

# CLUSTER CATEGORIES AND RATIONAL CURVES

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**ABSTRACT.** We study rational curves on smooth complex Calabi–Yau threefolds via noncommutative algebra. By the general theory of derived noncommutative deformations due to Efimov, Lunts and Orlov, the structure sheaf of a rational curve in a smooth CY 3-fold  $Y$  is pro-represented by a nonpositively graded dg algebra  $\Gamma$ . The curve is called *nc rigid* if  $H^0\Gamma$  is finite dimensional. When  $C$  is contractible,  $H^0\Gamma$  is isomorphic to the contraction algebra defined by Donovan and Wemyss. More generally, one can show that there exists a  $\Gamma$  pro-representing the (derived) multi-pointed deformation (defined by Kawamata) of a collection of rational curves  $C_1, \dots, C_t$  with  $\dim(\mathrm{Hom}_Y(\mathcal{O}_{C_i}, \mathcal{O}_{C_j})) = \delta_{ij}$ . The collection is called *nc rigid* if  $H^0\Gamma$  is finite dimensional. We prove that  $\Gamma$  is a homologically smooth bimodule 3CY algebra. As a consequence, we define a (2CY) cluster category  $\mathcal{C}_\Gamma$  for such a collection of rational curves in  $Y$ . It has finite-dimensional morphism spaces iff the collection is nc rigid. When  $\bigcup_{i=1}^t C_i$  is (formally) contractible by a morphism  $\widehat{Y} \rightarrow \widehat{X}$ , then  $\mathcal{C}_\Gamma$  is equivalent to the singularity category of  $\widehat{X}$  and thus categorifies the contraction algebra of Donovan and Wemyss. The Calabi–Yau structure on  $Y$  determines a canonical class  $[w]$  (defined up to right equivalence) in the zeroth Hochschild homology of  $H^0\Gamma$ . Using our previous work on the noncommutative Mather–Yau theorem and singular Hochschild cohomology, we prove that the singularities underlying a 3-dimensional smooth flopping contraction are classified by the derived equivalence class of the pair  $(H^0\Gamma, [w])$ . We also give a new necessary condition for contractibility of rational curves in terms of  $\Gamma$ .

## 1. INTRODUCTION

The study of rational curves in algebraic varieties lies at the core of birational geometry. A smooth rational curve  $C$  in a quasi-projective variety  $Y$  is called *rigid* if the component of the Hilbert scheme of curves containing  $C$  is a finite scheme. Note that this is weaker than the notion of *infinitesimally rigid*, which says that  $\mathrm{Ext}_Y^1(\mathcal{O}_C, \mathcal{O}_C) = 0$ . If a curve is not rigid then we call it *movable*. When  $Y$  is a smooth projective surface, a smooth rational curve  $C \subset Y$  is rigid if and only if its normal bundle  $N_{C/Y}$  is negative. And if  $C$  is rigid then it is *contractible*, i.e. for the formal completion  $\widehat{Y}$  of  $Y$  along  $C$  there exists a birational morphism  $f : \widehat{Y} \rightarrow \widehat{X}$  to a normal surface  $\widehat{X}$  that contracts  $C$ . The definition of contractibility in general can be found in Definition 2.9.

In this article, we will focus on the case when  $Y$  is a smooth complex Calabi–Yau threefold, i.e.  $\omega_Y$  is trivial. The situation is much more complicated than the surface case. We call a rational curve  $C \subset Y$  of type  $(a, b)$  if it has normal bundle  $\mathcal{O}(a) \oplus \mathcal{O}(b)$ . By the adjunction formula, we have  $a + b = -2$ . A  $(-1, -1)$ -curve is contractible. The underlying singular variety  $\widehat{X}$  is equivalent to the singular hypersurface  $x^2 + y^2 + u^2 + v^2 = 0$ . There exists a different resolution  $\widehat{Y}^+ \rightarrow \widehat{X}$  and the birational map  $\widehat{Y} \dashrightarrow \widehat{Y}^+$  is called the *Atiyah flop*. In [60], Reid proves that a  $(0, -2)$ -curve is either contractible or movable. The contractible case corresponds to the *Pagoda flops*. Laufer proves that a contractible curve is of the types  $(-1, -1)$ ,  $(0, -2)$  or  $(1, -3)$  (c.f. [64]). Katz and Morrison show that any simple flopping contraction (see definition in Section 2.5) can be constructed as base change of a universal contraction [35]. In general, it is not true that all rigid curves are contractible. A counter example was constructed by Clemens [15].

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*Key words and phrases.* Contractible curve, cluster category.

We study the contractibility of rational curves in Calabi-Yau 3-folds via noncommutative methods. In general, given a rational curve  $C \subset Y$  the problem is two-fold:

- (1) Find infinitesimal criteria for the contractibility of  $C$ .
- (2) If  $C$  is contractible, determine the underlying singularity of the contraction.

Our research is motivated by a remarkable paper of Donovan and Wemyss. In [16], Donovan and Wemyss considered the algebra  $\Lambda$  that represents the noncommutative deformation functor of  $\mathcal{O}_C$  for a contractible rational curve  $C \subset Y$ . They prove that  $\Lambda$  is finite dimensional and call it the *contraction algebra*. Indeed, the contraction algebra can be defined in a more general context where  $Y$  may be neither CY nor smooth, and the birational morphism may contract a divisor containing  $C$ . However we will focus on the special case when  $Y$  is a smooth CY 3-fold and the contraction is not divisorial. Donovan and Wemyss conjectured that the 3-dimensional simple flops are classified by the isomorphism types of the contraction algebras (cf. Conjecture 5.10).

In order to deal with the case of general flops where the exceptional fiber can have multiple irreducible components, Kawamata proposes to study the multi-pointed noncommutative deformation of a *semi-simple collection* (see definition in Section 2)<sup>1</sup> of sheaves  $\mathcal{E}_1, \dots, \mathcal{E}_t$ . A case of special interest is when the collection is  $\mathcal{O}_{C_1}, \dots, \mathcal{O}_{C_t}$  where  $C_i$  are irreducible components of the reduced exceptional fiber of a contraction (see Example 6.5 of [36]). We consider the derived noncommutative deformation theory of  $\mathcal{E} := \bigoplus_{i=1}^t \mathcal{E}_i$  for a semi-simple collection of sheaves  $\{\mathcal{E}_i\}_{i=1}^t$  on a smooth CY 3-fold  $Y$ . By a result of Efimov, Lunts and Orlov (cf. Theorem 2.2), such deformation functor is pro-represented by a nonpositively graded dg algebra  $\Gamma$ . We call  $\Gamma$  the *derived deformation algebra* of the semi-simple collection  $\{\mathcal{E}_i\}_{i=1}^t$ . We call a semi-simple collection  $\{\mathcal{E}_i\}_{i=1}^t$  *nc rigid* (“nc” stands for noncommutative) if  $H^0\Gamma$  is finite dimensional. Given a collection of smooth rational curve  $C_1, \dots, C_t$  such that  $\{\mathcal{O}_{C_i}\}_{i=1}^t$  form a semi-simple collection, we call  $\{C_i\}_{i=1}^t$  a *nc-rigid* collection of rational curves if  $H^0\Gamma$  is finite dimensional. If  $t = 1$  and  $C = C_1$  is nc rigid rational curve then the abelianization of  $H^0\Gamma$  represents the commutative deformation functor of  $\mathcal{O}_C$ . Therefore, a nc rigid curve is in particular rigid. Our first result is:

**Theorem A.** (Corollary 3.13) Let  $C_1, \dots, C_t$  be a collection of rational curves in a smooth quasi-projective Calabi-Yau 3-fold  $Y$  such that  $\{\mathcal{O}_{C_i}\}_{i=1}^t$  form a semi-simple collection. The derived deformation algebra  $\Gamma$  of  $\bigoplus_{i=1}^t \mathcal{O}_{C_i}$  is a non positive pseudo-compact dg algebra that is

- (1) homologically smooth;
- (2) bimodule 3CY.

Moreover,  $\Gamma$  is exact 3CY in either one of the following cases

- (a)  $Y$  is projective;
- (b) there is a (formal) contraction  $\widehat{f} : \widehat{Y} \rightarrow \widehat{X}$  such that  $\text{Ex}(\widehat{f}) = \bigcup_{i=1}^t C_i$ , where  $\text{Ex}(\widehat{f})$  stands for the reduced exceptional fiber of  $\widehat{f}$ .

This theorem establishes a link between birational geometry and the theory of cluster categories. We consider the triangle quotient  $\mathcal{C}_\Gamma := \text{per}(\Gamma)/D_{fd}(\Gamma)$  (see Section 3.3). It is Hom-finite if and only if  $\{C_i\}_{i=1}^t$  is nc rigid. By a result of Amiot [3], it is then a 2CY category. When  $C$  is contractible by a morphism  $\widehat{Y} \rightarrow \widehat{X}$ , then  $H^0\Gamma$  is isomorphic to the contraction algebra  $\Lambda$  defined in [16] and  $\mathcal{C}_\Gamma$  is equivalent to the singularity category of  $\widehat{X}$  and thus categorifies the contraction algebra of Donovan and Wemyss. If  $\Gamma$  is exact 3CY, Van den Bergh proved that it is quasi-isomorphic to a (complete) Ginzburg algebra  $\mathfrak{D}(Q, w)$  for some finite quiver  $Q$  and a potential  $w$ . If we fix the CY structure on  $Y$ , then there is a canonical class  $[w]$ , defined up to right equivalence, in the zeroth Hochschild homology of  $H^0\Gamma$  (see Proposition 4.8). The class  $[w]$  can be viewed as the “classical shadow” of the Calabi-Yau structure on  $Y$ . Our second result is:

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<sup>1</sup>Kawamata calls it a *simple collection*.

**Theorem B.** (Theorem 5.11) Let  $\widehat{f} : \widehat{Y} \rightarrow \widehat{X}$  and  $\widehat{f}' : \widehat{Y}' \rightarrow \widehat{X}'$  be two formal flopping contractions with reduced exceptional fibers  $\text{Ex}(\widehat{f}) = \bigcup_{i=1}^t C_i$  and  $\text{Ex}(\widehat{f}') = \bigcup_{i=1}^s C'_i$ . Denote respectively by  $\Gamma$  and  $\Gamma'$  the derived deformation algebras of  $\bigoplus_{i=1}^t \mathcal{O}_{C_i}$  and  $\bigoplus_{i=1}^s \mathcal{O}_{C'_i}$ , and by  $[w] \in \text{HH}_0(H^0\Gamma)$  and  $[w'] \in \text{HH}_0(H^0\Gamma')$  the canonical classes. Suppose there is a triangle equivalence

$$\overset{\mathbb{L}}{\otimes}_{H^0\Gamma} Z : D(H^0\Gamma) \xrightarrow{\sim} D(H^0\Gamma')$$

given by a dg bimodule  $Z$  such that  $\text{HH}_0(Z)$  (as defined in [41]) takes  $[w]$  to  $[w']$  in  $\text{HH}_0(H^0\Gamma') = \text{HH}_0(\Gamma')$ . Then  $\widehat{X}$  is isomorphic to  $\widehat{X}'$ . In particular,  $s$  is equal to  $t$ .

For general (non-simple) flopping contractions, there exist derived equivalent algebras  $H^0\Gamma$  and  $H^0\Gamma'$  that are non-isomorphic. August proves that the isomorphism classes of such algebras in a fixed derived equivalence class of  $H^0\Gamma$  are precisely the contraction algebras for the iterated flops of  $\widehat{Y}$  (see Theorem 1.4 of [4]). Different contraction algebras in the same derived equivalence class are related by the iterated mutations of the tilting objects. The mutations are the homological counterpart of flops between different minimal models. We refer to [79] for the general framework of the homological minimal model program.

Theorem B says that the underlying singularity type of the smooth minimal models is determined by the derived equivalence class of the pair  $(H^0\Gamma, [w])$ . We sketch the idea of the proof. From 3-dimensional birational geometry we know that the underlying singularity of a smooth flopping contraction is a hypersurface (see Section 2.5). It is a classical theorem of Mather and Yau that up to isomorphism, a germ of hypersurface singularity is determined by its Tyurina algebra (see [58] for the analytic case with isolatedness assumption and [26] for the formal case without the isolatedness assumption). Next we prove that the derived equivalence class of  $H^0\Gamma$  together with the canonical class  $[w]$  recovers the Tyurina algebra of the singularity. We solve this problem in two steps. First, we prove that the Tyurina algebra, therefore the isomorphism class of the hypersurface singularity, can be recovered from the ( $\mathbb{Z}$ -graded dg enhanced) cluster category  $\mathcal{C}_\Gamma$ . This result, proved in Section 5, should have independent interest. Secondly, we show that the isomorphism class of the Ginzburg algebra  $\mathfrak{D}(Q, w)$  that is quasi-isomorphic to  $\Gamma$  can be recovered from the data  $(H^0\Gamma, [w])$ . The proof uses a result of the first author and Gui-song Zhou in noncommutative differential calculus of potentials with finite-dimensional Jacobi-algebras [31]. Finally, we prove (in Section 4.4 and 4.5) that any derived Morita equivalence  $D(H^0\Gamma) \simeq D(H^0\Gamma')$  preserving the canonical class yields a derived Morita equivalence  $\text{per}(\Gamma) \simeq \text{per}(\Gamma')$ .

Note that in [29], the first author and Toda gave an alternative definition of the contraction algebra associated to a flopping contraction using the category of matrix factorizations. In this definition, the contraction algebra carries an additional (compared with the definition in [16])  $\mathbb{Z}/2$ -graded  $A_\infty$ -structure. In [30], the first author proved that the Tyurina algebra of the singularity can be recovered from the  $\mathbb{Z}/2$ -graded  $A_\infty$ -structure. Our proof of Theorem B shows that the  $\mathbb{Z}/2$ -graded  $A_\infty$ -structure on the contraction algebra can be recovered from the class  $[w]$ . Theorem B without the condition on the preservation of the canonical class is precisely the generalization of the conjecture by Donovan and Wemyss stated by August in Conjecture 1.3 of [4]. See Conjecture 5.10 for the original conjecture of Donovan and Wemyss, which is for simple flopping contractions. The geometric meaning of the class  $[w]$  remains to be understood. It is believed that the vanishing of  $[w]$  is closely related to the condition that  $\widehat{X}$  is quasi-homogeneous.

Our third result is a necessary condition on the contractibility of a nc rigid rational curve in a smooth CY 3-fold. Let  $u$  be a variable of degree 2.

**Theorem C.** (Proposition 6.9) Let  $C$  be a nc rigid rational curve in a smooth CY 3-fold  $Y$ . If  $C$  is contractible, then its derived deformation algebra  $\Gamma$  is  $k[u^{-1}]$ -enhanced (see definition in Section 6). Moreover,  $H^0\Gamma$  is a symmetric Frobenius algebra.

We conjecture that a nc rigid rational curve  $C$  is contractible if and only if  $\Gamma$  is  $k[u^{-1}]$ -enhanced (see Conjecture 6.8).

The paper is organized as follows. In Section 2 we review basics on derived noncommutative deformation theory, noncommutative crepant resolutions and flopping contractions. Concerning derived deformations, we complement the results of [23] by explaining the link to classical deformations in abelian categories in subsection 2.3. In Section 3, we discuss various notions of Calabi-Yau structures in geometry and algebra and prove Theorem A. The notion of cluster category is introduced in Section 3.3. When the curve is contractible, there are two cluster categories associated to it: one via derived deformation and the other via the NCCR. We prove that these two cluster categories are algebraically equivalent. In Section 4, we recall the definition of Ginzburg algebras and several results in noncommutative differential calculus including the noncommutative Mather-Yau theorem and noncommutative Saito theorem. We further show that for a contractible curve in a CY 3-fold, there exists a Ginzburg algebra weakly equivalent to the derived deformation algebra whose potential is canonically defined up to right equivalence. Then we establish a relation between the silting theory of a non positive dg algebra and the silting theory of its zeroth homology. In Section 5, we study the relation between the cluster category associated to the contractible curves and their underlying singularities via Hochschild cohomology. In particular, Theorem B is proved. In Section 6, we introduce the notion of  $k[u^{-1}]$ -enhancement of dg algebras. For derived deformation algebra, we establish a link between the existence of  $k[u^{-1}]$ -enhancement and contractibility of rational curve and prove Theorem C.

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## 2. PRELIMINARIES

**2.1. Notation and conventions.** Throughout  $k$  will be a ground field unless stated otherwise. Unadorned tensor products are over  $k$ . Let  $V$  be a  $k$ -vector space. We denote its dual vector space by  $DV$ . When  $V$  is graded,  $DV$  is understood as the the dual in the category of graded vector spaces. For a subspace  $V'$  of a complete topological vector space  $V$ , we denote the closure of  $V'$  in  $V$  by  $(V')^c$ . By definition a *pseudo-compact  $k$ -vector space* is a linear topological vector space which is complete and whose topology is generated by subspaces of finite codimension. Following [76] we will denote the corresponding category by  $PC(k)$ . We have inverse dualities

$$\mathbb{D} : Mod(k) \rightarrow PC(k)^{op} : V \mapsto \text{Hom}_k(V, k)$$

$$\mathbb{D} : PC(k) \rightarrow Mod(k)^{op} : W \mapsto \text{Hom}_{PC(k)}(W, k)$$

where we recall that for  $V \in Mod(k)$  the topology on  $\mathbb{D}V$  is generated by the kernels of  $\mathbb{D}V \rightarrow \mathbb{D}V'$  where  $V'$  runs through the finite dimensional subspaces of  $V$ . Similarly, if  $V$  is graded then  $\mathbb{D}$  is understood in the graded sense. For the definition of Hom-space and tensor product in  $PC(k)$ , we refer to Section 4 of [76]. Using the tensor product in  $PC(k)$ , we define the pseudo-compact dg algebras, modules and bimodules to be the corresponding objects in the category of graded objects

of  $PC(k)$ . Let  $A$  be pseudo-compact dg  $k$ -algebra. Denote by  $PC(A^e)$  the category of pseudo-compact  $A$ -bimodules. We will sometimes take a finite dimensional separable  $k$ -algebra  $l$  to be the ground ring. The definition of the duality functor  $\mathbb{D}$  on  $PC(l^e)$  requires some extra care due to the noncommutativity of  $l$ . We refer to Section 5 of [76] for the detailed discussion.

Denote by  $\text{PCAlgc}(l)$  the category of augmented pseudo-compact dg algebras  $A$  whose underlying graded algebras have their augmentation ideal equal to their Jacobson radical (cf. Proposition 4.3 and section 6 of [76]). Our main interest is in the case when  $l \cong ke_1 \times ke_2 \times \dots \times ke_n$  for central orthogonal idempotents  $(e_i)_i$ . For an object  $A \in \text{PCAlgc}(l)$ , we use Hom-spaces and tensor products in  $PC(l^e)$  to define the Hochschild and cyclic (co)homology. For details, we refer to Section 7 and Appendix A of [76]. If  $A$  is an  $l$ -algebra in  $\text{PCAlgc}(l)$ , we use  $\text{HH}_*(A)$ ,  $\text{HH}^*(A)$ ,  $\text{HC}_*(A)$  to denote the continuous Hochschild homology, cohomology and cyclic homology of  $A$ . Because for a pseudo-compact dg algebra, we will only consider continuous Hochschild homology, cohomology and cyclic homology, there is no risk of confusion. By an abuse of notations, for  $A \in \text{PCAlgc}(l)$  we denote by  $D(A)$  the pseudo-compact derived category of  $A$ . Its subcategories  $\text{per}(A)$  and  $D_{fd}(A)$  are defined as the thick subcategory generated by the free  $A$ -module  $A$  and as the full subcategory of all objects with homology of finite total dimension. Similar to the algebraic case, the notion of homological smoothness can be defined in the pseudo-compact setting. We refer to the Appendix of [51] for a careful treatment. For the bar-cobar formalism and Koszul duality of pseudo-compact dg algebras, we refer to Appendix A and D of [75].

**2.2. Derived deformation theory.** We briefly recall the setup of derived noncommutative deformation theory of Efimov, Lunts and Orlov. In this section, we fix a field  $k$ . We refer to [44] for foundational material on dg categories. For a dg category  $\mathcal{A}$ , we denote by  $D(\mathcal{A})$  the derived category of right dg  $\mathcal{A}$ -modules. Fix a positive integer  $n$  and let  $l$  be the separable  $k$ -algebra  $ke_1 \times \dots \times ke_n$ . An  $l$ -algebra  $A$  is a  $k$ -algebra together with a morphism of  $k$ -algebras  $l \rightarrow A$  (note that  $l$  is *not* necessarily central in  $A$ ). An equivalent datum is that of the  $k$ -category with  $n$  objects  $1, \dots, n$ , whose morphism space from  $i$  to  $j$  is given by  $e_j A e_i$ . An  $l$ -augmented (dg) algebra is a (dg)  $l$ -algebra  $\mathcal{R}$  together with an  $l$ -algebra morphism  $\mathcal{R} \rightarrow l$  such that the composition  $l \rightarrow \mathcal{R} \rightarrow l$  is the identity morphism. Its *augmentation ideal* is the kernel of the augmentation morphism  $\mathcal{R} \rightarrow l$ . An artinian  $l$ -algebra is an augmented  $l$ -algebra whose augmentation ideal is finite-dimensional and nilpotent. A dg  $l$ -algebra is *artinian* if it is an augmented dg  $l$ -algebra whose augmentation ideal is finite-dimensional and nilpotent. Denote by  $\mathbf{Art}_l$  and  $\mathbf{cArt}_l$  the categories of artinian  $l$ -algebras and of commutative artinian  $l$ -algebras. Denote by  $\mathbf{dgArt}_l$  the category of artinian dg algebras and by  $\mathbf{dgArt}_l^-$  the subcategory of  $\mathbf{dgArt}_l$  consisting of dg algebras concentrated in nonpositive degrees.

Fix a dg category  $\mathcal{A}$  and a dg  $\mathcal{A}$ -module  $E$  with a decomposition  $E = E_1 \oplus \dots \oplus E_n$ . We view  $E$  as an  $l^{op} \otimes \mathcal{A}$ -module in the natural way. The *dg endomorphism  $l$ -algebra* of  $E$  is the dg endomorphism algebra over  $\mathcal{A}$  of the sum  $E$  viewed as an  $l$ -algebra in the natural way. We are going to define a pseudo-functor  $\text{Def}(E)$  from  $\mathbf{dgArt}_l$  to the category  $\mathbf{Gpd}$  of groupoids. This pseudo-functor assigns to an artinian dg  $l$ -algebra  $\mathcal{R}$  the groupoid  $\text{Def}_{\mathcal{R}}(E)$  of  $\mathcal{R}$ -deformations of  $E$  in the derived category  $D(\mathcal{A})$ . We will mostly follow the notations of [22] and identify  $\mathcal{R}$  with the dg category with  $n$  objects  $1, \dots, n$ , where the morphism complex from  $i$  to  $j$  is given by  $e_j \mathcal{R} e_i$ . Denote the dg category  $\mathcal{R}^{op} \otimes \mathcal{A}$  by  $\mathcal{A}_{\mathcal{R}}$ . The augmentation  $\varepsilon : \mathcal{R} \rightarrow l$  yields the functor of extension of scalars  $\varepsilon^*$  taking a dg  $\mathcal{R}$ -module  $S$  to the dg  $l^{op} \otimes \mathcal{A}$ -module

$$\varepsilon^*(S) = l^{\mathbb{L}} \otimes_{\mathcal{R}} S.$$

**Definition 2.1.** [22, Definition 10.1] Fix an artinian augmented dg  $l$ -algebra  $\mathcal{R}$ . An object of the groupoid  $\text{Def}_{\mathcal{R}}(E)$  is a pair  $(S, \sigma)$ , where  $S$  is an object of  $D(\mathcal{A}_{\mathcal{R}})$  and

$$\sigma : \varepsilon^*(S) \rightarrow E$$

is an isomorphism in  $D(l^{op} \otimes \mathcal{A})$ . A morphism  $f : (S, \sigma) \rightarrow (T, \tau)$  between two  $\mathcal{R}$ -deformations of  $E$  is an isomorphism  $f : S \rightarrow T$  in  $D(\mathcal{A}_{\mathcal{R}})$  such that

$$\tau \circ \varepsilon^*(f) = \sigma.$$

This defines the groupoid  $\text{Def}_{\mathcal{R}}(E)$ . A homomorphism of augmented artinian dg  $l$ -algebras  $\phi : \mathcal{R} \rightarrow \mathcal{Q}$  induces the functor

$$\phi^* : \text{Def}_{\mathcal{R}}(E) \rightarrow \text{Def}_{\mathcal{Q}}(E)$$

given by  $\mathcal{Q} \overset{\mathbb{L}}{\otimes}_{\mathcal{R}} ?$ . Thus we obtain a pseudo-functor

$$\text{Def}(E) : \mathbf{dgArt}_l \rightarrow \mathbf{Gpd}.$$

We call  $\text{Def}(E)$  the *pseudo-functor of derived deformations of  $E$* . We denote by  $\text{Def}_-(E)$  the restrictions of the pseudo-functor  $\text{Def}(E)$  to the subcategory  $\mathbf{dgArt}_l^-$ .

The category of augmented dg  $l$ -algebras can be naturally enhanced to a weak 2-category. We refer to Definition 11.1 of [23] for the precise definition of the 2-category structure. In particular, we denote the corresponding 2-categorical enhancements of  $\mathbf{dgArt}_l$ ,  $\mathbf{dgArt}_l^-$  and  $\mathbf{Art}_l$  by  $2\text{-dgArt}_l$ ,  $2\text{-dgArt}_l^-$  and  $2\text{-Art}_l$  (in [23], they are denoted by  $2'\text{-dgArt}_l$  etc.). By Proposition 11.4 of [23], there exists a pseudo-functor  $\text{DEF}(E)$  from  $2\text{-dgArt}_l$  to  $\mathbf{Gpd}$  and which is an extension to  $2\text{-dgArt}_l$  of the pseudo-functor  $\text{Def}(E)$ . Similarly, there exists a pseudo-functor  $\text{DEF}_-(E)$  extending  $\text{Def}_-(E)$ .

The main theorem of [23] is:

**Theorem 2.2.** [23, Theorem 15.1, 15.2] *Let  $E_1, \dots, E_n$  be a collection of objects in  $D(\mathcal{A})$ . Let  $E$  be the direct sum of the  $E_i$  and  $C$  the extension algebra  $\text{Ext}_{\mathcal{A}}^*(E, E)$  considered as a graded  $l$ -algebra. Assume that we have*

- (a)  $C^p = 0$  for all  $p < 0$ ;
- (b)  $C^0 = l$ ;
- (c)  $\dim_k C^p < \infty$  for all  $p$  and  $\dim_k C^p = 0$  for all  $p \gg 0$ .

Denote by  $\mathcal{C}$  the dg endomorphism  $l$ -algebra of  $E$ . Let  $A$  be a strictly unital minimal model of  $\mathcal{C}$ . Then the pseudo-functor  $\text{DEF}_-(E)$  is pro-representable by the dg  $l$ -algebra  $\Gamma = \mathbb{D}BA$ , where  $B$  denotes the bar construction. That is, there exists an equivalence of pseudo-functors  $\text{DEF}_-(E) \simeq h_{\Gamma}$  from  $2\text{-dgArt}_l^-$  to  $\mathbf{Gpd}$ , where  $h_{\Gamma}$  denotes the groupoid of 1-morphisms  $1\text{-Hom}(\Gamma, ?)$ .

In the case where the dg category  $\mathcal{A}$  is given by an algebra  $A$  concentrated in degree 0 and  $E$  is a one-dimensional  $A$ -module, Booth [11, Theorem 3.5.9] obtains an analogous prorepresentability result for the set-valued framed deformation functor  $\text{Def}_A^{\text{fr}, \leq 0}(E)$  without having to impose the finiteness condition (c).

Let  $Y$  be a smooth algebraic variety. A collection of compactly supported coherent sheaves  $\mathcal{E}_1, \dots, \mathcal{E}_t$  on  $Y$  is called *semi-simple* if  $\text{Hom}_Y(\mathcal{E}_i, \mathcal{E}_i) \cong k$  for all  $i$  and  $\text{Hom}_Y(\mathcal{E}_i, \mathcal{E}_j) = 0$  for all  $i \neq j$ . The finiteness assumption in Theorem 2.2 is satisfied by any semi-simple collection. Let  $\mathcal{E}$  be the direct sum of such a collection of coherent sheaves on  $Y$ . We may denote the completion of  $Y$  along the support of  $\mathcal{E}$  by  $\widehat{Y}$ .

**Corollary 2.3.** *Given a semi-simple collection  $\{\mathcal{E}_i\}_{i=1}^t$  in  $D^b(\text{coh } \widehat{Y})$ , denote by  $\mathcal{C}$  the dg endomorphism  $l$ -algebra of  $\mathcal{E} := \bigoplus_{i=1}^t \mathcal{E}_i$ . Let  $A$  be a strictly unital minimal model of  $\mathcal{C}$  and  $\Gamma = \mathbb{D}BA$ . Then there is an equivalence*

$$\text{DEF}_-(\mathcal{E}) \simeq h_{\Gamma}.$$

We call  $\Gamma$  the *derived deformation algebra* of the collection  $\{\mathcal{E}_i\}_{i=1}^t$  in  $Y$ . When we want to emphasize the dependence on  $Y$  and  $\{\mathcal{E}_i\}_{i=1}^t$ ,  $\Gamma$  is replaced by  $\Gamma_{\mathcal{E}}^Y$ . The semi-simple collection  $\{\mathcal{E}_i\}_{i=1}^t$  is called *nc rigid* if  $\dim_k H^0(\Gamma_{\mathcal{E}}^Y) < \infty$ . In this paper, we are mainly interested in the

case when  $\{\mathcal{E}_i\}_{i=1}^t$  is (the structure sheaves) a collection of smooth rational curves  $C_1, \dots, C_t$  that satisfies the condition that  $\text{Hom}_Y(\mathcal{O}_{C_i}, \mathcal{O}_{C_j}) = 0$  for  $i \neq j$ . For such a collection of rational curves  $C := \{C_i\}_{i=1}^n$  where we write  $\Gamma_C^Y$  for  $\Gamma_{\{\mathcal{O}_{C_i}\}_{i=1}^t}^Y$ .

In the context of classical noncommutative deformation theory, the representability of noncommutative deformations of contractible rational curves was proved by Donovan and Wemyss.

**Theorem 2.4.** (*Proposition 3.1, Corollary 3.3 [16]*) *Let  $f : Y \rightarrow X$  be a simple flopping contraction of 3-folds (see definition in Section 2.5) and let  $C$  be the reduced exceptional fiber of  $f$ . The functor*

$$\pi_0(\text{Def}^{cl}(\mathcal{O}_C)) : \mathbf{Art}_k \rightarrow \mathbf{Set}$$

*is representable. The artinian algebra  $\Lambda$  representing it is called the contraction algebra associated to  $f : Y \rightarrow X$ .*

The definition of the classical deformation functor  $\text{Def}^{cl}$  is recalled in the next section (cf. Section 2 of [16]). If  $\Gamma$  is the derived deformation algebra of  $C$  (with  $t = 1$ ), it follows from the above Theorem and Theorem 2.5 below that the contraction algebra  $\Lambda$  is isomorphic to  $H^0\Gamma$ . Indeed, they both represent the same deformation functor and this determines them up to (non unique) isomorphism (cf. the proof of Theorem 2.14 in [70]).

**2.3. Link to classical deformations.** Let  $\mathcal{A}$  be a dg category. Let  $\mathcal{H} \subset D(\mathcal{A})$  be the heart of a  $t$ -structure on  $D(\mathcal{A})$ . We assume that  $\mathcal{H}$  is *faithful*, i.e. the higher extension groups computed in  $\mathcal{H}$  are canonically isomorphic to those computed in  $D(\mathcal{A})$ .

Let  $\mathcal{R}$  be an augmented artinian  $l$ -algebra. By an  $\mathcal{R}$ -module in  $\mathcal{H}$ , we mean an object  $M$  of  $\mathcal{H}$  endowed with an algebra homomorphism  $\mathcal{R} \rightarrow \text{End}(M)$ . Given such an  $\mathcal{R}$ -module, we denote by  $? \otimes_{\mathcal{R}} M$  the unique right exact functor  $\text{mod } \mathcal{R} \rightarrow \mathcal{H}$  extending the obvious additive functor  $\text{proj } \mathcal{R} \rightarrow \mathcal{H}$  taking  $\mathcal{R}$  to  $M$ . Here we denote by  $\text{proj } \mathcal{R}$  the category of finitely generated projective (right)  $\mathcal{R}$ -modules and by  $\text{mod } \mathcal{R}$  the category of finitely generated  $\mathcal{R}$ -modules. Notice that  $? \otimes_{\mathcal{R}}$  is a left Kan extension and thus unique up to unique isomorphism and functorial in  $M$ . It can be computed using projective resolutions. It is obvious how to define morphisms of  $\mathcal{R}$ -modules in  $\mathcal{H}$ .

Let  $E$  be the direct sum of a collection of  $n$  objects  $E_1, \dots, E_n$  of  $\mathcal{H}$ . We view  $E$  as an  $l$ -module in  $\mathcal{H}$  in the natural way. For an augmented artinian  $l$ -algebra  $\mathcal{R}$ , we define the groupoid  $\text{Def}_{\mathcal{R}}^{cl}(E)$  of classical deformations of  $E$  as follows: Its objects are pairs  $(M, \mu)$  where  $M$  is an  $\mathcal{R}$ -module in  $\mathcal{H}$  such that the functor  $? \otimes_{\mathcal{R}} M$  is exact and  $\mu : l \otimes_{\mathcal{R}} M \xrightarrow{\sim} E$  is an isomorphism of  $l$ -modules in  $\mathcal{H}$ . A morphism  $(L, \lambda) \rightarrow (M, \mu)$  is an isomorphism  $f : L \rightarrow M$  of  $\mathcal{R}$ -modules in  $\mathcal{H}$  such that  $\mu \circ (l \otimes_{\mathcal{R}} f) = \lambda$ .

For an augmented  $l$ -algebra  $A$  and an augmented artinian  $l$ -algebra  $\mathcal{R}$ , we define  $G(A, \mathcal{R})$  to be the groupoid whose objects are the morphisms  $A \rightarrow \mathcal{R}$  of augmented  $l$ -algebras and whose morphisms  $\phi_1 \rightarrow \phi_2$  are the invertible elements  $r$  of  $\mathcal{R}$  such that  $\phi_2(a) = r\phi_1(a)r^{-1}$  for all  $a$  in  $A$ .

**Theorem 2.5.** *Suppose that in addition to the above assumptions,  $E$  satisfies the hypotheses of Theorem 2.2. Let  $\Gamma$  be the pseudo-compact dg  $l$ -algebra defined there. Let  $\mathcal{R}$  be an augmented artinian  $l$ -algebra. Then  $H^0\Gamma$  represents the classical deformations of  $E$  in the sense that there is an equivalence of groupoids*

$$\text{Def}_{\mathcal{R}}^{cl}(E) \xrightarrow{\sim} G(H^0\Gamma, \mathcal{R}).$$

*Proof.* By Theorem 2.2, we have an equivalence of groupoids

$$\text{DEF}_-(E)(\mathcal{R}) \xrightarrow{\sim} \text{1-Hom}(\Gamma, \mathcal{R}),$$

where 1-Hom denotes the groupoid of 1-morphisms in 2-dgArt. We will show that  $\text{DEF}_-(E)(\mathcal{R})$  is equivalent to  $\text{Def}_{\mathcal{R}}^{cl}(E)$  and  $\text{1-Hom}(\Gamma, \mathcal{R})$  is equivalent to  $G(H^0\Gamma, \mathcal{R})$ . We start with the second equivalence. By Definition 11.1 of [23], an object of  $\text{1-Hom}(\Gamma, \mathcal{R})$  is a pair  $(M, \theta)$  consisting of

- a dg bimodule  $M$  in  $D(\Gamma^{op} \otimes \mathcal{R})$  such that the restriction to  $\mathcal{R}$  of  $M$  is isomorphic to  $\mathcal{R}$  in  $D(\mathcal{R})$  and
- an isomorphism  $\theta : M \xrightarrow{\mathbb{L}} \mathcal{R} l \rightarrow l$  in  $D(\Gamma^{op})$ .

A 2-morphism  $f : (M_1, \theta_1) \rightarrow (M_2, \theta_2)$  is an isomorphism  $f : M_1 \rightarrow M_2$  in  $D(\Gamma^{op} \otimes \mathcal{R})$  such that  $\theta_2 \circ (f \otimes_{\mathcal{R}} l) = \theta_1$ . We define a functor  $F : G(H^0 \Gamma, \mathcal{R}) \rightarrow \text{1-Hom}(\Gamma, \mathcal{R})$  as follows: Let  $\phi : H^0(\Gamma) \rightarrow \mathcal{R}$  be a morphism of augmented  $l$ -algebras. Since  $\Gamma$  is concentrated in degrees  $\leq 0$ , we have a canonical algebra morphism  $\Gamma \rightarrow H^0(\Gamma)$ . By composing it with  $\phi$  we get a morphism of augmented dg  $l$ -algebras  $\Gamma \rightarrow \mathcal{R}$ . It defines a structure of dg bimodule  $M$  on  $\mathcal{R}$ . We put  $F\phi = (M, \theta)$ , where  $\theta : \mathcal{R} \otimes_{\mathcal{R}} l \rightarrow l$  is the canonical isomorphism. Now let  $\phi_1$  and  $\phi_2$  be two morphisms of augmented algebras  $H^0(\Gamma) \rightarrow \mathcal{R}$ . Put  $(M_i, \theta_i) = F\phi_i$ ,  $i = 1, 2$ . Let  $r$  be an invertible element of  $\mathcal{R}$  such that  $\phi_2(a) = r\phi_1(a)r^{-1}$  for all  $a$  in  $H^0(\Gamma)$ . Then it is clear that the left multiplication with  $r$  defines an isomorphism of bimodules  $M_1 \rightarrow M_2$  compatible with the  $\theta_i$ . Recall that  $\Gamma^{op} \otimes \mathcal{R}$  is concentrated in degrees  $\leq 0$  so that its derived category has a canonical  $t$ -structure. Since the  $M_i$  live in the heart of this  $t$ -structure on  $D(\Gamma^{op} \otimes \mathcal{R})$ , it is also clear that  $F$  is fully faithful. It remains to be checked that  $F$  is essentially surjective. So let  $(M, \theta)$  be given. Since  $M$  is quasi-isomorphic to  $\mathcal{R}$  when restricted to  $\mathcal{R}$ , its homology is concentrated in degree 0. We can therefore replace  $M$  with  $H^0(M)$ , which is an ordinary  $H^0(\Gamma)$ - $\mathcal{R}$ -bimodule isomorphic to  $\mathcal{R}$  as a right  $\mathcal{R}$ -module (we also consider it as a left  $\Gamma$ -module via the canonical morphism  $\Gamma \rightarrow H^0(\Gamma)$ ). In particular,  $M$  is right projective and so  $M \xrightarrow{\mathbb{L}} \mathcal{R} ? = M \otimes_{\mathcal{R}} ?$ . We choose an isomorphism  $f : M \xrightarrow{\sim} \mathcal{R}$  of right  $\mathcal{R}$ -modules. After multiplying  $f$  with an invertible element of  $l$ , we may assume that  $f \otimes_{\mathcal{R}} l = \theta$ . The left  $\Gamma$ -module structure on  $M$  yields an algebra morphism

$$\phi : H^0(\Gamma) \rightarrow \text{End}_{\mathcal{R}}(M) \xrightarrow{\sim} \text{End}_{\mathcal{R}}(\mathcal{R}) = \mathcal{R}.$$

It is clear that  $f$  yields an isomorphism between  $(M, \theta)$  and  $F\phi$ .

We now construct an equivalence from  $\text{DEF}_-(E)(\mathcal{R})$  to  $\text{Def}_{\mathcal{R}}^{cl}(E)$ . Recall from Proposition 11.4 of [23] that the groupoid  $\text{DEF}_-(E)(\mathcal{R})$  equals the groupoid  $\text{Def}_{\mathcal{R}}(E)$  of Definition 2.1 (but  $\text{DEF}_-$  has enhanced 2-functoriality). Let  $P \rightarrow E$  be a cofibrant resolution of  $E$ . Since the graded algebra  $\text{Ext}^*(E, E)$  has vanishing components in degree  $-1$  and in all sufficiently high degrees, we can apply Theorem 11.8 of [23] to conclude that the groupoid  $\text{Def}_{\mathcal{R}}(E)$  is equivalent to the groupoid  $\text{Def}_{\mathcal{R}}^h(P)$  of homotopy deformations of Definition 4.1 of [22]. We now construct an equivalence  $F$  from  $\text{Def}_{\mathcal{R}}^h(P)$  to  $\text{Def}_{\mathcal{R}}^{cl}(E)$ . Let  $(S, \sigma)$  be an object of  $\text{Def}_{\mathcal{R}}^h(P)$ . We may assume that  $S = \mathcal{R} \otimes_l P$  as a graded bimodule and that  $\sigma$  is the canonical isomorphism  $l \otimes_{\mathcal{R}} (\mathcal{R} \otimes P) \xrightarrow{\sim} P$ . Let  $I$  denote the augmentation ideal of  $\mathcal{R}$ . Then  $S$  has a finite filtration by the dg submodules  $I^p S$ ,  $p \geq 0$ , and each subquotient is isomorphic to a summand of a finite sum of copies of  $l \otimes_{\mathcal{R}} S = P$ . Thus, the underlying dg  $\mathcal{A}$ -module  $M$  of  $S$  is isomorphic in  $D(\mathcal{A})$  to a finite iterated extension of objects of  $\text{add}(E)$ , the subcategory of direct factors of finite direct sums of copies of  $E$ . Therefore,  $M$  still lies in the heart  $\mathcal{H}$ . Note that as shown in the proof of Theorem 11.8 of [23],  $S$  is cofibrant over  $\mathcal{R}^{op} \otimes \mathcal{A}$ . Therefore,  $M$  is cofibrant over  $\mathcal{A}$ . The left  $\mathcal{R}$ -module structure on  $S$  yields an algebra homomorphism  $\mathcal{R} \rightarrow \text{End}(M)$ . Since each object of  $\text{mod } \mathcal{R}$  is a finite iterated extension of one-dimensional  $l$ -modules, the functor  $? \otimes_{\mathcal{R}} S : D(\mathcal{R}) \rightarrow D(\mathcal{A})$  takes  $\text{mod } \mathcal{R}$  to  $\mathcal{H}$ . Since  $? \otimes_{\mathcal{R}} S$  is a triangle functor, the induced functor  $\text{mod } \mathcal{R} \rightarrow \mathcal{H}$  is exact. Clearly it restricts to the natural functor  $\text{proj } \mathcal{R} \rightarrow \mathcal{H}$  and is therefore isomorphic to  $? \otimes_{\mathcal{R}} M : \text{mod } \mathcal{R} \rightarrow \mathcal{H}$ . Finally, the isomorphism  $l \otimes_{\mathcal{R}} S \xrightarrow{\sim} E$  yields an isomorphism  $\mu : l \otimes_{\mathcal{R}} M \xrightarrow{\sim} E$ . In this way, to an object  $(S, \sigma)$  of  $\text{Def}_{\mathcal{R}}^h(P)$ , we have associated an object  $F(S, \sigma) = (M, \mu)$  of  $\text{Def}_{\mathcal{R}}^{cl}(E)$ . Notice that by what we have just shown, we may also describe  $M \in \mathcal{H}$  as the zeroth homology  $H_{\mathcal{H}}^0(S)$  with respect to the  $t$ -structure associated with  $\mathcal{H}$ , that we have an isomorphism  $l \otimes_{\mathcal{R}} M \xrightarrow{\sim} H_{\mathcal{H}}^0(l \otimes_{\mathcal{R}} S)$  and that the isomorphism  $\mu : l \otimes_{\mathcal{R}} M \xrightarrow{\sim} E$  is induced by

$\sigma : l \otimes_{\mathcal{R}} S \xrightarrow{\mathbb{L}} E$ . Recall that a morphism  $(S_1, \sigma_1) \rightarrow (S_2, \sigma_2)$  of  $\text{Def}_{\mathcal{R}}^h(P)$  is a class of isomorphisms  $S_1 \rightarrow S_2$  of dg  $\mathcal{R}^{op} \otimes \mathcal{A}$ -modules compatible with the  $\sigma_i$  modulo homotopies compatible with the  $\sigma_i$ . Since the functor  $\text{Def}_{\mathcal{R}}^h(P) \rightarrow \text{Def}_{\mathcal{R}}(E)$  is an equivalence, these morphisms are in bijection with the isomorphisms  $S_1 \rightarrow S_2$  of  $D(\mathcal{R}^{op} \otimes \mathcal{A})$  compatible with the  $\sigma_i$ . Clearly each such morphism induces an isomorphism  $(M_1, \lambda_1) \rightarrow (M_2, \lambda_2)$ , where  $(M_i, \lambda_i) = F(S_i, \sigma_i)$ ,  $i = 1, 2$ . It follows from Lemma 2.6 below that this assignment is a bijection. It remains to be shown that  $F : \text{Def}_{\mathcal{R}}^h(P) \rightarrow \text{Def}_{\mathcal{R}}^{cl}(E)$  is essentially surjective. Since we have an equivalence  $\text{Def}_{\mathcal{R}}^h(P) \xrightarrow{\sim} \text{Def}_{\mathcal{R}}(E)$ , it suffices to lift a given object  $(M, \mu)$  of  $\text{Def}_{\mathcal{R}}^{cl}(E)$  to an object  $(S, \sigma)$  of  $\text{Def}_{\mathcal{R}}(E)$ . Let  $A$  denote the dg endomorphism  $l$ -algebra  $\mathbf{R}\text{Hom}_{\mathcal{A}}(M, M)$ . Then  $M$  becomes canonically an object of  $D(A^{op} \otimes \mathcal{A})$ . Now since  $M$  is in the heart of a  $t$ -structure, its negative self-extension groups vanish and we have a quasi-isomorphism  $\tau_{\leq 0} A \xrightarrow{\sim} \text{End}_{\mathcal{H}}(M)$ . Thus, in the homotopy category of dg  $l$ -algebras, we have a morphism

$$\mathcal{R} \rightarrow \text{End}_{\mathcal{H}}(M) \xrightarrow{\sim} \tau_{\leq 0} A \rightarrow A.$$

By tensoring with  $\mathcal{A}$  we obtain a morphism  $\mathcal{R}^{op} \otimes \mathcal{A} \rightarrow A^{op} \otimes \mathcal{A}$  in the homotopy category of dg categories. The associated restriction functor  $D(A^{op} \otimes \mathcal{A}) \rightarrow D(\mathcal{R}^{op} \otimes \mathcal{A})$  sends  $M$  to an object  $S$  of  $D(\mathcal{R}^{op} \otimes \mathcal{A})$ . By construction, the restriction of  $S$  to  $\mathcal{A}$  is isomorphic to  $M$  in  $D(\mathcal{A})$  and the left action of  $\mathcal{R}$  on  $S$  induces the given algebra morphism  $\mathcal{R} \rightarrow \text{End}_{\mathcal{H}}(M)$ . Since  $\mathcal{H}$  is the heart of a  $t$ -structure, we have a canonical realization functor  $D^b(\mathcal{H}) \rightarrow D(\mathcal{A})$  extending the inclusion  $\mathcal{H} \rightarrow D(\mathcal{A})$ , cf. Section 3.1.10 of [6] or Section 3.2 of [48]. Moreover, since  $\mathcal{H}$  is faithful, the realization functor is fully faithful. Since we only know how to compare tensor functors, we use a different construction to extend the inclusion  $\mathcal{H} \rightarrow D(\mathcal{A})$  to a triangle functor  $D^b(\mathcal{H}) \rightarrow D(\mathcal{A})$ . Let  $\mathcal{H}_{dg}$  be the full subcategory of the dg category of right  $\mathcal{A}$ -modules formed by cofibrant resolutions of the objects of  $\mathcal{H}$ . We have an equivalence of  $k$ -categories  $\mathcal{H} \xrightarrow{\sim} H^0(\mathcal{H}_{dg})$ . Since  $\mathcal{H}$  is the heart of a  $t$ -structure, the homology of the dg category  $\mathcal{H}_{dg}$  is concentrated in degrees  $\geq 0$ . Thus, we have quasi-equivalences  $\tau_{\leq 0} \mathcal{H}_{dg} \xrightarrow{\sim} H^0(\mathcal{H}_{dg}) \xrightarrow{\sim} \mathcal{H}$ . Therefore, in the homotopy category of dg categories, we obtain a morphism

$$\mathcal{H} \rightarrow \tau_{\leq 0}(\mathcal{H}_{dg}) \rightarrow \mathcal{H}_{dg} \rightarrow D_{dg}(\mathcal{A})$$

where  $D_{dg}(\mathcal{A})$  denotes the dg category of cofibrant dg  $\mathcal{A}$ -modules. It gives rise to an  $\mathcal{H}$ - $\mathcal{A}$ -bimodule  $R$ . Let  $\mathcal{H}^b(\mathcal{H})$  denote the category modulo homotopy of bounded complexes of objects of  $\mathcal{H}$ . Using the fact that short exact sequences of  $\mathcal{H}$  give rise to triangles in  $D(\mathcal{A})$ , one checks that the induced functor  $? \otimes_{\mathcal{H}} R : \mathcal{H}^b(\mathcal{H}) \rightarrow D(\mathcal{A})$  vanishes on the bounded acyclic complexes and therefore induces a triangle functor  $D^b(\mathcal{H}) \rightarrow D(\mathcal{A})$  still denoted by  $? \otimes_{\mathcal{H}} R$ . Since  $\mathcal{H}$  is a faithful heart, one obtains that this triangle functor is fully faithful. We claim that we have a square of triangle functors commutative up to isomorphism

$$\begin{array}{ccc} D^b(\text{mod } \mathcal{R}) & \xrightarrow{? \otimes_{\mathcal{R}} M} & D^b(\mathcal{H}) \\ \downarrow & & \downarrow ? \otimes_{\mathcal{H}} R \\ D(\mathcal{R}) & \xrightarrow{? \otimes_{\mathcal{A}} S} & D(\mathcal{A}). \end{array}$$

To check this, one has to check that the bimodules  $S$  and  $M \otimes_{\mathcal{H}} R$  are isomorphic in  $D(\mathcal{R}^{op} \otimes \mathcal{A})$ . This is easy using Lemma 2.6 below. Since  $? \otimes_{\mathcal{R}} M : \text{mod } \mathcal{R} \rightarrow \mathcal{H}$  is exact, the given isomorphism  $l \otimes_{\mathcal{R}} M \xrightarrow{\sim} E$  yields an isomorphism  $l \otimes_{\mathcal{R}} M \xrightarrow{\mathbb{L}} E$  in  $D^b(\mathcal{H})$  and thus an isomorphism  $\sigma : l \otimes_{\mathcal{R}} S \xrightarrow{\mathbb{L}} E$  in  $D(\mathcal{A})$ . It is now clear that we can recover  $M$  as  $H_{\mathcal{R}}^0(S)$  and that  $\mu : l \otimes_{\mathcal{R}} M \xrightarrow{\sim} E$  is the morphism induced by  $\sigma : l \otimes_{\mathcal{R}} S \xrightarrow{\sim} E$  in  $H_{\mathcal{R}}^0$ . By the description of  $F$  given above, this shows that we do have  $F(S, \sigma) \xrightarrow{\sim} (M, \mu)$ .  $\square$

Let  $\mathcal{A}$  be a dg category and  $B$  an ordinary  $k$ -algebra. Let  $X$  and  $Y$  be objects of  $D(B^{op} \otimes \mathcal{A})$  and let  $\text{res}(X)$  be the restriction of  $X$  to  $\mathcal{A}$ . The left action of  $B$  on  $X$  defines an algebra morphism

$$\alpha_X : B \rightarrow \text{End}_{D(\mathcal{A})}(\text{res}(X)).$$

Let  $\mathcal{M}(X, Y)$  be the space of all morphisms  $f : \text{res}(X) \rightarrow \text{res}(Y)$  in  $D(\mathcal{A})$  such that

$$f \circ \alpha_X(b) = \alpha_Y(b) \circ f$$

for all  $b \in B$ . The restriction functor induces a natural map

$$\Phi : \text{Hom}_{D(B^{op} \otimes \mathcal{A})}(X, Y) \rightarrow \mathcal{M}(X, Y).$$

**Lemma 2.6.** *If we have*

$$\text{Hom}_{D(\mathcal{A})}(\text{res}(X), \Sigma^{-n} \text{res}(Y)) = 0$$

for all  $n > 0$ , the map  $\Phi$  is bijective.

*Proof.* We adapt the argument of Section 5 of [43]. We may suppose that  $X$  is cofibrant over  $B^{op} \otimes \mathcal{A}$  and in particular cofibrant over  $\mathcal{A}$ . Then the sum total dg module of the bar resolution

$$\dots \longrightarrow B \otimes B^{\otimes p} \otimes X \longrightarrow \dots \longrightarrow B \otimes B \otimes X \longrightarrow B \otimes X \longrightarrow 0$$

is still cofibrant over  $B^{op} \otimes \mathcal{A}$  and quasi-isomorphic to  $X$ . We use it to compute  $\text{Hom}_{D(B^{op} \otimes \mathcal{A})}(X, Y)$ . By applying  $\text{Hom}_{B^{op} \otimes \mathcal{A}}(?, Y)$  to the bar resolution, we get a double complex  $D$  of the form

$$\text{Hom}_{\mathcal{A}}(X, Y) \longrightarrow \text{Hom}_{\mathcal{A}}(B \otimes X, Y) \longrightarrow \dots \longrightarrow \text{Hom}_{\mathcal{A}}(B^{\otimes p} \otimes X, Y) \longrightarrow \dots$$

We have to compute  $H^0$  of the product total complex  $\text{Tot}^\Pi D$ . Let  $D_{\geq 0}$  be the double complex obtained by applying the intelligent truncation functor  $\tau_{\geq 0}$  to each column of  $D$ . Let  $D_{< 0}$  be the kernel of  $D \rightarrow D_{\geq 0}$ . We claim that the product total complex of  $D_{< 0}$  is acyclic. Indeed, the homology of the  $p$ th column of  $D_{< 0}$  in degree  $-q$  is isomorphic to

$$\text{Hom}_{D(\mathcal{A})}(B^{\otimes p} \otimes X, \Sigma^{-q} Y).$$

It vanishes for  $-q < 0$  by our assumption. To show that the product total complex of  $D_{< 0}$  is acyclic, we consider the column filtration  $F_p D_{< 0}$ . Then  $D_{< 0}$  is the inverse limit of the  $F_p D_{< 0}$ . By induction on  $p$ , each  $F_p D_{< 0}$  has an acyclic total complex. Moreover, the transition maps  $F_{p+1} D_{< 0} \rightarrow F_p D_{< 0}$  induce componentwise surjections in the total complexes. It follows that the inverse limit of the total complexes of the  $F_p D_{< 0}$  is still acyclic and this inverse limit is the product total complex of  $D_{< 0}$ . So it is enough to compute  $H^0$  of  $\text{Tot}^\Pi D_{\geq 0}$ . For this, let us denote by

$$\overline{\text{Hom}}_{\mathcal{A}}^0(X, Y)$$

the quotient of  $\text{Hom}_{\mathcal{A}}^0(X, Y)$  by the nullhomotopic morphisms and similarly for  $\overline{\text{Hom}}_{\mathcal{A}}^0(B \otimes X, Y)$ . The space we have to compute is the homology in degree 0 of the total complex of the double complex

$$\begin{array}{ccc} \text{Hom}_{\mathcal{A}}^1(X, Y) & \longrightarrow & \text{Hom}_{\mathcal{A}}^1(B \otimes X, Y) , \\ \uparrow & & \uparrow \\ \overline{\text{Hom}}_{\mathcal{A}}^0(X, Y) & \longrightarrow & \overline{\text{Hom}}_{\mathcal{A}}^0(B \otimes X, Y) \end{array}$$

where the lower left corner is in bidegree  $(0, 0)$ . This equals the intersection of the kernels of the maps from bidegree  $(0, 0)$  to bidegrees  $(0, 1)$  and  $(1, 0)$ . The kernels of the vertical maps are respectively  $\text{Hom}_{D(\mathcal{A})}(X, Y)$  and  $\text{Hom}_{D(\mathcal{A})}(B \otimes X, Y)$  and the space we have to compute thus identifies with the kernel of the map

$$\text{Hom}_{D(\mathcal{A})}(X, Y) \rightarrow \text{Hom}_{D(\mathcal{A})}(B \otimes X, Y)$$

which is easily seen to take  $f : X \rightarrow Y$  to  $f \circ \alpha_X - \alpha_Y \circ f$ . Thus, the homology to be computed is isomorphic to  $\mathcal{M}(X, Y)$ .  $\square$

#### 2.4. Noncommutative crepant resolutions.

**Definition 2.7.** Let  $(R, \mathfrak{m})$  be a complete commutative Noetherian local Gorenstein  $k$ -algebra of Krull dimension  $n$  with isolated singularity and with residue field  $k$ . Denote the category of maximal Cohen–Macaulay (MCM) modules by  $\text{CM}_R$  and its stable category by  $\underline{\text{CM}}_R$ . Let  $N_0 = R, N_1, N_2, \dots, N_t$  be pairwise non-isomorphic indecomposables in  $\text{CM}_R$  and  $A := \text{End}_R(\bigoplus_{i=0}^t N_i)$ . We call  $A$  a *noncommutative resolution* (NCR) of  $R$  if it has finite global dimension. A NCR is called a *noncommutative crepant resolution* (NCCR) if  $A$  further satisfies that

- (a)  $A \in \text{CM}_R$ ;
- (b)  $\text{gldim}(A) = n$ .

If  $A$  is a NCCR, we call  $\bigoplus_{i=0}^t N_i$  a *tilting module*. Under the above conditions, Iyama shows that  $\bigoplus_{i=1}^t N_i$  is a *cluster tilting object* (see definition in Section 3.3) in  $\underline{\text{CM}}_R$ . Denote  $\bar{l}$  for  $A/\text{rad}A$  and  $e_0$  for the idempotent given by the projection  $R \oplus \bigoplus_{i=1}^t N_i \rightarrow R$ . Let  $S_0, S_1, \dots, S_t$  be the simple  $A$ -modules with  $S_0$  corresponding to the summand  $R$  of  $R \oplus \bigoplus_{i=1}^t N_i$ . De Thanhoffer de Völzey and Van den Bergh prove that  $\underline{\text{CM}}_R$  admits an explicit dg model in this case.

**Theorem 2.8.** ([73, Theorem 1.1]) *There exists a finite dimensional graded  $\bar{l}$ -bimodule  $V$  and a minimal model  $(\widehat{T}_{\bar{l}}V, d) \xrightarrow{\sim} A$  for  $A$ , where  $\widehat{T}_{\bar{l}}V$  is the graded completion of the ordinary tensor algebra  $T_{\bar{l}}V$  with respect to the two sided ideal generated by  $V$ . Put  $\Gamma = \widehat{T}_{\bar{l}}V/\widehat{T}_{\bar{l}}Ve_0\widehat{T}_{\bar{l}}V$ . Then one has*

$$\underline{\text{CM}}_R \cong \text{per}(\Gamma)/\text{thick}(S_1, \dots, S_t)$$

and furthermore  $\Gamma$  has the following properties

- (1)  $\Gamma$  has finite dimensional cohomology in each degree;
- (2)  $\Gamma$  is concentrated in negative degrees;
- (3)  $H^0\Gamma = A/Ae_0A$ ;
- (4) As a graded algebra  $\Gamma$  is of the form  $\widehat{T}_lV^0$  for  $V^0 = (1 - e_0)V(1 - e_0)$  with  $l := \bar{l}/ke_0$ .

#### 2.5. Flopping contraction.

**Definition 2.9.** A smooth rational curve  $C$  in a normal variety  $Y$  is called *contractible* if there exists an open subscheme  $Y^\circ \subset Y$  containing  $C$  and a proper birational morphism  $f^\circ : Y^\circ \rightarrow X^\circ$  such that

- (1)  $X^\circ$  is normal,
- (2) the exceptional locus  $\text{Ex}(f^\circ)$  contains  $C$ ,
- (3)  $f^\circ$  is an isomorphism in codimension one.

The above definition of contractibility is more restrictive than the standard one since it rules out the divisorial contraction (by the last condition). If  $Y$  is a 3-fold (which is our main interest), then  $\text{Ex}(f)$  must have dimension one by condition (3). However, it may contain other components besides  $C$ . If  $C$  is a contractible curve in  $Y$ , denote by  $\widehat{X}$  the formal completion of  $X^\circ$  along the exceptional subscheme, i.e. where  $f^\circ$  is not an isomorphism. Consider the Cartesian diagram

$$\begin{array}{ccc} \widehat{Y} & \longrightarrow & Y^\circ \\ \downarrow \widehat{f} & & \downarrow f^\circ \\ \widehat{X} & \longrightarrow & X^\circ \end{array}$$

where  $\widehat{Y}$  is the fiber product. We call  $\widehat{f} : \widehat{Y} \rightarrow \widehat{X}$  the *formal contraction* associate to the contraction  $f^\circ : Y^\circ \rightarrow X^\circ$ .

The following definition is a special case of Definition 6.10 of [52].

**Definition 2.10.** Let  $Y$  be a normal variety of dimension 3. A *flopping contraction* is a proper birational morphism  $f : Y \rightarrow X$  to a normal variety  $Y$  such that  $f$  is an isomorphism in codimension one, and  $K_Y$  is  $f$ -trivial. If  $Y$  is smooth, then we call  $f$  a smooth flopping contraction.

In this paper, we only consider smooth flopping contractions unless stated otherwise. Given a 3-dimensional flopping contraction  $f : Y \rightarrow X$ , let  $D$  be a divisor on  $Y$  such that  $-(K_Y + D)$  is  $f$ -ample. By Theorem 6.14 of [52], there exists a  $D$ -flop  $f^+ : Y^+ \rightarrow X$ . To be more precise,  $f^+$  is a proper birational morphism that is an isomorphism in codimension one, and  $K_{Y^+} + D^+$  is  $f^+$ -ample where  $D^+$  is the birational transform of  $D$  on  $Y^+$ . In particular,  $X$  is Gorenstein terminal. Without loss of generality, we may work locally near the exceptional fiber of  $f$ . By the classification theorem of 3-dimensional terminal singularities,  $X$  has an isolated cDV singularity (see Corollary 5.38 of [52]). Recall that a 3-fold singularity  $(X, 0)$  is called cDV if a generic hypersurface section  $0 \in H \subset X$  is a Du Val singularity. Because  $H$  has embedded dimension 3,  $X$  has embedded dimension 4, i.e.  $X$  is a hypersurface.

Denote by  $\text{Ex}(f)$  the reduced exceptional fiber of  $f$ . It is well known that  $\text{Ex}(f)$  is a tree of rational curves

$$\text{Ex}(f) = \bigcup_{i=1}^t C_i$$

with normal crossings such that  $C_i \cong \mathbb{P}^1$  (c.f. Lemma 3.4.1 of [74]). We call a 3-dimensional flopping contraction  $f : Y \rightarrow X$  *simple* if  $\text{Ex}(f) \cong \mathbb{P}^1$ . Let  $p$  be the singular point of  $X$ . By the remark above,  $R := \widehat{\mathcal{O}}_{X,p}$  is a complete local ring of the form  $k[[x, y, u, v]]/(g)$ . Now we set  $\widehat{X} := \text{Spec } R$  and  $\widehat{Y} := \widehat{X} \times_X Y$  and denote by  $\widehat{f}$  the base change of  $f$ . We call the triple  $(\widehat{Y}, \widehat{f}, R)$  the *formal contraction* associated to the flopping contraction  $f : Y \rightarrow X$ . Note that  $\widehat{Y}$  is Calabi-Yau.

Let  $(\widehat{Y}, \widehat{f}, R)$  be a formal flopping contraction. Now we consider the NCCR associated to a three dimensional flopping contraction  $\widehat{f} : \widehat{Y} \rightarrow \widehat{X}$  constructed as follows. For  $i = 1, \dots, t$ , let  $\widehat{\mathcal{L}}_i$  be a line bundle on  $\widehat{Y}$  such that  $\deg_{C_j} \widehat{\mathcal{L}}_i = \delta_{ij}$ . Define  $\widehat{\mathcal{N}}_i$  to be given by the maximal extension

$$(2.1) \quad 0 \longrightarrow \widehat{\mathcal{L}}_i^{-1} \longrightarrow \widehat{\mathcal{N}}_i \longrightarrow \mathcal{O}_{\widehat{Y}}^{\oplus r_i} \longrightarrow 0$$

associated to a minimal set of  $r_i$  generators of  $H^1(\widehat{Y}, \widehat{\mathcal{L}}_i^{-1})$ . Set  $N_i := \mathbf{R}f_* \widehat{\mathcal{N}}_i = f_* \widehat{\mathcal{N}}_i$  for  $i = 1, \dots, t$ . We set

$$(2.2) \quad A := \text{End}_{\widehat{Y}}(\mathcal{O}_{\widehat{Y}} \oplus \widehat{\mathcal{N}}_1 \oplus \dots \oplus \widehat{\mathcal{N}}_t) \cong \text{End}_R(R \oplus N_1 \oplus \dots \oplus N_t).$$

The second isomorphism can be proved as follows. It is clear that the natural morphism from  $A$  to  $\text{End}_R(R \oplus N_1 \oplus \dots \oplus N_t)$  is an isomorphism away from the exceptional locus, which has codimension 3. Since both are reflexive  $R$ -modules, it must be an isomorphism.

For simplicity, we denote by  $\widehat{\mathcal{N}}$  the direct sum  $\bigoplus_{i=1}^t \widehat{\mathcal{N}}_i$  and denote by  $N$  the direct sum  $\bigoplus_{i=1}^t N_i$ .

**Theorem 2.11.** ([74, Corollary 3.2.10]) *The functor  $\widehat{F} := \mathbf{R}\text{Hom}_{\widehat{Y}}(\mathcal{O}_{\widehat{Y}} \oplus \widehat{\mathcal{N}}, -)$  defines a triangle equivalence between  $D^b(\text{coh } \widehat{Y})$  and  $D^b(\text{Mod } A)$ , with quasi-inverse  $F^{-1} := (-) \otimes_A (\mathcal{O}_{\widehat{Y}} \oplus \widehat{\mathcal{N}})$ . In addition,  $A$  is itself Cohen-Macaulay.*

It follows that  $A$  is a NCCR.

**Corollary 2.12.** *Let  $(\widehat{Y}, \widehat{f}, R)$  be a 3-dimensional formal flopping contraction. Then*

- (1) *The structure sheaves of the irreducible components  $\{C_i\}_{i=1}^t$  of  $\text{Ex}(\widehat{f})$  form a semi-simple collection in  $\text{coh } (\widehat{Y})$ .*
- (2) *The derived deformation algebra  $\Gamma_C^{\widehat{Y}}$  of the collection  $C := \{C_i\}_{i=1}^t$  is linked by quasi-isomorphisms to  $\Gamma$  in Theorem 2.8.*
- (3) *For any  $i = 1, \dots, t$ ,  $C_i$  is nc rigid.*

*Proof.* To prove that  $\text{Hom}_{\widehat{Y}}(\mathcal{O}_{C_i}, \mathcal{O}_{C_j}) = 0$  for  $i \neq j$ , we simply need to use the condition that  $\text{Ex}(\widehat{f})$  is a tree of rational curves with normal crossings.

For any  $i = 1, \dots, t$ , it is easy to check that  $S_i \cong \widehat{F}(\Sigma \mathcal{O}_{C_i}(-1))$ . Note that  $(\bigoplus_{i=1}^t \mathcal{O}_{C_i}) \otimes (\bigotimes_{i=1}^t \mathcal{L}_i^{-1}) \cong \bigoplus_{i=1}^t \mathcal{O}_{C_i}(-1)$ . By Theorem 2.11,  $\widehat{F}$  induces an isomorphism of  $A_\infty$  algebras

$$(2.3) \quad \text{Ext}_{\widehat{Y}}^*(\bigoplus_{i=1}^t \mathcal{O}_{C_i}, \bigoplus_{i=1}^t \mathcal{O}_{C_i}) \cong \text{Ext}_A^*(\bigoplus_{i=1}^t S_i, \bigoplus_{i=1}^t S_i).$$

Let  $\bar{l} = A/\text{rad}(A)$  and  $l = \bar{l}/ke_0$ . Then there is a natural isomorphism of  $A$ -modules  $l \cong \bigoplus_{i=0}^t S_i$ , where  $S_0$  is the simple  $A$ -module that corresponds to the summand  $R$  of  $R \oplus \bigoplus_{i=1}^t N_i$ . By Lemma 4.1 of [73], the vector space  $V$  in Theorem 2.8 can be chosen as  $D(\Sigma \text{Ext}_A^{\geq 1}(\bar{l}, \bar{l}))$ . Therefore,  $\Gamma := \widehat{T}_l V / \widehat{T}_l V e_0 \widehat{T}_l V \cong \mathbb{D}B(\text{Ext}_A^*(l, l))$  represents the noncommutative deformations of semi-simple collection  $\{S_i\}_{i=1}^t \in \text{D}^b(\text{mod } A)$ . Part (2) follows from the isomorphism 2.3. Denote by  $\Gamma_{C_i}^{\widehat{Y}}$  the derived deformation algebra of  $\mathcal{O}_{C_i}$ . We have

$$\Gamma_{C_i}^{\widehat{Y}} \cong \Gamma / \sum_{j \neq i} \Gamma e_j \Gamma, \text{ and } H^0 \Gamma_{C_i}^{\widehat{Y}} \cong H^0 \Gamma / \sum_{j \neq i} H^0 \Gamma e_j H^0 \Gamma.$$

Then part (3) follows from (1) of Theorem 2.8. □

### 3. CALABI-YAU STRUCTURE AND CLUSTER CATEGORY

In this section, we first review several notions of Calabi-Yau property for triangulated categories, for homologically smooth dg algebras and for proper dg algebras. Then we recall geometric versions of the Calabi-Yau property and translate them into algebraic notions for endomorphism algebras of generators respectively for derived deformation algebras. Finally, we classify Calabi-Yau structures for 3-dimensional flopping contractions and review the cluster category.

#### 3.1. CY structures.

3.1.1. *CY triangulated categories.* Let  $\mathcal{T}$  be a Hom-finite  $k$ -linear triangulated category.

**Definition 3.1.** A *right Serre functor* for  $\mathcal{T}$  is a triangle functor  $S: \mathcal{T} \rightarrow \mathcal{T}$  such that there are bifunctorial isomorphisms

$$\mathcal{T}(Y, SX) \rightarrow D\mathcal{T}(X, Y)$$

for all  $X, Y \in \mathcal{T}$ . It is a *Serre functor* if it is an autoequivalence.

One can show that a right Serre functor exists if and only if for each object  $X$  of  $\mathcal{T}$ , the functor  $D\mathcal{T}(X, ?)$  is representable and in this case, the right Serre functor is unique up to canonical isomorphism of triangle functors [8, 75]. Let  $d$  be an integer. The triangulated category  $\mathcal{T}$  is  $d$ -Calabi-Yau if it admits a Serre functor isomorphic to  $\Sigma^d$ .

**3.1.2. CY smooth dg algebras.** A dg-\$k\$-algebra \$\Gamma\$ is called *homologically smooth* if \$\Gamma\$ is perfect in \$\mathrm{D}(\Gamma^e)\$. Then one checks that \$\mathrm{D}\_{fd}(\Gamma)\$, the subcategory of \$\mathrm{D}(\Gamma)\$ consisting of the modules whose homology is of finite total dimension, is contained in the perfect derived category \$\mathrm{per}(\Gamma)\$. Put

$$\Theta_\Gamma = \mathbf{R}\mathrm{Hom}_{\Gamma^e}(\Gamma, \Gamma^e).$$

Then we have a canonical isomorphism

$$\mathrm{HH}_d(\Gamma) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\Gamma^e)}(\Theta_\Gamma, \Sigma^{-d}\Gamma).$$

**Definition 3.2.** The dg algebra \$\Gamma\$ is called *bimodule dCY* if it is homologically smooth and there is an isomorphism in \$\mathrm{D}(\Gamma^e)\$

$$\eta : \Theta_\Gamma \xrightarrow{\sim} \Sigma^{-d}\Gamma.$$

A class \$\eta \in \mathrm{HH}\_d(\Gamma)\$ is called a *dCY structure* if the corresponding morphism \$\eta : \Theta\_\Gamma \rightarrow \Sigma^{-d}\Gamma\$ is an isomorphism in \$\mathrm{D}(\Gamma^e)\$. A dCY structure \$\eta\$ is called *exact* if there exists a class \$\xi \in \mathrm{HC}\_{d-1}(\Gamma)\$ such that \$B\xi = \eta\$, where \$B\$ is the Connes morphism. A choice of \$\xi\$ will be called an *exact lifting* of the dCY structure \$\eta\$. We call a bimodule dCY algebra \$\Gamma\$ an *exact dCY algebra* if the dCY structure is exact in addition.

**Definition 3.3.** The dg algebra \$\Gamma\$ is said to satisfy the *relative dCY property* if for \$L \in \mathrm{D}\_{fd}(\Gamma)\$ and \$M \in \mathrm{per}(\Gamma)\$, we have a bifunctorial isomorphism

$$\mathbf{DR}\mathrm{Hom}_{\mathrm{D}(\Gamma)}(L, M) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{D}(\Gamma)}(M, \Sigma^d L).$$

**Remark 3.4.** If \$\Gamma\$ is a topologically homologically smooth pseudo-compact dg algebra in \$\mathrm{PCAlg}(l)\$, we call \$\Gamma\$ a bimodule dCY if \$\eta\$ is an isomorphism in the pseudo-compact derived category of bimodules. The isomorphism \$\eta\$ represents a class in the continuous Hochschild homology \$\mathrm{HH}\_d(\Gamma)\$. Exactness is defined similarly by taking the continuous cyclic homology. We call a bimodule dCY pseudo-compact dg algebra \$\Gamma\$ in \$\mathrm{PCAlg}(l)\$ an *exact dCY algebra* if the dCY structure is exact in addition. Similarly, for a pseudo-compact algebra \$\Gamma\$ in \$\mathrm{PCAlg}(l)\$, the relative dCY property is defined by replacing \$\mathrm{D}(\Gamma), \mathrm{per}(\Gamma)\$ and \$\mathrm{D}\_{fd}(\Gamma)\$ with their pseudo-dompact counter-parts.

Given a homologically smooth dg algebra \$\Gamma\$, it follows from Lemma 3.4 in [46] that we have the implications

$$\begin{aligned} \Gamma \text{ is bimodule dCY} &\Rightarrow \Gamma \text{ satisfies the relative dCY property} \\ &\Rightarrow \mathrm{D}_{fd}(\Gamma) \text{ is a Hom-finite dCY triangulated category.} \end{aligned}$$

A similar chain of implications holds in the pseudo-compact case.

**3.1.3. CY proper dg algebras.** Let \$A\$ be a dg algebra. Suppose that \$A\$ is *proper*, i.e. that its homology is of finite total dimension. Then the category \$\mathrm{per}(A)\$ is Hom-finite. The proper dg algebra \$A\$ is called *perfectly dCY* if there is an isomorphism

$$DA \xrightarrow{\sim} \Sigma^d A$$

in \$\mathrm{D}(A^e)\$. By the following lemma, the triangulated category \$\mathrm{per}(A)\$ is then dCY.

**Lemma 3.5.** *Let \$A\$ be an arbitrary dg algebra. Then for \$P\$ perfect and \$M\$ arbitrary in \$\mathrm{D}(A)\$, there is a canonical isomorphism*

$$\mathbf{R}\mathrm{Hom}_A(M, P \overset{\mathbb{L}}{\otimes}_A DA) \xrightarrow{\sim} \mathbf{DR}\mathrm{Hom}_A(P, M).$$

*Proof.* We have a canonical isomorphism

$$P \overset{\mathbb{L}}{\otimes}_A \mathrm{Hom}_k(A, k) \rightarrow \mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_A(P, A), k)$$

which takes a tensor  $p \otimes f$  to the map taking  $g : P \rightarrow A$  to  $\pm f(g(p))$ . It induces an isomorphism

$$\mathbf{R}\mathrm{Hom}_A(M, P \overset{\mathbb{L}}{\otimes}_A DA) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_A(M, \mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_A(P, A), k)).$$

The last term is canonically isomorphic to

$$\mathrm{Hom}_k(M \overset{\mathbb{L}}{\otimes}_A \mathbf{R}\mathrm{Hom}_A(P, A), k) = D\mathbf{R}\mathrm{Hom}_A(P, M).$$

□

**Lemma 3.6.** *Let  $B$  be a smooth dg algebra and  $L$  a dg  $B$ -module whose homology is of finite total dimension. Let  $A = \mathbf{R}\mathrm{Hom}_B(L, L)$  be the dg endomorphism algebra of  $L$ .*

- a)  *$A$  is proper.*
- b) *If  $B$  is bimodule dCY, then  $A$  is perfectly dCY.*

*Proof.* To clarify the notation, let  $M$  be another dg  $B$ -module whose homology is of finite total dimension (take  $M = L$  for example). Then by Lemma 3.4 of [46], the complex  $\mathbf{R}\mathrm{Hom}_A(L, M)$  has its homology of finite total dimension. This implies a). For b), we have to produce an isomorphism

$$\Sigma^d A \xrightarrow{\sim} \mathrm{Hom}_k(A, k)$$

in  $D(A^e)$ . Let us put  $A_L = \mathbf{R}\mathrm{Hom}_B(L, L)$  and  $A_M = \mathbf{R}\mathrm{Hom}_B(M, M)$ . We construct more generally an isomorphism

$$\Sigma^d \mathbf{R}\mathrm{Hom}_B(L, M) \xrightarrow{\sim} \mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_B(M, L), k)$$

in  $D(A_L^{op} \otimes A_M)$ . For this, we choose a quasi-isomorphism  $P \rightarrow B$  with cofibrant  $P$  in the category of dg  $B^e$ -modules (for example, for  $P$  we can take the bimodule bar resolution of  $B$ ). We assume, as we may, that  $L$  and  $M$  are cofibrant as a dg  $B$ -modules so that we may choose  $A_L = \mathrm{Hom}_B(L, L)$  and  $A_M = \mathrm{Hom}_B(M, M)$  and  $\mathbf{R}\mathrm{Hom}_B(L, M) = \mathrm{Hom}_B(L, M)$ . The homotopy equivalence  $L \otimes_B P \rightarrow L$  induces a quasi-isomorphism of  $A_L$ - $A_M$ -bimodules

$$\mathrm{Hom}_B(L, M) \xrightarrow{\sim} \mathrm{Hom}_B(L \otimes_B P, M).$$

Let us choose a representative

$$\Sigma^{-d} P \rightarrow \mathrm{Hom}_{B^e}(P, B^e)$$

of the isomorphism  $\Sigma^{-d} B \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{B^e}(B, B^e)$  in  $D(B^e)$  giving the  $d$ -Calabi-Yau structure on  $B$ . We have the following chain of quasi-isomorphisms of  $A_L$ - $A_M$ -bimodules

$$\begin{aligned} M \otimes_B \Sigma^{-d} P \otimes_B DL &\rightarrow \mathrm{Hom}_k(L, M) \otimes_{B^e} \Sigma^{-d} P \\ &\rightarrow \mathrm{Hom}_k(L, M) \otimes_{B^e} \mathrm{Hom}_{B^e}(P, B^e) \\ &\rightarrow \mathrm{Hom}_{B^e}(P, \mathrm{Hom}_k(L, M)) \\ &= \mathrm{Hom}_B(L \otimes_B P, M). \end{aligned}$$

Here, for the first quasi-isomorphism, we use that the homology of  $L$  is of finite total dimension, for the second one, the lift of the Calabi-Yau isomorphism, for the third one that  $P$  is perfect and the last equality is a canonical isomorphism of dg modules. By passing to the  $k$ -dual, we deduce a quasi-isomorphism of  $A_M$ - $A_L$ -bimodules

$$D\mathrm{Hom}_B(L \otimes_B P, M) \rightarrow \mathrm{Hom}_k(M \otimes_B \Sigma^{-d} P \otimes_B DL, k) = \mathrm{Hom}_B(M \otimes_B \Sigma^{-d} P, \mathrm{Hom}_k(DL, k)).$$

Since  $L$  has homology of finite total dimension, the canonical morphism  $L \rightarrow DDL$  is invertible so that we get a quasi-isomorphism

$$\mathrm{Hom}_B(M \otimes_B \Sigma^{-d} P, L) \rightarrow \mathrm{Hom}_B(M \otimes_B \Sigma^{-d} P, \mathrm{Hom}_k(DL, k)).$$

Since the left hand term represents  $\Sigma^d \mathbf{R}\mathrm{Hom}_B(M, L)$  in  $D(A_M \otimes A_L^{op})$ , we are finished. □

**Definition 3.7.** Let  $k$  be a field of characteristic zero. Given a finite-dimensional  $A_\infty$ -algebra  $A$ , a *cyclic  $A_\infty$ -structure* of degree  $d$  on  $A$  is a non degenerate symmetric bilinear form

$$(\cdot, \cdot) : A \times A \rightarrow \Sigma^d A$$

of degree  $d$  such that

$$(m_n(a_1, \dots, a_n), a_{n+1}) = (-1)^n (-1)^{|a_1|(|a_2| + \dots + |a_{n+1}|)} (m_n(a_2, \dots, a_{n+1}), a_1)$$

In this case, we have in particular an isomorphism  $DA \xrightarrow{\sim} \Sigma^d A$  in the derived category of  $A$ -bimodules. Thus, if a dg algebra is quasi-isomorphic to an  $A_\infty$ -algebra admitting a cyclic  $A_\infty$ -structure of degree  $d$ , then it is perfectly dCY.

**Lemma 3.8.** *Let  $T \geq 1$  be an integer,  $l$  a product of  $t$  copies of  $k$  and  $A$  a dg algebra augmented over  $l$  and such that  $A$  belongs to the triangulated subcategory of  $D(A \otimes A^{op})$  generated by  $l \otimes l^{op}$  (in particular, the dg algebra  $A$  is proper). Let  $\Gamma$  be the pseudo-compact dg algebra  $\mathbb{D}BA$ , where  $B$  denotes the bar construction over  $l$ . Then  $\Gamma$  is topologically homologically smooth. Moreover, if  $A$  is perfectly  $d$ -Calabi–Yau, then  $\Gamma$  is bimodule  $d$ -Calabi–Yau.*

*Proof.* We refer to [40] for a brief summary of the bar-cobar formalism that we are going to use and to [55] and Appendix A of [76] for in-depth treatments. Let  $C$  be the augmented dg coalgebra  $BA$  and  $\tau : C \rightarrow A$  the canonical twisting cochain. Since  $\tau$  is acyclic, the canonical morphism

$$C \rightarrow C \otimes_\tau A \otimes_\tau C$$

is a weak equivalence of dg  $C$ -bicomodules (cf. [40]). Recall (for example from Appendix A of [76]) that a dg comodule is fibrant if its underlying graded comodule is cofree. Clearly, this holds for  $C \otimes_\tau A \otimes_\tau C$ . Since  $A$  belongs to the triangulated subcategory of the derived category of  $A \otimes A^{op}$  generated by  $l \otimes l$ , the dg bicomodule  $C \otimes_\tau A \otimes_\tau C$  belongs to the triangulated subcategory of the coderived category of  $C \otimes C^{op}$  generated by  $C \otimes C^{op}$ . By applying the duality  $\mathbb{D}$  we obtain that the morphism

$$(3.1) \quad \Gamma \otimes_\tau \mathbb{D}A \otimes_\tau \Gamma \rightarrow \Gamma$$

is a weak equivalence and that the object on the left is cofibrant. Moreover, we see that  $\Gamma$  belongs to the perfect derived category of  $\Gamma \otimes \Gamma^{op}$ . This means that  $\Gamma$  is topologically homologically smooth. Now suppose that  $A$  is perfectly  $d$ -Calabi–Yau. Since  $A$  is proper, it is weakly equivalent to its pseudo-compact completion  $\widehat{A}$ . By the Calabi–Yau property, we have an isomorphism  $\mathbb{D}A \xrightarrow{\sim} \Sigma^d \widehat{A}$  in the pseudo-compact derived category of dg  $\widehat{A}$ -bimodules. Now we compute the inverse dualizing complex of  $\Gamma$  using the resolution 3.1. We have isomorphisms in the pseudo-compact derived category of  $\Gamma$ -bimodules

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_{\Gamma^e}(\Gamma, \Gamma^e) &= \mathrm{Hom}_{\Gamma^e}(\Gamma \otimes_\tau \mathbb{D}A \otimes_\tau \Gamma, \Gamma^e) \\ &= \mathrm{Hom}_{psc}^{\tau, \tau}(\mathbb{D}A, \Gamma^e) \\ &= \mathrm{Hom}_{psc}^{\tau, \tau}(\Sigma^d \widehat{A}, \Gamma^e) \\ &= \Sigma^{-d} \Gamma \otimes_\tau \mathbb{D}A \otimes_\tau \Gamma \\ &= \Sigma^{-d} \Gamma. \end{aligned}$$

Here  $\mathrm{Hom}_{psc}^{\tau, \tau}$  denotes the space of morphisms in the category of pseudo-compact vector spaces twisted twice by  $\tau$ . This shows that  $\Gamma$  is topologically homologically bimodule  $d$ -Calabi–Yau.  $\square$

3.1.4. *CY structures in geometry, algebraic consequences.* We let  $k = \mathbb{C}$  be the field of complex numbers unless specified otherwise.

**Definition 3.9.** Let  $Y$  be a  $d$ -dimensional smooth quasi-projective  $\mathbb{C}$ -variety. We call  $Y$  a  $d$ -dimensional Calabi–Yau variety if there is an isomorphism  $\omega_Y := \Omega_Y^d \cong \mathcal{O}_Y$ , i.e. there exists a nowhere vanishing  $d$ -form. We call a nowhere vanishing section  $\eta : \mathcal{O}_Y \rightarrow \omega_Y$  a *dCY structure* on  $Y$ . We call the dCY structure *exact* if the the  $d$ -form  $\eta$  is exact, i.e. there exists a  $(d-1)$ -form  $\xi \in \Omega_Y^{d-1}$  such that  $d\xi = \eta$ . A choice of  $\xi$  will be called an *exact lifting* of the dCY structure  $\eta$ . If  $\widehat{Y}$  is a smooth formal scheme, we may define dCY structure in a similar way by considering de Rham complex of formal scheme  $\Omega_{\widehat{Y}}^*$ .

Recall from Theorem 3.1.1 of [9] that the derived category of quasi-coherent sheaves on a quasi-compact separated scheme admits a single perfect generator.

**Lemma 3.10.** *Let  $X$  be a quasi-projective variety,  $G$  a perfect generator of  $D(\mathrm{Qcoh} X)$  and  $A$  its derived endomorphism dg algebra. Then for  $L \in \mathrm{per}(X)$ , we have a functorial isomorphism in the derived category  $D(A^{op})$*

$$\mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_X(G, L), A) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_X(G^\vee, L^\vee),$$

where on the left,  $A$  acts on the left on the second argument  $A$  and on the right,  $A$  acts on the right on the first argument  $G^\vee$ .

*Proof.* We may and will assume that  $G$  is a left bounded complex with injective components and that  $L$  belongs to the thick subcategory generated by  $G$  in the homotopy category of complexes with injective components over  $\mathrm{Qcoh} X$ . We define  $A$  to be the differential graded endomorphism algebra  $\mathrm{Hom}_X(G, G)$  in the dg category of complexes over  $\mathrm{Qcoh} X$ , where we write  $\mathrm{Hom}_X(., .)$  for the morphism complexes of this category. By definition, we have

$$\mathbf{R}\mathrm{Hom}_X(G, L) = \mathrm{Hom}_X(G, L)$$

and this (right) dg  $A$ -module lies in the thick subcategory of the homotopy category of dg  $A$ -modules generated by  $A = \mathrm{Hom}_X(G, G)$ . Thus, we may compute

$$\mathbf{R}\mathrm{Hom}_A(\mathbf{R}\mathrm{Hom}_X(G, L), A) = \mathrm{Hom}_A(\mathrm{Hom}_X(G, L), A) = \mathrm{Hom}_A(\mathrm{Hom}_X(G, L), \mathrm{Hom}_X(G, G)).$$

Since  $\mathbf{R}\mathrm{Hom}(G, ?)$  is an equivalence, we have a quasi-isomorphism of dg  $A^{op}$ -modules (=left dg  $A$ -modules)

$$\mathrm{Hom}_X(L, G) \rightarrow \mathrm{Hom}_A(\mathrm{Hom}_X(G, L), \mathrm{Hom}_X(G, G)).$$

Let  $\mathcal{O} \rightarrow I$  be an injective resolution and define  $L^\vee$  using the sheaf-valued Hom with values in  $I$ . Since  $?^\vee$  is an equivalence  $\mathrm{per}(X)^{op} \xrightarrow{\sim} \mathrm{per}(X)$ , we have a quasi-isomorphism

$$A^{op} = \mathrm{Hom}_X(G, G)^{op} \rightarrow \mathrm{Hom}_X(G^\vee, G^\vee)$$

and a quasi-isomorphism of left dg  $A$ -modules

$$\mathrm{Hom}_X(L, G) \rightarrow \mathrm{Hom}_X(G^\vee, L^\vee).$$

We thus obtain quasi-isomorphisms of left dg  $A$ -modules

$$\mathrm{Hom}_A(\mathrm{Hom}_X(G, L), A) = \mathrm{Hom}_A(\mathrm{Hom}_X(G, L), \mathrm{Hom}_X(G, G)) \leftarrow \mathrm{Hom}_X(L, G) \rightarrow \mathrm{Hom}_X(G^\vee, L^\vee)$$

and this proves the claim.  $\square$

**Proposition 3.11.** *Let  $Y$  be a smooth quasi-projective Calabi–Yau  $d$ -fold.*

- a) *Let  $G$  be a perfect generator of  $D(\mathrm{Qcoh} Y)$  and  $B$  its derived endomorphism dg algebra. Then  $B$  is bimodule  $d$ -Calabi–Yau.*

- b) Let  $L$  be a bounded complex of coherent sheaves with compact support on  $Y$  and  $A$  its derived endomorphism dg algebra. Then  $A$  is perfectly  $d$ -Calabi-Yau.

*Proof.* a) In this proof, for simplicity, we write ordinary functors for derived functors. For a perfect object  $P$ , let  $P^\vee$  denote its dual (the derived functor of the sheaf valued Hom-functor with values in  $\mathcal{O}_Y$ ). The functor taking  $P$  to  $P^\vee$  is an equivalence  $\text{per}(Y)^{op} \rightarrow \text{per}(Y)$ . Therefore, the object  $G^\vee$  is a classical generator of  $\text{per}(Y)$  and thus another generator of  $D(\text{Qcoh } Y)$ . By Lemma 3.4.1 of [9], the object  $G \boxtimes G^\vee$  is a perfect generator of  $D(\text{Qcoh}(Y \times Y))$  and clearly its derived endomorphism dg algebra is  $B \otimes B^{op}$ . Let  $F^e$  denote the equivalence  $\text{Hom}(G \boxtimes G^\vee, ?)$  from  $D(\text{Qcoh}(Y \times Y))$  to  $D(B^e)$ . Let  $\Delta : Y \rightarrow Y \times Y$  denote the diagonal embedding. By Lemma 3.10 of [27], the image under  $F^e \Delta_*$  of the inverse of the canonical bundle of  $Y$  is  $\Sigma^d \Theta_B$ , where  $\Theta_B = \text{Hom}_{B^e}(B, B^e)$  is the inverse dualizing complex of  $B$ . Since the Lemma in [27] is stated and proved in the context of a smooth and *projective* variety, we repeat the argument. We first check that  $F^e(\Delta_* \mathcal{O}_Y)$  is isomorphic to the identity bimodule  $B$  in  $D(B^e)$ . Indeed, the equivalence

$$\text{Hom}(G \boxtimes G^\vee, ?) : D(\text{Qcoh}(Y \times Y)) \xrightarrow{\sim} D(B \otimes B^{op})$$

is the one introduced at the beginning of the proof of theorem 8.9 of [71]. It shows in particular that we have a bijection between the isomorphism classes of Fourier-Mukai kernels on  $Y$  and the isomorphism classes of dg  $B$ -bimodules (in their respective derived categories). Moreover, by Corollary 8.12 of [loc. cit.] both of these are in natural bijection with the continuous morphisms  $D_{dg}(\text{Qcoh } Y) \rightarrow D_{dg}(\text{Qcoh } Y)$  in the homotopy category of (large) dg categories. The explicit description of the bijection given after Corollary 8.12 of [loc. cit.] also shows that it is compatible with compositions of Fourier-Mukai kernels respectively the derived tensor product of dg bimodules. Thus, the identity Fourier-Mukai kernel  $\Delta_* \mathcal{O}_Y$  has to correspond to the identity bimodule  $B$  in  $D(B \otimes B^{op})$ .

For an object  $M$  of  $D(B^e)$ , we put  $M^\vee = \text{Hom}_{B^e}(M, B^e)$  and consider this as an object in the derived category  $D((B^e)^{op})$  of left  $B^e$ -modules. If we apply Lemma 3.10 to  $X = Y \times Y$ , the generator  $G \boxtimes G^\vee$  and an object  $L \in \text{per}(Y \times Y)$ , we obtain a canonical isomorphism

$$F^e(L^\vee) = \mathbf{R}\text{Hom}_{B^e}(\mathbf{R}\text{Hom}_{Y \times Y}(G \boxtimes G^\vee, L), B^e) \xrightarrow{\sim} \mathbf{R}\text{Hom}_{Y \times Y}((G \boxtimes G^\vee)^\vee, L^\vee)$$

in  $D((B^e)^{op})$ , where on the left,  $B^e$  acts by left multiplication on itself and on the right,  $B^e$  acts from the right on  $(G \boxtimes G^\vee)^\vee$ . Let us write  $B^e = B \otimes B^{op} = B_1 \otimes B_2^{op}$  and  $(G \boxtimes G^\vee)^\vee = (G_1 \boxtimes G_2^\vee)^\vee$ . Then the canonical isomorphism

$$(G_1 \boxtimes G_2^\vee)^\vee \xrightarrow{\sim} G_2 \boxtimes G_1^\vee$$

is compatible with the given action on the left and the canonical action twisted by the flip

$$\tau : (B_1 \otimes B_2^{op})^{op} \xrightarrow{\sim} B_2 \otimes B_1^{op}$$

on the right. Thus, we obtain an isomorphism

$$\mathbf{R}\text{Hom}_{B^e}(\mathbf{R}\text{Hom}_{Y \times Y}(G \boxtimes G^\vee, L), B^e) \xrightarrow{\sim} \mathbf{R}\text{Hom}_{Y \times Y}(G \boxtimes G^\vee, L^\vee)$$

in  $D((B^e)^{op})$ , where the action on the left is given by left multiplication on  $B^e$  and the action on the right is the natural one on  $G \boxtimes G^\vee$  twisted by the flip.

Let  $\omega_Y$  denote the canonical sheaf of  $Y$ . To conclude that  $\Sigma^{-d} \Delta_*(\omega_Y^{-1})$  corresponds to the inverse dualizing complex  $B^\vee$  through the equivalence  $F^e$ , it now suffices to show that we have an isomorphism

$$(\Delta_* \mathcal{O}_Y)^\vee \cong \Sigma^{-d} \Delta_* \omega_Y^{-1}.$$

Now since  $Y$  is smooth, we can apply Grothendieck duality and find

$$(\Delta_* \mathcal{O}_Y)^\vee = \mathcal{H}\text{om}(\Delta_* \mathcal{O}_Y, \mathcal{O}_{Y \times Y}) \cong \Delta_* \mathcal{H}\text{om}(\mathcal{O}_Y, (\Delta^* \mathcal{O}_{Y \times Y}) \otimes \Sigma^{-d} \omega_\Delta),$$

where  $d = \dim Y$  equals the relative dimension of  $\Delta$  and

$$\omega_\Delta = \omega_Y \otimes \Delta^* \omega_{Y \times Y}^{-1}$$

is the relative dualizing bundle of  $\Delta$ . Now  $\mathcal{H}om(\mathcal{O}_Y, ?)$  is the identity functor and  $\Delta^* \mathcal{O}_{Y \times Y}$  is  $\mathcal{O}_Y$ . Thus, we have

$$\Delta_* \mathcal{H}om(\mathcal{O}_Y, (\Delta^* \mathcal{O}_{Y \times Y}) \otimes \Sigma^{-d} \omega_\Delta) \cong \Delta_*(\omega_Y \otimes \Delta^* \Sigma^{-d} \omega_{Y \times Y}^{-1})$$

Finally, the adjunction formula for  $\Delta : Y \rightarrow Y \times Y$  implies that  $\omega_Y$  is isomorphic to the tensor product of  $\Delta^* \omega_{Y \times Y}$  with the top exterior power of the normal bundle, which identifies with the canonical bundle. We deduce that  $\Delta^* \omega_{Y \times Y}$  is isomorphic to  $\omega_Y^{\otimes 2}$ . This implies the result.

b) Under the equivalence  $\mathbf{R}\mathrm{Hom}(G, ?) : D(\mathrm{Qcoh} Y) \rightarrow D(B)$ , the object  $L$  is taken to a dg module whose homology is of finite total dimension. Since  $B$  is smooth and bimodule  $d\mathrm{CY}$ , the claim follows from Lemma 3.6.  $\square$

From the relative Calabi–Yau property for the dg algebra  $B$  constructed in part a) of the Proposition, one can deduce a relative Calabi–Yau property for sheaves on  $Y$ . For completeness, we include a direct geometric proof via Grothendieck duality.

**Lemma 3.12.** *Let  $Y$  be a quasi-projective smooth CY  $d$ -fold. For  $L \in D_c^b(\mathrm{coh} Y)$  a bounded complex of coherent sheaves with compact support and  $M \in D(\mathrm{Qcoh} Y)$ , there is a bifunctorial isomorphism*

$$\mathbf{DR}\mathrm{Hom}_{D(\mathrm{Qcoh} Y)}(L, M) \simeq \mathbf{R}\mathrm{Hom}_{D(\mathrm{Qcoh} Y)}(M, \Sigma^d L).$$

*Proof.* Since we work over characteristic zero, there exists a smooth projective variety  $\bar{Y}$  compactifying  $Y$ . Denote the natural embedding by  $i$ . Because  $\bar{Y}$  is Noetherian, we may identify  $D(\mathrm{coh} \bar{Y})$  with the subcategory  $D_{\mathrm{coh}}(\mathrm{Qcoh} \bar{Y}) \subset D(\mathrm{Qcoh} \bar{Y})$  consisting of complexes with coherent cohomologies (adding isomorphic objects). There exists an extension  $\bar{M} \in D(\mathrm{Qcoh} \bar{Y})$  of  $M$ , i.e.  $i^* \bar{M} = M$ . Since the cohomology of  $L$  has compact support,  $i_* L \in D_{\mathrm{coh}}^b(\mathrm{Qcoh} \bar{Y})$  which is in particular perfect.

We recall the Grothendieck duality for unbounded derived category due to Neeman [59]. Let  $f : X \rightarrow Y$  be a morphism of Noetherian proper schemes. The derived functor  $f_* : D(\mathrm{Qcoh} X) \rightarrow D(\mathrm{Qcoh} Y)$  admits a left adjoint  $f^*$  and a right adjoint  $f^!$ . Consider the special case  $f : \bar{Y} \rightarrow \mathrm{Spec} k$ . Because  $\bar{Y}$  is projective, conditions (a) and (b) of Fact 0.2 (ii) of Neeman ([59]) are satisfied (We can verify Conjecture 4.16 of Neeman, see Fact 0.8 (i) and (ii)). Then it follows from Fact 0.2 (ii) that  $f^! k = \Sigma^d \omega_{\bar{Y}}$ . Let  $\bar{L}$  be a perfect complex on  $\bar{Y}$  and  $\bar{N}$  be an arbitrary object in  $D(\mathrm{Qcoh} \bar{Y})$ . The Grothendieck duality implies that

$$\mathbf{R}\mathrm{Hom}_{D(k)}(f_*(\bar{L}^\vee \otimes \bar{N}), k) \simeq \mathbf{R}\mathrm{Hom}_{D(\mathrm{Qcoh} \bar{Y})}(\bar{L}^\vee \otimes \bar{N}, f^! k),$$

i.e.

$$\mathbf{DR}\mathrm{Hom}_{D(\mathrm{Qcoh} \bar{Y})}(\bar{L}, \bar{N}) \simeq \mathbf{R}\mathrm{Hom}_{D(\mathrm{Qcoh} \bar{Y})}(\bar{N}, \bar{L} \otimes \Sigma^d \omega_{\bar{Y}})$$

by Neeman's theorem (Fact 0.2 (ii) [59]).

Let  $Z$  be the scheme theoretical support of  $L$ , i.e.  $L = j_* L^\circ$  for  $L^\circ \in D_c^b(\mathrm{coh} Z)$  where  $j : Z \rightarrow Y$  is the canonical immersion. Set  $\bar{j} = i \circ j : Z \rightarrow \bar{Y}$  and  $\bar{L} = \bar{j}_* L$ . Since  $j$  is proper and  $i$  is étale,  $j^! M = j^! i^* \bar{M} \cong \bar{j}^! \bar{M}$ . We have

$$\begin{aligned} \mathbf{DR}\mathrm{Hom}_Y(L, M) &\simeq \mathbf{DR}\mathrm{Hom}_Z(L^\circ, j^! M) \\ &\simeq \mathbf{DR}\mathrm{Hom}_Z(L^\circ, \bar{j}^! \bar{M}) \\ &\simeq \mathbf{DR}\mathrm{Hom}_{\bar{Y}}(\bar{L}, \bar{M}) \\ &\simeq \mathbf{R}\mathrm{Hom}_{\bar{Y}}(\bar{M}, \Sigma^d \bar{L} \otimes \omega_{\bar{Y}}) \\ &\simeq \mathbf{R}\mathrm{Hom}_Y(M, \Sigma^d L) \end{aligned}$$

The fourth equivalence uses Grothendieck duality on  $\overline{Y}$  and the last uses  $i^*\omega_{\overline{Y}} \cong \omega_Y \cong \mathcal{O}_Y$ . The lemma is proved.  $\square$

**Lemma 3.13.** *Let  $\{\mathcal{E}_i\}_{i=1}^t$  be a semi-simple collection of compactly supported sheaves in a smooth quasi-projective CY  $d$ -fold  $Y$ . Write  $\Gamma := \Gamma_{\mathcal{E}}^Y$  for the derived deformation algebra of  $\mathcal{E} := \bigoplus_{i=1}^t \mathcal{E}_i$ . Then  $\Gamma$  is topologically homologically smooth and bimodule  $d$ -Calabi-Yau.*

*Proof.* Let  $A$  be the derived endomorphism algebra of  $\mathcal{E}$ . Since  $Y$  is smooth and  $\mathcal{E}$  has compact support, it is proper. Moreover, it can be chosen augmented. Clearly, it has its homology concentrated in non negative degrees and  $H^0(A)$  is isomorphic to a product of  $t$  copies of  $k$ , which is  $l$ . By Corollary 4.1 of [50], the dg algebra  $A$  belongs to the triangulated subcategory of its derived category generated by the semi-simple object  $l$ . We know that  $\Gamma$  is quasi-isomorphic to  $\mathbb{D}BA$ . Thus, by Lemma 3.8,  $\Gamma$  is topologically homologically smooth. Since  $Y$  is smooth and  $d$ -Calabi-Yau,  $A$  is perfectly  $d$ -Calabi-Yau by part b) of Proposition 3.11. Hence  $\Gamma$  is bimodule  $d$ -Calabi-Yau by Lemma 3.8.  $\square$

**Remark 3.14.** If we assume that  $Y$  is smooth and projective, then one can show that  $\text{Ext}_Y^*(\mathcal{E}, \mathcal{E})$  is a cyclic  $A_\infty$ -algebra. This can be proved by reducing to the analytic case and applying the holomorphic Chern-Simons theory (see Example 10.2.7 [53]). An algebraic proof of the case when  $Y$  is a projective Gorenstein CY curve can be found in Polishchuk's upcoming paper [68].

**Proposition 3.15.** *Let  $\{\mathcal{E}_i\}_{i=1}^t$  be a semi-simple collection of sheaves in a smooth projective CY  $d$ -fold  $Y$ . Write  $\Gamma := \Gamma_{\mathcal{E}}^Y$  for the derived deformation algebra of  $\mathcal{E} := \bigoplus_{i=1}^t \mathcal{E}_i$ . Then  $\Gamma$  is an exact  $d$ -CY-algebra.*

*Proof.* As in the proof of Lemma 3.13, we see that  $\Gamma$  is topologically homologically smooth and bimodule  $d$ -CY. The exactness of the CY-structure follows from Theorem 12.1 of [76] and the remark above.  $\square$

**Theorem 3.16.** (Iyama-Reiten) *Let  $R$  be an equi-codimensional Gorenstein normal domain of dimension  $d$  over an algebraically closed field  $k$ , and let  $A$  be an NCCR. Then  $A$  satisfies the relative dCY property. Moreover, if  $R$  is complete local then  $A$  is bimodule  $d$ -CY.*

*Proof.* The fact that  $A$  satisfies the relative dCY property is proved in Theorem 4.23 of [78]. We have  $A = \text{End}_R(R \oplus N)$  for the Cohen-Macaulay module  $N = \bigoplus_{i=1}^t N_i$  with  $N_1, \dots, N_t$  indecomposable. Denote by  $\bar{l}$  the algebra  $k^{t+1}$ . By Theorem 1.1 of [73], the algebra  $A$  is quasi-isomorphic to a pseudo-compact dg algebra  $\tilde{\Gamma} := (\widehat{T}_{\bar{l}}(V), d)$  in  $\text{PCAlg}(l)$  for a finite-dimensional graded  $\bar{l}$ -bimodule  $V$  concentrated in degrees  $\leq 0$  and a differential  $d$  taking  $V$  into the square of the kernel of the augmentation  $\widehat{T}_{\bar{l}}(V) \rightarrow \bar{l}$ . Since  $A$  is of finite global dimension,  $\tilde{\Gamma}$  is homologically smooth and by the first part, we have bifunctorial isomorphisms

$$\text{Hom}_{D(\tilde{\Gamma})}(M, P) = D\text{Hom}_{D(\tilde{\Gamma})}(P, \Sigma^d M)$$

for  $M$  in  $D_{fd}(\tilde{\Gamma})$  and  $P$  in  $\text{per}(\tilde{\Gamma})$ . Let

$$\Theta = \mathbf{R}\text{Hom}_{\tilde{\Gamma}^e}(\tilde{\Gamma}, \tilde{\Gamma}^e)$$

be the inverse dualizing complex of  $\tilde{\Gamma}$ . By Lemma 4.1 of [45], we have bifunctorial isomorphisms

$$\text{Hom}_{D(\tilde{\Gamma})}(L \overset{\mathbb{L}}{\otimes}_{\tilde{\Gamma}} \Theta, M) = D\text{Hom}_{D(\tilde{\Gamma})}(M, L)$$

for  $M$  in  $D_{fd}(\tilde{\Gamma})$  and an arbitrary object  $L$  of  $D(\tilde{\Gamma})$ . By combining these with the previous isomorphisms we find

$$\text{Hom}_{D(\tilde{\Gamma})}(P \overset{\mathbb{L}}{\otimes}_{\tilde{\Gamma}} \Theta, M) = \text{Hom}_{D(\tilde{\Gamma})}(\Sigma^{-d} P, M)$$

for  $P$  in  $\text{per}(\tilde{\Gamma})$  and  $M$  in  $D_{fd}(\tilde{\Gamma})$ . Since an object  $L$  of  $D(\tilde{\Gamma})$  is perfect if and only if  $\text{Hom}_{D(\tilde{\Gamma})}(L, M)$  is finite-dimensional for each  $M$  in  $D_{fd}(\tilde{\Gamma})$ , we see that  $P \otimes_{\tilde{\Gamma}}^{\mathbb{L}} \Theta$  is perfect for each perfect  $P$ . Now by taking  $P = \tilde{\Gamma}$  and  $M = \Sigma^n S_i$ , where  $n \in \mathbb{Z}$  and  $S_i$  is one of the  $t+1$  simple  $\tilde{\Gamma}$ -modules, we see that as a right  $\tilde{\Gamma}$ -module,  $\Theta$  is quasi-isomorphic to  $\Sigma^{-d}\tilde{\Gamma}$ . For the rest of the argument, let us replace  $\tilde{\Gamma}$  by the quasi-isomorphic pseudocompact algebra  $G = H^0(\tilde{\Gamma})$ , which is isomorphic to the original algebra  $A$ . Since  $G$  and  $\tilde{\Gamma}$  have canonically equivalent derived categories and derived categories of bimodules, we can view  $\Theta$  as a  $G$ -bimodule complex concentrated in degree  $d$ . After replacing it with a quasi-isomorphic bimodule complex, we may assume that it is an actual  $G$ -bimodule concentrated in degree  $d$ . Moreover, we know that as a right module, it is isomorphic to  $G$ . Thus, there is an automorphism  $\sigma$  of  $G$  such that  $\Sigma^d\Theta$  is isomorphic to  ${}_\sigma G$  as a bimodule. Since each object  $L$  of  $D_{fd}(G)$  is perfect, we have

$$\text{Hom}_{D(G)}(L_\sigma, M) = \text{Hom}_{D(G)}(L, M)$$

for all  $L$  and  $M$  in  $D_{fd}(G)$ , which shows that there is a functorial isomorphism  $L \xrightarrow{\sim} L_\sigma$  for each  $L$  in  $D_{fd}(G)$ . In particular, for  $L$ , we can take the finite-dimensional quotients of  $G$ . We deduce that in each finite-dimensional quotient of  $G$ , the automorphism  $\sigma$  induces an inner automorphism. Thus,  $\sigma$  itself is inner and  ${}_\sigma G$  is isomorphic to  $G$  as a bimodule. This shows that  $\Theta$  is quasi-isomorphic to  $\Sigma^{-d}\tilde{\Gamma}$  as a bimodule.  $\square$

**Corollary 3.17.** *Let  $R$  be a local complete equi-codimensional Gorenstein normal domain of dimension  $d$  over an algebraically closed field  $k$  of characteristic zero, and let  $A$  be an NCCR. Let  $\Gamma$  be the dg algebra constructed in Theorem 2.8. Then  $\Gamma$  is topologically homologically smooth and bimodule  $dCY$ .*

**3.2. Classification of CY structures for 3-dimensional flopping contractions.** If  $(\hat{Y}, \hat{f}, R)$  is a 3-dimensional formal flopping contraction, then  $R$  is a hypersurface. The natural isomorphism  $\hat{f}_*\omega_{\hat{Y}} \cong \omega_R$  identifies a CY structure  $\eta$  on  $\hat{Y}$  with a nonzero section  $\hat{f}(\eta)$  of  $\omega_R$ . By the Gorenstein property,  $\hat{f}(\eta)$  defines an isomorphism  $R \cong \omega_R$ .

**Theorem 3.18.** *Let  $(\hat{Y}, \hat{f}, R)$  be a 3-dimensional formal flopping contraction. The space of 3CY structures can be identified with  $R^\times$ . Moreover, every Calabi-Yau structure on  $\hat{Y}$  is exact. The space of all exact liftings of a 3CY structure can be identified with the cohomology group  $H^1(\hat{Y}, \Omega_{\hat{Y}}^1)$ .*

*Proof.* Assume that  $C := \text{Ex}(f)$  has  $t$  irreducible components  $C_1, \dots, C_t$ . Because  $R$  has rational singularities,  $H^0(\hat{Y}, \Omega_{\hat{Y}}^3) \cong H^0(\hat{Y}, \mathcal{O}_{\hat{Y}}) \cong R$ . Then the first claim follows. The Hodge-to-de Rham spectral sequence with  $E_1$  term

$$E_1^{pq} = H^q(\hat{Y}, \Omega_{\hat{Y}}^p),$$

converges to  $H_{DR}^{p+q}(\hat{Y}, \mathbb{C})$ . We claim that  $H^1(\Omega_{\hat{Y}}^1) \cong \mathbb{C}^t$ . Because the first Chern classes of  $\hat{\mathcal{L}}_i$  for  $i = 1, \dots, t$  are linearly independent,  $\dim_{\mathbb{C}} H^1(\Omega_{\hat{Y}}^1) \geq t$ . We write  $\hat{X} := \text{Spec } R$ . By the Leray spectral sequence

$$H^p(\hat{X}, \mathbf{R}^q \hat{f}_*\Omega^1) \Rightarrow H^{p+q}(\hat{Y}, \Omega^1),$$

we have an exact sequence

$$0 \rightarrow H^0(\hat{X}, \mathbf{R}^1 \hat{f}_*\Omega^1) \rightarrow H^1(\hat{Y}, \Omega^1) \rightarrow H^1(\hat{X}, \hat{f}_*\Omega^1) \rightarrow 0$$

The right most term vanishes since  $\hat{X}$  is affine. Since  $\hat{Y}$  is a small resolution of 3-dimensional Gorenstein singularities, the normal bundle of  $C_i$  is  $\mathcal{O}(a) \oplus \mathcal{O}(b)$  with  $(a, b) = \{(-1, -1), (0, -2), (1, -3)\}$  ([64, Theorem 4]). By the exact sequence

$$0 \rightarrow \mathcal{O}(-a) \oplus \mathcal{O}(-b) \rightarrow \Omega^1|_{C_i} \rightarrow \mathcal{O}(-2) \rightarrow 0$$

we have  $H^1(C_i, \Omega^1|_{C_i}) \cong H^1(C_i, \mathcal{O}(-2)) = \mathbb{C}$ . By the normal crossing condition, there is a short exact sequence of sheaves

$$0 \longrightarrow \Omega^1|_C \rightarrow \Omega^1|_{C_1} \oplus \Omega^1|_{\cup_{i=2}^t C_i} \longrightarrow \Omega^1|_p \longrightarrow 0$$

where  $p = C_1 \cap (\cup_{i=2}^t C_i)$ . Then we get a surjection

$$H^1(C, \Omega^1|_C) \rightarrow H^1(C_1, \Omega^1|_{C_1}) \oplus H^1(\cup_{i=2}^t C_i, \Omega^1_{\cup_{i=2}^t C_i}).$$

By induction,  $\dim_{\mathbb{C}} H^1(C, \Omega^1|_C) \leq t$ . Then the conclusion follows from the formal function theorem and the Lerray spectral sequence.

The term  $E_2^{30}$  of the Hodge-to-de Rham spectral sequence is the quotient  $H^0(\widehat{Y}, \Omega^3)/dH^0(\widehat{Y}, \Omega^2)$ . Recall that  $H^1(\widehat{Y}, \Omega^1)$  admits a  $\mathbb{C}$ -basis by  $c_1$  of the line bundles  $\widehat{\mathcal{L}}_i$  with  $i = 1, \dots, t$ . Because  $(1, 1)$  classes are  $d$ -closed, the differential  $H^1(\widehat{Y}, \Omega^1) \rightarrow H^0(\widehat{Y}, \Omega^3)/dH^0(\widehat{Y}, \Omega^2)$  is zero. Moreover since  $H^2(\mathcal{O}_{\widehat{Y}}) = 0$ , we have  $E_r^{30} = E_2^{30}$  for  $r \geq 2$ . Because  $H_{DR}^3(\widehat{Y}, \mathbb{C}) \cong H_{DR}^3(C, \mathbb{C}) = 0$ ,  $E_2^{30}$  must vanish. Therefore, all 3-forms are exact.

Denote by  $\Omega_{\widehat{Y}}^{\leq i}$  the stupid truncation  $\sigma_{\leq i} \Omega_{\widehat{Y}}^*$  of the de Rham complex. There is a long exact sequence of hypercohomology

$$\dots \xrightarrow{B} H^{i-n}(\widehat{Y}, \Omega_{\widehat{Y}}^i) \xrightarrow{I} \mathbb{H}^{2i-n}(\widehat{Y}, \Omega_{\widehat{Y}}^{\leq i}) \xrightarrow{S} \mathbb{H}^{2i-n}(\widehat{Y}, \Omega_{\widehat{Y}}^{\leq i-1}) \xrightarrow{B} H^{i-n+1}(\widehat{Y}, \Omega_{\widehat{Y}}^i) \longrightarrow \dots$$

Take  $i = 3$  and  $n = 4$ . Let leftmost term vanishes and  $\mathbb{H}^2(\widehat{Y}, \Omega_{\widehat{Y}}^{\leq 3}) = H_{DR}^2(\widehat{Y}) \cong H^1(\widehat{Y}, \Omega_{\widehat{Y}}^1)$ . So the last claim is proved.  $\square$

**Corollary 3.19.** *Let  $R$  be a complete local equi-codimensional Gorenstein normal domain of dimension  $d$  over an algebraically closed field of characteristic zero, and let  $A$  be an NCCR. Then every dCY structure on  $A$  is exact.*

*Proof.* This is an immediate consequence of Theorem 3.16, Theorem 2.11 and Corollary 9.3 of [76].  $\square$

The following proposition follows immediately from the Hochschild-Kostant-Rosenberg theorem.

**Proposition 3.20.** *Let  $(\widehat{Y}, \widehat{f}, R)$  be a 3-dimensional formal flopping contraction. Let  $A = \text{End}_R(R \oplus N)$  be the corresponding NCCR. Then there is a bijective correspondence between the space of 3CY structures (resp. exact 3CY structures) on  $\widehat{Y}$  and that of  $A$ .*

**3.3. Cluster category.** Let  $\Gamma$  be a dg  $k$ -algebra. Suppose that  $\Gamma$  has the following properties:

- (1)  $\Gamma$  is homologically smooth, i.e.  $\Gamma$  is a perfect  $\Gamma^e$ -module;
- (2) for each  $p > 0$ , the space  $H^p \Gamma$  vanishes;
- (3)  $H^0 \Gamma$  is finite dimensional;
- (4)  $\Gamma$  satisfies the relative 3CY property.

By property (1),  $D_{fd}(\Gamma)$  is a subcategory of the perfect derived category  $\text{per}(\Gamma)$ . The *generalized cluster category*  $\mathcal{C}_\Gamma$  is defined to be the triangle quotient  $\text{per}(\Gamma)/D_{fd}(\Gamma)$ . We denote by  $\pi$  the canonical projection functor  $\pi : \text{per}(\Gamma) \rightarrow \mathcal{C}_\Gamma$ . For simplicity, we will omit the adjective ‘generalized’ and call  $\mathcal{C}_\Gamma$  the cluster category associated to  $\Gamma$ . An object  $T \in \mathcal{C}_\Gamma$  is called a *cluster-tilting object* if

- (1)  $\text{Ext}_{\mathcal{C}_\Gamma}^1(T, T) = 0$ ;
- (2) For any object  $X$  such that  $\text{Ext}_{\mathcal{C}_\Gamma}^1(T, X) = 0$ , one has  $X \in \text{add}(T)$ .

Amiot has proved that  $\pi(\Gamma)$  is a cluster-tilting object in  $\mathcal{C}_\Gamma$  (Theorem 2.1 [3]). We call  $H^0 \Gamma$  the *CY tilted algebra* associated to the cluster category  $\mathcal{C}_\Gamma$ , cf. [61].

**Remark 3.21.** If  $\Gamma$  is a pseudo-compact dg  $l$ -algebra in  $\text{PCAlg}(l)$ , we may define a continuous version of cluster category. Condition (1) is replaced by

(1')  $\Gamma$  is topologically homologically smooth,

and the *topological* cluster category  $\mathcal{C}_\Gamma$  is defined to be the triangle quotient  $\text{per}(\Gamma)/D_{fd}(\Gamma)$  where  $\text{per}(\Gamma)$  and  $D_{fd}(\Gamma)$  are considered as subcategories of the pseudo-compact derived category. We refer to the Appendix of [51] for the details.

**Remark 3.22.** If we drop the assumption that  $H^0\Gamma$  is finite dimensional, the quotient category  $\mathcal{C}_\Gamma = \text{per}(\Gamma)/D_{fd}(\Gamma)$  is no longer Hom-finite. The Calabi–Yau property only holds when one restricts to suitable subcategories, cf. Proposition 2.16 of [66].

**Theorem 3.23.** [3, Theorem 2.1]<sup>2</sup> *Let  $\Gamma$  be a dg  $k$ -algebra with the above properties. Then the cluster category  $\mathcal{C}_\Gamma$  is Hom-finite and 2CY as a triangulated category. Moreover, the object  $\pi(\Gamma)$  is a cluster tilting object. Its endomorphism algebra is isomorphic to  $H^0\Gamma$ .*

**Definition 3.24.** Let  $\{C_i\}_{i=1}^t$  be a collection of smooth rational curves in a smooth quasi-projective CY 3-fold  $Y$  with fixed CY-structure  $\eta : \mathcal{O}_Y \xrightarrow{\sim} \omega_Y$ , such that  $\{\mathcal{O}_{C_i}\}$  form a semi-simple collection. Denote by  $\mathcal{C}(Y, \{C_i\}_{i=1}^t, \eta)$  for the (topological) cluster category associated to the derived deformation algebra of  $\bigoplus_{i=1}^t \mathcal{O}_{C_i}$ . We call  $\mathcal{C}(Y, \{C_i\}_{i=1}^t, \eta)$  the *cluster category associated to the triple  $(Y, \eta, \{C_i\}_{i=1}^t)$* .

**Definition 3.25.** Let  $R$  be a complete local equicodimensional Gorenstein normal domain of dimension 3 over an algebraically closed field  $k$  of characteristic zero, and let  $A$  be the NCCR associated to the collection of indecomposables  $R, N_1, \dots, N_t$ . Fix a 3CY structure  $\eta \in \text{HH}_3(A, A)$ . Denote by  $\mathcal{C}(R, \{N_i\}_{i=1}^t, \eta)$  the cluster category associated to the dg algebra  $\Gamma$  constructed in Theorem 2.8, and call it *the cluster category associated to the triple  $(R, \{N_i\}_{i=1}^t, \eta)$* .

A priori, the dg algebra  $\Gamma$  constructed in Theorem 2.8 is pseudo-compact. However, if  $D(\Gamma)$  denotes the ordinary derived category and  $D_{pc}(\Gamma)$  the pseudo-compact derived category, then the natural functor  $D_{pc}(\Gamma) \rightarrow D(\Gamma)$  induces equivalences in the perfect derived categories and in the subcategories of objects with finite-dimensional total homology. Therefore, the two candidates for the cluster category are equivalent.

The following result is an immediate consequence of Corollary 2.12.

**Corollary 3.26.** *Let  $(\hat{Y}, \hat{f}, R)$  be a 3-dimensional formal flopping contraction, and let  $A$  be the NCCR associated to the collection of indecomposables  $R, N_1, \dots, N_t$  constructed in Section 2.5. Fix a 3CY structure  $\eta$  on  $\hat{Y}$  and denote its counter part on  $A$  by the same symbol. Then there is a triangle equivalence between the cluster categories*

$$\mathcal{C}(Y, \{C_i\}_{i=1}^t, \eta) \simeq \mathcal{C}(R, \{N_i\}_{i=1}^t, \eta).$$

#### 4. GINZBURG ALGEBRAS

In this section we introduce the notion of Ginzburg (dg) algebra and prove several properties of it. The cluster category can be defined via the Ginzburg algebra, which provides an effective tool to do computations.

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<sup>2</sup>In the original statement of [3], the author assumed that  $\Gamma$  is bimodule 3CY. However, the proof is still valid under the weaker assumption that  $\Gamma$  satisfies the relative 3CY property.

**4.1. Definitions.** Fix a commutative ring  $k$ . Let  $Q$  be a finite quiver, possibly with loops and 2-cycles. Denote by  $Q_0$  and  $Q_1$  the set of nodes and arrows of  $Q$  respectively. Denote by  $kQ$  the path algebra and by  $\widehat{kQ}$  the complete path algebra with respect to the two-sided ideal generated by arrows. For each  $a \in Q_1$ , we define the cyclic derivative  $D_a$  with respect to  $a$  as the unique linear map

$$D_a : kQ/[kQ, kQ] \rightarrow kQ$$

which takes the class of a path  $p$  to the sum  $\sum_{p=uav} vu$  taken over all decompositions of the path  $p$ . The definition can be extended to  $\widehat{kQ}_{\text{cyc}} := \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]^c$  where the superscript  $c$  stands for the completion with respect to the adic topology defined above. An element  $w$  in  $\widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]^c$  is called a *potential* on  $Q$ . It is given by a (possibly infinite) linear combination of cycles in  $Q$ .

**Definition 4.1.** (Ginzburg) Let  $Q$  be a finite quiver and  $w$  a potential on  $Q$ . Let  $\overline{Q}$  be the graded quiver with the same vertices as  $Q$  and whose arrows are

- the arrows of  $Q$  (of degree 0);
- an arrow  $a^* : j \rightarrow i$  of degree  $-1$  for each arrow  $a : i \rightarrow j$  of  $Q$ ;
- a loop  $t_i : i \rightarrow i$  of degree  $-2$  for each vertex  $i$  of  $Q$ .

The (complete) Ginzburg (dg)-algebra  $\mathfrak{D}(Q, w)$  is the dg  $k$ -algebra whose underlying graded algebra is the completion (in the category of graded vector spaces) of the graded path algebra  $k\overline{Q}$  with respect to the two-sided ideal generated by the arrows of  $\overline{Q}$ . Its differential is the unique linear endomorphism homogeneous of degree 1 satisfying the Leibniz rule, and which takes the following values on the arrows of  $\overline{Q}$ :

- $da = 0$  for  $a \in Q_1$ ;
- $d(a^*) = D_a w$  for  $a \in Q_1$ ;
- $d(t_i) = e_i(\sum_{a \in Q_1} [a, a^*])e_i$  for  $i \in Q_0$  where  $e_i$  is the idempotent associated to  $i$ .

Denote by  $l$  the product  $\prod_{i \in Q_0} ke_i$ . Then  $\widehat{kQ}$  is isomorphic to the complete tensor algebra  $\widehat{T}_l V$  with  $V$  being the vector space spanned by arrows of  $Q$ .

**Remark 4.2.** In most references, the above definition corresponds to the *complete Ginzburg algebra* while the algebra without taking the graded completion is called *Ginzburg algebra*. The complete Ginzburg algebra  $\mathfrak{D}(Q, w)$  is considered as an object of  $\text{PCAlg}(l)$ . Because we only consider complete Ginzburg algebra in this paper, we will call it the Ginzburg algebra for simplicity.

**Definition 4.3.** Let  $Q$  be a finite quiver and  $w$  a potential on  $Q$ . The *Jacobi algebra*  $\Lambda(Q, w)$  is the zeroth homology of  $\mathfrak{D}(Q, w)$ , which is the quotient algebra

$$\widehat{kQ}/((D_a w, a \in Q_1))^c$$

where  $((D_a w, a \in Q_1))^c$  is the closed two-sided ideal generated by  $D_a w$ . A Ginzburg algebra  $\mathfrak{D}(Q, w)$  is called *Jacobi-finite* if  $\dim_k \Lambda(Q, w) < \infty$ .

Van den Bergh showed the following result.

**Theorem 4.4.** (Van den Bergh)[46, Appendix] *Let  $Q$  be finite quiver and  $w$  be a potential. Then  $\mathfrak{D}(Q, w)$  is topologically homologically smooth and bimodule 3CY.*

The above theorem was first proved by Van den Bergh in the algebraic setting in [46]. But the same proof can be adapted to the pseudo-compact case (cf. [76]). Given a Jacobi-finite Ginzburg algebra  $\Gamma := \mathfrak{D}(Q, w)$ , there is an associated cluster category  $\mathcal{C}_\Gamma := \text{per}(\Gamma)/\text{D}_{fd}(\Gamma)$ .

**Remark 4.5.** There exists a canonical exact CY structure on  $\Gamma = \mathfrak{D}(Q, w)$ . We follow the notation of [76] to write  $M_l := M/[l, M]$  for a  $l$ -bimodule  $M$ . Because the reduced cyclic homology of  $\Gamma$  is

equal to the homology of  $(\Gamma/l + [\Gamma, \Gamma])_l$  (see Proof of Theorem 11.2.1 of [75]), a class of  $\text{HC}_2(\Gamma, \Gamma)$  is represented by a degree  $-2$  element  $\chi$  of  $\Gamma$  such that  $d\chi \in l + [\Gamma, \Gamma]$ . By the definition of  $d$  of  $\Gamma$ ,  $\chi := \sum_{i \in Q_0} t_i$  represents a class in  $\text{HC}_2^{\text{red}}(\Gamma, \Gamma)$ . Because  $\Gamma$  is cofibrant, by Proposition 7.2.1 of [75] the Hochschild chain complex of  $\Gamma$  is quasi-isomorphic to the mapping cone of

$$\Omega_l^1 \Gamma / [\Gamma, \Omega_l^1 \Gamma] \xrightarrow{\partial_1} \Gamma / [l, \Gamma]$$

with differential defined by  $\partial_1(aDb) = [a, b]$ , where  $Db = 1 \otimes b - b \otimes 1$ . In other words, a class in  $\text{HH}_3(\Gamma, \Gamma)$  is represented by a pair of elements  $(\omega, a)$  of degree  $(-2, -3)$  satisfying  $\partial_1(\omega) = da$  and  $d\omega = 0$ . Because  $d$  and  $D$  commute,  $(D\chi, 0)$  represents a class in  $\text{HH}_3(\Gamma, \Gamma)$ , which is the image of  $\chi$  under the Connes map. In [75], a class in  $\text{HH}_d(\Gamma, \Gamma)$  is called *non-degenerate* if the corresponding morphism  $\Theta_\Gamma \rightarrow \Sigma^{-d}\Gamma$  is an isomorphism. By Lemma 11.1.2 of [75],  $(D\chi, 0)$  is non-degenerate.

For a Ginzburg algebra  $\Gamma = \mathfrak{D}(Q, w)$ , denote  $\Lambda(Q, w)$  by  $\Lambda$  for short. The image of  $w$  under the canonical map  $\text{HH}_0(\widehat{kQ}, \widehat{kQ}) = \widehat{kQ}_{\text{cyc}} \rightarrow \text{HH}_0(\Lambda, \Lambda) = \Lambda_{\text{cyc}}$ , denoted by  $[w]$ , is a canonical class associated to the Ginzburg algebra  $\mathfrak{D}(Q, w)$ . Therefore, we see that starting from a Ginzburg algebra  $\Gamma = \mathfrak{D}(Q, w)$  we get not only a triangulated category  $\mathcal{C}_\Gamma$  but an additional piece of information that is a canonical class  $[w]$  in the 0-th Hochschild homology of the CY tilted algebra. We will show in the next section that this class is determined by the CY structure up to right equivalences.

**4.2. Existence and uniqueness of potential.** The definition of Ginzburg algebra is not homotopically invariant. It is important to know when a bimodule 3CY dg algebra admits a model given by a Ginzburg algebra. The following theorem is due to Van den Bergh.

**Theorem 4.6.** [76, Theorem 10.2.2] *Let  $k$  be a field and  $l$  be a finite dimensional commutative separable  $k$ -algebra. Assume that  $\Gamma$  is a pseudo-compact dg  $l$ -algebra in  $\text{PCAlg}(l)$  concentrated in nonpositive degrees. Then the following are equivalent*

- (1)  $\Gamma$  is exact 3CY.
- (2)  $\Gamma$  is weakly equivalent to a Ginzburg algebra  $\mathfrak{D}(Q, w)$  for some finite quiver  $Q$  with  $w$  contains only cubic terms and higher.

The following result of Van den Bergh provides a lot of examples of dg algebras whose 3CY structures can be lifted to exact ones.

**Theorem 4.7.** [76, Corollary 9.3] *Assume that  $k$  has characteristic zero and let  $\Gamma$  be a pseudo-compact dg algebra in  $\text{PCAlg}(l)$  concentrated in degree zero. Then  $\Gamma$  is bimodule dCY if and only if it is exact dCY.*

By putting the above two theorems together, we see that if  $\Gamma$  is a pseudo-compact dg  $l$ -algebra in  $\text{PCAlg}(l)$  concentrated in degree zero that is bimodule 3CY then it is quasi-isomorphic to a Ginzburg algebra  $\mathfrak{D}(Q, w)$  for some finite quiver  $Q$  and potential  $w$ .

Given a pseudo-compact  $l$ -algebra  $A$ , two classes  $[w]$  and  $[w']$  in  $A/[A, A]^c$  are called *right equivalent* if there exists an automorphism  $\gamma$  of  $A$  such that  $\gamma_*[w] = [w']$ . Now we assume that a bimodule 3CY dg algebra  $\Gamma$  in  $\text{PCAlg}(l)$  is exact. So it admits a model given by  $\mathfrak{D}(Q, w)$ . Note that being bimodule CY and exact CY are homotopically invariant properties. The next proposition shows that the right equivalence class of  $[w]$  in  $\text{HH}_0(\Lambda(Q, w))$  for such a dg algebra is indeed a homotopy invariant. The proof is implicitly contained in Van den Bergh's proof of Theorem 4.6 (cf. proof of Theorem 11.2.1 of [76]). We recall it for completeness. See Remark 4.9 for a conceptional explanation of Van den Bergh's result.

**Proposition 4.8.** (Van den Bergh) *Let  $k$  be a field of characteristic zero and  $l = ke_1 \times \dots \times ke_n$ . Let  $\Gamma$  be a pseudo-compact 3CY dg  $l$ -algebra in  $\text{PCAlg}(l)$  with a fixed exact 3CY structure  $\tau \in$*

$\mathrm{HH}_3(\Gamma, \Gamma)$ . Suppose there are two pairs  $(Q, w)$  and  $(Q', w')$  such that  $\Gamma$  is quasi-isomorphic to  $\mathfrak{D}(Q, w)$  and  $\mathfrak{D}(Q', w')$  respectively. Assume that under these quasi-isomorphisms we have  $B\chi$  and  $B\chi'$  identified with  $\tau$  where  $\chi, \chi'$  are the canonical classes in  $\mathrm{HC}_2$  defined in Remark 4.5. Then  $Q = Q'$  and  $w$  is right equivalent to  $w'$ . In particular, the classes  $[w], [w'] \in H^0\Gamma/[H^0\Gamma, H^0\Gamma]^c$  are right equivalent.

*Proof.* In Theorem 11.2.1 of [75], it is proved that there is a weak equivalence  $(\widehat{T}_l V, d) \rightarrow \Gamma$  where  $V = \Sigma^{-1}(\mathrm{DExt}_{\Gamma}^*(l, l))_{\leq 0}$  such that

- (1)  $V = V_c + lz$  with  $V_c = \Sigma^{-1}(\mathrm{DExt}_{\Gamma}^1(l, l) \oplus \mathrm{DExt}_{\Gamma}^2(l, l))$  and  $z$  being an  $l$ -central element of degree  $-2$ .
- (2)  $dz = \sigma' \eta \sigma''$  with  $\eta \in (V_c \otimes_l V_c)_l$  being a non-degenerate and anti-symmetric element. Here  $\sigma' \otimes \sigma'' \in l \otimes_k l$  is the Casimir element corresponds to the trace  $l \rightarrow k$  (see Section 5 of [75]).

Since  $Q$  (resp.  $Q'$ ) depends only on  $l$ -bimodule structure on  $\mathrm{Ext}_{\Gamma}^1(l, l)$  (see [74, Section 10.3]), we have  $Q = Q'$ . Note that the perfect pairing on  $V_c$  is determined by the bimodule 3CY structure  $\tau$  but does not depend on the exact lifting. Using the perfect pairing, any non-degenerate and anti-symmetric element in  $(V_c \otimes_l V_c)_l$  can be reduced to a canonical form by choosing appropriate basis on  $\mathrm{Ext}_{\Gamma}^1(l, l)$ . The element  $\eta$  defines a bisymplectic form  $\omega_{\eta}$  of degree  $-1$  on  $\widehat{T}_l V$  (see definition in Section 10.1 of [75]). By Lemma 11.3.1 of [75] there exists a potential  $w \in \widehat{T}_l V / [\widehat{T}_l V, \widehat{T}_l V]$  of degree zero such that for any  $f \in \widehat{T}_l V$

$$df = \{w, f\}_{\omega_{\eta}}$$

where  $\{-, -\}_{\omega_{\eta}}$  is the Poisson bracket associate to the bisymplectic form  $\omega_{\eta}$ . Since  $w$  does not have constant terms, it is uniquely determined by the derivation  $d = \{w, -\}_{\omega_{\eta}}$ . In other words, any two potentials  $w$  and  $w'$  without constant terms satisfying the above equation differ by an automorphism  $\widehat{T}_l V \rightarrow \widehat{T}_l V$ . Moreover, since  $w$  and  $w'$  are of degree zero, therefore does not involve variables in  $\mathrm{Ext}_{\Gamma}^{>2}(l, l)$ , this automorphism is precisely a formal change of variables on  $\mathrm{Ext}_{\Gamma}^1(l, l)$ . Such a formal change of variable induces an isomorphism from  $\mathfrak{D}(Q, w)$  to  $\mathfrak{D}(Q, w')$  (see [31, Theorem 4.3]), therefore an automorphism of the complete path algebra of  $Q$  and an automorphism of  $H^0\Gamma$ .  $\square$

**Remark 4.9.** Theorem 11.2.1 of [75], can be viewed as a Darboux-Weinstein theorem in noncommutative formal symplectic geometry. On  $\mathrm{Ext}_{\Gamma}^*(l, l)$ , the cyclic  $A_{\infty}$ -structure can be interpreted as a symplectic structure. The symplectic structure restricts to the truncation  $\mathrm{Ext}_{\Gamma}^1(l, l) \oplus \mathrm{Ext}_{\Gamma}^2(l, l)$  so that  $\mathrm{Ext}_{\Gamma}^1(l, l)$  is a (graded) Lagrangian. Then Theorem 11.2.1 of [75] says that there exists a coordinate on  $\mathrm{Ext}_{\Gamma}^1(l, l)$  under which the symplectic form can be normalized so that it has constant coefficients, which is in particular exact. The differential  $d$  of  $\Gamma$  can be interpreted as a homological vector field of degree 1. Then the contraction of the normalized symplectic form by  $d$  is the exterior derivative of a potential  $w$  of degree 0. Note that a different choice of Darboux coordinates can only differ by a change of variables on  $\mathrm{Ext}_{\Gamma}^1(l, l)$ , which leads to the above proposition.

**Corollary 4.10.** Let  $(\widehat{Y}, \widehat{f}, R)$  be a 3-dimensional formal flopping contraction with reduced exceptional fiber  $\mathrm{Ex}(f) = \bigcup_{i=1}^t C_i$ , and let  $A = \mathrm{End}_R(\bigoplus_{i=1}^t N_i \oplus R)$  be the NCCR associated to it. Fix a 3CY structure  $\eta$  on  $\widehat{Y}$ , and therefore on  $A$ . Denote the CY tilted algebra of  $\mathcal{C}(Y, \{C_i\}_{i=1}^t, \eta) \simeq \mathcal{C}(R, \{N_i\}_{i=1}^t, \eta)$  by  $\Lambda$ . Then there exists a canonical class, defined up to right equivalence, on  $\mathrm{HH}_0(\Lambda) = \Lambda / [\Lambda, \Lambda]^c$  represented by a potential.

The canonical class  $[w]$  in the 0-th Hochschild homology of  $H^0\Gamma$  is part of the ‘‘classical shadow’’ of the CY structure. The class plays a crucial role in the geometric applications. When  $\Gamma$  is weakly equivalent to a Jacobi-finite Ginzburg algebra  $\mathfrak{D}(\widehat{F}, w)$  for a complete free algebra  $\widehat{F} = k\langle\langle x_1, \dots, x_n \rangle\rangle$ , then this class vanishes if and only if  $w$  is right equivalent to a weighted homogeneous

noncommutative polynomial (see Theorem 4.16). Therefore, the quasi-homogeneity of a potential is indeed a homotopy invariant of the CY algebra. This motivates the following definition.

**Definition 4.11.** Let  $k$  be a field and  $l$  be a finite dimensional commutative separable  $k$ -algebra, and let  $\Gamma$  be a pseudo-compact dg  $l$ -algebra in  $\text{PCAlg}(l)$  concentrated in nonpositive degrees. Assume that  $\Gamma$  is exact 3CY. Then  $\Gamma$  is called *quasi-homogeneous* if the canonical class  $[w]$  is (right equivalent to) zero.

The notion of quasi-homogeneity is expected to be independent of choices of CY structure. In the case of simple flopping contractions, the first author and Gui-song Zhou have conjectured that this notion of quasi-homogeneity is indeed equivalent to the quasi-homogeneity of the underlying hypersurface singularity  $R$  (see Conjecture 4.18 [31]).

**4.3. Properties of Jacobi-finite Ginzburg algebras.** In this section, we collect several results about Jacobi-finite Ginzburg algebras. We take  $k$  to be the field of complex numbers, though some of the results are valid more generally.

**Theorem 4.12.** [31, Theorem 3.16] *Let  $Q$  be a finite quiver and  $w$  be a potential in  $\widehat{\mathbb{C}Q}_{\text{cyc}}$ . Assume that the Jacobi algebra  $\Lambda(Q, w)$  is finite dimensional. Then  $w$  is right equivalent to a formal series with only finitely many nonzero terms.*

As a consequence, we may assume the potential is a noncommutative polynomial to begin with if the Jacobi algebra is known to be finite dimensional.

**Theorem 4.13.** (Noncommutative Mather–Yau theorem)[31, Theorem 3.5] *Let  $Q$  be a finite quiver and let  $w, w' \in \widehat{\mathbb{C}Q}_{\text{cyc}}$  be two potentials with only cubic terms and higher. Suppose that the Jacobi algebras  $\Lambda(Q, w)$  and  $\Lambda(Q, w')$  are both finite dimensional. Then the following two statements are equivalent:*

- (1) *There is an algebra isomorphism  $\gamma : \Lambda(Q, w) \cong \Lambda(Q, w')$  so that  $\gamma_*([w]) = [w']$  in  $\Lambda(Q, w')_{\text{cyc}}$ .*
- (2)  *$w$  and  $w'$  are right equivalent in  $\widehat{\mathbb{C}Q}_{\text{cyc}}$ .*

The noncommutative Mather–Yau theorem has an immediate application to Ginzburg algebras.

**Corollary 4.14.** [31, Theorem 4.3] *Fix a finite quiver  $Q$ . Let  $w, w' \in \widehat{\mathbb{C}Q}_{\text{cyc}}$  be two potentials with only cubic terms and higher, such that the Jacobi algebras  $\Lambda(Q, w)$  and  $\Lambda(Q, w')$  are both finite dimensional. Assume there is an algebra isomorphism  $\gamma : \Lambda(Q, w) \rightarrow \Lambda(Q, w')$  so that  $\gamma_*([w]) = [w']$ . Then there exists a dg algebra isomorphism*

$$\beta : \mathfrak{D}(Q, w) \xrightarrow{\cong} \mathfrak{D}(Q, w')$$

such that  $\beta(t_i) = t_i$  for any  $i \in Q_0$ .

**Definition 4.15.** Fix  $\widehat{F}$  to be the complete free associative algebra  $\mathbb{C}\langle\langle x_1, \dots, x_n \rangle\rangle$ . Let  $(r_1, \dots, r_n)$  be a tuple of rational numbers. A potential  $w \in \widehat{F}_{\text{cyc}} := \widehat{F}/[\widehat{F}, \widehat{F}]^c$  is said to be *weighted-homogeneous of type*  $(r_1, \dots, r_n)$  if it has a representative which is a linear combination of monomials  $x_{i_1} x_{i_2} \cdots x_{i_p}$  such that  $r_{i_1} + r_{i_2} + \dots + r_{i_p} = 1$ .

**Theorem 4.16.** (Noncommutative Saito theorem) *Let  $w \in \widehat{F}_{\text{cyc}}$  be a potential with only cubic terms and higher such that the Jacobi algebra associated to  $w$  is finite dimensional. Then  $[w] = 0$  if and only if  $w$  is right equivalent to a weighted-homogenous potential of type  $(r_1, \dots, r_n)$  for some rational numbers  $r_1, \dots, r_n$  which lie strictly between 0 and  $1/2$ . Moreover, in this case, all such types  $(r_1, \dots, r_n)$  agree with each other up to permutations on the indices  $1, \dots, n$ .*

Recall that  $\mathcal{C}_\Gamma$  is constructed as the Verdier quotient of the perfect derived category of  $\Gamma$  by its full subcategory of dg modules whose homology is of finite total dimension. The category of perfect dg  $\Gamma$ -modules has a canonical dg enhancement and we obtain a natural dg enhancement  $(\mathcal{C}_\Gamma)_{dg}$  for  $\mathcal{C}_\Gamma$  using the Drinfeld quotient of the dg category of strictly perfect dg  $\Gamma$ -modules by its full subcategory on the dg modules whose homology is of finite total dimension.

**Theorem 4.17.** *Let  $Q$  be a finite quiver and  $w$  a Jacobi-finite potential on  $Q$ . Let  $\Gamma$  be the complete Ginzburg algebra associated with  $(Q, w)$ . Denote by  $\Lambda_{dg}$  the dg endomorphism algebra of  $\Gamma$  in the canonical dg enhancement of the cluster category  $\mathcal{C}_\Gamma$ . Then there is a canonical isomorphism in the homotopy category of dg algebras*

$$\Gamma \xrightarrow{\sim} \tau_{\leq 0} \Lambda_{dg}.$$

*Proof.* There is a canonical morphism

$$\Gamma = \mathbf{R}\mathrm{Hom}_\Gamma(\Gamma, \Gamma) \rightarrow (\mathcal{C}_\Gamma)_{dg}(\Gamma, \Gamma) = \Lambda_{dg}$$

where the right hand side denotes the dg endomorphism algebra of  $\Gamma$  in the canonical dg enhancement  $(\mathcal{C}_\Gamma)_{dg}$  of  $\mathcal{C}_\Gamma$ . It suffices to show that the canonical map

$$H^{-p}(\Gamma) = \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{-p}\Gamma) \rightarrow \mathrm{Hom}_{\mathcal{C}_\Gamma}(\Gamma, \Sigma^{-p}\Gamma)$$

is invertible for  $p \geq 0$ . By Proposition 2.8 of [3], we have

$$\mathrm{Hom}_{\mathcal{C}_\Gamma}(\Gamma, \Sigma^{-p}\Gamma) = \mathrm{colim}_n \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \tau_{\leq n}(\Sigma^{-p}\Gamma)).$$

We have

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \tau_{\leq n}(\Sigma^{-p}\Gamma)) = \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \Sigma^{-p}\Gamma).$$

Consider the canonical triangle

$$\tau_{\leq n}\Gamma \longrightarrow \Gamma \longrightarrow \tau_{>n}\Gamma \longrightarrow \Sigma(\tau_{\leq n}\Gamma).$$

Recall that by Lemma 2.5 of [3], the spaces  $H^p(\Gamma)$  are finite-dimensional for all  $p \in \mathbb{Z}$ . Therefore, the object  $\tau_{>n}\Gamma$  belongs to  $D_{fd}(\Gamma)$  and  $\Sigma^{-p}\Gamma$  belongs to  $\mathrm{per}(\Gamma)$ .

By the 3-Calabi-Yau property, we have

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Sigma^{-1}\tau_{>n}\Gamma, \Sigma^{-p}\Gamma) = D\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{p+2}\tau_{>n}\Gamma))$$

which vanishes because  $\tau_{>n}\Gamma$  has no homology in degrees  $> 0$ . Similarly, we have

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{>n}\Gamma, \Sigma^{-p}\Gamma) = D\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{p+3}(\tau_{>n}\Gamma))$$

which vanishes for the same reason. Thus we have

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \Sigma^{-p}\Gamma) = \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{-p}\Gamma).$$

□

**Corollary 4.18.** *Let  $Q$  be a quiver with one node and arbitrary number of loops and  $\Gamma = \mathfrak{D}(Q, w)$  a Jacobi-finite Ginzburg algebra. Then  $H^0\Gamma$  is self-injective and there is an isomorphism*

$$\Sigma^2\Gamma \xrightarrow{\sim} \tau_{\leq -1}\Gamma$$

*in the derived category of dg  $\Gamma$ -modules. In particular, we have  $H^i(\Gamma) = 0$  for odd  $i$  and  $H^i(\Gamma) \cong H^0(\Gamma)$  for even  $i \leq 0$ .*

*Proof.* By [3], the cluster category  $\mathcal{C}_\Gamma$  is a Hom-finite 2-Calabi-Yau category and the image  $T$  of  $\Gamma$  in  $\mathcal{C}_\Gamma$  is a cluster-tilting object in  $\mathcal{C}_\Gamma$ . By Theorem 4.1 of [1], the cluster-tilting objects of  $\mathcal{C}_\Gamma$  are in bijection with the support  $\tau$ -tilting modules over  $\text{End}(T)$ . Since  $\text{End}(T) = H^0(\Gamma)$  is local, the only support  $\tau$ -tilting modules over  $\text{End}(T)$  are 0 and  $\text{End}(T)$ , by example 6.1 of [loc. cit.]. Thus, the only cluster-tilting objects of  $\mathcal{C}_\Gamma$  are  $T$  and  $\Sigma T$ . In particular,  $\Sigma^2 T$  has to be isomorphic to  $T$  (since  $\text{Hom}(\Sigma T, \Sigma^2 T) = 0$  and  $\Sigma^2 T$  must be a cluster-tilting object). This implies that  $H^0\Gamma = \text{End}(T)$  is self-injective, since, by the 2-Calabi-Yau property, we have an isomorphism of right  $\text{End}(T)$ -modules

$$D\text{Hom}(T, T) = \text{Hom}(T, \Sigma^2 T) = \text{Hom}(T, T).$$

Let  $\phi : \Sigma^2\Gamma \rightarrow \Gamma$  be a lift of an isomorphism  $\Sigma^2 T \rightarrow T$  in  $\mathcal{C}_\Gamma$ . Let  $p \geq 2$ . In the commutative square

$$\begin{array}{ccc} \text{Hom}_{\text{per}(\Gamma)}(\Sigma^p\Gamma, \Sigma^2\Gamma) & \longrightarrow & \text{Hom}_{\mathcal{C}_\Gamma}(\Sigma^p T, \Sigma^2 T) \\ \downarrow & & \downarrow \phi_* \\ \text{Hom}_{\text{per}(\Gamma)}(\Sigma^p\Gamma, \Gamma) & \longrightarrow & \text{Hom}_{\mathcal{C}_\Gamma}(\Sigma^p T, T) \end{array}$$

the horizontal arrows are isomorphisms by Theorem 4.17 and the right vertical arrow  $\phi_*$  is an isomorphism. Thus, the morphism  $\phi : \Sigma^2\Gamma \rightarrow \Gamma$  induces isomorphisms in  $H^i$  for  $i \leq -2$ . Moreover, we have  $H^{-1}(\Gamma) = \text{Hom}(T, \Sigma^{-1}T) = \text{Hom}(T, \Sigma T) = 0$  since  $\Sigma^{-1}T$  is isomorphic to  $\Sigma T$ . It follows that  $\phi$  induces an isomorphism

$$\Sigma^2\Gamma \xrightarrow{\sim} \tau_{\leq -1}\Gamma.$$

□

**Remark 4.19.** For the pair  $(Q, w)$  associated to 3-dimensional flopping contractions, one can show that  $H^0\Gamma$  is indeed symmetric (see Theorem 6.5). In the context of general contraction with one dimensional fiber, Kawamata has proved that the classical (multi-pointed) deformation algebra of the reduced exceptional fiber is always self-injective (see Proposition 6.3 [36]). So in particular it is Gorenstein. This result overlaps with the above corollary in the case of simple flopping contractions. For a general finite quiver  $Q$ , the 0-th homology of a Jacobi-finite Ginzburg algebra  $\mathfrak{D}(Q, w)$  is not self-injective. Moreover, Kawamata proves that the deformation algebra is always isomorphic to its opposite algebra (see Corollary 6.3 of [37]).

**Corollary 4.20.** *Let  $\Gamma$  be the Ginzburg algebra of a Jacobi-finite quiver with potential. Let  $T$  be the image of  $\Gamma$  in the cluster category  $\mathcal{C}_\Gamma$ .*

- a)  $H^0\Gamma$  is selfinjective if and only if  $H^{-1}\Gamma$  vanishes if and only if  $T$  is isomorphic to  $\Sigma^2 T$  in  $\mathcal{C}_\Gamma$ .
- b) If the identity functor of  $\mathcal{C}_\Gamma$  is isomorphic to  $\Sigma^2$ , then  $H^0\Gamma$  is symmetric and there is an isomorphism of graded algebras  $H^0(\Gamma) \otimes k[u^{-1}] \xrightarrow{\sim} H^*(\Gamma)$ , where  $u$  is of degree 2.

*Proof.* a) By Theorem 4.17, the space  $H^{-1}\Gamma$  is isomorphic to  $\mathcal{C}_\Gamma(T, \Sigma^{-1}T)$  and  $H^0\Gamma$  is isomorphic to the endomorphism algebra of  $T$ . By Proposition 3.6 of [34], the endomorphism algebra is selfinjective if and only if  $\mathcal{C}_\Gamma(T, \Sigma^{-1}T)$  vanishes if and only if  $T$  is isomorphic to  $\Sigma^2 T$  in  $\mathcal{C}_\Gamma$ .

b) By combining the functorial isomorphism from  $T$  to  $\Sigma^2 T$  with the Calabi-Yau property we get an isomorphism of bimodules over the endomorphism algebra of  $T$

$$\mathcal{C}_\Gamma(T, T) \xrightarrow{\sim} \mathcal{C}_\Gamma(T, \Sigma^2 T) \xrightarrow{\sim} D\mathcal{C}_\Gamma(T, T).$$

Since  $H^0\Gamma$  is in particular selfinjective, the space  $H^{-1}\Gamma$  vanishes by a). We get an isomorphism of graded algebras

$$k[u, u^{-1}] \otimes_k \mathcal{C}_\Gamma(T, T) \xrightarrow{\sim} \bigoplus_{p \in \mathbb{Z}} \mathcal{C}_\Gamma(T, \Sigma^p T)$$

where  $u$  is of degree 2 by sending  $u$  to the functorial isomorphism  $T \xrightarrow{\sim} \Sigma^2 T$ . Thanks to Theorem 4.17, by truncation, we get an isomorphism of graded algebras

$$k[u^{-1}] \otimes_k H^0 \Gamma \xrightarrow{\sim} H^* \Gamma.$$

□

**4.4. Silting theory for a non positive dg algebra and its zeroth homology.** Let  $\mathcal{T}$  be a triangulated category. Recall that a *tilting object* for  $\mathcal{T}$  is a classical generator  $T$  of  $\mathcal{T}$  such that  $\mathcal{T}(T, \Sigma^p T)$  vanishes for all  $p \neq 0$ . A *silting object* [49] for  $\mathcal{T}$  is a classical generator  $T$  of  $\mathcal{T}$  such that  $\mathcal{T}(T, \Sigma^p T)$  vanishes for all  $p > 0$ . The advantage of silting objects over tilting objects is that (under suitable finiteness assumptions) they are stable under mutation [2].

We recall fundamental definitions and results from [2]. Assume from now on that  $\mathcal{T}$  is  $k$ -linear, Hom-finite and has split idempotents. In particular, it is a Krull–Schmidt category, i.e. indecomposables have local endomorphism rings and each object is a finite direct sum of indecomposables (which are then unique up to isomorphism and permutation). An object of  $\mathcal{T}$  is *basic* if it is a direct sum of pairwise non isomorphic indecomposables. If  $X$  is an object of  $\mathcal{T}$  and  $\mathcal{U}$  a full additive subcategory stable under retracts, a *left  $\mathcal{U}$ -approximation* of  $X$  is a morphism  $f : X \rightarrow U$  to an object of  $\mathcal{U}$  such that each morphism  $X \rightarrow V$  to an object of  $\mathcal{U}$  factors through  $f : X \rightarrow U$ . It is *minimal* if each endomorphism  $g : U \rightarrow U$  such that  $g \circ f = f$  is an isomorphism. Notice that the morphism  $f : X \rightarrow U$  is a minimal left  $\mathcal{U}$ -approximation iff the morphism  $f^* : \mathcal{U}(U, ?) \rightarrow \mathcal{T}(X, ?)|_{\mathcal{U}}$  is a projective cover in the category of left  $\mathcal{U}$ -modules. In particular, minimal left approximations are unique up to non unique isomorphism when they exist. Existence is automatic if  $\mathcal{U}$  has finitely many indecomposables  $U_1, \dots, U_n$  (which is the case in our applications) because then the functor  $\mathcal{T}(X, ?)|_{\mathcal{U}}$  corresponds to a finite-dimensional left module over the finite-dimensional endomorphism algebra of the sum of the  $U_i$ . A *(minimal) right  $\mathcal{U}$ -approximation* is defined dually. For an object  $X$  of  $\mathcal{T}$ , we denote by  $\text{add}X$  the full subcategory formed by all direct factors of finite direct sums of copies of  $X$ .

Let  $M$  be a basic silting object of  $\mathcal{T}$  and  $X$  an indecomposable direct summand of  $M$ . Denote by  $M/X$  the object such that  $M \cong X \oplus M/X$ . By definition, the *left mutation*  $\mu_X(M)$  of  $M$  at  $X$  is the silting object  $M/X \oplus Y$ , where  $Y$  is defined by a triangle

$$X \longrightarrow E \longrightarrow Y \longrightarrow \Sigma X$$

and  $X \rightarrow E$  is a minimal left  $\text{add}(M/X)$ -approximation. It is not hard to show that then  $E \rightarrow Y$  is a minimal right  $\text{add}(M/X)$ -approximation, which implies that  $Y$  is indecomposable. Indeed, let us recall the argument: Let  $M' = M/X$ . Since  $M$  is silting, we have an exact sequence of  $\text{End}(M')$ -modules

$$\text{Hom}(M', E) \rightarrow \text{Hom}(M', Y) \rightarrow 0,$$

where  $\text{Hom}(M', E)$  is projective over  $\text{End}(M')$ . Saying that  $E \rightarrow Y$  is a minimal right  $\text{add}(M/X)$ -approximation is equivalent to saying that  $\text{Hom}(M', E) \rightarrow \text{Hom}(M', Y)$  is a projective cover. If  $Y$  was decomposable, the morphism  $\text{Hom}(M', E) \rightarrow \text{Hom}(M', Y)$  would therefore split into a direct sum of two surjective morphisms and this splitting would be induced by a splitting of the morphism  $E \rightarrow Y$  as a direct sum of two non trivial morphisms  $E' \rightarrow Y'$  and  $E'' \rightarrow Y''$ . But then  $X$  would be decomposable, a contradiction. The right mutation  $\mu_X^-(M)$  is defined dually. The right mutation of  $\mu_X(M)$  at  $Y$  is isomorphic to  $M$ .

**Example 4.21.** Suppose that  $\Gamma$  is a Jacobi-finite Ginzburg algebra associated with a finite quiver and a potential not containing cycles of length  $\leq 2$ . Then  $A = \Gamma$  satisfies our assumptions and  $\Gamma$  is a basic silting object in  $\text{per}(\Gamma)$ . Let  $M$  be a silting object in  $\text{per}(\Gamma)$  and  $\Gamma'$  the derived endomorphism algebra of  $M$ . Then the homologies  $H^p(\Gamma')$  are finite-dimensional and vanish in degrees  $p > 0$ . Since  $M$  generates  $\text{per}(\Gamma)$ , the  $\Gamma'$ – $\Gamma$ -bimodule  $M$  yields an algebraic triangle equivalence  $D(\Gamma') \xrightarrow{\sim} D(\Gamma)$ .

Conversely, if we start from a dg algebra  $\Gamma'$  whose homologies are finite-dimensional and vanish in degrees  $> 0$  and from an algebraic triangle equivalence  $D(\Gamma') \xrightarrow{\sim} D(\Gamma)$ , then the image  $M$  of  $\Gamma'$  in  $\text{per}(\Gamma)$  is a silting object. In any case, the dg algebra  $\Gamma'$  is an exact bimodule 3-Calabi-Yau and has its homology concentrated in degrees  $\leq 0$ . By Van den Bergh's theorem [76], the dg algebra  $\Gamma'$  is again a Jacobi-finite Ginzburg algebra (up to weak equivalence). In particular, for  $M$  we can take the mutation  $M' = \mu_X \Gamma$ , where  $X = e_i \Gamma$  for a vertex  $i$  of the quiver of  $\Gamma$ . We define the associated Ginzburg algebra  $\Gamma'$  to be the *left mutation of  $\Gamma$  at  $i$* . Notice that by construction, we have a canonical derived equivalence from  $\Gamma'$  to  $\Gamma$ . In the same way, we can define the right mutation  $\Gamma''$  of  $\Gamma$  at  $i$  using the right mutation  $M'' = \mu_X^-(\Gamma)$  of  $\Gamma$  at  $X$ . The right mutation  $\Gamma''$  turns out to be quasi-isomorphic to the left mutation  $\Gamma'$ . Indeed, by Theorem 4.17, these algebras are the  $\tau_{\leq 0}$ -truncations of the derived endomorphism algebras of the images  $\pi(M')$  and  $\pi(M'')$  in the cluster category  $\mathcal{C}_\Gamma$ . Now we have  $\pi(M') \cong \pi(M'')$  because they are the left resp. right mutation in the sense of Iyama-Yoshino [33] of the cluster-tilting object  $\pi(\Gamma)$  at  $\pi(X)$  and for cluster-tilting objects in 2-Calabi-Yau triangulated categories, right and left mutation coincide up to isomorphism.

Now let  $A$  be a dg  $k$ -algebra whose homologies  $H^p A$  vanish in all degrees  $p > 0$ . An object of  $\text{per}(A)$  is called *2-term* if it is the cone over a morphism of  $\text{add}(A)$ . We write  $2\text{-per}(A)$  for the full subcategory of  $\text{per}(A)$  formed by the 2-term objects. We write  $2\text{silt}(A)$  for the set of isomorphism classes of *2-term silting objects*. Our aim is to compare  $2\text{silt}(A)$  with  $2\text{silt}(H^0 A)$ . Note that by our assumption on  $A$ , we have a canonical morphism  $A \rightarrow H^0 A$  in the homotopy category of dg algebras. We write  $F : \text{per}(A) \rightarrow \text{per}(H^0 A)$  for the derived tensor product over  $A$  with  $H^0 A$ . Part a) of the following theorem is due to Brüstle–Yang [13].

**Proposition 4.22.**    a) *The functor  $F$  induces a bijection  $2\text{silt}(A) \xrightarrow{\sim} 2\text{silt}(H^0 A)$ .*  
b) *Suppose that we have  $H^{-1}(A) = 0$ . Then the functor  $F$  restricts to an equivalence*

$$2\text{-per}(A) \xrightarrow{\sim} 2\text{-per}(H^0 A).$$

*In particular, for each 2-term object  $T$ , the functor  $F$  induces an isomorphism*

$$\text{End}_A(T) \xrightarrow{\sim} \text{End}_{H^0 A}(FT).$$

*Proof.* Part a) is Proposition A.3 of [13]. For part b), using the assumption and the 5-lemma, we check successively that  $F$  induces the following bijections:

1) For  $P, Q \in \text{add}(A)$  and  $p \geq -1$

$$\text{Hom}(P, \Sigma^p Q) \xrightarrow{\sim} \text{Hom}(FP, \Sigma^p FQ).$$

2) For  $P \in \text{add}(A)$ ,  $M \in 2\text{-per}(A)$  and  $p \in \{-1, 0\}$

$$\text{Hom}(P, \Sigma^p M) \xrightarrow{\sim} \text{Hom}(FP, \Sigma^p FM).$$

3) For  $M, M' \in 2\text{-per}(A)$

$$\text{Hom}(M, M') \xrightarrow{\sim} \text{Hom}(FM, FM').$$

□

Now let  $A$  be a pseudocompact dg algebra in  $\text{PCAlg}(l)$  strictly concentrated in degrees  $\leq 0$ . Let  $e$  be an idempotent of  $H^0 A$  and  $A'$  the derived endomorphism algebra of the image of  $A$  in the Verdier quotient of  $\text{per}(A)$  by the thick subcategory generated by  $eA$ . Then  $A'$  is concentrated in degrees  $\leq 0$  and we have a canonical morphism  $A \rightarrow A'$  in the homotopy category of  $\text{PCAlg}(l)$ . If  $A$  is of the form  $(\widehat{T}_l(V), d)$  for a pseudocompact  $l$ -bimodule  $V$  concentrated in degrees  $\leq 0$ , where  $\widehat{T}_l(V)$  is the completed tensor algebra, then  $A'$  is quasi-isomorphic to the quotient of  $A$  by the two-sided closed ideal generated by  $e$  (cf. [12]). Put  $A_0 = H^0 A'$  so that we have a canonical morphism

$p : A \rightarrow A_0$ . Let  $B$  and  $B_0$  be pseudocompact dg algebras in  $\text{PCAlg}(l)$ ,  $X \in \text{D}(A^{\text{op}} \otimes B)$  such that  $X_B$  is perfect and  $Q \in \text{D}(B^{\text{op}} \otimes B_0)$  such that  $Q_{B_0}$  is perfect.

**Proposition 4.23.** *Suppose that  $eX \overset{\mathbb{L}}{\otimes}_B Q$  vanishes and that the object  $X \overset{\mathbb{L}}{\otimes}_B Q$  of  $\text{D}(B_0)$  has no selfextensions in degrees  $p < 0$ . Then there is an object  $Y$  of  $\text{D}(A_0^{\text{op}} \otimes B_0)$ , unique up to isomorphism, such that we have an isomorphism*

$$X \overset{\mathbb{L}}{\otimes}_B Q \xrightarrow{\sim} A_0 \overset{\mathbb{L}}{\otimes}_{A_0} Y$$

in  $\text{D}(A^{\text{op}} \otimes B_0)$ . Thus, the square

$$\begin{array}{ccc} \text{D}(A) & \xrightarrow{X} & \text{D}(B) \\ A_0 \downarrow & & \downarrow Q \\ \text{D}(A_0) & \xrightarrow{Y} & \text{D}(B_0) \end{array}$$

is commutative up to isomorphism, where we write dg bimodules instead of derived tensor products by dg bimodules.

**Remark 4.24.** In our applications in this article, the idempotent  $e$  will be 0. We state and prove the proposition in the general case because it provides an alternative approach to the problem of relating the tilting theory of maximal modification algebras [79] to that of the associated contraction algebras as treated by August in [5]. Let  $R$  be a complete local cDV singularity and  $M$  a maximal basic rigid object in the category of Cohen-Macaulay modules over  $R$  containing  $R$  as a direct summand. We can take  $A = \text{End}_R(M)$  and  $e$  the idempotent corresponding to the projection on  $R$ . Then  $A_0 = H^0 A'$  is isomorphic to the stable endomorphism algebra of  $M$ , i.e. the contraction algebra associated with  $M$ . Let  $N$  be another maximal basic rigid object containing  $R$  as a direct summand,  $B$  its endomorphism algebra and  $B_0$  the associated contraction algebra. Then  $X = \text{Hom}_R(N, M)$  yields a derived equivalence  $? \overset{\mathbb{L}}{\otimes}_A X : \text{D}(A) \xrightarrow{\sim} \text{D}(B)$  taking  $eA$  to  $eB$ . Moreover, the complex  $X \overset{\mathbb{L}}{\otimes}_B B_0$  is a 2-term silting object of  $\text{per}(B_0)$  (as it follows from silting reduction [2] combined with part a) of Proposition 4.22) and hence a tilting object since  $B_0$  is symmetric. Thus, the hypotheses of the proposition hold and there is a canonical two-sided tilting complex  $Y$  in  $\text{D}(A_0^{\text{op}} \otimes B_0)$ . Clearly, the construction is compatible with compositions via derived tensor products.

*Proof of the Proposition.* Put  $U = X \overset{\mathbb{L}}{\otimes}_B Q$  viewed as an object in  $\text{D}(A^{\text{op}} \otimes B_0)$ . The morphism  $A \rightarrow A'$  induces the Verdier quotient

$$\text{per}(A) \rightarrow \text{per}(A') = \text{per}(A)/\mathbf{thick}(eA)$$

and is therefore a dg quotient. By the universal property of the dg quotient, there is an object  $Z$  in  $\text{D}(A'^{\text{op}} \otimes B_0)$  unique up to isomorphism such that the restriction of  $Z$  along  $A^{\text{op}} \otimes B_0 \rightarrow A'^{\text{op}} \otimes B_0$  is isomorphic to  $U$  in  $\text{D}(A^{\text{op}} \otimes B_0)$ . We have a canonical morphism in the homotopy category of dg algebras

$$A' \longrightarrow \mathbf{R}\text{Hom}_{B_0}(Z, Z).$$

Since by assumption  $\mathbf{R}\text{Hom}_{B_0}(Z, Z)$  is concentrated in degrees  $\geq 0$  and  $A'$  in degrees  $\leq 0$ , this morphism factors uniquely through a morphism

$$A_0 = H^0 A' \rightarrow \mathbf{R}\text{Hom}_{B_0}(Z, Z)$$

in the homotopy category of dg algebras. Let us show how to refine this argument so as to obtain an object  $Y$  of  $\text{D}(A_0^{\text{op}} \otimes B_0)$  which restricts to  $Z \in \text{D}(A'^{\text{op}} \otimes B_0)$ . We may and will assume that  $A'$  is

cofibrant and strictly concentrated in degrees  $\leq 0$ . We may and will also assume that  $Z$  is cofibrant as a dg  $A'$ - $B_0$ -bimodule. The left  $A'$ -module structure on  $Z$  then yields a morphism of dg algebras

$$A' \longrightarrow \text{Hom}_{B_0}(Z, Z).$$

Since  $A'$  is strictly concentrated in non positive degrees, it factors uniquely through a morphism of dg algebras

$$A' \longrightarrow \tau_{\leq 0} \text{Hom}_{B_0}(Z, Z).$$

Since  $Z_{B_0}$  has no negative selfextensions, we have a surjective quasi-isomorphism of dg algebras

$$\tau_{\leq 0} \text{Hom}_{B_0}(Z, Z) \longrightarrow H^0 \text{Hom}_{B_0}(Z, Z).$$

The composition

$$A' \longrightarrow \tau_{\leq 0} \text{Hom}_{B_0}(Z, Z) \longrightarrow H^0 \text{Hom}_{B_0}(Z, Z)$$

uniquely factors through an algebra morphism  $A_0 = H^0(A') \rightarrow H^0 \text{Hom}_{B_0}(Z, Z)$ . We thus obtain a commutative square of dg algebra morphisms

$$\begin{array}{ccc} A' & \xrightarrow{\quad} & \text{Hom}_{B_0}(Z, Z) \\ \downarrow & \searrow & \uparrow \\ & \tau_{\leq 0} \text{Hom}_{B_0}(Z, Z) & \\ \downarrow & & \downarrow \\ A_0 = H^0(A') & \longrightarrow & H^0 \text{Hom}_{B_0}(Z, Z). \end{array}$$

We factor the morphism  $A' \rightarrow A_0$  as the composition  $A' \rightarrow \tilde{A}_0 \rightarrow A_0$  of an acyclic fibration with a cofibration. We consider the diagram

$$\begin{array}{ccc} A' & \xrightarrow{\quad} & \text{Hom}_{B_0}(Z, Z) \\ \downarrow & \searrow & \uparrow \\ \tilde{A}_0 & \xrightarrow{\quad} & \tau_{\leq 0} \text{Hom}_{B_0}(Z, Z) \\ \downarrow & & \downarrow \\ A_0 = H^0(A') & \longrightarrow & H^0 \text{Hom}_{B_0}(Z, Z). \end{array}$$

Here the morphism represented by a dotted arrow exists so that the diagram becomes commutative because  $A' \rightarrow \tilde{A}_0$  is a cofibration and  $\tau_{\leq 0} \text{Hom}_{B_0}(Z, Z) \rightarrow H^0 \text{Hom}_{B_0}(Z, Z)$  is an acyclic fibration. Thus, we obtain a structure of dg  $\tilde{A}_0$ - $B_0$ -bimodule on  $Z$  which restricts to the given structure of dg  $A'$ - $B_0$ -bimodule. Since we have the quasi-isomorphism  $\tilde{A}_0 \rightarrow A_0$ , we can find a bimodule  $Y$  in  $D(A_0^{op} \otimes B_0)$  unique up to isomorphism and which restricts (up to isomorphism) to  $Z$  in  $D(A'^{op} \otimes B_0)$ .

Now suppose we have a second object  $Y'$  in  $D(A_0^{op} \otimes B_0)$  which becomes isomorphic to  $Z$  in  $D(A'^{op} \otimes H^0 B)$ . We have a chain of isomorphisms

$$\begin{aligned} \text{Hom}_{D(A'^{op} \otimes B_0)}(Y, Y') &= \text{Hom}_{D(A'^e)}(A', \mathbf{R}\text{Hom}_{B_0}(Y, Y')) \\ &= \text{Hom}_{D(A'^e)}(H^0 A', H^0 \mathbf{R}\text{Hom}_{B_0}(Y, Y')) \\ &= \text{Hom}_{H^0(A')^e}(H^0 A', H^0 \mathbf{R}\text{Hom}_{B_0}(Y, Y')) \\ &= \text{Hom}_{D(A_0^e)}(A_0, H^0 \mathbf{R}\text{Hom}_{B_0}(Y, Y')) \\ &= \text{Hom}_{D(A_0^e)}(A_0, \mathbf{R}\text{Hom}_{B_0}(Y, Y')) \\ &= \text{Hom}_{D(A_0^{op} \otimes B_0)}(Y, Y'). \end{aligned}$$

Clearly, the composition

$$\text{Hom}_{D(A_0^{op} \otimes B_0)}(Y, Y') \xrightarrow{\sim} \text{Hom}_{D(A'^{op} \otimes B_0)}(Y, Y')$$

of these isomorphisms is given by the restriction of scalars functor

$$D(A^0 \otimes H^0 B) \rightarrow D(A' \otimes H^0 B).$$

Now any restriction of scalars functor reflects isomorphisms because it is compatible with the forgetful functors to the derived category of vector spaces. Thus, isomorphisms are preserved. This shows the uniqueness.  $\square$

Let  $B$  be a dg  $k$ -algebra whose homologies are finite-dimensional and vanish in degrees  $> 0$ . Let  $C$  be a finite-dimensional basic  $k$ -algebra (i.e.  $C$  is basic as a right module over itself) and let  $Z$  be an object of  $D(C^{op} \otimes H^0 B)$  such that

$$? \otimes_C Z : D(C) \rightarrow D(H^0 B)$$

is an equivalence. Notice that  $Z_{H^0 B}$  is a tilting object in  $\text{per}(H^0 B)$  and in particular a silting object, which is basic by our assumption on  $C$ .

**Theorem 4.25.** *Assume that  $Z_{H^0 B}$  is a 2-term silting object and we have  $H^{-1}(B) = 0$ . Then there is a dg algebra  $A$  whose homologies  $H^p A$  are finite-dimensional and whose components vanish in degrees  $p > 0$ , a derived equivalence  $? \otimes_A X : D(A) \xrightarrow{\sim} D(B)$ , an isomorphism of algebras  $\phi : H^0 A \xrightarrow{\sim} C$  and an isomorphism*

$${}_\phi Z \xrightarrow{\sim} X \otimes_B H^0 B$$

in  $D(A^{op} \otimes H^0 B)$ , where the left  $A$ -module structure on  ${}_\phi Z$  is defined via the composition  $A \rightarrow H^0 A \rightarrow C$ . In particular, we have a diagram, commutative up to isomorphism

$$\begin{array}{ccc} D(A) & \xrightarrow{X} & D(B) \\ {}^{H^0 A} \swarrow & & \downarrow {}^{H^0 B} \\ D(H^0 A) & \xrightarrow[{}_\phi C]{} & D(C) \xrightarrow[Z]{} D(H^0 B), \end{array}$$

where we write dg bimodules instead of derived tensor products by dg bimodules.

*Proof.* By part a) of Proposition 4.22, there is a 2-term silting object  $M$  of  $\text{per}(B)$  such that  $M \otimes_B H^0 B$  is isomorphic to  $Z_{H^0 B}$ . Since  $M$  is silting, its derived endomorphism algebra has its homology concentrated in non positive degrees and we define

$$A = \tau_{\leq 0} \mathbf{R}\text{Hom}(M, M).$$

We let  $X \in D(A^{op} \otimes B)$  be the dg bimodule given by  $M$  with its canonical left  $A$ -action. Since  $X \overset{\mathbb{L}}{\otimes}_B H^0 B$  is isomorphic to the tilting object  $Z_{H^0 B}$ , it has no self-extensions in degree  $< 0$ . Therefore, Proposition 4.23 yields an object  $Y$  of  $D(H^0(A)^{op} \otimes H^0(B))$  and an isomorphism

$$\psi : X \overset{\mathbb{L}}{\otimes}_B H^0 B \xrightarrow{\sim} Y|_{A^{op} \otimes H^0 B}$$

in  $D(A^{op} \otimes H^0 B)$ , where the left  $A$ -module structure on  $Y$  comes from the canonical morphism  $A \rightarrow H^0(A)$ . By part b) of Proposition 4.22, we have an isomorphism  $\text{End}(M) \xrightarrow{\sim} \text{End}(M \overset{\mathbb{L}}{\otimes}_B H^0 B)$ . By construction, we have an isomorphism  $H^0 A \xrightarrow{\sim} \text{End}(M)$  or equivalently  $H^0 A \xrightarrow{\sim} \text{End}(X_B)$ . Thus, the composition of  $? \overset{\mathbb{L}}{\otimes}_B H^0 B$  with  $? \overset{\mathbb{L}}{\otimes}_A X$  induces an isomorphism

$$H^0 A \xrightarrow{\sim} \text{End}_{D(H^0 B)}(X \overset{\mathbb{L}}{\otimes}_B H^0 B).$$

Via  $\psi$ , we get an isomorphism

$$H^0 A \xrightarrow{\sim} \text{End}_{D(H^0 B)}(Y)$$

given by the left action of  $H^0 A$  on  $Y$ . We choose an isomorphism  $Y_{H^0 B} \xrightarrow{\sim} Z_{H^0 B}$  in  $D(H^0 B)$  and define  $\phi : H^0 A \xrightarrow{\sim} C$  so as to make the following square commutative

$$\begin{array}{ccc} H^0 A & \longrightarrow & \text{End}(Y_{H^0 B}) \\ \phi \downarrow & & \downarrow \\ C & \longrightarrow & \text{End}(Z_{H^0 B}). \end{array}$$

By Lemma 2.6, the chosen isomorphism  $Y_{H^0 B} \xrightarrow{\sim} Z_{H^0 B}$  lifts to an isomorphism  $Y \xrightarrow{\sim} {}_\phi Z$  in  $D(H^0(A)^{op} \otimes H^0(B))$ . Whence a composed isomorphism

$$X \overset{\mathbb{L}}{\otimes}_B H^0 B \xrightarrow{\sim} Y|_{A^{op} \otimes H^0 B} \xrightarrow{\sim} {}_\phi Z|_{A^{op} \otimes H^0 B}$$

□

**4.5. Cyclic homology and preservation of the canonical class.** Let  $k$  be a field of characteristic 0 and  $l$  a finite product of copies of  $k$ . Let  $V$  be a pseudocompact  $l$ -bimodule and  $d$  a continuous differential on the completed tensor algebra  $\widehat{T}_l(V)$ . Put  $A = (\widehat{T}_l(V), d)$ . We define  $\Omega_l A$  by the short exact sequence

$$0 \longrightarrow \Omega^1 A \longrightarrow A \otimes_l A \xrightarrow{\mu} A \longrightarrow 0,$$

where  $\mu$  is the multiplication of  $A$ . Then the morphism

$$A \otimes_l V \otimes_l A \rightarrow \Omega^1 A$$

taking  $a \otimes v \otimes b$  to  $av \otimes b - a \otimes vb$  is an isomorphism of graded  $l$ -bimodules, cf. Example 3.10 of [69]. We can describe the induced differential on  $A \otimes_l V \otimes_l A$  as follows (cf. Proposition 3.7 of [46]): Let  $D : A \rightarrow A \otimes_l V \otimes_l A$  be the unique continuous bimodule derivation which restricts to the map  $v \mapsto 1 \otimes v \otimes 1$  on  $V$ . We have

$$D(v_1 \dots v_n) = 1 \otimes v_1 \otimes (v_2 \dots v_n) + \sum_{i=2}^{n-1} v_1 \dots v_{i-1} \otimes v_i \otimes v_{i+1} \dots v_n + (v_1 \dots v_{n-1}) \otimes v_n \otimes 1.$$

Then the induced differential on  $A \otimes_l V \otimes_l A$  sends  $a \otimes v \otimes b$  to

$$(-1)^{|a|} a D(dv)b + (da) \otimes v \otimes b + (-1)^{(|v|+|a|)} a \otimes v \otimes (db).$$

For an  $l$ -bimodule  $M$ , we write  $M_l$  for the coinvariant module  $M/[l, M]$ . For an  $A$ -bimodule  $M$ , we let  $M_{\natural}$  be the coinvariant module  $M/[A, M]$ . We have an isomorphism of graded  $A$ -modules

$$(A \otimes_l V \otimes_l A)_{\natural} \xrightarrow{\sim} (V \otimes_l A)_l$$

taking  $a \otimes v \otimes b$  to  $(-1)^{|a|(|v|+|b|)} v \otimes ba$ . The induced differential on the right hand side is given as follows: If  $D(dv) = \sum_i a_i \otimes v_i \otimes b_i$ , then

$$d(v \otimes a) = (-1)^{|v|} v \otimes (da) + \sum_i (-1)^{|a_i|(|v_i|+|b_i|+|a|)} v_i \otimes b_i a a_i.$$

Following section 3 of [69] we define morphisms of complexes

$$\partial_1 : (V \otimes_l A)_l \rightarrow A_l \quad \text{and} \quad \partial_0 : A_l \rightarrow (V \otimes_l A)_l$$

as follows:  $\partial_1$  sends  $v \otimes a$  to  $va - (-1)^{|v||a|} av$  and  $\partial_0$  sends  $v_1 \dots v_n$  to

$$\sum_i \pm v_i \otimes v_{i+1} \dots v_n v_1 \dots v_{i-1},$$

where the sign is determined by the Koszul sign rule. We then have  $\partial_0 \partial_1 = 0 = \partial_1 \partial_0$ .

The (continuous) Hochschild homology of  $A$  is computed by the total complex of

$$(V \otimes_l A)_l \xrightarrow{\partial_1} A_l$$

and the (continuous) cyclic homology of  $A$  is computed by the product total complex of

$$\dots \longrightarrow (V \otimes_l A)_l \xrightarrow{\partial_1} A_l \xrightarrow{\partial_0} (V \otimes_l A)_l \xrightarrow{\partial_1} A_l \dots$$

Since  $k$  is of characteristic 0, the morphism  $A_l \rightarrow A/([A, A] + l)$  induces a quasi-isomorphism from this complex to  $A/([A, A] + l)$ . The ISB-sequence

$$\dots \longrightarrow \mathrm{HH}_n \xrightarrow{I} \mathrm{HC}_n \xrightarrow{S} \mathrm{HC}_{n-2} \xrightarrow{B} \mathrm{HH}_{n-1} \longrightarrow \dots$$

is induced by the following sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & (V \otimes_l A)_l \xrightarrow{\partial_1} A_l \\ & & \downarrow & & \downarrow & & \parallel \\ \dots & \longrightarrow & (V \otimes_l A)_l & \xrightarrow{\partial_1} & A_l & \xrightarrow{\partial_0} & (V \otimes_l A)_l \xrightarrow{\partial_1} A_l \\ & & \parallel & & \parallel & & \parallel \\ \dots & \longrightarrow & (V \otimes_l A)_l & \xrightarrow{\partial_1} & A_l & \longrightarrow & 0 \longrightarrow 0 \\ & & \downarrow & & \downarrow \partial_0 & & \downarrow \\ \dots & \longrightarrow & 0 & \longrightarrow & (V \otimes_l A)_l & \xrightarrow{\partial_1} & A_l \end{array}$$

Notice that the first three rows form a short exact sequence and that the composition of the last vertical morphism with the second last vertical morphism is only homotopic to zero.

Now let  $Q$  be a finite quiver,  $l$  the product over the  $ke_i$ , where  $i$  runs through the vertices of  $Q$ ,  $w$  a potential on  $Q$  and  $A = \Gamma$  the associated complete Ginzburg algebra with generators the arrows  $\alpha$  of  $Q$  in degree 0, the reversed arrows  $\alpha^*$  in degree  $-1$  and the loops  $t_i$  in degree  $-2$ . Then  $A$  is

of the form  $(\widehat{T}_l(V), d)$ , where  $V$  is the  $l$ -bimodule with basis given by the arrows  $\alpha$ ,  $\alpha^*$  and  $t_i$ . Let  $t$  be the sum of the  $t_i$ . By definition, we have

$$d(t) = \sum_{\alpha} [\alpha, \alpha^*].$$

Thus,  $t$  defines an element in  $\mathrm{HC}_2(A)$ .

**Lemma 4.26.** *The image of the class of  $t$  under  $S : \mathrm{HC}_2(A) \rightarrow \mathrm{HC}_0(A)$  is the canonical class  $[w]$ , i.e. the image of  $w$  under the projection  $\mathrm{HC}_0(T_l V) \rightarrow \mathrm{HC}_0(A)$ .*

*Proof.* We compute  $S(t)$  using the above description of  $S$ . We need to lift  $t$  to an element of the total complex computing cyclic homology. We have

$$d(t) = \partial_1 \left( \sum_{\alpha} \alpha \otimes \alpha^* \right).$$

We have

$$d \left( \sum_{\alpha} \alpha \otimes \alpha^* \right) = \sum_{\alpha} \alpha \otimes d(\alpha^*) = \sum_{\alpha} \alpha \otimes D_{\alpha}(w).$$

Thus, we have

$$d \left( \sum_{\alpha} \alpha \otimes \alpha^* \right) = \partial_0(w)$$

and  $S(t)$  is the image of  $w$  in  $H^0(A/([A, A] + l) = \mathrm{HC}_0(\Gamma)$ . Notice that  $BS(t) = B(w)$  is indeed a boundary in the Hochschild complex: It is the differential of

$$\sum_{\alpha} \alpha \otimes \alpha^* - t.$$

□

**Corollary 4.27.** *Let  $\Gamma' = \Gamma(Q', w')$  be a Ginzburg algebra and  $A$  a pseudo-compact dg algebra in  $\mathrm{PCAlg}(l)$  concentrated in degrees  $\leq 0$ . Let  $X$  be a dg  $A\text{-}\Gamma'$ -bimodule such that  $? \overset{\mathbb{L}}{\otimes}_A X : \mathrm{D}(A) \rightarrow \mathrm{D}(\Gamma')$  is an equivalence. Then there is a quiver with potential  $(Q'', w'')$  and a weak equivalence  $s : \Gamma(Q'', w'') \rightarrow A$  such that for the restriction  $_s X$  along  $s$ , the isomorphism  $\mathrm{HC}_0(_s X)$  takes the class  $[w'']$  to  $[w']$ .*

*Proof.* We know that the class  $[t'] \in \mathrm{HC}_2(\Gamma')$  is non degenerate in the sense that  $B[t'] \in \mathrm{HH}_3(\Gamma')$  defines an isomorphism  $\Sigma^3 \Theta_{\Gamma'} \xrightarrow{\sim} \Gamma'$  in  $\mathrm{D}(\Gamma'^e)$ , where  $\Theta_{\Gamma'}$  is the inverse dualizing complex. Thus the image  $\tau$  of  $[t']$  under  $\mathrm{HC}_2(X)^{-1}$  is a non degenerate element of  $\mathrm{HC}_2(A)$ . The proof of Theorem 10.2.2 in [76] then shows that there is a quiver  $Q''$ , a potential  $w''$  and a weak equivalence  $s : \Gamma(Q'', w'') \rightarrow A$  which takes  $[t'']$  to  $\tau$ . Thus, the composition  $\mathrm{HC}_2(X) \circ \mathrm{HC}_2(s) = \mathrm{HC}_2(_s X)$  takes  $[t'']$  to  $[t']$  and the isomorphism  $\mathrm{HC}_0(_s X)$  takes  $[w''] = S[t'']$  to  $[w'] = S[t']$ . □

## 5. CY TILTED ALGEBRAS AND SINGULARITIES

**5.1. Basics on Hochschild cohomology.** Let  $k$  be a commutative ring and  $A$  be a unital  $k$ -algebra projective over  $k$ . Denote by  $\overline{A}$  the quotient  $A/k \cdot 1$ . Define the *normalized bar complex* associated to  $A$  to be the complex  $B_k A := A \otimes_k T \Sigma \overline{A} \otimes_k A$  with differential  $\sum_{i=0}^{n-1} (-1)^i b_i : A \otimes \overline{A}^{\otimes n-1} \otimes A \rightarrow A \otimes \overline{A}^{\otimes n-2} \otimes A$

$$b_i(a_0, \dots, a_n) = (a_0, \dots, a_i a_{i+1}, \dots, a_n).$$

It is a projective bimodule resolution of  $A$ . Let  $M$  be an  $A$ -bimodule. The *Hochschild cochain complex* with coefficients in the bimodule  $M$  is defined to be the complex  $C^*(A, M) := \mathrm{Hom}_{A^e}(B_k(A), M)$  with differential

$$\delta(f) = -(-1)^n f \circ b$$

for  $f : A \otimes \bar{A}^{\otimes n} \otimes A \rightarrow M$ . The  $i$ -th *Hochschild cohomology of the algebra  $A$  with coefficients in the bimodule  $M$*  is defined to be  $\mathrm{HH}^i(A, M) := H^i(C^*(A, M), \delta)$ .

Let  $A$  be an augmented dg  $k$ -algebra. Denote by  $\bar{A}$  the kernel of the augmentation. Then the bar complex  $B_k A$  is equipped with a second differential induced from the differential  $d_A$  on  $A$ . Given an  $A$ -bimodule  $M$ , the Hochschild cochain complex  $C^*(A, M)$  is equipped with a second differential  $d$  induced by  $d_A$  and the internal differential  $d_M$  on  $M$ . The  $i$ th *Hochschild cohomology of the dg algebra  $A$  with coefficients in the bimodule  $M$*  is defined to be  $\mathrm{HH}^i(A, M) := H^i(C^*(A, M), d + \delta)$ .

It is well-known that  $\mathrm{HH}^i(A, M)$  is isomorphic to  $\mathrm{Ext}_{A^e}^i(A, M)$ . When  $A$  is a smooth commutative  $k$ -algebra,  $\mathrm{HH}^*(A, A)$  is isomorphic to the polyvector fields on  $\mathrm{Spec} A$  by the Hochschild-Kostant-Rosenberg theorem. For non-smooth algebras, there exist different variants of Hochschild cohomology.

Let  $A$  be an associative  $k$ -algebra projective over  $k$ . Define the module of *Kähler differentials*  $\Omega_A$  to be the kernel of the multiplication map  $\mu : A \otimes A \rightarrow A$ . Clearly,  $\Omega_A$  inherits a bimodule structure from  $A \otimes A$ . It is easy to show that  $\Omega_A$  is generated as a bimodule by the elements of the form  $xdy := xy \otimes 1 - x \otimes y$ . The left and right module structure are given by

$$a(xdy) = (ax)dy, \quad (xdy)a = xd(ya) - xyda.$$

Define the module of  $n$ -forms to be the  $n$ -fold tensor product

$$\Omega_A^n := \Omega_A \otimes_A \Omega_A \otimes \dots \otimes_A \Omega_A.$$

Using the above identities, one can check that  $\Omega_A^n$  is generated as a bimodule by the elements of the form  $a_0 da_1 da_2 \dots da_n$ . There is an isomorphism of bimodules  $\Omega_A^n \cong A \otimes_k \bar{A}^{\otimes n}$  defined by

$$a_0 da_1 da_2 \dots da_n \mapsto a_0 \otimes a_1 \otimes \dots \otimes a_n.$$

Set  $\Omega_A^0 = A$  and  $\Omega_A^1 = \Omega_A$ . Write  $\Omega_A^*$  for  $\bigoplus_{n \geq 0} \Omega_A^n$ . The bimodule structure on  $\Omega_A$  naturally extends to an associative algebra structure on  $\Omega_A^*$ . The obvious differential

$$D : a_0 da_1 da_2 \dots da_n \mapsto da_0 da_1 da_2 \dots da_n$$

makes  $\Omega_A^*$  into a differential graded algebra.

For  $m \in \mathbb{Z}$ ,

$$C^m(A, \Sigma^n \Omega_A^n) = \mathrm{Hom}_k(\bar{A}^{\otimes(n+m)}, A \otimes_k \bar{A}^{\otimes n}).$$

Consider the chain maps  $\theta_n : C^*(A, \Sigma^n \Omega_A^n) \rightarrow C^*(A, \Sigma^{n+1} \Omega_A^{n+1})$  between the Hochschild cochain complexes defined by  $f \mapsto f \otimes \mathrm{Id}_{\Sigma \bar{A}}$ .

**Definition 5.1.** Let  $A$  be an associative  $k$ -algebra. Then the *singular Hochschild cochain complex* of  $A$ , denoted by  $C_{sg}^*(A, A)$ , is defined as the colimit of the inductive system in the category of cochain complexes of  $k$ -modules,

$$0 \longrightarrow C^*(A, A) \xrightarrow{\theta_0} C^*(A, \Sigma \Omega_A^1) \xrightarrow{\theta_1} \dots \longrightarrow C^*(A, \Sigma^n \Omega_A^n) \xrightarrow{\theta_n} \dots$$

Namely,  $C_{sg}^*(A, A) := \mathrm{colim}_n C^*(A, \Sigma^n \Omega_A^n)$ . Its cohomology groups are denoted by  $\mathrm{HH}_{sg}^*(A, A)$ .

By construction, we have a natural chain morphism from  $C^*(A, A)$  to  $C_{sg}^*(A, A)$ , which induces a natural morphism from  $\mathrm{HH}^*(A, A)$  to  $\mathrm{HH}_{sg}^*(A, A)$ .

Let  $A$  be a Noetherian  $k$ -algebra. Define  $D_{sg}(A)$  to be the Verdier quotient of  $D^b(A)$  by the subcategory  $\mathrm{per}(A)$ . We denote the extension group in  $D_{sg}(A)$  by  $\underline{\mathrm{Ext}}_A^i(?, ?)$ . The singular Hochschild cohomology groups are related to the extension groups in  $D_{sg}(A^e)$ .

**Proposition 5.2.** (*Theorem 3.6 [77]*) Let  $A$  be a Noetherian  $k$ -algebra. Then there exists a natural isomorphism

$$\mathrm{HH}_{sg}^*(A, A) \xrightarrow{\sim} \underline{\mathrm{Ext}}_{A^e}^*(A, A),$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Ext}_{A^e}^*(A, A) & \longrightarrow & \underline{\mathrm{Ext}}_{A^e}^*(A, A) \\ \cong \uparrow & & \cong \uparrow \\ \mathrm{HH}^*(A, A) & \longrightarrow & \mathrm{HH}_{sg}^*(A, A) \end{array}$$

From Wang's result, we see that the singular Hochschild cohomology admits a structure of graded commutative algebra. Notice that this is not immediate from generalities about monoidal triangulated categories because the singularities category of  $A^e$  does not have any obvious monoidal structure.

**5.2. Hochschild cohomology of Gorenstein algebras.** A (not necessarily commutative) Noetherian ring  $A$  is called *Gorenstein* if it has finite injective dimension both as a left and right  $A$ -module. As in the commutative case, we denote by  $\mathrm{CM}_A$  the category of maximal Cohen-Macaulay (left)  $A$ -modules and denote by  $\underline{\mathrm{CM}}_A$  its stable category. Buchweitz proved that if  $A$  is Gorenstein, then  $\underline{\mathrm{CM}}_A$  is equipped with a structure of triangulated category and  $\underline{\mathrm{CM}}_A \cong \mathrm{D}_{sg}(A)$ .

We recall a fundamental result on extension groups in the stable category over Gorenstein rings due to Buchweitz.

**Proposition 5.3.** (*Corollary 6.3.4 of [14]*) Let  $A$  be a Gorenstein ring and let  $X$  and  $Y$  be objects in  $\mathrm{D}^b(A)$ . There exists a positive integer  $m$  depending on  $A$ ,  $X$  and  $Y$  such that the natural morphism  $\mathrm{Ext}_A^i(X, Y) \rightarrow \underline{\mathrm{Ext}}_A^i(X, Y)$  is surjective for  $i = m$  and is an isomorphism for  $i > m$ .

Combining Proposition 5.2 and Proposition 5.3, we obtain the following result.

**Corollary 5.4.** Let  $R$  be a commutative noetherian Gorenstein  $k$ -algebra. If  $R \otimes R$  is noetherian, there exists a positive integer  $m$  such that for  $i > m$ , the natural morphism

$$\mathrm{HH}^i(R, R) \rightarrow \mathrm{HH}_{sg}^i(R, R)$$

is an isomorphism.

*Proof.* By Definition 5.1, we have a morphism  $\mathrm{HH}^i(R, R) \rightarrow \mathrm{HH}_{sg}^i(R, R)$  for all  $i$ . In order to apply Proposition 5.3 to show that it is an isomorphism, we need to check that  $R \otimes R$  is Noetherian. This follows from Theorem 1.6 of [72].  $\square$

A commutative local complete Gorenstein  $k$ -algebra  $\hat{R}$  is called a hypersurface algebra if  $\hat{R} \cong k[[x_1, \dots, x_n]]/(g)$ . We say that  $\hat{R}$  is a hypersurface algebra with isolated singularities if  $g$  has an isolated critical point.

**Theorem 5.5.** (*Theorem 3.2.7 [25]*) Let  $R = k[x_1, \dots, x_n]/(g)$  be a hypersurface algebra with isolated singularities. Denote by  $M_g$  the Milnor algebra  $k[x_1, \dots, x_n]/(\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n})$ , and by  $K_g$  and  $T_g$  the kernel and cokernel of the endomorphism of  $M_g$  defined by multiplication with  $g$ . Then for  $r \geq n$ , there is an isomorphism of  $R$ -modules

$$\begin{cases} \mathrm{HH}^r(R, R) \cong T_g & r \text{ is even,} \\ \mathrm{HH}^r(R, R) \cong K_g & r \text{ is odd.} \end{cases}$$

*Proof.* The proof in [25] shows that in degrees  $r \geq n$ , the Hochschild cohomology  $\mathrm{HH}^r(R, R)$  is isomorphic to the homology in degree  $r$  of the complex

$$k[u] \otimes K(R, \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}),$$

where  $u$  is of degree 2 and  $K$  denotes the Koszul complex. Put  $P = k[x_1, \dots, x_n]$ . Since  $R$  is quasi-isomorphic to  $K(P, g)$  and the  $(\partial g)/(\partial x_i)$  form a regular sequence in  $P$ , the Koszul complex is quasi-isomorphic to  $K(M_g, g)$ .  $\square$

Note that  $T_g$  is the *Tyurina algebra*  $k[x_1, \dots, x_n]/(g, \frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n})$ . Since  $K_g$  is the kernel of the multiplication map  $g : M_g \rightarrow M_g$  it is a module over  $T_g$ .

**Lemma 5.6.** *Let  $A$  be a commutative  $k$ -algebra such that  $A$  and  $A^e$  are noetherian. Let  $S \subset A$  be a multiplicative subset. If  $M$  is a finitely generated  $A$ -module and  $L$  an  $A$ -module, we have a canonical isomorphism*

$$\mathbf{R}\mathrm{Hom}_{A^e}(L, M)_S \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{A_S^e}(L_S, M_S).$$

*Proof.* Since  $L$  is finitely generated over  $A$ , it is finitely generated over  $A^e$ . Since  $A^e$  is noetherian, we have a projective resolution  $P \rightarrow L$  with finitely generated components. This implies that we have isomorphisms

$$\mathbf{R}\mathrm{Hom}_{A^e}(L, M)_S = \mathrm{Hom}_{A^e}(P, M)_S = \mathrm{Hom}_{A^e}(P, M_S).$$

Since  $A_S \otimes_A P \otimes_A A_S \rightarrow L_S$  is a projective resolution over  $A_S^e$ , we find

$$\mathrm{Hom}_{A^e}(P, M_S) = \mathrm{Hom}_{A_S^e}(A_S \otimes_A P \otimes_A A_S, M_S) = \mathbf{R}\mathrm{Hom}_{A_S^e}(L_S, M_S).$$

$\square$

**Remark 5.7.** In the setting of Theorem 5.5, assume that  $g$  has isolated singularities and that the origin is a singular point of the vanishing locus of  $g$ . If we denote by  $\mathfrak{m}$  the maximal ideal  $(x_1, \dots, x_n) \subset k[x_1, \dots, x_n]$ , then  $g \in \mathfrak{m}$ . Denote by  $M_{g,\mathfrak{m}}$ ,  $T_{g,\mathfrak{m}}$  and  $K_{g,\mathfrak{m}}$  the localizations of  $M_g$ ,  $T_g$  and  $K_g$ . It follows from the lemma that Theorem 5.5 still holds if one replaces  $R$  by  $R_{\mathfrak{m}} := k[x_1, \dots, x_n]_{\mathfrak{m}}/(g)$  and replaces  $T_g$  and  $K_g$  by  $T_{g,\mathfrak{m}}$  and  $K_{g,\mathfrak{m}}$ .

For a Noetherian  $k$ -algebra  $A$ , the derived category of singularities  $D_{sg}(A)$  is equipped with a canonical dg enhancement, obtained from its construction as a Verdier quotient of two canonically enhanced triangulated categories [42, 18]. Instead of  $\mathrm{HH}_{sg}^*(A, A)$ , one may also consider the Hochschild cohomology of the dg category  $D_{sg}(A)$ , which we will denote by  $\mathrm{HH}^*(D_{sg}(A))$ .

**Theorem 5.8.** (*Keller [47]*) *There is a canonical morphism of graded algebras*

$$\mathrm{HH}_{sg}^*(A, A) \rightarrow \mathrm{HH}^*(D_{sg}(A)).$$

*It is invertible if  $D_{dg}^b(\mathrm{mod} A)$  is smooth.*

Now we establish the main result of the subsection.

**Theorem 5.9.** *Let  $\hat{R} = k[[x_1, \dots, x_n]]/(g)$  be a hypersurface algebra with isolated singularity. Denote by  $\hat{T}_g$  the Tyurina algebra of  $g$ . Then there is an isomorphism of  $k$ -algebras*

$$\mathrm{HH}^0(D_{sg}(\hat{R})) \cong \hat{T}_g.$$

*Moreover, if  $\hat{R}' = k[[x_1, \dots, x_n]]/(g')$  is another hypersurface algebra with isolated singularity such that  $D_{sg}(\hat{R}')$  is quasi-equivalent with  $D_{sg}(\hat{R})$  as dg-categories, then  $\hat{R}'$  is isomorphic to  $\hat{R}$ .*

*Proof.* Because  $g$  has an isolated critical point, we may assume that  $g$  is a polynomial without loss of generality. Denote by  $R$  the algebra  $k[x_1, \dots, x_n]_{\mathfrak{m}}/(g)$ . Notice that  $R$  has an isolated singularity at the origin and that its completion identifies with  $\hat{R}$ . By Theorem 5.7 of [20], the triangulated category  $D_{sg}(\hat{R})$  is the Karoubi envelope of  $D_{sg}(R)$ . Therefore, the two dg categories have equivalent derived categories and there is a natural isomorphism  $\mathrm{HH}^*(D_{sg}(R)) \xrightarrow{\sim} \mathrm{HH}^*(D_{sg}(\hat{R}))$ . Orlov proved in [62] that  $D_{sg}(R)$  is triangle equivalent with the homotopy category of matrix factorizations

$\text{MF}(k[x_1, \dots, x_n]_{\mathfrak{m}}, g)$ . The triangle equivalence is lifted to an equivalence of dg categories by the work of Blanc, Robalo, Toën and Vezzosi [7]. Therefore, the dg category  $D_{sg}(R)$  is 2-periodic and so is its Hochschild cohomology. So there exists a natural isomorphism of  $\hat{R}$ -modules

$$\text{HH}^0(D_{sg}(R)) \xrightarrow{\sim} \text{HH}^{2r}(D_{sg}(R))$$

for all  $r \in \mathbb{Z}$ . By Theorem B of [24], the bounded dg derived category  $D_{dg}^b(\text{mod } R)$  is smooth. Thus, by Theorem 5.8, we have

$$\text{HH}^{2r}(D_{sg}(R)) \xrightarrow{\sim} \text{HH}_{sg}^{2r}(R, R).$$

By Corollary 5.4, for  $r \gg 0$ , we have

$$\text{HH}^{2r}(R, R) \xrightarrow{\sim} \text{HH}_{sg}^{2r}(R, R)$$

and by Theorem 5.5 and Remark 5.7, we have

$$\text{HH}^{2r}(R, R) \xrightarrow{\sim} T_g.$$

Because  $g$  has an isolated critical point, there is an isomorphism

$$\hat{T}_g \cong T_g.$$

Then the first claim follows. The second claim follows from the formal version of the Mather-Yau theorem (see Theorem 1.1 [26]).  $\square$

**5.3. Classification of 3-dimensional smooth flops.** Let  $(\hat{Y}, \hat{f}, R)$  be a 3-dimensional formal flopping contraction with  $\text{Ex}(f) = \bigcup_{i=1}^t C_i$ , and let  $A = \text{End}_R(\bigoplus_{i=1}^t N_i \oplus R)$  be the NCCR associated to it. We have associated to it an exact 3CY algebra: the derived deformation algebra  $\Gamma$  of the semi-simple collection  $\mathcal{O}_{C_1}, \dots, \mathcal{O}_{C_t}$ , and the cluster category  $\mathcal{C}_\Gamma$ . There are two relaxations of the above context.

First we take  $Y$  a CY 3-fold with a semi-simple collection of rational curves  $\mathcal{O}_{C_1}, \dots, \mathcal{O}_{C_t}$ . We may still define the derived deformation algebra  $\Gamma$  and the cluster category  $\mathcal{C}_\Gamma$ . However, in general  $\Gamma$  is only bimodule CY and  $\mathcal{C}_\Gamma$  may not be Hom-finite. If we assume that the  $Y$  is projective then  $\Gamma$  will be exact. However, most  $\hat{Y}$  are not expected to have CY compactifications. The second relaxation is to take a 3-dimensional hypersurface ring  $R$  with isolated singularities. Associate to it is the derived category of singularity  $D_{sg}(R)$ . This is a Hom-finite CY category. One may ask when is it possible to express it as a cluster category of a CY algebra.

In the case of flopping contractions, these two relaxations are related by Theorem 2.8. By [74],  $\hat{Y}$  admits a tilting bundle,  $\Gamma$  is exact and has finite dimensional cohomology. By Corollary 2.12, Theorem 4.6 and 4.7,  $\Gamma$  is weakly equivalent to a Ginzburg algebra  $\mathfrak{D}(Q, w)$  with  $t$  nodes. On the other hand if  $R$  admits a NCCR  $A$  then  $D_{sg}(R)$  is equivalent to  $\mathcal{C}_\Gamma$  for the CY algebra  $\Gamma$ . By Theorem 2.8,  $A$  admits a minimal model  $(\widehat{T}_l V, d)$  with the dual space of  $\Sigma^{-1} \text{Ext}_A^{\geq 1}(\bigoplus_{i=0}^t S_i, \bigoplus_{i=0}^t S_i)$  (see section 4 of [73]). Since  $A$  is Calabi-Yau,  $d$  can be derived from a potential, i.e.  $(\widehat{T}_l V, d)$  is a Ginzburg algebra (see [73, Proposition 1.2]). By the derived equivalence theorem 2.11 of Van den Bergh, the derived deformation algebra of the semi-simple collection  $\mathcal{O}_{C_1}, \dots, \mathcal{O}_{C_t}$  is isomorphic to the quotient  $\widehat{T}_l V / \widehat{T}_l V e_0 \widehat{T}_l V$  in Theorem 2.8. It is natural to expect that the deformation theory of the exceptional curves and the singularity theory of  $R$  should determine each other since both are governed by the CY algebra  $\Gamma$ .

Recall that the CY tilted algebra  $\Lambda := H^0 \Gamma$  is isomorphic to  $\text{End}_{\mathcal{C}_\Gamma}(\pi(\Gamma))$  (c.f. Theorem 3.23). Donovan and Wemyss conjectured that  $\Lambda$  alone can already determine the analytic type of  $R$ .

**Conjecture 5.10.** *Let  $(\hat{Y}, \hat{f}, R)$  and  $(\hat{Y}', \hat{f}', R')$  be two 3-dimensional simple formal flopping contractions with associated CY tilted algebras  $\Lambda$  and  $\Lambda'$ . Then the following are equivalent*

- (1)  $R$  is isomorphic to  $R'$ .

(2)  $\Lambda$  is isomorphic to  $\Lambda'$ .

Donovan and Wemyss have extended this conjecture to the case of not necessarily simple formal flopping contractions by replacing (2) with

(2')  $\Lambda$  is derived equivalent to  $\Lambda'$ ,

cf. Conjecture 1.3 in [4]. In this situation, the implication from (1) to (2) is known to be true by iterating a construction of Dugas [19]. The implication from (2') to (1) is one of the main open problems in the homological minimal model program for 3-folds. In this section, we will prove a slightly weaker version of this implication.

The exactness of  $\Gamma$  poses a strong constraint on  $\Lambda$ , i.e. the relations of  $\Lambda$  can be written as cyclic derivative of a potential  $w$  by Theorem 4.6. If we fix the exact CY structure then  $w$  is uniquely determined up to right equivalences (Proposition 4.8).

**Theorem 5.11.** *Let  $(\widehat{Y}, \widehat{f}, R)$  and  $(\widehat{Y}', \widehat{f}', R')$  be formal flopping contractions. Given exact CY structures  $\eta$  and  $\eta'$  on  $\widehat{Y}$  and  $\widehat{Y}'$  respectively, denote by  $(\Lambda, [w])$  and  $(\Lambda', [w'])$  the associated CY tilted algebras and the canonical classes of potentials. If there exists a derived equivalence from  $\Lambda$  to  $\Lambda'$  given by a bimodule complex  $Z$  such that the induced map  $\mathrm{HH}_0(Z)$  takes  $[w]$  to  $[w']$ , then  $R$  is isomorphic to  $R'$ .*

The proof of this theorem will take up the rest of the section. Here we highlight major components of the proof. First we prove that the cluster category  $\mathcal{C}_\Gamma$  is dg equivalent to the  $D_{sg}(R)$  with its canonical  $\mathbb{Z}$ -graded dg structure (Lemma 5.12). The second step is to establish the result that analytic type of isolated hypersurface singularity (with fixed embedded dimension) is determined by its  $\mathbb{Z}$ -graded dg category of singularities (Theorem 5.9). In the last step, we prove that  $\Gamma$  can be reconstructed from the CY tilted algebra  $\Lambda$  together with the class  $[w] \in \mathrm{HH}_0(\Lambda)$  represented by the potential.

Let  $(R, \mathfrak{m})$  be a complete commutative Noetherian local Gorenstein  $k$ -algebra of Krull dimension  $n$  with isolated singularity and with residue field  $k$ . Suppose that  $R$  admits a NCCR, then  $D_{sg}(R)$  has another dg model via the triangle equivalence  $D_{sg}(R) \simeq \mathcal{C}_\Gamma$  (c.f. Theorem 2.8). We first prove that these two models are dg quasi-equivalent.

**Lemma 5.12.** *In the homotopy category of dg categories, there is an isomorphism between  $\mathcal{C}_\Gamma = \mathrm{per}(\Gamma)/D_{fd}(\Gamma)$  and the category of singularities  $D_{sg}(R) = D^b(R)/K^b(\mathrm{proj}(R))$ , both equipped with their canonical dg enhancements.*

*Proof.* Let  $\mathcal{A}$  and  $\mathcal{B}$  be two pretriangulated dg categories. We call a triangle functor  $F : H^0(\mathcal{A}) \rightarrow H^0(\mathcal{B})$  algebraic if there is a dg  $\mathcal{A}$ - $\mathcal{B}$ -bimodule  $X$  such that we have a square of triangle functors, commutative up to isomorphism

$$\begin{array}{ccc} H^0(\mathcal{A}) & \xrightarrow{F} & H^0(\mathcal{B}) \\ \downarrow & & \downarrow \\ D(\mathcal{A}) & \xrightarrow{\quad ? \otimes_{\mathcal{A}} X \quad} & D(\mathcal{B}), \end{array}$$

where the vertical arrows are induced by the Yoneda functors. We know from [71] that morphisms  $\mathcal{A} \rightarrow \mathcal{B}$  in the homotopy category of dg categories are in bijection with isomorphism classes of right quasi-representable  $\mathcal{A}$ - $\mathcal{B}$ -bimodules in the derived category of bimodules. Thus, it suffices to show that the triangle equivalence  $\mathcal{C}_\Gamma \xrightarrow{\sim} D_{sg}(R)$  is algebraic. We use the notation of subsection 2.4 and put  $N = N_1 \oplus \dots \oplus N_t$ . Let  $\mathcal{F}$  denote the thick triangulated subcategory of  $\mathrm{per}(\mathcal{A})$  generated by the simples  $S_1, \dots, S_t$ . Let us recall from Proposition 3 of [63] that we have a diagram of

triangle functors, commutative up to isomorphism and whose rows and columns are exact sequences of triangulated categories

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \text{per}(A)/\text{per}(R) & \longrightarrow & \underline{\text{CM}}_R \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \text{per}(A) & \longrightarrow & D^b(\underline{\text{CM}}_R) \longrightarrow 0 \\
 & & & \uparrow & & \uparrow & \\
 & & \text{per}(R) & \xlongequal{\quad} & \text{per}(R) & & \\
 & & \uparrow & & \uparrow & & \\
 0 & & & & 0 & &
 \end{array}$$

Here the category  $D^b(\underline{\text{CM}}_R)$  is the bounded derived category of the exact category  $\underline{\text{CM}}_R$ , the functor  $\text{per}(R) \rightarrow D^b(\underline{\text{CM}}_R)$  is induced by the inclusion  $\text{proj}(R) \rightarrow \underline{\text{CM}}_R$ , the functor  $\text{per}(A) \rightarrow D^b(\underline{\text{CM}}_R)$  is induced by  $? \otimes_A^{\mathbb{L}} (R \oplus N)$  and  $\text{per}(R) \rightarrow \text{per}(A)$  is induced by the inclusion

$$\text{add}(R) \rightarrow \text{add}(R \oplus N) = \text{proj}(A),$$

where the categories  $\text{add}(R)$  and  $\text{add}(R \oplus N)$  are full subcategories of  $\underline{\text{CM}}_R$  and the last equality denotes the equivalence given by the functor  $\text{Hom}(R \oplus N, ?)$ . We endow  $\text{per}(A)/\text{per}(R)$  with the dg enhancement given by the dg quotient [42, 18]. It is then clear that the triangle functors of the middle row and of the middle column are algebraic. Let us show that the functor  $D^b(\underline{\text{CM}}_R) \rightarrow \underline{\text{CM}}_R$  is algebraic. The canonical dg enhancement of  $\underline{\text{CM}}_R$  is given by the triangle equivalence from the homotopy category  $H^0(\mathcal{A})$  of the dg category  $\mathcal{A}$  of acyclic complexes over  $\text{proj}(R)$  to  $\underline{\text{CM}}_R$  taking an acyclic complex  $P$  to its zero cycles  $Z^0(P)$ . Let  $\mathcal{B}$  be the dg enhanced derived category  $D_{dg}^b(\underline{\text{CM}}_R)$ . We define a  $\mathcal{B}$ - $\mathcal{A}$ -bimodule  $X$  by putting

$$X(P, M) = \text{Hom}(P, \Sigma M)$$

where  $P$  is an acyclic complex of finitely generated projective  $R$ -modules and  $M$  a bounded complex over  $\underline{\text{CM}}_R$ . For  $M \in \mathcal{B}$ , denote by  $M^\wedge$  the representable dg functor  $\mathcal{B}(?, M)$ . We have  $M^\wedge \otimes_{\mathcal{B}} X = X(?, M)$ . If  $M$  is concentrated in degree 0, we have canonical isomorphisms

$$H^0 X(P, M) = \text{Hom}_{\underline{\text{CM}}_R}(Z^0(P), M) \quad \text{and} \quad H^p X(P, M) = \text{Hom}_{\underline{\text{CM}}_R}(Z^0(P), \Sigma^p M),$$

which shows that  $X(?, M)$  is quasi-representable by a complete resolution of  $M$ . By dévissage, it follows that  $X(?, M)$  is quasi-representable for any bounded complex  $M$  and one checks easily that the (derived=non derived) tensor product with  $X$  induces the canonical triangle functor  $D^b(\underline{\text{CM}}_R) \rightarrow \underline{\text{CM}}_R$ . It follows that, at the level of dg categories,  $\underline{\text{CM}}_R$  identifies with the dg quotient of  $D^b(\underline{\text{CM}}_R)$  by  $\text{per}(R)$ . In other words, the canonical equivalence

$$D^b(\underline{\text{CM}}_R)/\text{per}(R) \xrightarrow{\sim} \underline{\text{CM}}_R$$

is algebraic. Therefore, the induced functor  $\text{per}(A)/\text{per}(R) \rightarrow \underline{\text{CM}}_R$  is algebraic. Thus, the whole diagram is made up of algebraic functors. Now notice that the inclusion  $\underline{\text{CM}}_R \subset \text{mod } R$  induces algebraic equivalences  $D^b(\underline{\text{CM}}_R) \xrightarrow{\sim} D^b(\text{mod } R)$  so that we get algebraic equivalences

$$\underline{\text{CM}}_R \xleftarrow{\sim} D^b(\underline{\text{CM}}_R)/\text{per}(R) \xrightarrow{\sim} D^b(\text{mod } R)/\text{per}(R)$$

Thus, it will suffice to prove that the equivalence between the cluster category of  $\Gamma$  and the stable category  $\underline{\text{CM}}_R$  is algebraic.

Now using the notations of Theorem 2.8 put  $\tilde{\Gamma} = (\widehat{T}_l V, d)$  so that we have a quasi-isomorphism  $\tilde{\Gamma} \xrightarrow{\sim} A$ . It induces an algebraic equivalence  $\text{per}(\tilde{\Gamma}) \xrightarrow{\sim} \text{per}(A)$ . This equivalence induces an algebraic equivalence  $\text{tria}(e_0 \tilde{\Gamma}) \rightarrow \text{per}(R)$ , where  $\text{tria}(e_0 \tilde{\Gamma})$  is the triangulated subcategory generated by the  $\tilde{\Gamma}$ -module  $e_0 \tilde{\Gamma}$ . The quotient map  $\tilde{\Gamma} \rightarrow \Gamma$  induces an algebraic triangle functor  $\text{per}(\tilde{\Gamma}) \rightarrow \text{per}(\Gamma)$  and we know from Lemma 7.2 of [38] that it is a localization with kernel  $\text{tria}(e_0 \tilde{\Gamma})$ . We obtain a diagram of triangle functors, commutative up to isomorphism, whose vertical arrows are equivalences and whose rows are exact.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{tria}(e_0 \tilde{\Gamma}) & \longrightarrow & \text{per}(\tilde{\Gamma}) & \longrightarrow & \text{per}(\Gamma) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{per}(R) & \longrightarrow & \text{per}(A) & \longrightarrow & \text{per}(A)/\text{per}(R) \longrightarrow 0. \end{array}$$

By passage to the dg quotient, the rightmost vertical arrow is an algebraic triangle equivalence. By composing the algebraic inclusion  $\mathcal{F} \rightarrow \text{per}(A)/\text{per}(R)$  with a quasi-inverse of the algebraic equivalence  $\text{per}(\Gamma) \xrightarrow{\sim} \text{per}(A)/\text{per}(R)$ , we obtain an algebraic inclusion  $\mathcal{F} \rightarrow \text{per}(\Gamma)$  whose image identifies with the thick subcategory of  $\text{per}(\Gamma)$  generated by the simple  $H^0(\Gamma)$ -modules. This subcategory equals  $D_{fg}(\Gamma)$ . Indeed, clearly it is contained in  $D_{fd}(\Gamma)$  and conversely, it contains the category  $\text{mod } H^0(\Gamma)$  of finite-dimensional  $H^0(\Gamma)$ -modules since  $H^0(\Gamma)$  is finite-dimensional so that every finite-dimensional  $H^0(\Gamma)$ -module is a finite iterated extension of simples. Since  $\text{mod } H^0(\Gamma)$  is the heart of a bounded  $t$ -structure on  $D_{fd}(\Gamma)$ , the image of  $\mathcal{F}$  equals  $D_{fd}(\Gamma)$ . This yields the first row of the following diagram with exact rows whose vertical arrows are equivalences.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \text{per}(\Gamma) & \longrightarrow & \mathcal{C}_\Gamma \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \text{per}(A)/\text{per}(R) & \longrightarrow & \underline{\text{CM}}_R \longrightarrow 0. \end{array}$$

Again by passage to the dg quotient, the rightmost vertical arrow is an algebraic triangle equivalence.  $\square$

**Proof of Theorem 5.11.** Let  $(Q, w)$  and  $(Q', w')$  be the quivers with potential constructed from the formal flopping contractions and  $\Gamma$  and  $\Gamma'$  the associated Ginzburg dg algebras. Let  $\Lambda = H^0(\Gamma)$  and  $\Lambda' = H^0(\Gamma')$  be the associated contraction algebras. Recall that these algebras are symmetric so that tilting objects coincide with silting objects in their derived categories. We will construct a quiver with potential  $(Q'', w'')$  with associated Ginzburg algebra  $\Gamma''$ , a dg  $\Gamma''$ - $\Gamma'$ -bimodule  $\tilde{Z}$  and an isomorphism  $\psi$  from  $\Lambda'' = H^0(\Gamma'')$  to  $C$  such that the square

$$\begin{array}{ccc} \Gamma'' & \xrightarrow{\tilde{Z}} & \Gamma' \\ \Lambda'' \downarrow & & \downarrow \Lambda' \\ \Lambda'' & \xrightarrow{\psi Z} & \Lambda' \end{array}$$

is commutative and the isomorphism  $HC_0(\tilde{Z})$  takes the class  $[w'']$  to  $[w']$ . Here we write dg bimodules instead of derived tensor products, algebras instead of their derived categories and the top and bottom arrows are equivalences. Notice that by [41], Hochschild homology is functorial with respect to right perfect dg bimodules so that the notation  $HC_0(\tilde{Z})$  does make sense.

To construct the above square, we may assume that  $Z$  is a 2-term silting object since by part (3) of Theorem 7.2 of [5], the standard derived equivalence given by  $Z$  is the composition of equivalences given by 2-term tilting complexes and their inverses. Let  $A$  be the dg algebra obtained by applying Theorem 4.25 to  $B = \Gamma'$ ,  $H^0(B) = \Lambda'$ ,  $C = \Lambda$  and  $Z = Z$ . Its homologies  $H^p(A)$  are finite-dimensional and vanish for  $p > 0$ . Thus, it is quasi-isomorphic to a dg algebra of the form  $(\widehat{T}_l V, d)$ , where  $V$  is a graded bimodule whose components vanish in degrees  $> 0$  and are finite-dimensional in all degrees  $\leq 0$ . So we may assume that  $A$  is in  $\text{PCAlg}(l)$ . By Theorem 4.25, there is moreover a dg  $A$ - $\Gamma'$ -bimodule  $X$  yielding an equivalence  $? \otimes_A X : \mathbf{D}(A) \xrightarrow{\sim} \mathbf{D}(\Gamma')$  and an isomorphism  $\phi : H^0(A) \rightarrow \Lambda$  such that we have an isomorphism

$${}_\phi Z \xrightarrow{\sim} X \overset{\mathbb{L}}{\otimes}_{\Gamma'} \Lambda'$$

in the derived category  $\mathbf{D}(A^{op} \otimes \Lambda')$  and in particular a square, commutative up to isomorphism

$$\begin{array}{ccc} A & \xrightarrow{X} & \Gamma' \\ H^0 A \downarrow & & \downarrow \Lambda' \\ H^0(A) & \xrightarrow[\phi]{\sim} & \Lambda \xrightarrow[Z]{} \Lambda'. \end{array}$$

By Corollary 4.27, there is a quiver with potential  $(Q'', w'')$  and a weak equivalence  $s : \Gamma'' \rightarrow A$  from the associated Ginzburg algebra  $\Gamma''$  to  $A$  such that the isomorphism  $HC_0(sX)$  takes the class  $[w'']$  to  $[w']$ . We define  $\tilde{Z} = {}_s X$  and  $\psi = \phi \circ H^0(s)$  to obtain the diagram

$$\begin{array}{ccccc} \Gamma'' & \xrightarrow{s} & A & \xrightarrow{X} & \Gamma' \\ \downarrow & & H^0 A \downarrow & & \downarrow \Lambda' \\ H^0(\Gamma'') & \xrightarrow[H^0 s]{\sim} & H^0(A) & \xrightarrow[\phi]{\sim} & \Lambda \xrightarrow[Z]{} \Lambda'. \end{array}$$

So by construction, the isomorphism  $HH_0({}_s X) = HH_0(\psi Z)$  takes  $[w'']$  to  $[w']$ . We may assume all potentials to contain no cycles of length  $\leq 2$  and then it follows that  $\psi = \phi \circ H^0(s)$  induces an isomorphism of quivers  $Q'' \xrightarrow{\sim} Q$ . Indeed, it induces an isomorphism in the Jacobian algebras and the vertices  $i$  of the quiver  $Q$  are in bijection with the isomorphism classes of simple modules  $S_i$  of the pseudocompact algebra  $\Lambda = H^0(\Gamma)$  and the number of arrows from  $i$  to  $j$  equals the dimension of the space of extensions  $\text{Ext}_\Lambda^1(S_j, S_i)$ . By Corollary 4.14, there is an isomorphism  $\beta : \Gamma'' \rightarrow \Gamma$ . The dg bimodule  ${}_{s\beta^{-1}} X$  yields an algebraic triangle equivalence  $\text{per}(\Gamma) \xrightarrow{\sim} \text{per}(\Gamma')$ . Such an equivalence induces an equivalence between the subcategories of dg modules with finite-dimensional homologies because their objects  $M$  are characterized as those for which  $\text{Hom}(P, M)$  is finite-dimensional for any object  $P$ . Thus, the algebraic triangle equivalence  $\text{per}(\Gamma) \xrightarrow{\sim} \text{per}(\Gamma')$  induces an algebraic triangle equivalence in the cluster categories. Hence, by Lemma 5.12, it induces an algebraic triangle equivalence  $\mathbf{D}_{sg}(R) \xrightarrow{\sim} \mathbf{D}_{sg}(R')$  and therefore an algebra isomorphism  $\text{HH}^0(\mathbf{D}_{sg}(R)) \cong \text{HH}^0(\mathbf{D}_{sg}(R'))$ . By Theorem 5.9, we get an isomorphism  $R \cong R'$ .  $\square$

## 6. CONTRACTIBILITY OF RATIONAL CURVE

**6.1. dg  $k[u^{-1}]$ -algebras.** In this section, we define dg- $k[u^{-1}]$ -algebras and study their properties. All the definitions and results can be adapted to the pseudo-compact case, with notations appropriately replaced by their pseudo-compact counterparts.

**Definition 6.1.** Let  $S$  be a commutative dg algebra. Denote by  $\mathcal{C}(S)$  the category of complexes of  $S$ -modules with the monoidal structure given by the tensor product over  $S$ . A dg  $S$ -algebra is an algebra in  $\mathcal{C}(S)$ , i.e. a dg  $S$ -module with an  $S$ -bilinear multiplication and a unit.

Let  $k$  be a field,  $S$  a commutative dg  $k$ -algebra and  $\Gamma$  a dg  $S$ -algebra. By restriction, each dg  $\Gamma$ -module becomes a dg  $S$ -module and the morphism complexes between dg  $\Gamma$ -modules are naturally dg  $S$ -modules. Thus, the derived category  $D(\Gamma)$  is naturally enriched over  $D(S)$ .

**Definition 6.2.** Let  $S = k[u^{-1}]$  be the commutative dg algebra with  $\deg(u) = 2, \deg(u^{-1}) = -2$  and zero differential. We call a dg  $k$ -algebra  $A$   $k[u^{-1}]$ -enhanced if  $A$  is isomorphic to a dg  $k[u^{-1}]$ -algebra in the homotopy category of dg  $k$ -algebras.

Let us put  $S = k[u^{-1}]$  and  $K = k[u, u^{-1}]$ . For a dg  $S$ -module  $M$ , we have a canonical isomorphism

$$H^*(M \otimes_S K) = H^*(M) \otimes_S K.$$

We call  $M$  a *torsion module* if  $M \otimes_S K$  is acyclic. This happens if and only if  $H^*(M)$  is a torsion module, i.e. for each  $m$  in  $H^*(M)$ , there exists a  $p \gg 0$  such that  $mu^{-p} = 0$ .

Let  $A$  be a dg  $S$ -algebra. The functor taking a dg  $A$ -module  $M$  to the dg  $A \otimes_S K$ -module  $M \otimes_S K$  preserves quasi-isomorphisms. Thus, it induces a functor  $? \otimes_S K : D(A) \rightarrow D(A \otimes_S K)$ . The kernel of this functor consists of the dg  $A$ -modules which are torsion as dg  $S$ -modules. We write  $D(A)_{u^{-1}\text{-tor}}$  for the kernel and  $\text{per}(A)_{u^{-1}\text{-tor}}$  for its intersection with the perfect derived category  $\text{per}(A)$ .

**Lemma 6.3.** *We have exact sequences of triangulated categories*

$$0 \longrightarrow D(A)_{u^{-1}\text{-tor}} \longrightarrow D(A) \longrightarrow D(A \otimes_S K) \longrightarrow 0$$

and

$$0 \longrightarrow \text{per}(A)_{u^{-1}\text{-tor}} \longrightarrow \text{per}(A) \longrightarrow \text{per}(A \otimes_S K) \longrightarrow 0.$$

*Proof.* The restriction along  $A \rightarrow A \otimes_S K$  induces a fully faithful right adjoint to  $? \otimes_S K : D(A) \rightarrow D(A \otimes_S K)$ . Thus, the latter functor is a localization functor. By definition, its kernel is  $D(A)_{u^{-1}\text{-tor}}$  so that we obtain the first sequence. To deduce the second one, it suffices to show that the kernel of  $D(A) \rightarrow D(A \otimes_S K)$  is compactly generated. Indeed, let  $P$  be the cone over the morphism

$$A \rightarrow \Sigma^{-2} A$$

given by the multiplication by  $u^{-1}$ . Clearly  $P$  is compact and we claim that it generates the kernel. For this, it suffices to show that the right orthogonal of  $P$  in the kernel vanishes. Indeed, let  $M$  be in the kernel. If  $\mathbf{R}\text{Hom}_A(P, M)$  vanishes, then the morphism  $\Sigma^2 M \rightarrow M$  given by the multiplication by  $u^{-1}$  is a quasi-isomorphism. Thus,  $u^{-1}$  acts in  $H^*(M)$  by an isomorphism. But on the other hand,  $H^*(M)$  is torsion. So  $H^*(M)$  vanishes and  $M$  is acyclic as was to be shown.  $\square$

A dg  $S$ -algebra  $A$  concentrated in non positive degrees is *non degenerate* if the morphism  $A \rightarrow \Sigma^{-2} A$  given by the multiplication by  $u^{-1}$  induces isomorphisms  $H^n(A) \xrightarrow{\sim} H^{n-2}(A)$  for all  $n \leq 0$ .

**Lemma 6.4.** *Let  $A$  be a dg  $k$ -algebra concentrated in nonpositive degrees. Assume that  $A$  is homologically smooth, that  $H^n(A)$  is finite-dimensional for each  $n \in \mathbb{Z}$  and that  $A$  admits a non degenerate  $k[u^{-1}]$ -enhancement. Then the subcategory  $D_{fd}(A) \subset \text{per}(A)$  coincides with  $\text{per}(A)_{u^{-1}\text{-tor}}$ .*

*Proof.* Since  $A$  is homologically smooth,  $D_{fd}(A)$  is contained in  $\text{per}(A)$  and clearly it consists of torsion modules. Conversely, we know from the proof of Lemma 6.3 that  $\text{per}(A)_{u^{-1}\text{-tor}}$  is the thick subcategory of  $D(A)$  generated by the cone  $P$  over the morphism  $A \rightarrow \Sigma^{-2} A$  given by the multiplication by  $u^{-1}$ . Since  $A$  is non degenerate, the object  $P$  lies in  $D_{fd}(A)$ .  $\square$

**Proposition 6.5.** *Let  $Q$  be a finite quiver with potential  $w$  such that the associated Ginzburg algebra  $\Gamma = \mathfrak{D}(Q, w)$  has finite-dimensional Jacobi algebra. Assume that  $\Gamma$  is equipped with a nondegenerate  $k[u^{-1}]$ -enhancement. Then  $\mathcal{C}_\Gamma$  is a  $\mathbb{Z}/2$ -graded 0CY triangulated category, equivalent with the category of perfect modules over  $\Gamma \otimes_{k[u^{-1}]} k[u, u^{-1}]$  as  $k[u, u^{-1}]$ -enhanced triangulated categories. In particular, the Jacobi algebra  $H^0(\Gamma)$  is a symmetric Frobenius algebra.*

*Proof.* We may assume that  $\Gamma$  itself is a differential graded  $k[u^{-1}]$ -algebra. The multiplication with  $u^{-1}$  yields a functorial morphism  $\text{Id} \rightarrow \Sigma^{-2}$  of triangle functors  $D(\Gamma) \rightarrow D(\Gamma)$  and  $\text{per}(\Gamma) \rightarrow \text{per}(\Gamma)$ . This induces a functorial morphism of triangle functors  $\text{Id} \rightarrow \Sigma^{-2}$  in the cluster category  $\mathcal{C}_\Gamma$ . To check that it is invertible, it is enough to check that its action on the cluster-tilting object  $\Gamma \in \mathcal{C}_\Gamma$  is invertible (since  $\mathcal{C}_\Gamma$  equals its thick subcategory generated by  $\Gamma$ ). Now by our assumption, the morphism  $u^{-1} : \Sigma^2 \Gamma \rightarrow \Gamma$  induces isomorphisms in  $H^n$  for  $n \leq -2$  and the 0-map for  $n \geq -1$ . Thus, it induces a quasi-isomorphism  $\Sigma^2 \Gamma \rightarrow \tau_{\leq -2} \Gamma$ . We claim that the canonical morphism  $\tau_{\leq -2} \Gamma \rightarrow \Gamma$  becomes invertible in the cluster category. Indeed, the homology of its cone is of finite total dimension since  $H^p \Gamma$  is of finite dimension for all integers  $p$  by Lemma 2.5 of [3]. It follows that  $u$  induces an isomorphism  $\Sigma^2 \Gamma \xrightarrow{\sim} \Gamma$  in  $\mathcal{C}_\Gamma$ . The rest follows because we have isomorphisms of  $H^0(\Gamma)$ -bimodules

$$H^0(\Gamma) \xrightarrow{\sim} \mathcal{C}_\Gamma(\Gamma, \Gamma) \xrightarrow{\sim} \mathcal{C}_\Gamma(\Gamma, \Sigma^2 \Gamma) \xrightarrow{\sim} D\mathcal{C}_\Gamma(\Gamma, \Gamma).$$

□

**6.2.  $k[u^{-1}]$ -structures on Ginzburg algebras associated to contractible curves.** Recall that for a Jacobi-finite Ginzburg algebra  $\Gamma := \mathfrak{D}(Q, w)$  associate to quiver  $Q$  with only one vertex (and multiple loops),  $H^0(\Gamma)$  is self-injective. On the other hand, if  $\Gamma$  is non-degenerately  $k[u^{-1}]$ -enhanced then  $H^0(\Gamma)$  is symmetric.

The simplest case is when  $\Gamma = k[t]$  with zero differential. It is  $k[u^{-1}]$ -enhanced by setting  $u^{-1} = t$ . This is the derived deformation algebra for  $\mathcal{O}_C$  of a  $(-1, -1)$ -curve, which is always contractible.

**Proposition 6.6.** *Let  $F = k\langle\langle x \rangle\rangle$  be the complete free algebra of one generator and  $w \in F$  be an element with no constant term. Then  $\Gamma := \mathfrak{D}(F, w)$  is  $k[u^{-1}]$ -enhanced.*

*Proof.* A general element  $w \in F$  is of the form:

$$w = x^{n+1} + \text{higher order terms}.$$

When  $n = 1$  we are in the case of  $(-1, -1)$  curve. We assume that  $n \geq 2$ . The Jacobi algebra of  $\mathfrak{D}(F, w)$  is isomorphic to  $k[[x]]/(x^n)$ . It is always finite dimensional. Because  $w = x^{n+1} \cdot u$  for some unit  $u \in k[[x]]$ ,  $[w] = 0$  in  $k[[x]]/(x^n)$ . By Theorem 4.16,  $w$  is right equivalent to  $x^{n+1}$ . Without loss of generality, we may assume  $w = \frac{x^{n+1}}{n+1}$  to begin with. Then the Ginzburg algebra  $\mathfrak{D}(F, w)$  is isomorphic to  $k\langle\langle x, \theta, t \rangle\rangle$  with  $dt = [x, \theta]$  and  $d\theta = x^n$ . It is easy to check that the two-sided differential ideal  $(t, [x, \theta])$  is acyclic. As a consequence, the quotient morphism

$$\Gamma = (k\langle\langle x, \theta, t \rangle\rangle, d) \rightarrow \Gamma' := (k\langle\langle x, \theta, t \rangle\rangle/(t, [x, \theta]), d)$$

is a quasi-isomorphism of dg algebras. Note that  $\Gamma'$  is isomorphic to the complex

$$\dots k[[x]]\theta^3 \xrightarrow{d} k[[x]]\theta^2 \xrightarrow{d} k[[x]]\theta \xrightarrow{d} k[[x]] \longrightarrow 0$$

where

$$d(\theta^{2k}) = 0, \quad d(\theta^{2k+1}) = x^n \theta^{2k}.$$

Define the action of  $u^{-1}$  on  $\Gamma'$  by multiplication by  $\theta^2$ . It is easy to check it makes  $\Gamma'$  a dg  $k[u^{-1}]$ -algebra. □

From the above proposition, we see that the Ginzburg algebras associated to the “one-loop quiver” are essentially classified by the dimension of their Jacobi algebras. Moreover, they all admit  $k[u^{-1}]$ -enhancements. If  $\dim_k(H^0 \Gamma) = n$  for  $n > 1$ , then  $\Gamma$  is equivalent to the derived deformation algebra of a floppable  $(0, -2)$  curve of width  $n$  (see [60] for the geometric definition of width). The following corollary can be viewed as a noncommutative counter part of the classification theorem of Reid [60].

**Corollary 6.7.** *Let  $C$  be a rational curve in a quasi-projective smooth CY 3-fold  $Y$  of normal bundle  $\mathcal{O}_C \oplus \mathcal{O}_C(-2)$ . Denote its derived deformation algebra by  $\Gamma$ . Then*

- (1)  $C$  is movable if and only if  $\Gamma$  has infinite dimensional Jacobi algebra;
- (2) If  $C$  is rigid then it is contractible. The dimension of  $H^0\Gamma$  is equal to  $n$  for  $n > 1$  if and only if the underlying singularity is isomorphic to the germ of hypersurface  $x^2 + y^2 + u^2 + v^{2n} = 0$  at the origin.

It is proved by Laufer (c.f. [64]) that a contractible rational curve in a CY 3-fold must have normal bundle of types  $(-1, -1)$ ,  $(0, -2)$  or  $(1, -3)$ . Donovan and Wemyss give an example of a rigid rational curve of type  $(1, -3)$  that is not nc rigid (see Example 6.4 in [17]). In their example, there exists a birational morphism that contracts a divisor containing the  $(1, -3)$  curve. Kawamata asked whether it is true that  $C$  is contractible if it is nc rigid (see Question 6.6 of [37]). We formulate a conjecture in terms of the derived deformation algebra.

**Conjecture 6.8.** *Let  $C \subset Y$  be a nc rigid rational curve in a smooth quasi-projective CY 3-fold. Denote its associated derived deformation algebra by  $\Gamma_C^Y$ . Then  $C$  is contractible if and only if  $\Gamma_C^Y$  is  $k[u^{-1}]$ -enhanced.*

Note that one direction of the conjecture follows from our Theorem 4.17.

**Proposition 6.9.** *Let  $C \subset Y$  be a contractible rational curve in a smooth quasi-projective CY 3-fold. Then  $\Gamma_C^Y$  is  $k[u^{-1}]$ -enhanced.*

*Proof.* Denote  $R$  for the ring of formal functions on the singularity underlying the contraction. For simplicity, we denote the derived deformation algebra  $\Gamma_C^Y$  by  $\Gamma$ . By Proposition 5.12,  $\mathcal{C}_\Gamma$  is quasi-equivalent to  $D_{sg}(R)$  as dg categories. Under the equivalence, the projection image of  $\Gamma$  is identified with the Cohen-Macaulay module  $N \in D_{sg}(R) \cong \underline{\text{CM}}_R$ . By Theorem 4.17,  $\Gamma$  is isomorphic to  $\tau_{\leq 0}\Lambda_{dg}$  where  $\Lambda_{dg}$  is the dg endomorphism algebra of  $N$  in  $\underline{\text{CM}}_R$ . Because  $R$  is a hypersurface ring, the dg category  $\underline{\text{CM}}_R$  carries a canonical  $\mathbb{Z}/2$ -graded structure (equivalently  $k[u, u^{-1}]$ -structure) by Eisenbud's theorem [21]. Therefore,  $\Gamma$  is  $k[u^{-1}]$ -enhanced.  $\square$

We have already seen that in  $(-1, -1)$  and  $(0, -2)$  cases, the  $k[u^{-1}]$ -structure on  $\Gamma$  can be computed explicitly. However, we don't have any explicit construction for the  $(1, -3)$  case even though we know it must exist. We do have an explicit formula for the symmetric Frobenius structure on the CY tilted algebra  $H^0\Gamma$  in term of the residue map of matrix factorizations (see [29]).

## REFERENCES

- [1] T. Adachi, O. Iyama, I. Reiten,  *$\tau$ -tilting theory*, Compos. Math. 150 (2014), 415–452.
- [2] T. Aihara and O. Iyama, *Silting mutation in triangulated categories*, J. Lond. Math. Soc. (2) 85 (2012), no. 3, 633–668.
- [3] C. Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*. (English, French summary) Ann. Inst. Fourier (Grenoble) 59, no. 6 (2009): 2525–2590.
- [4] J. August, *On the Finiteness of the Derived Equivalence Classes of some Stable Endomorphism Rings*, arXiv:1801.05687 [math.RT].
- [5] J. August, *The Tilting Theory of Contraction Algebras*, Adv. Math. 374 (2020), 107372, 56 pp.
- [6] A. A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers. (French) [Perverse sheaves] Analysis and topology on singular spaces, I*, (Luminy, 1981), 5–171, Astérisque, 100, Soc. Math. France, Paris, 1982.
- [7] A. Blanc, M. Robalo, B. Toën, G. Vezzosi, *Motivic realizations of singularity categories and vanishing cycles*, arXiv:1607.03012 [math.AG].
- [8] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and re- constructions*, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 6, 1183–1205, 1337.
- [9] A. I. Bondal and M. Van den Bergh, *Generators and representability of functors in commutative and non-commutative geometry*, Mosc. Math. J. 3 (2003), 1–36.
- [10] M. V. Bondarko, *Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes (for motives and in general)*, J. K-Theory 6 (2010), no. 3, 387–504.
- [11] M. Booth, *Noncommutative deformation theory, the derived quotient, and DG singularity categories*, arXiv:1810.10060 [math.AG].

- [12] C. Braun, J. Chuang and A. Lazarev, *Derived localisation of algebras and modules*, Adv. Math. 328 (2018), 555–622.
- [13] T. Brüstle, D. Yang, *Ordered exchange graphs*, in: Advances in representation theory of algebras, 135–193, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2013.
- [14] R.-O. Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings*, preprint (1986): <http://hdl.handle.net/1807/16682>
- [15] H. Clemens, *The infinitesimal Abel-Jacobi mapping and moving the  $\mathcal{O}(2) + \mathcal{O}(-4)$  curve* Duke Mathematical Journal 59, no. 1 (1989): 233–240.
- [16] W. Donovan, M. Wemyss, *Noncommutative deformations and flops*, Duke Math. J. 165, no. 8 (2016): 1397–1474.
- [17] W. Donovan, M. Wemyss, *Contractions and deformations*, arXiv:1511.00406v2.
- [18] V. Drinfeld, *DG quotients of DG categories*, J. Algebra 272 (2004), no. 2, 643–691.
- [19] A. Dugas, *A construction of derived equivalent pairs of symmetric algebras*, Proc. Amer. Math. Soc. 143 (2015), no. 6, 2281–2300.
- [20] T. Dyckerhoff, *Compact generators in categories of matrix factorizations*, Duke Math. J. vol 159, no. 2 (2011), 233–274.
- [21] D. Eisenbud *Homological algebra on a complete intersection, with an application to group representations* Transactions of the American Mathematical Society 260, no. 1 (1980): 35–64.
- [22] A. I. Efimov, V. A. Lunts, D. O. Orlov, *Deformation theory of objects in homotopy and derived categories. I. General theory*. Adv. Math. 222 (2009), no. 2, 359–401.
- [23] A. I. Efimov, V. A. Lunts, D. O. Orlov, *Deformation theory of objects in homotopy and derived categories. II. Pro-representability of the deformation functor*. Adv. Math. 224 (2010), no. 1, 45–102.
- [24] Alexey Elagin, Valery A. Lunts, and Olaf M. Schnürer, *Smoothness of derived categories of algebras*, arXiv:1810.07626 [math.AG].
- [25] J. A. Guccione, J. J. Guccione, M. J. Redondo, O. E. Villamayor, *Hochschild and cyclic homology of hypersurfaces* Advances in Mathematics 95, no. 1 (1992): 18–60.
- [26] G.-M. Greuel, T. H. Pham, *Mather–Yau theorem in positive characteristic*, J. Algebraic Geom. 26 (2017): 347–355.
- [27] A. Ikeda, Y. Qiu,  *$\mathbb{X}$ -stability conditions on Calabi–Yau– $\mathbb{X}$  categories and twisted periods*, preprint, arXiv:1807.00469.
- [28] R. Hartshorne, *On the De Rham cohomology of algebraic varieties*. Publications Mathématiques de l’Institut des Hautes études Scientifiques 45, no. 1 (1975): 6–99.
- [29] Z. Hua, Y. Toda, *Contraction algebra and invariants of singularities*, Int. Math. Res. Notices (2017): <https://doi-org.eproxy2.lib.hku.hk/10.1093/imrn/rnw333>
- [30] Z. Hua, *Contraction algebra and singularity of three-dimensional flopping contraction*, Mathematische Zeitschrift (2018): 1–13.
- [31] Z. Hua, G.-S. Zhou, *Noncommutative Mather–Yau theorem and its applications to Calabi–Yau algebras and homological minimal model program*. arXiv preprint arXiv:1803.06128 (2018).
- [32] Z. Hua, G.-S. Zhou, *Quasi-homogeneity of potentials* J. Noncommut. Geom. 15 (2021), no. 2, pp. 399–422.
- [33] O. Iyama and Y. Yoshino, *Mutation in triangulated categories and rigid Cohen–Macaulay modules*, Invent. Math. 172 (2008), no. 1, 117–168.
- [34] O. Iyama and S. Oppermann, *Stable categories of higher preprojective algebras*, Adv. Math. 244 (2013), 23–68.
- [35] S. Katz and D. R. Morrison, *Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups*, J. Algebraic Geom. 1 (1992): 449–530.
- [36] Kawamata, Yujiro. *On multi-pointed non-commutative deformations and Calabi–Yau threefolds* Compositio Mathematica 154, no. 9 (2018): 1815–1842.
- [37] Y. Kawamata, *noncommutative deformations of simple objects in a category of perverse coherent sheaves* arXiv preprint arXiv:1806.04858 (2018).
- [38] M. Kalck, D. Yang, *Relative singularity categories II: DG models*, arXiv preprint arXiv:1803.0819 (2018).
- [39] B. Keller, *Deriving DG categories*, Ann. Scient. Ec. Norm. Sup. 27 (1994), 63–102.
- [40] B. Keller, *Koszul duality and coderived categories (after K. Lefèvre)*, preprint, 2003, <https://webusers.imj-prg.fr/~bernhard.keller/publ/kdc.pdf>
- [41] B. Keller, *Invariance and localization for cyclic homology of DG algebras*, J. Pure Appl. Algebra 123 (1998), no. 1–3, 223–273.
- [42] B. Keller, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra 136 (1999), no. 1, 1–56.
- [43] B. Keller, *Bimodule complexes via strong homotopy actions*. Algebr. Represent. Theory 3 (2000), no. 4, 357–376.

- [44] B. Keller, *On differential graded categories*, International Congress of Mathematicians. Vol. II, 151–190, Eur. Math. Soc., Zürich, 2006.
- [45] B. Keller, *Calabi–Yau triangulated categories*, Trends in representation theory of algebras and related topics, 467–489, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008.
- [46] B. Keller, *Deformed Calabi-Yau completions*, J. Reine Angew. Math. 654 (2011): 125–180. With an appendix by Michel Van den Bergh.
- [47] B. Keller, *Singular Hochschild cohomology via the singularity category*, arXiv:1809.05121 [math.RT].
- [48] B. Keller, D. Vossieck, *Sous les catégories dérivées*, C. R. Acad. Sci. Paris 305 (1987), 225–228.
- [49] B. Keller and D. Vossieck, *Aisles in derived categories*, Deuxième Contact Franco-Belge en Algèbre (Faulx-les-Tombes, 1987), Bull. Soc. Math. Belg. Sér. A 40 (1988), no. 2, 239–253.
- [50] B. Keller, P. Nicolás, *Weight structures and simple dg modules for positive dg algebras*, IMRN 2013, no. 5, 1028–1078.
- [51] B. Keller, D. Yang, *Derived equivalences from mutations of quivers with potentials*, Adv. Math. 226 (2011): 2118–2168.
- [52] J. Kollar, S. Mori, *Birational geometry of algebraic varieties*, Vol. 134. Cambridge university press, 2008.
- [53] M. Kontsevich, Y. Soibelman, *Notes on  $A_\infty$ -algebras,  $A_\infty$ -categories and noncommutative geometry*. In Homological mirror symmetry, pp. 1–67. Springer, Berlin, Heidelberg, 2008.
- [54] H. Krause, *Completing perfect complexes*, with appendices by Tobias Barthel, Bernhard Keller and Henning Krause, Math. Z. 296 (2020), no. 3–4, 1387–1427.
- [55] K. Lefèvre-Hasegawa, *Sur les  $A_\infty$ -catégories*, Ph. D. thesis, Université Paris 7, 2003, arXiv:math/0310337.
- [56] V. Lunts, D. Orlov, *Uniqueness of enhancement for triangulated categories*, Journal of the American Mathematical Society 23, no. 3 (2010): 853–908.
- [57] J.-L. Loday, *Cyclic homology* Vol. 301. Springer Science and Business Media, 2013.
- [58] J. N. Mather, S. S.-T. Yau, *Classification of isolated hypersurfaces singularities by their moduli algebras*, Invent. Math. 69 (1982): 243–251.
- [59] A. Neeman, *Derived categories and Grothendieck duality*, Triangulated categories, 290–350, London Math. Soc. Lecture Note Ser., 375, Cambridge Univ. Press, Cambridge, 2010.
- [60] M. Reid, *Minimal models of canonical 3-folds*, Algebraic Varieties and Analytic Varieties, Tokyo, 1981 (1983).
- [61] I. Reiten, *2-Calabi-Yau tilted algebras*, São Paulo J. Math. Sci., 4 (2010), no. 3, 529–545.
- [62] D. Orlov, *Triangulated categories of singularities and D-branes in Landau–Ginzburg models*, Tr. Mat. Inst. Steklova, 246 (2004), 240–262.
- [63] Y. Palu, *Grothendieck group and generalized mutation rule for 2-Calabi-Yau triangulated categories*, J. Pure Appl. Algebra 213 (2009), 1438–1449.
- [64] H. C. Pinkham, *Factorization of birational maps in dimension 3*, in Proceedings of Symposia in Pure mathematics, vol. 40, pp. 343–371. 201 Charles st, providence, RI 02940-2213: Amer Mathematical Soc, 1983.
- [65] D. Pauksztello, *Compact cogenerated objects in triangulated categories and co-t-structures*, Cent. Eur. J. Math. 6 (2008), no. 1, 25–42.
- [66] P.-G. Plamondon, *Cluster characters for cluster categories with infinite-dimensional morphism spaces*, Adv. Math. 227 (2011), no. 1, 1–39.
- [67] P.-G. Plamondon, *Generic bases for cluster algebras from the cluster category*, Int. Math. Res. Not. IMRN 2013, no. 10, 2368–2420.
- [68] A. Polishchuk,  *$A_\infty$ -structures associated with pairs of 1-spherical objects and noncommutative orders over curves*, Transactions of the American Mathematical Society 373, no. 9 (2020): 6029–6093.
- [69] Daniel Quillen, *Cyclic cohomology and algebra extensions*, K-Theory 3 (1989), no. 3, 205–246.
- [70] E. Segal, *The  $A_\infty$ -deformation theory of a point and the derived categories of local Calabi-Yaus*, J. Algebra 320 (2008), no. 8, 3232–3268.
- [71] B. Toën, *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math. 167 (2007), 615–667.
- [72] M. Tousi, S. Yassemi, *Tensor products of some special rings*, Journal of Algebra, 268(2), 2003, 672–676.
- [73] de Thanhoffer de Völcsey, Louis, and Michel van den Bergh, *Explicit models for some stable categories of maximal Cohen–Macaulay modules*, Mathematical Research Letters 23, no. 5 (2016): 1507–1526.
- [74] M. Van den Bergh, *Three-dimensional flops and noncommutative rings*, Duke Math. J. 122, no. 3 (2004): 423–455.
- [75] M. Van den Bergh, *The signs of Serre duality*, Appendix A to R. Bocklandt, *Graded Calabi-Yau algebras of dimension 3*, Journal of Pure and Applied Algebra 212 (2008), 14–32.
- [76] M. Van den Bergh, *Calabi-Yau algebras and superpotentials*, Selecta Mathematica 21 (2015): 555–603.

- [77] Z. Wang, *Gerstenhaber algebra and Deligne's conjecture on Tate-Hochschild cohomology*, arXiv:1801.07990 [math.RT].
- [78] M. Wemyss, *Lectures on noncommutative resolutions*, MSRI Summer School, June 2012, available at the author's home page.
- [79] M. Wemyss, *Flops and clusters in the homological minimal model programme*, Invent. Math. 211 (2018), no. 2, 435–521.

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