Contemporary Mathematics

A remark on Hochschild cohomology and Koszul duality

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Dedicated to the José Antonio de la Peña on the occasion of his sixtieth birthday

Abstract. Applying recent results by Lowen–Van den Bergh we show that Hochschild cohomology is preserved under Koszul–Moore duality as a Gerstenhaber algebra. More precisely, the corresponding Hochschild complexes are linked by a quasi-isomorphism of B_∞-algebras.

1. Introduction

Consider the following statement:

Hochschild cohomology is preserved under Koszul duality.

This is clearly wrong. Indeed, if V is a non zero finite-dimensional vector space over a field k, then the center (=zeroth Hochschild cohomology) of the symmetric algebra SV is SV but the center of the Koszul dual exterior algebra $\Lambda(V^*)$ is finite-dimensional (for a study of the Hochschild cohomology of $\Lambda(V^*)$, cf. [11]). To try and save the statement, let us recall that $\Lambda(V^*)$ is in fact the Yoneda algebra

$$\mathsf{Ext}^*_{SV}(k,k)$$

and that, with the zero differential, it is even quasi-isomorphic to the derived endomorphism algebra $\mathsf{RHom}_{SV}(k,k)$. Thus, it is natural to endow $\Lambda(V^*)$ with the grading so that V^* sits in degree 1 and with the zero differential. With this new interpretation of $\Lambda(V^*)$ as a dg (=differential graded) algebra, we find that its zeroth Hochschild cohomology is the *completion* $\widehat{SV}$ of the symmetric algebra at the augmentation ideal. It turns out that in order to get exactly SV, it suffices to replace the dg algebra $\Lambda(V^*)$ with the k-dual dg *coalgebra* ΛV. In fact, we then get an algebra isomorphism

$$HH^*(SV,SV) \xrightarrow{\sim} HH^*(\Lambda V,\Lambda V).$$

Our main results are that this isomorphism generalizes from SV to any augmented dg k-algebra (when we replace ΛV with the Koszul–Moore dual dg coalgebra) and that it lifts to an isomorphism between the corresponding Hochschild cochain complexes in the homotopy category of B_∞-algebras. In particular, the isomorphism in Hochschild cohomology is an isomorphism of Gerstenhaber algebras. The algebra isomorphism for augmented dg algebras follows from the setup of Koszul–Moore duality, which we recall in section 2. Our first proof of the lift to the B_∞-level is now superseded by recent work of Lowen–Van den Bergh [8], which we use in the very short proof of Theorem 2.6.

Previous results relating the Hochschild cohomologies of Koszul(-Moore) dual algebras can be found in [3], [5] [4], [2] and [1] cf. remark 2.7.

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2. Reminder on Koszul–Moore duality

We follow Lefèvre–Hasegawa [7] and Positselski [9], *cf.* also [6] and Appendix A to [10]. Let k be a field and A a dg k-algebra. Thus, A is a $\mathbb{Z}$-graded associative algebra with 1

$$A = \bigoplus_{p\in\mathbb{Z}} A^p$$

endowed with a homogeneous linear endomorphism d of degree 1, the differential, such that $d^2 = 0$ and we have the Leibniz rule

$$d(ab) = (da)b + (-1)^p a(db)$$

for all $a \in A^p$ and all $b \in A$. Let $\varepsilon : A \to k$ be an augmentation (a morphism of dg algebras). For example, if V is a vector space (concentrated in degree 0), we can consider $A = SV$ (concentrated in degree 0 with $d = 0$). Dually, let C be a dg coalgebra and $\varepsilon : k \to C$ a co-augmentation. For example, we may consider $C = \Lambda V$, where V is concentrated in degree -1 and $d = 0$. Denote by $\mathsf{Hom}_k(C, A)$ the graded vector space whose nth component is formed by the homogeneous k-linear maps $f : C \to A$ of degree n. We make $\mathsf{Hom}_k(C, A)$ into a differential graded algebra by setting

$$d(f) = d \circ f - (-1)^n f \circ d$$

for f homogeneous of degree n and

$$f * g = \mu \circ (f \otimes g) \circ \Delta$$

for homogeneous f and g, where μ is the multiplication of A and Δ the comultiplication of C. A *twisting cochain* is an element $\tau \in \mathsf{Hom}_k(C, A)$ of degree 1 such that

$$\varepsilon \circ \tau = 0 = \tau \circ \eta \text{ and } d(\tau) + \tau * \tau = 0.$$

For example, with the above notations, the composition of the natural projection and inclusion morphisms

$$\Lambda V \to V \to SV$$

is a twisting cochain. We denote by $\mathsf{Tw}(C, A)$ the set of twisting cochains. From now on, we assume that C is *cocomplete*, *i.e.* that $\overline{C} = \mathsf{cok}(\eta)$ is the union of the kernels of the maps induced by the iterated comultiplications

$$\Delta^{(n)} : C \to C^{\otimes n} \ , \ n \geqslant 2.$$

We denote by Alg the category of augmented dg algebras and by Coalg the category of cocomplete co-augmented dg coalgebras.

Proposition 2.1. a) *The functor* $\mathsf{Tw}(?, A) : \mathsf{Coalg}^{op} \to \mathsf{Set}$ *is representable,* i.e. *there is an object* $BA \in \mathsf{Coalg}$ *and a functorial bijection*

$$\mathsf{Tw}(C', A) \xrightarrow{\sim} \mathsf{Coalg}(C', BA).$$

b) *The functor* $\mathsf{Tw}(C, ?) : \mathsf{Alg} \to \mathsf{Set}$ *is co-representable,* i.e. *there is an object* $\Omega A \in \mathsf{Alg}$ *and a functorial bijection*

$$\mathsf{Tw}(C, A') \xrightarrow{\sim} \mathsf{Alg}(\Omega C, A').$$

The dg coalgebra BA is known as the *bar construction* and the dg algebra ΩC as the *cobar construction*. It is not hard to describe BA and ΩC explicitly but we will not need this. Notice that if we denote by DBA the k-dual dg algebra of BA, then we have a canonical isomorphism

$$DBA \xrightarrow{\sim} \mathsf{RHom}_A(k, k)$$

and that the latter is quasi-isomorphic to the Koszul dual $A^!$ (with the generators in differential degree 1) if A is a Koszul algebra concentrated in degree 0. We denote the category of dg right A-modules by $\mathsf{Mod}\, A$. Its localization with respect to all quasi-isomorphisms is the *derived category* $\mathcal{D}A$. A (right) dg comodule M is *cocomplete* if it is the union of the kernels of the maps

$$M \to M \otimes \overline{C}^{n-1} \ , \ n \geqslant 2 \, ,$$

induced by the iterated comultiplications. We denote by $\mathsf{Com}\, C$ the category of cocomplete right dg C-comodules. It becomes a Frobenius exact category when endowed with the conflations given by the exact sequences whose underlying sequences of graded comodules split. The *category up to homotopy* is the associated stable category. We denote by by $\mathcal{D}C$ the *co-derived category*, *i.e.* the localization of $\mathsf{Com}\, C$ at the class of all *co-quasi-isomorphisms*. Here, a morphism $s : L \to M$ of dg comodules is a co-quasi-isomorphism if its cone lies in the smallest triangulated subcategory of the category up to homotopy of dg comodules stable under coproducts and containing all totalizations of short exact sequences

$$0 \to X \to Y \to Z \to 0$$

of dg comodules. This definition is due to Positselski [9]. One can also characterize the co-quasi-isomorphisms using the cobar construction for dg comodules, *cf.* [7] and below. For the comparison between the two, *cf.* Appendix A of [10].

Let us fix a twisting cochain $\tau : C \to A$. For $M \in \mathsf{Mod}\, A$, let $M \otimes_\tau C$ be the graded comodule $M \otimes C$ endowed with the differential

$$d \otimes \mathbf{1} + \mathbf{1} \otimes d + d_\tau \ , \quad \text{where } d_\tau = (\mu \otimes \mathbf{1}) \circ (\mathbf{1} \otimes \tau \otimes \mathbf{1}) \circ (\mathbf{1} \otimes \Delta).$$

For $L \in \mathsf{Com}\, C$, let $L \otimes_\tau A$ be the graded A-module $L \otimes A$ endowed with the differential

$$d \otimes \mathbf{1} + \mathbf{1} \otimes d + d_\tau \ , \quad \text{where } d_\tau = (\mathbf{1} \otimes \mu) \circ (\mathbf{1} \otimes \tau \otimes \mathbf{1}) \circ (\Delta \otimes \mathbf{1}).$$

PROPOSITION 2.2. *The pair $(? \otimes_\tau A, ? \otimes_\tau C)$ is a pair of adjoint functors between $\mathsf{Com}\, C$ and $\mathsf{Mod}\, A$. It induces a pair of adjoint functors between $\mathcal{D}C$ and $\mathcal{D}A$.*

We define the twisting cochain $\tau : C \to A$ to be *acyclic* if the associated adjoint functors between $\mathcal{D}C$ and $\mathcal{D}A$ are equivalences. In this case, we say that A and C are *Koszul–Moore dual* to each other.

THEOREM 2.3 ([7]). *The following are equivalent:*

i) *τ is acyclic;*
ii) *τ induces a quasi-isomorphism $\Omega C \to A$;*
iii) *τ induces a weak equivalence $C \to BA$ (*i.e. *the induced morphism $\Omega C \to \Omega BA$ is a quasi-isomorphism);*
iv) *The natural morphism $A \otimes_\tau C \otimes_\tau A \to A$ is a quasi-isomorphism.*

For example, it follows from part iv) that SV and ΛV (with the above notations) are Koszul–Moore dual to each other. By part iii), we have Koszul–Moore duality between A and BA and therefore a triangle equivalence

$$\mathcal{D}(BA) \xrightarrow{\sim} \mathcal{D}A.$$

It is remarkable that the dg coalgebra BA determines all of $\mathcal{D}A$ whereas the dual dg algebra $DBA \xrightarrow{\sim} \mathsf{RHom}_A(k, k)$ a priori only determines the thick subcategory of $\mathcal{D}A$ generated by k.

Assume that $\tau : C \to A$ is an acylic cochain. Put $A^e = A \otimes A^{op}$ and $C^e = C \otimes C^{op}$ and let

$$\tau^e = \tau \otimes \eta + \eta \otimes \tau : C^e \to A^e.$$

Clearly τ^e is a twisting cochain.

PROPOSITION 2.4. *The twisting cochain τ^e is acyclic and the induced equivalence*

$$\mathcal{D}(C^e) \xrightarrow{\sim} \mathcal{D}(A^e)$$

takes the dg bicomodule C to the dg bimodule A. Thus we have an induced isomorphism of graded algebras

$$HH^*(C) = \mathsf{Ext}^*_{C^e}(C, C) \xrightarrow{\sim} \mathsf{Ext}^*_{A^e}(A, A) = HH^*(A).$$

Remark 2.5. *We see that Koszul–Moore duality preserves Hochschild cohomology as a graded algebra.*

Proof. Let $\varphi_A : A \otimes_\tau C \otimes_\tau A \to A$ be the canonical morphism. Then the canonical morphism

$$\varphi_{A^e} : A^e \otimes_{\tau^e} C^e \otimes_{\tau^e} A^e \to A^e$$

is the composition of the isomorphism

$$(A \otimes A^{op}) \otimes_{\tau^e} (C \otimes C^{op}) \otimes_{\tau^e} (A \otimes A^{op}) \xrightarrow{\sim} (A \otimes_\tau C \otimes_\tau A) \otimes (A^{op} \otimes_\tau C^{op} \otimes_\tau A^{op})$$

with the quasi-isomorphism

$$\varphi_A \otimes \varphi_{A^{op}} : (A \otimes_\tau C \otimes_\tau A) \otimes (A^{op} \otimes_\tau C^{op} \otimes_\tau A^{op}) \to A \otimes A^{op}.$$

Thus τ^e is an ayclic twisting cochain by Theorem 2.3. The induced equivalence takes C to

$$C \otimes_{\tau^e} (A \otimes A^{op}) = A \otimes_\tau C \otimes_\tau A$$

which is quasi-isomorphic to A since τ is acyclic. $\surd$

Theorem 2.6. *The isomorphism of the proposition lifts to an isomorphism in the homotopy category of B_∞-algebras between the corresponding Hochschild cochain complexes. In particular, it preserves the Gerstenhaber brackets.*

Remark 2.7. *In [3], Buchweitz related the Hochschild cohomology algebras of a Koszul algebra and its Koszul dual. In Theorem 3.5 of [5], we showed that for a* Koszul algebra A *(concentrated in differential degree* 0*), there is a canonical isomorphism in the category of Adams graded B_∞-algebras between the Hochschild complex of A and that of the Koszul dual algebra $A^!$ whose pth graded piece is put into bidegree $(p, -p)$, where the first component is the differential degree and the second component the Adams degree. Here, we get rid of the Koszulity assumption and the Adams grading by using dg coalgebras.*

In [4], the Félix–Menichi–Thomas show that for a simply connected coalgebra C, the Hochschild cohomologies of the dg algebras ΩC and $\mathsf{Hom}_k(C, k)$ are isomorphic as Gerstenhaber algebras. Another proof of this, under less stringent connectedness assumptions, is given in section 3 of Briggs–Gélinas' [2]. For Koszul A_∞–algebras, an isomorphism of the Hochschild cohomologies as weight graded A_∞-algebras (but not as Gerstenhaber algebras) is proved by Berglund–Börjeson in Theorem 3.2 of [1].

Proof. The twisting cochain induces a weak equivalence $BA \to C$. By the argument of [5], this yields an isomorphism in the homotopy category of B_∞-algebras between the Hochschild complexes of BA and C (these are sometimes called coHochschild complexes). Thus, we may assume that $C = BA$ and τ is the canonical twisting cochain. Let $\widetilde{B}A = A \otimes_\tau C \otimes_\tau A$. Then $\widetilde{B}A$ has a natural structure of dg coalgebra in the category $\mathsf{Mod}(A^e)$ of dg A-A-bimodules endowed with $\otimes_A$. We have a lax monoidal functor

$$F : \mathsf{Mod}(A^e) \to \mathcal{C}(k)$$

taking a bimodule M to $k \otimes_A M \otimes_A k$. The lax structure is given by the morphism

$$F(L \otimes_A M) = k \otimes_A (L \otimes_A M) \otimes_A k = k \otimes_A (L \otimes_A A \otimes_A M) \otimes_A k \to k \otimes_A L \otimes_A k \otimes_A M \otimes_A k = (FL) \otimes_k (FM).$$

The functor F sends the A^e-coalgebra $\widetilde{B}A$ to the k-coalgebra $BA = C$. It induces a morphism from the Hochschild complex of $\widetilde{B}A$ to that of C and this is easily seen to be a quasi-isomorphism compatible with the cup product and the brace operations. The claim follows by Theorem 5.1 of [8] which states that there is a canonical isomorphism in the homotopy category of B_∞-algebras between the Hochschild complex of $\widetilde{B}A$ and the Hochschild complex of A. It is not hard to check that in homology, it induces the isomorphism of Propostion 2.4. $\surd$

Example 2.8. *Suppose that k is of characteristic* 0 *and $\mathfrak{g}$ a finite-dimensional Lie algebra over k. Let $U\mathfrak{g}$ be its enveloping algebra and $\Lambda\mathfrak{g}$ the supersymmetric coalgebra on $\mathfrak{g}$ placed in degree -1 and endowed with the coalgebra differential whose (-2)-component is the bracket $\Lambda^2\mathfrak{g} \to \mathfrak{g}$. Thus, the underlying complex of $\Lambda\mathfrak{g}$ is the Chevalley–Eilenberg complex of $\mathfrak{g}$. Then the map $\tau : \Lambda\mathfrak{g} \to U\mathfrak{g}$*

which is the composition of the projection $\Lambda\mathfrak{g} \to \mathfrak{g}$ with the inclusion $\mathfrak{g} \to U\mathfrak{g}$ is an acyclic twisting cochain and we obtain an isomorphism of Gerstenhaber algebras

$$HH^*(\Lambda\mathfrak{g}) \xrightarrow{\sim} HH^*(U\mathfrak{g}).$$

It would be interesting to generalize this example to Lie algebroids.

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