and B be k-algebras. Let $X \in \mathsf{rep}(A, B)$ and

$$X^\vee = \mathrm{RHom}_B(X, B) \in \mathcal{D}(B^{op} \otimes A).$$

Note that in general, $X^\vee$ is *not perfect over* A and so does not belong to $\mathsf{rep}(B, A)$. Recall the canonical action and evaluation morphisms

$$\alpha : A \to X \overset{L}{\otimes}_B X^\vee \text{ in } \mathcal{D}(A^e) \quad \text{and} \quad \varepsilon : X^\vee \overset{L}{\otimes}_A X \to B \text{ in } \mathcal{D}(B^e).$$

Theorem 5.1. *We have a canonical morphism*

$$HH_*(X) : HH_*(A) \to HH_*(B)$$

It is an isomorphism if $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ *is an equivalence.*

Sketch of proof. Recall that $HH_*(A) = H_*(A \overset{L}{\otimes}_{A^e} A)$. We have a canonical morphism $\psi(X)$ defined as the composition

$$\begin{array}{ccccc} A \overset{L}{\otimes}_{A^e} A & \xrightarrow{\mathbf{1}\otimes\alpha} & A \overset{L}{\otimes}_{A^e} (X \overset{L}{\otimes}_B X^\vee) & = & A \overset{L}{\otimes}_{A^e} (X \otimes_k X^\vee) \overset{L}{\otimes}_{B^e} B \, . \\ & & & & \downarrow \text{flip} \\ B \overset{L}{\otimes}_{B^e} B & \xleftarrow[\mathbf{1}\otimes\varepsilon]{} & B \overset{L}{\otimes}_{B^e} (X^\vee \otimes_A X) & = & B \overset{L}{\otimes}_{B^e} (X^\vee \otimes_k X) \overset{L}{\otimes}_{A^e} A \end{array}$$

Then $HH_*(X)$ is $H_*(\psi(X))$. One shows that $\psi(A) = \mathbf{1}$ and for $X \in \mathsf{rep}(A, B)$ and $Y \in \mathsf{rep}(B, C)$, we have $\psi(X \overset{L}{\otimes}_B Y) = \psi(Y) \circ \psi(X)$, *cf.* Theorem 2.1 in [3]. This implies the second claim. $\surd$

Now recall that a fully faithful functor $F : \mathcal{A} \to \mathcal{B}$ between k-categories yields a restriction morphism $F^* : Z(\mathcal{B}) \to Z(\mathcal{A})$. Since Hochschild cohomology may be viewed as a 'derived center', the following theorem is quite natural.

Theorem 5.2. *Suppose we have $X \in \mathsf{rep}(A, B)$ such that the functor $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ is fully faithful. Then we have a canonical 'restriction' morphism*

$$HH^*(X) : HH^*(B) \to HH^*(A).$$

It is an isomorphism if $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ is an equivalence.

Below, we will show using other methods that even if X_B is not perfect but $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ is fully faithful, we still have such a restriction morphism.

Sketch of proof. It is not hard to show, without any hypothesis on $X \in \mathsf{rep}(A, B)$, that we have an adjoint pair

$$X^\vee \overset{L}{\otimes}_A ? \overset{L}{\otimes}_A X : \mathcal{D}A^e \rightleftarrows \mathcal{D}B^e : X \overset{L}{\otimes}_B ? \overset{L}{\otimes}_B X^\vee .$$

Now we construct a map from

$$HH^p(B) = \mathrm{Hom}_{\mathcal{D}B^e}(B, \Sigma^p B)$$

to $HH^p(A)$ as follows:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}B^e}(B, \Sigma^p B) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}B^e}(X^\vee \overset{L}{\otimes}_A X, \Sigma^p B) \\ & & \big\downarrow{\scriptstyle adj} \\ \mathrm{Hom}_{\mathcal{D}A^e}(A, \Sigma^p A) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{D}A^e}(A, \Sigma^p X \overset{L}{\otimes}_B X^\vee). \end{array}$$

$\surd$

Each morphism $f : A \to \Sigma^p A$ of $\mathcal{D}A^e$ induces a morphism

$$f \overset{L}{\otimes}_{A^e} \mathbf{1}_A : A \overset{L}{\otimes}_{A^e} A \to \Sigma^p A \overset{L}{\otimes}_{A^e} A.$$

In this way, we obtain an action

$$\cap : HH^*(A) \otimes HH_*(A) \to HH_*(A)$$

called the cap product (not to be confused with the cup product $\cup$ on Hochschild cohomology).

Theorem 5.3 ([2])**.** *Suppose that $X \in \mathsf{rep}(A, B)$ is such that the functor $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ is an equivalence. Then the induced isomorphisms*

$$HH_*(X) : HH_*(A) \xrightarrow{\sim} HH_*(B) \quad \textit{and} \quad HH^*(X) : HH^*(A) \xrightarrow{\sim} HH^*(B)$$

are compatible with the cap product.

6. Differential graded categories

Recall from Lemma 2.2, that for an algebra A, the restriction along the inclusion $A \subset \mathsf{Mod}\, A$ is an isomorphism

$$Z(\mathsf{Mod}\, A) \xrightarrow{\sim} Z(A).$$

It is natural to ask what the derived version of this fact is. In the derived version, Hochschild cohomology should replace the center and the derived category should replace the module category. So we would like to know how to recover Hochschild cohomology from the derived category $\mathcal{D}A$. Unfortunately, this seems to be an ill-posed question nobody knows how to answer. However, it is easy to recover Hochschild cohomology from the differential graded (=dg) version of $\mathcal{D}A$, namely the dg category $\mathcal{D}_{dg}A$. Our first aim in this section is to define the dg category $\mathcal{D}_{dg}A$ in an intrinsic way, via a universal property.

6.1. **Dg categories and their derived categories.** Recall that the category of complexes $\mathcal{C}k$ is *monoidal*, *i.e.* it is endowed with the bifunctor $(L, M) \mapsto L \otimes M$ given by

$$(L \otimes M)^n = \bigoplus_{p+q=n} L^p \otimes M^q, \quad d(l \otimes m) = (dl) \otimes m + (-1)^{|l|} l \otimes dm$$

enjoying a number of desirable properties. A *dg category* is a category $\mathcal{A}$ enriched in the monoidal category of complexes. Thus, the morphism spaces $\mathcal{A}(X, Y)$ are complexes and the compositions

$$\mathcal{A}(Y, Z) \otimes \mathcal{A}(X, Y) \to \mathcal{A}(X, Z)$$

and units $k \to \mathcal{A}(X, X)$ are morphisms of complexes.

For example, if B is an algebra, the dg category $\mathcal{C}_{dg}B$ has the same objects as $\mathcal{C}B$ and its morphism complexes are defined by

$$(\mathcal{C}_{dg}B)(L, M) = \mathrm{Hom}_B(L, M)\,,$$

cf. section 4. As in the case of k-categories, we identify dg categories with one object with dg algebras.

If $\mathcal{A}$ is a dg category, the *category* $H^0\mathcal{A}$ has the same objects as $\mathcal{A}$ and the morphism spaces $H^0(\mathcal{A}(X, Y))$ with the natural compositions. For example, this yields another viewpoint on the homotopy category $\mathcal{H}B$ via the equality of categories

$$H^0(\mathcal{C}_{dg}B) = \mathcal{H}B.$$

A *dg functor* $F : \mathcal{A} \to \mathcal{B}$ is a functor such that

$$F : \mathcal{A}(X, Y) \to \mathcal{B}(FX, FY)$$

is a morphism of complexes for all $X, Y \in \mathcal{A}$. It is a *quasi-equivalence* if $F : \mathcal{A}(X, Y) \to \mathcal{B}(FX, FY)$ is a quasi-isomorphism for all $X, Y \in \mathcal{A}$ and the induced functor $H^0F : H^0\mathcal{A} \to H^0\mathcal{B}$ is an equivalence. The *category* Hqe is the localization of the category dgcat of small dg categories at the class of all quasi-equivalences. For example, if $f : A \to B$ is a quasi-isomorphism between dg algebras, it may be viewed as a quasi-equivalence between dg categories with one object.

Theorem 6.2 (Tabuada [47]). *The category* dgcat *carries a (cofibrantly generated) Quillen model structure whose weak equivalences are the quasi-equivalences. In particular, the morphisms* $\mathcal{A} \to \mathcal{B}$ *in* Hqe *form a* set *for all small dg categories* $\mathcal{A}$, $\mathcal{B}$.

Thanks to the theorem, we can speak about representable functors on the category Hqe without having to 'enlarge the universe'.

Theorem 6.3 (1999 [34]). *Let $\mathcal{A}$ be a dg category and $\mathcal{N} \subseteq \mathcal{A}$ a full dg subcategory. Then there is a morphism $Q : \mathcal{A} \to \mathcal{A}/\mathcal{N}$ of* Hqe *which kills $\mathcal{N}$ (*i.e. *we have $\mathbf{1}_{QN} = 0$ in $H^0(\mathcal{A}/\mathcal{N})$ for all $N \in \mathcal{N}$) and which is universal in* Hqe *among the morphisms killing $\mathcal{N}$.*

We define the *dg quotient of $\mathcal{A}$ by $\mathcal{N}$* to be the dg category $\mathcal{A}/\mathcal{N}$ of the theorem. It is thus unique up to unique isomorphism in Hqe.

Theorem 6.4 (Drinfeld 2004 [13]). *The dg category $\mathcal{A}/\mathcal{N}$ is obtained from $\mathcal{A}$ by adjoining a contracting homotopy h_N for each object N on $\mathcal{N}$ (*i.e. *h_N is of degree -1 and $d(h_N) = \mathbf{1}_N$).*

For an algebra B, we define the *dg derived category of B* to be the dg quotient

$$\mathcal{D}_{dg}B = \mathcal{C}_{dg}B/\mathcal{A}c_{dg}B\ ,$$

where $\mathcal{A}c_{dg}B$ is the full dg category of $\mathcal{C}_{dg}B$ whose objects are the acyclic complexes.

Theorem 6.5 ([34, 13]). *We have a canonical equivalence $H^0(\mathcal{D}_{dg}B) \xrightarrow{\sim} \mathcal{D}\mathcal{B}$.*

Our next aim is to define the derived category $\mathcal{D}\mathcal{A}$ (as well as its dg version $\mathcal{D}_{dg}\mathcal{A}$) of a dg category $\mathcal{A}$. Define the *opposite dg category $\mathcal{A}^{op}$* to be the dg category with the same objects, with the morphism complexes $\mathcal{A}^{op}(X, Y) = \mathcal{A}(Y, X)$ and the compositions given by

$$f \circ_{\mathcal{A}^{op}} g = (-1)^{|f||g|} g \circ f$$

for all homogeneous $f \in \mathcal{A}^{op}(Y, Z)$ and $g \in \mathcal{A}^{op}(X, Y)$. For two dg functors $F, G : \mathcal{A} \to \mathcal{B}$, define the complex $\mathrm{Hom}(F, G)$ to be the subcomplex of

$$\prod_{X \in \mathcal{A}} \mathcal{B}(FX, GX)$$

formed by the families (φX) such that

$$(Gf) \circ (\varphi X) = (-1)^{|\varphi||f|} (\varphi Y) \circ (Ff)$$

for all $X, Y \in \mathcal{A}$ and $f : X \to Y$. In this way, the category $\mathrm{Fun}_{dg}(\mathcal{A}, \mathcal{B})$ of dg functors from $\mathcal{A}$ to $\mathcal{B}$ becomes a dg category. We define the dg category of dg right $\mathcal{A}$-modules to be

$$\mathcal{C}_{dg}\mathcal{A} = \mathrm{Fun}_{dg}(\mathcal{A}^{op}, \mathcal{C}_{dg}k)$$

and the category of right $\mathcal{A}$-modules to be $Z^0\mathcal{C}_{dg}\mathcal{A}$ (same objects as $\mathcal{C}_{dg}\mathcal{A}$ and morphism spaces $Z^0(\mathcal{C}_{dg}\mathcal{A})(L, M)$). We define the *homotopy category of dg $\mathcal{A}$-modules* as

$$\mathcal{H}\mathcal{A} = H^0\mathcal{C}_{dg}\mathcal{A}.$$

For example, for each object $X \in \mathcal{A}$, we have the *representable dg module*

$$X^\wedge = \mathcal{A}(?, X) : \mathcal{A}^{op} \to \mathcal{C}_{dg}k.$$

Whence the *dg Yoneda functor*

$$\mathcal{A} \to \mathcal{C}_{dg}\mathcal{A}\ ,\ X \mapsto X^\wedge.$$

As an exercise, the reader may want to prove the dg Yoneda lemma:

Lemma 6.6. *For X in $\mathcal{A}$ and M in $\mathcal{C}\mathcal{A}$, we have a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(X^\wedge, M) \xrightarrow{\sim} MX\ ,\ f \mapsto f(\mathbf{1}_X).$$

Notice that when B is an ordinary algebra and $\mathcal{A}$ the dg category whose endomorphism algebra is B (concentrated in degree 0), then $\mathcal{C}_{dg}\mathcal{A} = \mathcal{C}_{dg}B$

Let $\mathcal{A}$ be a small dg category. A morphism $s : L \to M$ of dg $\mathcal{A}$-modules is a *quasi-isomorphism* if

$$sX : LX \to MX$$

is a quasi-isomorphism for each $X \in \mathcal{A}$. We define the *derived category* $\mathcal{DA}$ to be the localization of $\mathcal{CA}$ (respectively $\mathcal{HA}$) at the class of quasi-isomorphisms and the *dg derived category* $\mathcal{D}_{dg}\mathcal{A}$ to be the dg quotient

$$\mathcal{C}_{dg}(\mathcal{A})/\mathcal{A}c_{dg}(\mathcal{A}).$$

As in Theorem 4.1, the quotient functor $\mathcal{HA} \to \mathcal{DA}$ admits a left adjoint $\mathbf{p}$ and a right adjoint $\mathbf{i}$, *cf.* [32], and the construction of the derived functors generalizes.

Let us give an example where we have a beautiful description of the derived category of a non trivial dg category: Let A be a right noetherian algebra (concentrated in degree 0) and $\mathsf{mod}\, A$ the abelian category of finitely generated (right) A-modules. Let $\mathcal{C}^b_{dg}(\mathsf{mod}\, A) \subset \mathcal{C}_{dg}A$ be the full dg subcategory of bounded complexes over $\mathsf{mod}\, A$ and

$$\mathcal{D}^b_{dg}(\mathsf{mod}\, A) = \mathcal{C}^b_{dg}(\mathsf{mod}\, A)/\mathcal{A}c^b_{dg}(\mathsf{mod}\, A).$$

Let $\mathsf{Inj}\, A$ denote the category of all injective modules and $\mathcal{H}\,\mathsf{Inj}\, A$ the homotopy category of (unbounded) complexes of injective modules.

Theorem 6.7 (Krause [37])**.** *We have a canonical triangle equivalence*

$$\mathcal{H}\,\mathsf{Inj}\, A \xrightarrow{\sim} \mathcal{D}(\mathcal{D}^b_{dg}(\mathsf{mod}\, A)).$$

6.8. **Hochschild (co)homology of dg categories.** Let $\mathcal{A}$ be a small dg category. We have the following generalization of Proposition 4.4. Recall that a *thick subcategory* of a triangulated category is a full triangulated subcategory stable under taking direct factors.

Proposition 6.9. *An object $P \in \mathcal{DA}$ is compact if and only if it is* perfect, i.e. *contained in the thick subcategory generated by the representable modules $X^\wedge$, $X \in \mathcal{A}$.*

Let $\mathcal{B}$ be another dg category. The *tensor product* $\mathcal{A}\otimes\mathcal{B}$ is the dg category whose objects are the pairs (X, Y), $X \in \mathcal{A}$, $Y \in \mathcal{B}$, and whose morphisms are given by

$$(\mathcal{A}\otimes\mathcal{B})((X, Y), (X', Y')) = \mathcal{A}(X, X') \otimes \mathcal{B}(Y, Y').$$

We define $\mathsf{rep}(\mathcal{A}, \mathcal{B})$ as the full subcategory of $\mathcal{D}(\mathcal{B}\otimes\mathcal{A}^{op})$ formed by the dg bimodules X whose restriction to $\mathcal{B}$ is perfect.

Let $\mathcal{A}^e = \mathcal{A}\otimes\mathcal{A}^{op}$. The *identity bimodule* $I_\mathcal{A}$ sends (X, Y) to $\mathcal{A}(X, Y)$, $X, Y \in \mathcal{A}$. We put

$$HH^*(\mathcal{A}) = H^*\mathrm{RHom}_{\mathcal{A}^e}(I_\mathcal{A}, I_\mathcal{A}) \quad \text{and} \quad HH_*(\mathcal{A}) = H_*(I_\mathcal{A} \otimes_{\mathcal{A}^e} I_\mathcal{A}).$$

These may also be computed as the (co)homologies of the complexes $C_*\mathcal{A}$ and $C^*\mathcal{A}$ constructed as follows: The complex $C_*\mathcal{A}$ is the sum total complex of the bicomplex

$$\ldots \longrightarrow \bigoplus \mathcal{A}(X_0, X_1) \otimes \mathcal{A}(X_1, X_0) \longrightarrow \bigoplus \mathcal{A}(X_0, X_0)$$

whose pth column ($p \geq 0$) is the sum

$$\bigoplus \mathcal{A}(X_p, X_0) \otimes \mathcal{A}(X_{p-1}, X_p) \otimes \cdots \otimes \mathcal{A}(X_0, X_1)$$

taken over all sequences of objects $X_0, X_1, \ldots, X_p$ of $\mathcal{A}$ and whose horizontal differential is given by formula (3.0.1) adjusted following the Koszul sign rule. The complex $C^*\mathcal{A}$ is the product total complex of the bicomplex

$$\prod \mathcal{A}(X_0, X_0) \longrightarrow \prod \mathrm{Hom}_k(\mathcal{A}(X_0, X_1), \mathcal{A}(X_0, X_1)) \longrightarrow \ldots$$

whose pth column ($p \geq 0$) is

$$\prod \mathrm{Hom}_k(\mathcal{A}(X_{p-1}, X_p) \otimes \ldots \otimes \mathcal{A}(X_0, X_1), \mathcal{A}(X_0, X_p))$$

where the product is taken over all sequences of objects $X_0, \ldots, X_p$ of $\mathcal{A}$ and whose horizontal differential is given by formula (3.0.2) adjusted following the Koszul sign rule.

Theorem 6.10 ([33, 30]). *Let $\mathcal{A}$ and $\mathcal{B}$ be dg categories and $X \in \mathcal{D}(\mathcal{B} \otimes \mathcal{A}^{op})$.*

a) *If $X \in \mathsf{rep}(\mathcal{A}, \mathcal{B})$ (*i.e.* X is right perfect), then there is a canonical induced morphism*

$$HH_*(X) : HH_*(\mathcal{A}) \to HH_*(\mathcal{B}).$$

It only depends on the class of X in $K_0(\mathsf{rep}(A, B))$ and is an isomorphism if the functor $? \overset{L}{\otimes}_A X : \mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{B}$ is an equivalence.

b) *Suppose that the functor $? \overset{L}{\otimes}_A X : \mathcal{D}\mathcal{A} \to \mathcal{D}\mathcal{B}$ is fully faithful (but X is not necessarily right perfect). Then there is a canonical restriction morphism*

$$\mathrm{res}_X : HH^*(\mathcal{B}) \to HH^*(\mathcal{A}).$$

It is an isomorphism if the functor $X \overset{L}{\otimes}_B ? : \mathcal{D}(\mathcal{B}^{op}) \to \mathcal{D}(\mathcal{A}^{op})$ is also fully faithful.

We will sketch a proof of b) after Theorem 7.5.

Corollary 6.11 (Lowen–Van den Bergh [40]). *Let $\mathcal{A}$ be a dg category. The restriction along the Yoneda functor $\mathcal{A} \to \mathcal{D}_{dg}\mathcal{A}$ induces an isomorphism*

$$HH^*(\mathcal{D}_{dg}\mathcal{A}) \xrightarrow{\sim} HH^*(\mathcal{A}).$$

A similar result was obtained by Toën in [48]. It should be viewed as the derived version of the isomorphism

$$Z(\mathsf{Mod}\, A) \xrightarrow{\sim} Z(A)$$

of Lemma 2.2 for an algebra A.

7. Higher structure

7.1. **Higher structure on Hochschild homology.** Let A be an algebra (for simplicity). Recall the Connes–Quillen cyclic bicomplex

$$\begin{array}{ccccccc}
\cdots & & \cdots & & \cdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
A^{\otimes 2} & \xleftarrow[1-t]{} & A^{\otimes 2} & \xleftarrow[N]{} & A^{\otimes 2} & \xleftarrow[1-t]{} & \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \\
A & \xleftarrow[1-t]{} & A & \xleftarrow[N]{} & A & \xleftarrow[1-t]{} & \cdots
\end{array}$$

Here we put

$$t(a_1 \otimes \ldots \otimes a_n) = (-1)^{n-1} a_2 \otimes \ldots \otimes a_n \otimes a_1 \quad \text{and} \quad N = 1 + t + \cdots + t^{n-1}.$$

The bicocomplex is 2-periodic in the horizontal direction. Its even columns are copies of the Hochschild chain complex and its odd columns copies of the bimodule bar resolution of A. The homology of the sum total complex is the cyclic homology of A. Let MA denote the cone over the subcomplex formed by the first two columns. Let us write d for its differential. Let $d' : MA \to MA$ be the homogeneous map of (cohomological) degree

-1 given by the projection onto the first column followed by the map N followed by the inclusion of the second column. We have

$$d^2 = 0\,,\; d'^2 = 0\,,\; dd' + d'd = 0.$$

This means that MA is a *mixed complex*, *i.e.* a dg module over the dg algebra $\Lambda = k[\varepsilon]/(\varepsilon^2)$, where ε is of degree -1 and $d = 0$. Of course, ε acts via d' in MA. We write $\mathcal{D}\mathrm{Mix}$ for the *mixed derived category*, *i.e.* the category $\mathcal{D}\Lambda$.

Notice that MA is functorial with respect to algebra morphisms which do not necessarily preserve the unit like the morphism

$$A \to M_2(A)\,,\; a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}.$$

The inclusion of the first column clearly yields an isomorphism $A \overset{L}{\otimes}_{A^e} A \xrightarrow{\sim} MA$ in $\mathcal{D}k$. Thus we have $HH_*(A) = H_*(MA)$. Moreover, the mixed complex MA contains the information on cyclic, negative cyclic and periodic cyclic homology as shown by the canonical isomorphisms (*cf.* [29])

$$HC_*(A) = H_*(MA\overset{L}{\otimes}_\Lambda k),\; HN_*(A) = H_*\mathrm{RHom}_\Lambda(k, MA),\; HP_*(A) = \mathrm{holim}_n(MA\overset{L}{\otimes}_\Lambda \Sigma^{-2n}k).$$

Here, the transition morphisms of the inverse system are induced by the morphism $k \to \Sigma^2 k$ of the canonical triangle

$$\Sigma k \longrightarrow \Lambda \longrightarrow k \longrightarrow \Sigma^2 k.$$

This triangle induces the ISB-sequence

$$HC_{*-1} \xrightarrow{B} HH_* \xrightarrow{I} HC_* \xrightarrow{S} HC_{*-2} \xrightarrow{B} HH_{*-1}\,.$$

Now let B be another algebra and $X \in \mathsf{rep}(A, B)$. We may and will assume that the restriction X_B is right bounded with projective components. Then we have $X^\wedge = \mathrm{RHom}_B(X, B) = \mathrm{Hom}_B(X, B)$. We have natural morphisms of differential graded algebras (not preserving the unit)

$$A \xrightarrow{\alpha} \begin{bmatrix} A & X \\ X^\wedge & B \end{bmatrix} \xleftarrow{\beta} B$$

taking a to $\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and b to $\left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right]$.

Lemma 7.2 ([33]). *The morphism $M\beta$ is invertible in $DMix$.*

We define $MX = (M\beta)^{-1} \circ M\alpha$ in $\mathcal{D}\mathrm{Mix}$.

Theorem 7.3 ([33]). a) *We have $M({}_AA_A) = \mathbf{1}_{MA}$ and for $X \in \mathsf{rep}(A, B)$ and $Y \in \mathsf{rep}(B, C)$, we have $M(X \overset{L}{\otimes}_B Y) = MY \circ MX$.*

b) *The morphism MX only depends on the class of X in $K_0(\mathsf{rep}(A, B))$.*

It is not hard to generalize the definitions and results of this subsection from algebras to dg categories.

7.4. **Higher structure on Hochschild cohomology.** Let A be an algebra (for simplicity) and C^*A its Hochschild cochain complex, *cf.* section 3. If c and u, v, ..., w are Hochschild cochains, one defines [28] the *brace operation* $c\{u, v, \ldots, w\}$ by substituting the cochains u, v, ..., w for some of the arguments of c, inserting suitable signs and summing over all possibilities of doing this, cf. Figure 1.

The complex C^*A together with the cup product $\cup$ and the brace operations is an example of a B_∞-algebra in the sense of Getzler–Jones [18], *i.e.* a graded vector space V

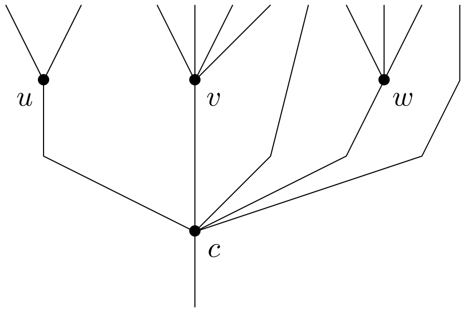


FIGURE 1. The brace operations

such that the tensor coalgebra $T^c(\Sigma V)$ (with the deconcatenation coproduct) is endowed with a dg bialgebra structure whose comultiplication is deconcatenation. Here the letter 'B' stands for 'Baues', in honour of Hans–Joachim Baues, who showed [5] that the singular cochain complex with integer coefficients of any topological space carries a natural B_∞-algebra structure. Notice that B_∞–structures are closely related to monoidal structures (*cf.* [39] and the references given there). In the case of the Hochschild cochain complex, the monoidal category is the derived category $\mathcal{D}(A^e)$ with the derived tensor product $\overset{L}{\otimes}_A$ over A; in the case of the singular cochain complex on a topological space X, it is the derived category of sheaves of abelian groups on X with the derived tensor product. The B_∞-algebra structure on C^*A contains in particular the information on the Gerstenhaber bracket, which may be recovered via

$$[c, u] = c\{u\} \mp u\{c\}.$$

By the *homotopy category of B_∞-algebras* we mean the localization of the category of B_∞-algebras with respect to all morphisms of B_∞-algebras inducing isomorphisms in homology.

Let B be another algebra and $X \in \mathcal{D}(B \otimes A^{op})$ a complex of A-B-bimodules.

Theorem 7.5 ([30])**.** *Suppose that the functor $? \overset{L}{\otimes}_A X : \mathcal{D}A \to \mathcal{D}B$ is fully faithful (but X_B is not necessarily perfect). Then there is a canonical restriction morphism*

$$\mathrm{res}_X : C^*B \to C^*A$$

in the homotopy category of B_∞-algebras. It is invertible if the functor $X \overset{L}{\otimes}_B ? : \mathcal{D}(B^{op}) \to \mathcal{D}(A^{op})$ is also fully faithful (for example if $? \overset{L}{\otimes}_A$ is an equivalence).

We sketch the proof: Let G be the 'glued' dg category with two objects 1 and 2 such that $G(1,1) = B$, $G(2,2) = A$, $G(1,2) = X$ and $G(2,1) = 0$. We have obvious forgetful (or 'restriction') maps

$$C^*B \xleftarrow{\mathrm{res}_B} C^*G \xrightarrow{\mathrm{res}_A} C^*A.$$

which clearly respect the B_∞-structure. It is a classical fact, *cf.* [25, 10, 42, 20, 11, 22, 6, 19], that we have a homotopy bicartesian square

$$\begin{array}{ccc} C^* & \xrightarrow{\mathrm{res}_A} & C^*A \\ {\scriptstyle \mathrm{res}_B}\downarrow & & \downarrow{\scriptstyle \alpha} \\ C^*B & \longrightarrow & \mathrm{RHom}_{B\otimes A^{op}}(X, X). \end{array}$$

We claim that α and hence res_B is invertible in $\mathcal{D}k$. Indeed, we have a commutative square in $\mathcal{D}k$

$$\begin{array}{ccc} C^*A & \xrightarrow{\sim} & \mathrm{RHom}_{A^e}(A,A) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ \mathrm{RHom}_{B\otimes A^{op}}(X,X) & \xrightarrow{\sim} & \mathrm{RHom}_{A^e}(A,\mathrm{RHom}_B(X,X)), \end{array}$$

where β is induced by the action morphism

$$A \to \mathrm{RHom}_B(X,X).$$

This is invertible by our assumption that the functor $? \overset{L}{\otimes}_A X$ is fully faithful. We put $\mathrm{res}_X = \mathrm{res}_A \circ \mathrm{res}_B^{-1}$.

Corollary 7.6. *For an algebra A, the isomorphism*

$$HH^*(\mathcal{D}_{dg}A) \xrightarrow{\sim} HH^*(A)$$

of Corollary 6.11 lifts to an isomorphism in the homotopy category of B_∞-algebras.

8. Tate–Hochschild cohomology

Let A be a right noetherian algebra and $\mathsf{mod}\,A$ the abelian category of finitely generated (right) A-modules. Let $\mathcal{D}^b(\mathsf{mod}\,A)$ be its bounded derived category. The perfect derived category $\mathsf{per}(A)$ is a thick subcategory of $\mathcal{D}^b(\mathsf{mod}\,A)$. They coincide if A is of finite global dimension. We define the *singularity category* $\mathsf{sg}(A)$ *to be the quotient* $\mathcal{D}^b(\mathsf{mod}\,A)/\,\mathsf{per}(A)$. This category was first considered by Buchweitz [7] in 1986 and rediscovered by Orlov [44] in 2003 in a geometric setting. It measures the non regularity of the algebra A.

As an example, let $S = k[x_1, \ldots, x_n]$ and let $A = S/(f)$ for a non zero $f \in S$. It follows from the work of Eisenbud [16] that the singularity category $\mathsf{sg}(A)$ is triangle equivalent to the homotopy category of matrix factorizations of f. By definition, this category equals $H^0(\mathsf{mf}_{dg}(f))$, where $\mathsf{mf}_{dg}(f)$ is the *dg category of matrix factorizations of* f: its objects are the 2-periodic diagrams (not complexes!)

$$\ldots \xrightarrow{d_0} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \ldots$$

where the P_i are finitely generated projective S-modules and d^2 is the multiplication with f. For two such objects P and Q, the *morphism complex* $\mathrm{Hom}(P,Q)$ has as its nth component the space of homogeneous S-linear maps $g : P \to Q$ of degree n. The differential is given by $d(g) = d \circ g - (-1)^n g \circ d$. We leave it as an exercise for the reader to check that $d^2(g) = 0$. It is an important point that the complexes $\mathrm{Hom}(P,Q)$ are also 2-periodic, so that we may also view $\mathsf{mf}_{dg}(f)$ as a differential $\mathbb{Z}/2\mathbb{Z}$–graded category!

Now suppose that $A^e = A \otimes A^{op}$ is also (right) noetherian. We define *Tate–Hochschild cohomology of* A to be the Yoneda algebra $\mathsf{Ext}_{\mathsf{sg}(A^e)}(A,A)$ of the identity bimodule in the singularity category of the enveloping algebra. By definition, it is an algebra and it is not hard to check directly that it is (graded) commutative. However, the singularity category $\mathsf{sg}(A^e)$ is *not monoidal* in any natural way. We may nevertheless ask whether Tate–Hochschild cohomology carries the same rich structure as classical Hochschild cohomology. This problem was open for some time and finally solved in the thesis of Zhengfang Wang:

Theorem 8.1 (Zhengfang Wang)**.**
a) *$HH^*_{sg}(A)$ carries a natural (but intricate!) Gerstenhaber bracket* [51].
b) *There is a natural B_∞-algebra $C^*_{sg}A$ computing $HH^*_{sg}(A)$ with its Gerstenhaber bracket* [50].

Thus, we see that there is a complete structural analogy between Tate–Hochschild cohomology and classical Hochschild cohomology. It is therefore natural to ask whether Tate–Hochschild cohomology is not an instance of classical Hochschild cohomology, i.e. whether the Tate–Hochschild cohomology of A is classical Hochschild cohomology of some more complicated object associated with A in analogy with Corollary 6.11. Recall that a dg category $\mathcal{A}$ is *smooth* if the identity bimodule $I_{\mathcal{A}} : (X, Y) \mapsto \mathcal{A}(X, Y)$ is perfect in the derived category $\mathcal{D}(\mathcal{A}^e)$ of bimodules.

Theorem 8.2 ([31])**.** *There is a canonical morphism of graded algebras*

$$HH^*_{sg}(A) \to HH^*(\mathsf{sg}_{dg}(A)).$$

It is an isomorphism if the dg category $\mathcal{D}^b_{dg}(\mathsf{mod}\, A)$ is smooth.

According to Theorem A of Elagin–Lunts–Schnürer's [15], the dg category $\mathcal{D}^b_{dg}(\mathsf{mod}\, A)$ is smooth if A is a finite-dimensional algebra over any field k such that $A/\operatorname{\mathsf{rad}}(A)$ is separable over k (which is automatic if k is perfect). By Theorem B of [loc. cit.], it also holds if the algebra A is right noetherian and finitely generated over its center and the center is a finitely generated algebra over k.

Conjecture 8.3. *The morphism of the theorem lifts to a morphism in the homotopy category of B_∞-algebras.*

Note that this morphism will be an isomorphism if the bounded dg derived category $\mathcal{D}^b_{dg}(\mathsf{mod}\, A)$ is smooth. In particular, this should hold for each finite-dimension algebra defined by a quiver with an admissible ideal of relations. The following theorem confirms the conjecture for radical square 0 algebras.

Theorem 8.4 (Chen–Li–Wang [9])**.** *The conjecture holds if $A = kQ/(Q_1)^2$, where Q is a finite quiver without sinks or sources and $(Q_1)^2$ the square of the ideal of the path algebra kQ generated by the arrows.*

To show why the conjecture is probably not easy to prove, let us sketch the construction of the isomorphism in Theorem 8.2. Let $\mathcal{M} = \mathcal{D}^b_{dg}(\mathsf{mod}\, A)$ and $\mathcal{S} = \mathsf{sg}_{dg}(A)$. We have natural dg functors

$$A \xrightarrow{i} \mathcal{M} \xrightarrow{p} \mathcal{S}$$

whose composition vanishes in the homotopy category of dg categories. We construct the following square (commutative up to isomorphism)

$$\begin{array}{ccccc} \mathcal{D}^b(A \otimes A^{op}) & \xrightarrow{(1\otimes i)^*} & \mathcal{D}(A \otimes A^{op}) & \xrightarrow{(i\otimes \mathbf{1})^!} & \mathcal{D}(\mathcal{M} \otimes \mathcal{M}^{op}) \\ \downarrow & & & & \downarrow {\scriptstyle (p\otimes p)^*} \\ \mathsf{sg}(A \otimes A^{op}) & - & - & - \to & \mathcal{D}(\mathcal{S} \otimes \mathcal{S}^{op}) \end{array}$$

Here, for a dg functor $f : \mathcal{A}_1 \to \mathcal{A}_2$, we denote by f^* the left adjoint and by $f^!$ the right adjoint of the restriction functor $f_* : \mathcal{D}\mathcal{A}_2 \to \mathcal{D}\mathcal{A}_1$. One checks that the dashed triangle functor exists, takes the identity bimodule A to the identity bimodule $\mathcal{S}(?, -)$ and induces an isomorphism between the Yoneda algebras of these objects. Since the functor is induced by the composition of a right derived with a left derived functor, it is hard to compute it explicitly and that is why the conjecture is not obvious.

9. Application: two reconstruction theorems

We apply the results of the preceding section to the reconstruction of singularities.

9.1. **Isolated hypersurface singularities.**

Theorem 9.2 (Hua-K [27])**.** *Let $S = \mathbb{C}[[x_1, \ldots, x_n]]$ and suppose that $R = S/(f)$ is an isolated singularity. Then R is determined up to isomorphism by its dimension and the dg singularity category $\mathsf{sg}_{dg}(R)$.*

Notice that because of Knörrer periodicity, the dg singularity category alone does not determine R.

Sketch of proof. We consider the center

$$Z(\mathsf{sg}_{dg}(R)) = HH^0(\mathsf{sg}_{dg}(R)).$$

By Theorem A of [15], the bounded dg derived category $\mathcal{D}^b(\mathsf{mod}\, R)$ is smooth. By Theorem 8.2, the algebra $HH^0(\mathsf{sg}_{dg}(R))$ is isomorphic to $HH^0_{sg}(R)$. Now since R is a hypersurface, the dg singularity category may be described by matrix factorizations and is therefore 2-periodic. Hence its Hochschild cohomology is 2-periodic and, again by Theoreom 8.2, so is $HH^*_{sg}(R)$. Thus, we have an isomorphism

$$HH^0_{sg}(R) \xrightarrow{\sim} HH^{2r}_{sg}(R).$$

Now by a theorem of Buchweitz [7], for Gorenstein algebras, in sufficiently high degrees, Tate–Hochschild cohomology agrees with classical Hochschild cohomology and we get an isomorphism

$$HH^{2r}_{sg}(R) \xrightarrow{\sim} HH^{2r}(R).$$

Thus, we are reduced to the computation of Hochschild cohomology of a hypersurface. Thanks to the results of [23], we find that $HH^{2r}(R)$ and $HH^0(R)$ are isomorphic to the Tyurina algebra

$$S/(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}).$$

Now the Tyurina algebra together with the dimension determine R by the Mather–Yau theorem [41], more precisely its formal series version proved in [21]. Notice that in this sketch, we have neglected the technical problems arising from the fact that R is a topological algebra. $\surd$

Notice that in the above computation, we have considered the dg singularity category $\mathsf{sg}_{dg}(R)$ as a differential $\mathbb{Z}$-graded category. If one considers it as a differential $\mathbb{Z}/2$-graded category, one obtains a different result for the center, namely the Milnor algebra

$$Z_{\mathbb{Z}/2}(\mathsf{sg}_{dg}(R)) = S/(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$$

as shown by Dyckerhoff [14].

9.3. **Compound Du Val singularities.** Let R be a complete local isolated compound Du Val singularity (thus, it is 3-dimensional, normal and a generic hyperplane section through the origin is a Kleinian surface singularity). Let

$$f : Y \to X = \mathrm{Spec}(R)$$

be a small crepant resolution (thus, it is birational, an isomorphism in codimension 1, an isomorphism outside the exceptional fibre and $f^*(\omega_X) \cong \omega_Y$). Then the reduced exceptional fibre $\mathcal{F}$ of f is a tree of rational curves $\mathbb{P}^1$. The morphism f contracts this tree to a point. Associated to this situation is the *contraction algebra* Λ introduced by Donovan–Wemyss [12]. It is a finite-dimensional algebra which represents the deformations with non commutative base of the exceptional fibre of f. It is known that numerous invariants of the singularity can be computed from the algebra Λ. This has lead Donovan–Wemyss to conjecture [4] that the derived equivalence class of Λ determines R up to isomorphism. We show a weakened version: Thanks to work of Van den Bergh and de Thanhoffer de Voelcsey [49], it is known that Λ is the Jacobian algebra of a quiver Q with potential W. By definition, the potential is an element of $HH_0(\mathbb{C}Q)$. Let $\overline{W}$ be its image in $HH_0(\Lambda)$.

Theorem 9.4 (Hua–K [27])**.** *The derived equivalence class of the pair $(\Lambda, \overline{W})$ determines R.*

The proof uses, among other things, Theorem 9.2 and silting theory.

Let us point out the link to cluster theory: It turns out that the singularity category $\mathsf{sg}(R)$ is triangle equivalent to a generalized cluster category in the sense of Amiot [1], namely the generalized cluster category $\mathcal{C}_{Q,W}$ associated with the quiver Q with potential W. This category has therefore the same main properties as the categories appearing in the (additive) categorification of Fomin–Zelevinsky cluster algebras, *cf.* [36]. However, the quivers that appear are quite different: Whereas the quivers in Donovan–Wemyss' theory have many loops and 2-cycles, the quivers appearing in cluster theory never have loops or 2-cycles.

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Université de Paris, UFR de Mathématiques, Institut de Mathématiques de Jussieu–PRG, UMR 7586 du CNRS, Case 7012, Bâtiment Sophie Germain, 75205 Paris Cedex 13, France
E-mail address: bernhard.keller@imj-prg.fr
URL: https://webusers.imj-prg.fr/~bernhard.keller/