CLUSTER CATEGORIES AND RATIONAL CURVES

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Abstract. We study rational curves on smooth complex Calabi–Yau threefolds via noncommutative algebra. By the general theory of derived noncommutative deformations due to Efimov, Lunts and Orlov, the structure sheaf of a rational curve in a smooth CY 3-fold Y is pro-represented by a nonpositively graded dg algebra Γ. The curve is called *nc rigid* if $H^0\Gamma$ is finite dimensional. When C is contractible, $H^0\Gamma$ is isomorphic to the contraction algebra defined by Donovan and Wemyss. More generally, one can show that there exists a Γ pro-representing the (derived) multi-pointed deformation (defined by Kawamata) of a collection of rational curves $C_1, \ldots, C_t$ so that $\dim(\mathrm{Hom}_Y(\mathcal{O}_{C_i}, \mathcal{O}_{C_j})) = \delta_{ij}$. The collection is called *nc rigid* if $H^0\Gamma$ is finite dimensional. We prove that Γ is a homologically smooth bimodule 3CY algebra. As a consequence, we define a (2CY) cluster category $\mathcal{C}_\Gamma$ for such a collection of rational curves in Y. It has finite-dimensional morphism spaces iff the collection is nc rigid. When $\bigcup_{i=1}^t C_i$ is (formally) contractible by a morphism $\widehat{Y} \to \widehat{X}$, then $\mathcal{C}_\Gamma$ is equivalent to the singularity category of $\widehat{X}$ and thus categorifies the contraction algebra of Donovan and Wemyss. The Calabi-Yau structure on Y determines a canonical class $[w]$ (defined up to right equivalence) in the zeroth Hochschild homology of $H^0\Gamma$. Using our previous work on the noncommutative Mather–Yau theorem and singular Hochschild cohomology, we prove that the singularities underlying a 3-dimensional smooth flopping contraction are classified by the derived equivalence class of the pair $(H^0\Gamma, [w])$. We also give a new necessary condition for contractibility of rational curves in terms of Γ.

1. Introduction

The study of rational curves in algebraic varieties lies at the core of birational geometry. A smooth rational curve C in a quasi-projective variety Y is called *rigid* if the component of the Hilbert scheme of curves containing C is a finite scheme. Note that this is weaker than the notion of *infinitesimally rigid*, which says that $\mathrm{Ext}^1_Y(\mathcal{O}_C, \mathcal{O}_C) = 0$. If a curve is not rigid then we call it *movable*. When Y is a smooth projective surface, a smooth rational curve $C \subset Y$ is rigid if and only if its normal bundle $N_{C/Y}$ is negative. And if C is rigid then it is *contractible*, i.e. for the formal completion $\widehat{Y}$ of Y along C there exists a birational morphism $f : \widehat{Y} \to \widehat{X}$ to a normal surface $\widehat{X}$ that contracts C. The definition of contractibility in general can be found in Definition 2.9.

In this article, we will focus on the case when Y is a smooth complex Calabi–Yau threefold, i.e. ω_Y is trivial. The situation is much more complicated than the surface case. We call a rational curve $C \subset Y$ of type (a, b) if it has normal bundle $\mathcal{O}(a) \oplus \mathcal{O}(b)$. By the adjunction formula, we have $a + b = -2$. A $(-1, -1)$-curve is contractible. The underlying singular variety $\widehat{X}$ is equivalent to the singular hypersurface $x^2 + y^2 + u^2 + v^2 = 0$. There exists a different resolution $\widehat{Y}^+ \to \widehat{X}$ and the birational map $\widehat{Y} \dashrightarrow \widehat{Y}^+$ is called the *Atiyah flop*. In [55], Reid proves that a $(0, -2)$-curve is either contractible or movable. The contractible case corresponds to the *Pagoda flops*. Laufer proves that a contractible curve is of the types $(-1, -1)$, $(0, -2)$ or $(1, -3)$ (c.f. [59]). Katz and Morrison show that any simple flopping contraction (see definition in Section 2.5) can be constructed as base change of a universal contraction [33]. In general, it is not true that all rigid curves are contractible. A counter example was constructed by Clemens [14].

Key words and phrases. Contractible curve, cluster category.

We study the contractibility of rational curves in Calabi-Yau 3-folds via noncommutative methods. In general, given a rational curve $C \subset Y$ the problem is two-fold:

(1) Find infinitesimal criteria for the contractibility of C.
(2) If C is contractible, determine the underlying singularity of the contraction.

Our research is motivated by a remarkable paper of Donovan and Wemyss. In [15], Donovan and Wemyss considered the algebra Λ that represents the noncommutative deformation functor of $\mathcal{O}_C$ for a contractible rational curve $C \subset Y$. They prove that Λ is finite dimensional and call it the *contraction algebra*. Indeed, the contraction algebra can be defined in a more general context where Y may be neither CY nor smooth, and the birational morphism may contract a divisor containing C. However we will focus on the special case when Y is a smooth CY 3-fold and the contraction is not divisorial. Donovan and Wemyss conjectured that the 3-dimensional simple flops are classified by the isomorphism types of the contraction algebras (cf. Conjecture 5.10).

In order to deal with the case of general flops where the exceptional fiber can have multiple irreducible components, Kawamata proposes to study the multi-pointed noncommutative deformation of a *semi-simple collection* (see definition in Section 2) [1] of sheaves $\mathcal{E}_1, \ldots, \mathcal{E}_t$. A case of special interest is when the collection is $\mathcal{O}_{C_1}, \ldots, \mathcal{O}_{C_t}$ where C_i are irreducible components of the reduced exceptional fiber of a contraction (see Example 6.5 of [34]). We consider the derived noncommutative deformation theory of $\mathcal{E} := \bigoplus_{i=1}^t \mathcal{E}_i$ for a semi-simple collection of sheaves $\{\mathcal{E}_i\}_{i=1}^t$ on a smooth CY 3-fold Y. By a result of Efimov, Lunts and Orlov (cf. Theorem 2.2), such deformation functor is pro-represented by a nonpositively graded dg algebra Γ. We call Γ the *derived deformation algebra* of the semi-simple collection $\{\mathcal{E}_i\}_{i=1}^t$. We call a semi-simple collection $\{\mathcal{E}_i\}_{i=1}^t$ *nc rigid* ("nc" stands for noncommutative) if $H^0\Gamma$ is finite dimensional. Given a collection of smooth rational curve $C_1, \ldots, C_t$ such that $\{\mathcal{O}_{C_i}\}_{i=1}^t$ form a semi-simple collection, we call $\{C_i\}_{i=1}^t$ a *nc-rigid* collection of rational curves if $H^0\Gamma$ is finite dimesional. If $t = 1$ and $C = C_1$ is nc rigid rational curve then the abelianization of $H^0\Gamma$ represents the commutative deformation functor of $\mathcal{O}_C$. Therefore, a nc rigid curve is in particular rigid. Our first result is:

Theorem A. (Corollary 3.12) Let $C_1, \ldots, C_t$ be a collection of rational curves in a smooth quasi-projective Calabi-Yau 3-fold Y such that $\{\mathcal{O}_{C_i}\}_{i=1}^t$ form a semi-simple collection. The derived deformation algebra Γ of $\bigoplus_{i=1}^t \mathcal{O}_{C_i}$ is a non positive pseudo-compact dg algebra that is

(1) homologically smooth;
(2) bimodule 3CY.

Moreover, Γ is exact 3CY in either one of the following cases

(a) Y is projective;
(b) there is a (formal) contraction $\widehat{f} : \widehat{Y} \to \widehat{X}$ such that $\mathrm{Ex}(\widehat{f}) = \bigcup_{i=1}^t C_i$, where $\mathrm{Ex}(\widehat{f})$ stands for the reduced exceptional fiber of $\widehat{f}$.

This theorem establishes a link between birational geometry and the theory of cluster categories. We consider the triangle quotient $\mathcal{C}_\Gamma := \mathrm{per}(\Gamma)/\mathrm{D}_{fd}(\Gamma)$ (see Section 3.3). It is Hom-finite if and only if $\{C_i\}_{i=1}^t$ is nc rigid. By a result of Amiot [3], it is then a 2CY category. When C is contractible by a morphism $\widehat{Y} \to \widehat{X}$, then $H^0\Gamma$ is isomorphic to the contraction algebra Λ defined in [15] and $\mathcal{C}_\Gamma$ is equivalent to the singularity category of $\widehat{X}$ and thus categorifies the contraction algebra of Donovan and Wemyss. If Γ is exact 3CY, Van den Bergh proved that it is quasi-isomorphic to a (complete) Ginzburg algebra $\mathfrak{D}(Q, w)$ for some finite quiver Q and a potential w. If we fix the CY structure on Y, then there is a canonical class $[w]$, defined up to right equivalence, in the zeroth Hochschild homology of $H^0\Gamma$ (see Proposition 4.8). The class $[w]$ can be viewed as the "classical shadow" of the Calabi–Yau structure on Y. Our second result is:

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[1] Kawamata calls it a *simple collection*.

Theorem B. (Theorem 5.11) Let $\widehat{f}:\widehat{Y}\to\widehat{X}$ and $\widehat{f}':\widehat{Y}'\to\widehat{X}'$ be two formal flopping contractions with reduced exceptional fibers $\operatorname{Ex}(\widehat{f})=\bigcup_{i=1}^{t}C_i$ and $\operatorname{Ex}(\widehat{f}')=\bigcup_{i=1}^{s}C_i'$. Denote respectively by Γ and Γ' the derived deformation algebras of $\bigoplus_{i=1}^{t}\mathcal{O}_{C_i}$ and $\bigoplus_{i=1}^{s}\mathcal{O}_{C_i'}$, and by $[w]\in\mathrm{HH}_0(H^0\Gamma)$ and $[w']\in\mathrm{HH}_0(H^0\Gamma')$ the canonical classes. Suppose there is a triangle equivalence

$$?\overset{\mathbb{L}}{\otimes}_{H^0\Gamma}Z:\mathrm{D}(H^0\Gamma)\xrightarrow{\sim}\mathrm{D}(H^0\Gamma')$$

given by a dg bimodule Z such that $\mathrm{HH}_0(Z)$ (as defined in [39]) takes $[w]$ to $[w']$ in $\mathrm{HH}_0(H^0\Gamma')=\mathrm{HH}_0(\Gamma')$. Then $\widehat{X}$ is isomorphic to $\widehat{X}'$. In particular, s is equal to t.

For general (non-simple) flopping contractions, there exist derived equivalent algebras $H^0\Gamma$ and $H^0\Gamma'$ that are non-isomorphic. August proves that the isomorphism classes of such algebras in a fixed derived equivalence class of $H^0\Gamma$ are precisely the contraction algebras for the iterated flops of $\widehat{Y}$ (see Theorem 1.4 of [4]). Different contraction algebras in the same derived equivalence class are related by the iterated mutations of the tilting objects. The mutations are the homological counterpart of flops between different minimal models. We refer to [73] for the general framework of the homological minimal model program.

Theorem B says that the underlying singularity type of the smooth minimal models is determined by the derived equivalence class of the pair $(H^0\Gamma,[w])$. We sketch the idea of the proof. From 3-dimensional birational geometry we know that the underlying singularity of a smooth flopping contraction is a hypersurface (see Section 2.5). It is a classical theorem of Mather and Yau that up to isomorphism, a germ of hypersurface singularity is determined by its Tyurina algebra (see [53] for the analytic case with isolatedness assumption and [24] for the formal case without the isolatedness assumption). Next we prove that the derived equivalence class of $H^0\Gamma$ together with the canonical class $[w]$ recovers the Tyurina algebra of the singularity. We solve this problem in two steps. First, we prove that the Tyurina algebra, therefore the isomorphism class of the hypersurface singularity, can be recovered from the ($\mathbb{Z}$-graded dg enhanced) cluster category $\mathcal{C}_\Gamma$. This result, proved in Section 5, should have independent interest. Secondly, we show that the isomorphism class of the Ginzburg algebra $\mathfrak{D}(Q,w)$ that is quasi-isomorphic to Γ can be recovered from the data $(H^0\Gamma,[w])$. The proof uses a result of the first author and Gui-song Zhou in noncommutative differential calculus of potentials with finite-dimensional Jacobi-algebras [29]. Finally, we prove (in Section 4.4 and 4.5) that any derived Morita equivalence $\mathrm{D}(H^0\Gamma)\simeq\mathrm{D}(H^0\Gamma')$ preserving the canonical class yields a derived Morita equivalence $\mathrm{per}(\Gamma)\simeq\mathrm{per}(\Gamma')$.

Note that in [27], the first author and Toda gave an alternative definition of the contraction algebra associated to a flopping contraction using the category of matrix factorizations. In this definition, the contraction algebra carries an additional (compared with the definition in [15]) $\mathbb{Z}/2$-graded A_∞-structure. In [28], the first author proved that the Tyurina algebra of the singularity can be recovered from the $\mathbb{Z}/2$-graded A_∞-structure. Our proof of Theorem B shows that the $\mathbb{Z}/2$-graded A_∞-structure on the contraction algebra can be recovered from the class $[w]$. Theorem B without the condition on the preservation of the canonical class is precisely the generalization of the conjecture by Donovan and Wemyss stated by August in Conjecture 1.3 of [4]. See Conjecture 5.10 for the original conjecture of Donovan and Wemyss, which is for simple flopping contractions. The geometric meaning of the class $[w]$ remains to be understood. It is believed that the vanishing of $[w]$ is closely related to the condition that $\widehat{X}$ is quasi-homogeneous.

Our third result is a necessary condition on the contractibility of a nc rigid rational curve in a smooth CY 3-fold. Let u be a variable of degree 2.

Theorem C. (Proposition 6.9) Let C be a nc rigid rational curve in a smooth CY 3-fold Y. If C is contractible, then its derived deformation algebra Γ is $k[u^{-1}]$-enhanced (see definition in Section 6). Moreover, $H^0\Gamma$ is a symmetric Frobenius algebra.

We conjecture that a nc rigid rational curve C is contractible if and only if Γ is $k[u^{-1}]$-enhanced (see Conjecture 6.8).

The paper is organized as follows. In Section 2 we review basics on derived noncommutative deformation theory, noncommutative crepant resolutions and flopping contractions. Concerning derived deformations, we complement the results of [22] by explaining the link to classical deformations in abelian categories in subsection 2.3. In Section 3, we discuss various notions of Calabi-Yau structures in geometry and algebra and prove Theorem A. The notion of cluster category is introduced in Section 3.3. When the curve is contractible, there are two cluster categories associated to it: one via derived deformation and the other via the NCCR. We prove that these two cluster categories are algebraically equivalent. In Section 4, we recall the definition of Ginzburg algebras and several results in noncommutative differential calculus including the noncommutative Mather-Yau theorem and noncommutative Saito theorem. We further show that for a contractible curve in a CY 3-fold, there exists a Ginzburg algebra weakly equivalent to the derived deformation algebra whose potential is canonically defined up to right equivalence. Then we establish a relation between the silting theory of a non positive dg algebra and the silting theory of its zeroth homology. In Section 5, we study the relation between the cluster category associated to the contractible curves and their underlying singularities via Hochschild cohomology. In particular, Theorem B is proved. In Section 6, we introduce the notion of $k[u^{-1}]$-enhancement of dg algebras. For derived deformation algebra, we establish a link between the existence of $k[u^{-1}]$-enhancement and contractibility of rational curve and prove Theorem C.

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2. Preliminaries

2.1. **Notation and conventions.** Throughout k will be a ground field unless stated otherwise. Unadorned tensor products are over k. Let V be a k-vector space. We denote its dual vector space by DV. When V is graded, DV is understood as the the dual in the category of graded vector spaces. For a subspace V' of a complete topological vector space V, we denote the closure of V' in V by $(V')^c$. By definition a *pseudo-compact k-vector space* is a linear topological vector space which is complete and whose topology is generated by subspaces of finite codimension. Following [70] we will denote the corresponding category by $PC(k)$. We have inverse dualities

$$\mathbb{D} : Mod(k) \to PC(k)^{op} : V \mapsto \mathrm{Hom}_k(V, k)$$

$$\mathbb{D} : PC(k) \to Mod(k)^{op} : W \mapsto \mathrm{Hom}_{PC(k)}(W, k)$$

where we recall that for $V \in Mod(k)$ the topology on $\mathbb{D}V$ is generated by the kernels of $\mathbb{D}V \to \mathbb{D}V'$ where V' runs through the finite dimensional subspaces of V. Similarly, if V is graded then $\mathbb{D}$ is understood in the graded sense. For the definition of Hom-space and tensor product in $PC(k)$, we refer to Section 4 of [70]. Using the tensor product in $PC(k)$, we define the pseudo-compact dg algebras, modules and bimodules to be the corresponding objects in the category of graded objects of $PC(k)$. Let A be pseudo-compact dg k-algebra. Denote by $PC(A^e)$ the category of pseudo-compact A-bimodules. We will sometimes take a finite dimensional separable k-algebra l to be the ground ring. The definition of the duality functor $\mathbb{D}$ on $PC(l^e)$ requires some extra care due to the noncommutativity of l. We refer to Section 5 of [70] for the detailed discussion.

Denote by $\mathrm{PCAlgc}(l)$ the category of augmented pseudo-compact dg algebras A whose underlying graded algebras have their augmentation ideal equal to their Jacobson radical (cf. Proposition 4.3 and section 6 of [70]). Our main interest is in the case when $l \cong ke_1 \times ke_2 \times \ldots \times ke_n$ for central orthogonal idempotents $(e_i)_i$. For an object $A \in \mathrm{PCAlgc}(l)$, we use Hom-spaces and tensor products in $PC(l^e)$ to define the Hochschild and cyclic (co)homology. For details, we refer to Section 7 and Appendix A of [70]. If A is an l-algebra in $\mathrm{PCAlgc}(l)$, we use $\mathrm{HH}_*(A), \mathrm{HH}^*(A), \mathrm{HC}_*(A)$ to denote the continuous Hochschild homology, cohomology and cyclic homology of A. Because for a pseudo-compact dg algebra, we will only consider continuous Hochschild homology, cohomology and cyclic homology, there is no risk of confusion. By an abuse of notations, for $A \in \mathrm{PCAlgc}(l)$ we denote by $\mathrm{D}(A)$ the pseudo-compact derived category of A. Its subcategories $\mathrm{per}(A)$ and $\mathrm{D}_{fd}(A)$ are defined as the thick subcategory generated by the free A-module A and as the full subcategory of all objects with homology of finite total dimension. Similar to the algebraic case, the notion of homological smoothness can be defined in the pseudo-compact setting. We refer to the Appendix of [48] for a careful treatment. For the bar-cobar formalism and Koszul duality of pseudo-compact dg algebras, we refer to Appendix A and D of [69].

2.2. **Derived deformation theory.** We briefly recall the setup of derived noncommutative deformation theory of Efimov, Lunts and Orlov. In this section, we fix a field k. We refer to [42] for foundational material on dg categories. For a dg category $\mathcal{A}$, we denote by $\mathrm{D}(\mathcal{A})$ the derived category of right dg $\mathcal{A}$-modules. Fix a positive integer n and let l be the separable k-algebra $ke_1 \times \ldots \times ke_n$. An *$l$-algebra* A is a k-algebra together with a morphism of k-algebras $l \to A$ (note that l is *not* necessarily central in A). An equivalent datum is that of the k-category with n objects $1, \ldots, n$, whose morphism space from i to j is given by e_jAe_i. An *l-augmented* (dg) algebra is a (dg) l-algebra $\mathcal{R}$ together with an l-algebra morphism $\mathcal{R} \to l$ such that the composition $l \to \mathcal{R} \to l$ is the identity morphism. Its *augmentation ideal* is the kernel of the augmentation morphism $\mathcal{R} \to l$. An *artinian* l-algebra is an augmented l-algebra whose augmentation ideal is finite-dimensional and nilpotent. A dg l-algebra is *artinian* if it is an augmented dg l-algebra whose augmentation ideal is finite-dimensional and nilpotent. Denote by $\mathbf{Art}_l$ and $\mathbf{cArt}_l$ the categories of artinian l-algebras and of commutative artinian l-algebras. Denote by $\mathbf{dgArt}_l$ the category of artinian dg algebras and by $\mathbf{dgArt}_l^-$ the subcategory of $\mathbf{dgArt}_l$ consisting of dg algebras concentrated in nonpositive degrees.

Fix a dg category $\mathcal{A}$ and a dg $\mathcal{A}$-module E with a decomposition $E = E_1 \oplus \cdots \oplus E_n$. We view E as an $l^{op} \otimes \mathcal{A}$-module in the natural way. The *dg endomorphism l-algebra* of E is the dg endomorphism algebra over $\mathcal{A}$ of the sum E viewed as an l-algebra in the natural way. We are going to define a pseudo-functor $\mathrm{Def}(E)$ from $\mathbf{dgArt}_l$ to the category $\mathbf{Gpd}$ of groupoids. This pseudo-functor assigns to an artinian dg l-algebra $\mathcal{R}$ the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$ of $\mathcal{R}$-deformations of E in the derived category $\mathrm{D}(\mathcal{A})$. We will mostly follow the notations of [21] and identify $\mathcal{R}$ with the dg category with n objects $1, \ldots, n$, where the morphism complex from i to j is given by $e_j\mathcal{R}e_i$. Denote the dg category $\mathcal{R}^{op} \otimes \mathcal{A}$ by $\mathcal{A}_{\mathcal{R}}$. The augmentation $\varepsilon : \mathcal{R} \to l$ yields the functor of extension of scalars ε^* taking a dg $\mathcal{R}$-module S to the dg $l^{op} \otimes \mathcal{A}$-module

$$\varepsilon^*(S) = l \overset{\mathbb{L}}{\otimes}_{\mathcal{R}} S.$$

Definition 2.1. [21, Definition 10.1] Fix an artinian augmented dg l-algebra $\mathcal{R}$. An object of the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$ is a pair (S, σ), where S is an object of $\mathrm{D}(\mathcal{A}_{\mathcal{R}})$ and

$$\sigma : \varepsilon^*(S) \to E$$

is an isomorphism in $\mathrm{D}(l^{op}\otimes\mathcal{A})$. A morphism $f:(S,\sigma)\to(T,\tau)$ between two $\mathcal{R}$-deformations of E is an isomorphism $f:S\to T$ in $\mathrm{D}(\mathcal{A}_{\mathcal{R}})$ such that

$$\tau\circ\varepsilon^*(f)=\sigma.$$

This defines the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$. A homomorphism of artinian dg l-algebras $\phi:\mathcal{R}\to\mathcal{Q}$ induces the functor

$$\phi^*:\mathrm{Def}_{\mathcal{R}}(E)\to\mathrm{Def}_{\mathcal{Q}}(E)$$

given by $\mathcal{Q}\overset{\mathbb{L}}{\otimes}_{\mathcal{R}}?$. Thus we obtain a pseudo-functor

$$\mathrm{Def}(E):\mathbf{dgArt}_l\to\mathbf{Gpd}.$$

We call $\mathrm{Def}(E)$ the *pseudo-functor of derived deformations of* E. We denote by $\mathrm{Def}_-(E)$ the restrictions of the pseudo-functor $\mathrm{Def}(E)$ to the subcategory $\mathbf{dgArt}_l^-$.

The category of augmented dg l-algebras can be naturally enhanced to a weak 2-category. We refer to Definition 11.1 of [22] for the precise definition of the 2-category structure. In particular, we denote the corresponding 2-categorical enhancements of $\mathbf{dgArt}_l$, $\mathbf{dgArt}_l^-$ and $\mathbf{Art}_l$ by 2-$\mathbf{dgArt}_l$, 2-$\mathbf{dgArt}_l^-$ and 2-$\mathbf{Art}_l$ (in [22], they are denoted by $2'$-$\mathbf{dgArt}_l$ etc.). By Proposition 11.4 of [22], there exists a pseudo-functor $\mathrm{DEF}(E)$ from 2-$\mathbf{dgArt}_l$ to $\mathbf{Gpd}$ and which is an extension to 2-$\mathbf{dgArt}_l$ of the pseudo-functor $\mathrm{Def}(E)$. Similarly, there exists a pseudo-functor $\mathrm{DEF}_-(E)$ extending $\mathrm{Def}_-(E)$.

The main theorem of [22] is:

Theorem 2.2. [22, Theorem 15.1, 15.2] *Let* $E_1, \ldots, E_n$ *be a collection of objects in* $\mathrm{D}(\mathcal{A})$. *Let* E *be the direct sum of the* E_i *and* C *the extension algebra* $\mathrm{Ext}^*_{\mathcal{A}}(E,E)$ *considered as a graded* l*-algebra. Assume that we have*

(a) $C^p=0$ *for all* $p<0$;
(b) $C^0=l$;
(c) $\dim_k C^p<\infty$ *for all* p *and* $\dim_k C^p=0$ *for all* $p\gg 0$.

Denote by $\mathcal{C}$ *the dg endomorphism* l*-algebra of* E. *Let* A *be a strictly unital minimal model of* $\mathcal{C}$. *Then the pseudo-functor* $\mathrm{DEF}_-(E)$ *is pro-representable by the dg* l*-algebra* $\Gamma=\mathbb{D}BA$, *where* B *denotes the bar construction. That is, there exists an equivalence of pseudo-functors* $\mathrm{DEF}_-(E)\simeq h_\Gamma$ *from* 2-$\mathbf{dgArt}_l^-$ *to* $\mathbf{Gpd}$, *where* h_Γ *denotes the groupoid of* 1*-morphisms* 1-$\mathrm{Hom}(\Gamma,?)$.

In the case where the dg category $\mathcal{A}$ is given by an algebra A concentrated in degree 0 and E is a one-dimensional A-module, Booth [11, Theorem 3.5.9] obtains an analogous prorepresentability result for the set-valued framed deformation functor $\mathrm{Def}_A^{\mathrm{fr},\leq 0}(E)$ without having to impose the finiteness condition (c).

Let Y be a smooth algebraic variety. A collection of coherent sheaves of compact support $\mathcal{E}_1,\ldots,\mathcal{E}_t$ on Y is called *semi-simple* if $\mathrm{Hom}_Y(\mathcal{E}_i,\mathcal{E}_i)\cong k$ for all i and $\mathrm{Hom}_Y(\mathcal{E}_i,\mathcal{E}_j)=0$ for all $i\neq j$. The finiteness assumption in Theorem 2.2 is satisfied by any semi-simple collection. Let $\mathcal{E}$ be the direct sum of such a collection of coherent sheaves on Y. We may denote the completion of Y along the support of $\mathcal{E}$ by $\widehat{Y}$.

Corollary 2.3. *Given a semi-simple collection* $\{\mathcal{E}_i\}_{i=1}^t$ *in* $\mathrm{D}^b(\mathrm{coh}\,\widehat{Y})$, *denote by* $\mathcal{C}$ *the dg endomorphism* l*-algebra of* $\mathcal{E}:=\bigoplus_{i=1}^t\mathcal{E}_i$. *Let* A *be a strictly unital minimal model of* $\mathcal{C}$ *and* $\Gamma=\mathbb{D}BA$. *Then there is an equivalence*

$$\mathrm{DEF}_-(\mathcal{E})\simeq h_\Gamma.$$

We call Γ the *derived deformation algebra* of the collection $\{\mathcal{E}_i\}_{i=1}^t$ in Y. When we want to emphasize the dependence on Y and $\{\mathcal{E}_i\}_{i=1}^t$, Γ is replaced by $\Gamma^Y_{\mathcal{E}}$. The semi-simple collection $\{\mathcal{E}_i\}_{i=1}^t$ is called *nc rigid* if $\dim_k H^0(\Gamma^Y_{\mathcal{E}})<\infty$. In this paper, we are mainly interested in the case when $\{\mathcal{E}_i\}_{i=1}^t$ is (the structure sheaves) a collection of smooth rational curves $C_1, \ldots, C_t$ that satisfies the condition that $\operatorname{Hom}_Y(\mathcal{O}_{C_i}, \mathcal{O}_{C_j}) = 0$ for $i \neq j$. For such a collection of rational curves $C := \{C_i\}_{i=1}^n$ where we write Γ_C^Y for $\Gamma^Y_{\{\mathcal{O}_{C_i}\}_{i=1}^t}$.

In the context of classical noncommutative deformation theory, the representability of noncommutative deformations of contractible rational curves was proved by Donovan and Wemyss.

Theorem 2.4. *(Proposition 3.1, Corollary 3.3 [15]) Let $f : Y \to X$ be a simple flopping contraction of 3-folds (see definition in Section 2.5) and let C be the reduced exceptional fiber of f. The functor*

$$\pi_0(\mathrm{Def}^{cl}(\mathcal{O}_C)) : \mathbf{Art}_k \to \mathbf{Set}$$

is representable. The artinian algebra Λ representing it is called the contraction algebra *associated to $f : Y \to X$.*

The definition of the classical deformation functor Def^{cl} is recalled in the next section (cf. Section 2 of [15]). If Γ is the derived deformation algebra of C (with $t = 1$), it follows from the above Theorem and Theorem 2.5 below that the contraction algebra Λ is isomorphic to $H^0\Gamma$. Indeed, they both represent the same deformation functor and this determines them up to (non unique) isomorphism (cf. the proof of Theorem 2.14 in [65]).

2.3. **Link to classical deformations.** Let $\mathcal{A}$ be a dg category. Let $\mathcal{H} \subset \mathrm{D}(\mathcal{A})$ be the heart of a t-structure on $\mathrm{D}(\mathcal{A})$. We assume that $\mathcal{H}$ is *faithful*, i.e. the higher extension groups computed in $\mathcal{H}$ are canonically isomorphic to those computed in $\mathrm{D}(\mathcal{A})$.

Let $\mathcal{R}$ be an augmented artinian l-algebra. By an *$\mathcal{R}$-module in $\mathcal{H}$*, we mean an object M of $\mathcal{H}$ endowed with an algebra homomorphism $\mathcal{R} \to \operatorname{End}(M)$. Given such an $\mathcal{R}$-module, we denote by $?\otimes_{\mathcal{R}}M$ the unique right exact functor $\operatorname{mod}\mathcal{R} \to \mathcal{H}$ extending the obvious additive functor $\operatorname{proj}\mathcal{R} \to \mathcal{H}$ taking $\mathcal{R}$ to M. Here we denote by $\operatorname{proj}\mathcal{R}$ the category of finitely generated projective (right) $\mathcal{R}$-modules and by $\operatorname{mod}\mathcal{R}$ the category of finitely generated $\mathcal{R}$-modules. It is obvious how to define morphisms of $\mathcal{R}$-modules in $\mathcal{H}$.

Let E be the direct sum of a collection of n objects $E_1, \ldots, E_n$ of $\mathcal{H}$. We view E as an l-module in $\mathcal{H}$ in the natural way. For an augmented artinian l-algebra $\mathcal{R}$, we define the groupoid $\mathrm{Def}^{cl}_{\mathcal{R}}(E)$ of classical deformations of E as follows: Its objects are pairs (M, μ) where M is an $\mathcal{R}$-module in $\mathcal{H}$ such that the functor $?\otimes_{\mathcal{R}}M$ is exact and $\mu : l\otimes_{\mathcal{R}}M \xrightarrow{\sim} E$ is an isomorphism of l-modules in $\mathcal{H}$. A morphism $(L, \lambda) \to (M, \mu)$ is an isomorphism $f : L \to M$ of $\mathcal{R}$-modules in $\mathcal{H}$ such that $\mu \circ (l\otimes_{\mathcal{R}} f) = \lambda$.

For an augmented l-algebra A and an augmented artinian l-algebra $\mathcal{R}$, we define $G(A, \mathcal{R})$ to be the groupoid whose objects are the morphisms $A \to \mathcal{R}$ of augmented l-algebras and whose morphisms $\phi_1 \to \phi_2$ are the invertible elements r of $\mathcal{R}$ such that $\phi_2(a) = r\phi_1(a)r^{-1}$ for all a in A.

Theorem 2.5. *Suppose that in addition to the above assumptions, E satisfies the hypotheses of Theorem 2.2. Let Γ be the pseudo-compact dg l-algebra defined there. Let $\mathcal{R}$ be an augmented artinian l-algebra. Then $H^0\Gamma$ represents the classical deformations of E in the sense that there is an equivalence of groupoids*

$$\mathrm{Def}^{cl}_{\mathcal{R}}(E) \xrightarrow{\sim} G(H^0\Gamma, \mathcal{R}).$$

Proof. By Theorem 2.2, we have an equivalence of groupoids

$$\mathrm{DEF}_{-}(E)(\mathcal{R}) \xrightarrow{\sim} 1\text{-}\mathrm{Hom}(\Gamma, \mathcal{R}),$$

where 1-Hom denotes the groupoid of 1-morphisms in 2-**dgArt**. We will show that $\mathrm{DEF}_{-}(E)(\mathcal{R})$ is equivalent to $\mathrm{Def}^{cl}_{\mathcal{R}}(E)$ and $1\text{-}\mathrm{Hom}(\Gamma, \mathcal{R})$ is equivalent to $G(H^0\Gamma, \mathcal{R})$. We start with the second equivalence. By Definition 11.1 of [22], an object of $1\text{-}\mathrm{Hom}(\Gamma, \mathcal{R})$ is a pair (M, θ) consisting of

- a dg bimodule M in $\mathrm{D}(\Gamma^{op}\otimes\mathcal{R})$ such that the restriction to $\mathcal{R}$ of M is isomorphic to $\mathcal{R}$ in $\mathrm{D}(\mathcal{R})$ and
- an isomorphism $\theta : M\overset{\mathbb{L}}{\otimes}_{\mathcal{R}}l \to l$ in $\mathrm{D}(\Gamma^{op})$.

A 2-morphism $f : (M_1,\theta_1) \to (M_2,\theta_2)$ is an isomorphism $f : M_1 \to M_2$ in $\mathrm{D}(\Gamma^{op}\otimes\mathcal{R})$ such that $\theta_2\circ(f\overset{\mathbb{L}}{\otimes}_{\mathcal{R}}l) = \theta_1$. We define a functor $F : G(H^0\Gamma,\mathcal{R}) \to$ 1-$\mathrm{Hom}(\Gamma,\mathcal{R})$ as follows: Let $\phi : H^0(\Gamma) \to \mathcal{R}$ be a morphism of augmented l-algebras. Since Γ is concentrated in degrees ≤ 0, we have a canonical algebra morphism $\Gamma \to H^0(\Gamma)$. By composing it with ϕ we get a morphism of augmented dg l-algebras $\Gamma \to \mathcal{R}$. It defines a structure of dg bimodule M on $\mathcal{R}$. We put $F\phi = (M,\theta)$, where $\theta : \mathcal{R}\overset{\mathbb{L}}{\otimes}_{\mathcal{R}}l \to l$ is the canonical isomorphism. Now let ϕ_1 and ϕ_2 be two morphisms of augmented algebras $H^0(\Gamma) \to \mathcal{R}$. Put $(M_i,\theta_i) = F\phi_i$, $i = 1,2$. Let r an be an invertible element of $\mathcal{R}$ such that $\phi_2(a) = r\phi_1(a)r^{-1}$ for all a in $H^0(\Gamma)$. Then it is clear that the left multiplication with r defines an isomorphism of bimodules $M_1 \to M_2$ compatible with the θ_i. Since the M_i live in the heart of the canonical t-structure on $\mathrm{D}(\Gamma^{op}\otimes\mathcal{R})$, it is also clear that F is fully faithful. It remains to be checked that F is essentially surjective. So let (M,θ) be given. Since M is quasi-isomorphic to $\mathcal{R}$ when restricted to $\mathcal{R}$, its homology is concentrated in degree 0 and it identifies with an ordinary $H^0(\Gamma)$-$\mathcal{R}$-bimodule isomorphic to $\mathcal{R}$ as a right $\mathcal{R}$-module. By choosing an isomorphism of right $\mathcal{R}$-modules $M \xrightarrow{\sim} \mathcal{R}$, we obtain an algebra morphism $\phi : H^0(\Gamma) \to \mathrm{End}_{\mathcal{R}}(M) \xrightarrow{\sim} \mathrm{End}_{\mathcal{R}}(\mathcal{R}) = \mathcal{R}$. In particular, M is right projective and so $M\overset{\mathbb{L}}{\otimes}_{\mathcal{R}}? = M\otimes_{\mathcal{R}}?$. We choose an isomorphism $f : M \xrightarrow{\sim} \mathcal{R}$ of right $\mathcal{R}$-modules. After multiplying f with an invertible element of l, we may assume that $f\otimes_{\mathcal{R}}l = \theta$. The left Γ-module structure on M yields an algebra morphism

$$\phi : H^0(\Gamma) \to \mathrm{End}_{\mathcal{R}}(M) \xrightarrow{\sim} \mathrm{End}_{\mathcal{R}}(\mathcal{R}) = \mathcal{R}.$$

It is clear that f yields an isomorphism between (M,θ) and $F\phi$.

We now construct an equivalence from $\mathrm{DEF}_{-}(E)(\mathcal{R})$ to $\mathrm{Def}^{cl}_{\mathcal{R}}(E)$. Recall from Proposition 11.4 of [22] that the groupoid $\mathrm{DEF}_{-}(E)(\mathcal{R})$ equals the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$ of Definition 2.1 (but DEF_{-} has enhanced 2-functoriality). Let $P \to E$ be a cofibrant resolution of E. Since the graded algebra $\mathrm{Ext}^*(E,E)$ has vanishing components in degree -1 and in all sufficiently high degrees, we can apply Theorem 11.8 of [22] to conclude that the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$ is equivalent to the groupoid $\mathrm{Def}^{h}_{\mathcal{R}}(P)$ of homotopy deformations of Definition 4.1 of [21]. We now construct an equivalence F from $\mathrm{Def}^{h}_{\mathcal{R}}(P)$ to $\mathrm{Def}^{cl}_{\mathcal{R}}(E)$. Let (S,σ) be an object of $\mathrm{Def}^{h}_{\mathcal{R}}(P)$. We may assume that $S = \mathcal{R}\otimes_l P$ as a graded bimodule and that σ is the canonical isomorphism $l\otimes_{\mathcal{R}}(\mathcal{R}\otimes P) \xrightarrow{\sim} P$. Let I denote the augmentation ideal of $\mathcal{R}$. Then S has a finite filtration by the dg submodules I^pS, $p \geq 0$, and each subquotient is isomorphic to a summand of a finite sum of copies of $l\otimes_{\mathcal{R}}S = P$. Thus, the underlying dg $\mathcal{A}$-module M of S is isomorphic in $\mathrm{D}(\mathcal{A})$ to a finite iterated extension of objects of $\mathrm{add}(E)$, the subcategory of direct factors of finite direct sums of copies of E. Therefore, M still lies in the heart $\mathcal{H}$. Note that as shown in the proof of Theorem 11.8 of [22], S is cofibrant over $\mathcal{R}^{op}\otimes\mathcal{A}$. Therefore, M is cofibrant over $\mathcal{A}$. The left $\mathcal{R}$-module structure on S yields an algebra homomorphism $\mathcal{R} \to \mathrm{End}(M)$. Since each object of $\mathrm{mod}\,\mathcal{R}$ is a finite iterated extension of one-dimensional l-modules, the functor $?\otimes_{\mathcal{R}}S : \mathrm{D}(\mathcal{R}) \to \mathrm{D}(\mathcal{A})$ takes $\mathrm{mod}\,R$ to $\mathcal{H}$. Since $?\otimes_{\mathcal{R}}S$ is a triangle functor, the induced functor $\mathrm{mod}\,R \to \mathcal{H}$ is exact. Clearly it restricts to the natural functor $\mathrm{proj}\,\mathcal{R} \to \mathcal{H}$ and is therefore isomorphic to $?\otimes_{\mathcal{R}}M : \mathrm{mod}\,\mathcal{R} \to \mathcal{H}$. Finally, the isomorphism $l\otimes_{\mathcal{R}}S \xrightarrow{\sim} E$ yields an isomorphism $\mu : l\otimes_{\mathcal{R}}M \xrightarrow{\sim} E$. In this way, to an object (S,σ) of $\mathrm{Def}^{h}_{\mathcal{R}}(P)$, we have associated an object $F(S,\sigma) = (M,\mu)$ of $\mathrm{Def}^{cl}_{\mathcal{R}}(E)$. Recall that a morphism $(S_1,\sigma_1) \to (S_2,\sigma_2)$ of $\mathrm{Def}^{h}_{\mathcal{R}}(E)$ is a class of isomorphisms $S_1 \to S_2$ of dg $\mathcal{R}^{op}\otimes\mathcal{A}$-modules compatible with the σ_i modulo homotopies compatible with the σ_i. Since the functor $\mathrm{Def}^{h}_{\mathcal{R}}(P) \to \mathrm{Def}_{\mathcal{R}}(E)$ is fully faithful, these morphisms are in bijection with the isomorphisms $S_1 \to S_2$ of $\mathrm{D}(\mathcal{R}^{op}\otimes\mathcal{A})$ compatible with the σ_i. Clearly each such morphism induces an isomorphism $(M_1, \lambda_1) \to (M_2, \lambda_2)$, where $(M_i, \lambda_i) = F(S_i, \sigma_i)$, $i = 1, 2$. It follows from Lemma 2.6 below that this assignment is a bijection. It remains to be shown that $F : \mathrm{Def}^h_{\mathcal{R}}(P) \to \mathrm{Def}^{cl}_{\mathcal{R}}(E)$ is essentially surjective. Since we have an equivalence $\mathrm{Def}^h_{\mathcal{R}}(P) \xrightarrow{\sim} \mathrm{Def}_{\mathcal{R}}(E)$, it suffices to lift a given object (M, μ) of $\mathrm{Def}^{cl}_{\mathcal{R}}(E)$ to an object (S, σ) of $\mathrm{Def}_{\mathcal{R}}(E)$. Let A denote the dg endomorphism l-algebra $\mathbf{R}\mathrm{Hom}_{\mathcal{A}}(M, M)$. Then M becomes canonically an object of $\mathrm{D}(A^{op} \otimes \mathcal{A})$. Now since M is in the heart of a t-structure, its negative self-extension groups vanish and we have a quasi-isomorphism $\tau_{\leq 0} A \xrightarrow{\sim} \mathrm{End}_{\mathcal{H}}(M)$. Thus, in the homotopy category of dg algebras, we have a morphism

$$\mathcal{R} \to \mathrm{End}_{\mathcal{H}}(M) \xleftarrow{\sim} \tau_{\leq 0} A \to A.$$

By tensoring with $\mathcal{A}$ we obtain a morphism $\mathcal{R}^{op} \otimes \mathcal{A} \to A^{op} \otimes \mathcal{A}$ in the homotopy category of dg categories. The associated restriction functor $\mathrm{D}(A^{op} \otimes \mathcal{A}) \to \mathrm{D}(\mathcal{R}^{op} \otimes \mathcal{A})$ sends M to an object S of $\mathrm{D}(\mathcal{R}^{op} \otimes \mathcal{A})$. By construction, the restriction of S to $\mathcal{A}$ is isomorphic to M in $\mathrm{D}(\mathcal{A})$ and the left action of $\mathcal{R}$ on S induces the given algebra morphism $\mathcal{R} \to \mathrm{End}_{\mathcal{H}}(M)$. Since $\mathcal{H}$ is the heart of a t-structure, we have a canonical realization functor $\mathrm{D}^b(\mathcal{H}) \to \mathrm{D}(\mathcal{A})$ extending the inclusion $\mathcal{H} \to \mathrm{D}(\mathcal{A})$, cf. Section 3.1.10 of [6] or Section 3.2 of [46]. Moreover, since $\mathcal{H}$ is faithful, the realization functor is fully faithful. Since we only know how to compare tensor functors, we use a different construction to extend the inclusion $\mathcal{H} \to \mathrm{D}(\mathcal{A})$ to a triangle functor $\mathrm{D}^b(\mathcal{H}) \to \mathrm{D}(\mathcal{A})$. Let $\mathcal{H}_{dg}$ be the full subcategory of the dg category of right $\mathcal{A}$-modules formed by cofibrant resolutions of the objects of $\mathcal{H}$. We have an equivalence of k-categories $\mathcal{H} \xrightarrow{\sim} H^0(\mathcal{H}_{dg})$. Since $\mathcal{H}$ is the heart of a t-structure, the homology of the dg category $\mathcal{H}_{dg}$ is concentrated in degrees ≥ 0. Thus, we have quasi-equivalences $\tau_{\leq 0} \mathcal{H}_{dg} \xrightarrow{\sim} H^0(\mathcal{H}_{dg}) \xleftarrow{\sim} \mathcal{H}$. Therefore, in the homotopy category of dg categories, we obtain a morphism

$$\mathcal{H} \to \tau_{\leq 0}(\mathcal{H}_{dg}) \to \mathcal{H}_{dg} \to \mathrm{D}_{dg}(\mathcal{A})$$

where $\mathrm{D}_{dg}(\mathcal{A})$ denotes the dg category of cofibrant dg $\mathcal{A}$-modules. It gives rise to an $\mathcal{H}$-$\mathcal{A}$-bimodule R. Using the fact that short exact sequences of $\mathcal{H}$ give rise to triangles in $\mathrm{D}(\mathcal{A})$, one checks that the induced functor $? \otimes_{\mathcal{H}} R : \mathcal{H}^b(\mathcal{H}) \to \mathrm{D}(\mathcal{A})$ vanishes on the bounded acyclic complexes and therefore induces a triangle functor $\mathrm{D}^b(\mathcal{H}) \to \mathrm{D}(\mathcal{A})$ still denoted by $? \otimes_{\mathcal{H}} R$. Since $\mathcal{H}$ is a faithful heart, one obtains that this triangle functor is fully faithful. We claim that we have a square of triangle functors commutative up to isomorphism

$$\begin{tikzcd}
\mathrm{D}^b(\mathrm{mod}\, \mathcal{R}) \arrow[r, "? \otimes_{\mathcal{R}} M"] \arrow[d] & \mathrm{D}^b(\mathcal{H}) \arrow[d, "? \otimes_{\mathcal{H}} R"] \\
\mathrm{D}(\mathcal{R}) \arrow[r, "? \overset{\mathbb{L}}{\otimes}_{\mathcal{A}} S"'] & \mathrm{D}(\mathcal{A}).
\end{tikzcd}$$

To check this, one has to check that the bimodules S and $M \otimes_{\mathcal{H}} R$ are isomorphic in $\mathrm{D}(\mathcal{R}^{op} \otimes \mathcal{A})$. This is easy using Lemma 2.6 below. Since $? \otimes_{\mathcal{R}} M : \mathrm{mod}\, \mathcal{R} \to \mathcal{H}$ is exact, the given isomorphism $l \otimes_{\mathcal{R}} M \xrightarrow{\sim} E$ yields an isomorphism $l \overset{\mathbb{L}}{\otimes}_{\mathcal{R}} M \xrightarrow{\sim} E$ in $\mathrm{D}^b(\mathcal{H})$ and thus an isomorphism $\sigma : l \overset{\mathbb{L}}{\otimes}_{\mathcal{R}} S \xrightarrow{\sim} E$ in $\mathrm{D}(\mathcal{A})$. □

Let $\mathcal{A}$ be a dg category and B an ordinary k-algebra. Let X and Y be objects of $\mathrm{D}(B^{op} \otimes \mathcal{A})$ and let $\mathrm{res}(X)$ be the restriction of X to $\mathcal{A}$. The left action of B on X defines an algebra morphism

$$\alpha_X : B \to \mathrm{End}_{\mathrm{D}\mathcal{A}}(\mathrm{res}(X)).$$

Let $\mathcal{M}(X, Y)$ be the space of all morphisms $f : \mathrm{res}(X) \to \mathrm{res}(Y)$ in $\mathrm{D}(\mathcal{A})$ such that

$$f \circ \alpha_X(b) = \alpha_Y(b) \circ f$$

for all $b \in B$. The restriction functor induces a natural map

$$\Phi : \mathrm{Hom}_{\mathrm{D}(B^{op}\otimes\mathcal{A})}(X,Y) \to \mathcal{M}(X,Y).$$

Lemma 2.6. *If we have*

$$\mathrm{Hom}_{\mathrm{D}\mathcal{A}}(\mathrm{res}(X), \Sigma^{-n}\mathrm{res}(Y)) = 0$$

for all $n > 0$, the map Φ is bijective.

Proof. We adapt the argument of Section 5 of [41]. We may suppose that X is cofibrant over $B^{op}\otimes\mathcal{A}$ and in particular cofibrant over $\mathcal{A}$. Then the sum total dg module of the bar resolution

$$\ldots \longrightarrow B\otimes B^{\otimes p}\otimes X \longrightarrow \ldots \longrightarrow B\otimes B\otimes X \longrightarrow B\otimes X \longrightarrow 0$$

is still cofibrant over $B^{op}\otimes\mathcal{A}$ and quasi-isomorpic to X. We use it to compute $\mathrm{Hom}_{\mathrm{D}(B^{op}\otimes\mathcal{A})}(X,Y)$. By applying $\mathrm{Hom}_{B^{op}\otimes\mathcal{A}}(?,Y)$ to the bar resolution, we get a double complex D of the form

$$\mathrm{Hom}_{\mathcal{A}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(B\otimes X,Y) \longrightarrow \ldots \longrightarrow \mathrm{Hom}_{\mathcal{A}}(B^{\otimes p}\otimes X,Y) \longrightarrow \ldots$$

We have to compute H^0 of the product total complex $\mathrm{Tot}^{\Pi}D$. Let $D_{\geq 0}$ be the double complex obtained by applying the intelligent truncation functor $\tau_{\geq 0}$ to each column of D. Let $D_{<0}$ be the kernel of $D \to D_{\geq 0}$. We claim that the product total complex of $D_{<0}$ is acyclic. Indeed, the homology of the pth column of $D_{<0}$ in degree $-q$ is isomorphic to

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(B^{\otimes p}\otimes X, \Sigma^{-q}Y).$$

It vanishes for $-q < 0$ by our assumption. To show that the product total complex of $D_{<0}$ is acyclic, we consider the column filtration $F_pD_{<0}$. Then $D_{<0}$ is the inverse limit of the $F_pD_{<0}$. By induction on p, each $F_pD_{<0}$ has an acyclic total complex. Moreover, the transition maps $F_{p+1}D_{<0} \to F_pD_{<0}$ induce componentwise surjections in the total complexes. It follows that the inverse limit of the total complexes of the $F_pD_{<0}$ is still acyclic and this inverse limit is the product total complex of $D_{<0}$. So it is enough to compute H^0 of $\mathrm{Tot}^{\Pi}D_{\geq 0}$. It is straightforward to check that this space canonically identifies with $\mathcal{M}(X,Y)$. □

2.4. Noncommutative crepant resolutions.

Definition 2.7. Let $(R,\mathfrak{m})$ be a complete commutative Noetherian local Gorenstein k-algebra of Krull dimension n with isolated singularity and with residue field k. Denote the category of maximal Cohen–Macaulay (MCM) modules by CM_R and its stable category by $\underline{\mathrm{CM}}_R$. Let $N_0 = R, N_1, N_2, \ldots, N_t$ be pairwise non-isomorphic indecomposables in CM_R and $A := \mathrm{End}_R(\bigoplus_{i=0}^t N_i)$. We call A a *noncommutative resolution* (NCR) of R if it has finite global dimension. A NCR is called a *noncommutative crepant resolution* (NCCR) if A further satisfies that

- (a) $A \in \mathrm{CM}_R$;
- (b) $\mathrm{gldim}(A) = n$.

If A is a NCCR, we call $\oplus_{i=0}^t N_i$ a *tilting module.* Under the above conditions, Iyama shows that $\oplus_{i=1}^t N_i$ is a *cluster tilting object* (see definition in Section 3.3) in $\underline{\mathrm{CM}}_R$. Denote $\bar{l}$ for $A/\mathrm{rad}A$ and e_0 for the idempotent given by the projection $R\oplus\bigoplus_{i=1}^t N_i \to R$. Let $S_0, S_1, \ldots, S_t$ be the simple A-modules with S_0 corresponding to the summand R of $R\oplus\bigoplus_{i=1}^t N_i$. De Thanhoffer de Völcsey and Van den Bergh prove that $\underline{\mathrm{CM}}_R$ admits an explicit dg model in this case.

Theorem 2.8. ([67, Theorem 1.1]) *There exists a finite dimensional graded vector space V and a minimal model $(\widehat{T}_{\bar{l}}V, d) \xrightarrow{\sim} A$ for A. Put $\Gamma = \widehat{T}_{\bar{l}}V/\widehat{T}_{\bar{l}}Ve_0\widehat{T}_{\bar{l}}V$. Then one has*

$$\underline{\mathrm{CM}}_R \cong \mathrm{per}(\Gamma)/\mathbf{thick}(S_1, \ldots, S_t)$$

and furthermore Γ has the following properties

(1) Γ *has finite dimensional cohomology in each degree;*
(2) Γ *is concentrated in negative degrees;*
(3) $H^0\Gamma = A/Ae_0A$*;*
(4) *As a graded algebra* Γ *is of the form* $\widehat{T}_l V^0$ *for* $V^0 = (1-e_0)V(1-e_0)$ *with* $l := \bar{l}/ke_0$*.*

2.5. Flopping contraction.

Definition 2.9. A smooth rational curve C in a normal variety Y is called *contractible* if there exists an open subscheme $Y^\circ \subset Y$ containing C and a proper birational morphism $f^\circ : Y^\circ \to X^\circ$ such that

(1) X° is normal,
(2) the exceptional locus $\mathrm{Ex}(f^\circ)$ contains C,
(3) f° is an isomorphism in codimension one.

The above definition of contractibility is more restrictive than the standard one since it rules out the divisorial contraction (by the last condition). If Y is a 3-fold (which is our main interest), then $\mathrm{Ex}(f)$ must have dimension one by condition (3). However, it may contain other components besides C. If C is a contractible curve in Y, denote by $\widehat{X}$ the formal completion of X° along the exceptional subscheme, i.e. where f° is not an isomorphism. Consider the Cartesian diagram

$$\begin{tikzcd}
\widehat{Y} \arrow[r] \arrow[d, "\widehat{f}"] & Y^\circ \arrow[d, "f^\circ"] \\
\widehat{X} \arrow[r] & X^\circ
\end{tikzcd}$$

where $\widehat{Y}$ is the fiber product. We call $\widehat{f} : \widehat{Y} \to \widehat{X}$ the *formal contraction* associate to the contraction $f^\circ : Y^\circ \to X^\circ$.

The following definition is a special case of Definition 6.10 of [49].

Definition 2.10. Let Y be a normal variety of dimension 3. A *flopping contraction* is a proper birational morphism $f : Y \to X$ to a normal variety Y such that f is an isomorphism in codimension one, and K_Y is f-trivial. If Y is smooth, then we call f a smooth flopping contraction.

In this paper, we only consider smooth flopping contractions unless stated otherwise. Given a 3-dimensional flopping contraction $f : Y \to X$, let D be a divisor on Y such that $-(K_Y + D)$ is f-ample. By Theorem 6.14 of [49], there exists a D-flop $f^+ : Y^+ \to X$. To be more precise, f^+ is a proper birational morphism that is an isomorphism in codimension one, and $K_{Y^+} + D^+$ is f^+-ample where D^+ is the birational transform of D on Y^+. In particular, X is Gorenstein terminal. Without loss of generality, we may work locally near the exceptional fiber of f. By the classification theorem of 3-dimensional terminal singularities, X has an isolated cDV singularity (see Corollary 5.38 of [49]). Recall that a 3-fold singularity $(X, 0)$ is called cDV if a generic hypersurface section $0 \in H \subset X$ is a Du Val singularity. Because H has embedded dimension 3, X has embedded dimension 4, i.e. X is a hypersurface.

Denote by $\mathrm{Ex}(f)$ the reduced exceptional fiber of f. It is well known that $\mathrm{Ex}(f)$ is a tree of rational curves

$$\mathrm{Ex}(f) = \bigcup_{i=1}^{t} C_i$$

with normal crossings such that $C_i \cong \mathbb{P}^1$ (c.f. Lemma 3.4.1 of [68]). We call a 3-dimensional flopping contraction $f : Y \to X$ *simple* if $\mathrm{Ex}(f) \cong \mathbb{P}^1$. Denote by $\widehat{Y}$ the formal completion of Y along $\mathrm{Ex}(f)$, by $\widehat{X}$ the formal completion of X at 0 and by $\widehat{f}$ the completion of f. By the remark above, we may assume $\widehat{X} = \operatorname{Spec} R$ for a complete local ring $R = k[[x, y, u, v]]/(g)$. We call the triple $(\widehat{Y}, \widehat{f}, R)$ the *formal contraction* associated to the flopping contraction $f : Y \to X$. Note that $\widehat{Y}$ is Calabi-Yau.

Let $(\widehat{Y}, \widehat{f}, R)$ be a formal flopping contraction. Now we consider the NCCR associated to a three dimensional flopping contraction $\widehat{f} : \widehat{Y} \to \widehat{X}$ constructed as follows. For $i = 1, \ldots, t$, let $\widehat{\mathcal{L}}_i$ be a line bundle on $\widehat{Y}$ such that $\deg_{C_j} \widehat{\mathcal{L}}_i = \delta_{ij}$. Define $\widehat{\mathcal{N}}_i$ to be given by the maximal extension

$$0 \longrightarrow \widehat{\mathcal{L}}_i^{-1} \longrightarrow \widehat{\mathcal{N}}_i \longrightarrow \mathcal{O}_{\widehat{Y}}^{\oplus r_i} \longrightarrow 0 \tag{2.1}$$

associated to a minimal set of r_i generators of $H^1(\widehat{Y}, \widehat{\mathcal{L}}_i^{-1})$. Set $N_i := \mathbf{R} f_* \widehat{\mathcal{N}}_i = f_* \widehat{\mathcal{N}}_i$ for $i = 1, \ldots, t$. We set

$$A := \operatorname{End}_{\widehat{Y}}(\mathcal{O}_{\widehat{Y}} \oplus \widehat{\mathcal{N}}_1 \oplus \ldots \widehat{\mathcal{N}}_t) \cong \operatorname{End}_R(R \oplus N_1 \oplus \ldots \oplus N_t). \tag{2.2}$$

For simplicity, we denote by $\widehat{\mathcal{N}}$ the direct sum $\bigoplus_{i=1}^t \widehat{\mathcal{N}}_i$ and denote by N the direct sum $\bigoplus_{i=1}^t N_i$.

Theorem 2.11. *([68, Corollary 3.2.10]) The functor $\widehat{F} := \mathbf{R}\mathrm{Hom}_{\widehat{Y}}(\mathcal{O}_{\widehat{Y}} \oplus \widehat{\mathcal{N}}, -)$ defines a triangle equivalence between $\mathrm{D}^b(\operatorname{coh} \widehat{Y})$ and $\mathrm{D}^b(\operatorname{Mod} A)$, with quasi-inverse $F^{-1} := (-) \otimes_A (\mathcal{O}_{\widehat{Y}} \oplus \widehat{\mathcal{N}})$. In addition, A is itself Cohen-Macaulay.*

It follows that A is a NCCR.

Corollary 2.12. *Let $(\widehat{Y}, \widehat{f}, R)$ be a 3-dimensional formal flopping contraction. Then*

1. *The irreducible components $\{C_i\}_{i=1}^t$ of $\operatorname{Ex}(\widehat{f})$ form a semi-simple collection.*
2. *The derived deformation algebra $\Gamma_C^{\widehat{Y}}$ of the collection $C := \{C_i\}_{i=1}^t$ is linked by quasi-isomorphisms to Γ in Theorem 2.8.*
3. *For any $i = 1, \ldots, t$, C_i is nc rigid.*

Proof. To prove that $\operatorname{Hom}_Y(\mathcal{O}_{C_i}, \mathcal{O}_{C_j}) = 0$ for $i \neq j$, we simply need to use the condition that $\operatorname{Ex}(\widehat{f})$ is a tree of rational curves with normal crossing.

For any $i = 1, \ldots, t$, it is easy to check that $S_i \cong \widehat{F}(\Sigma \mathcal{O}_{C_i}(-1))$. Note that $(\bigoplus_{i=1}^t \mathcal{O}_{C_i}) \otimes (\bigotimes_{i=1}^t \mathcal{L}_i^{-1}) \cong \bigoplus_{i=1}^t \mathcal{O}_{C_i}(-1)$. By Theorem 2.11, $\widehat{F}$ induces an isomorphism of A_∞ algebras

$$\operatorname{Ext}^*_{\widehat{Y}}(\bigoplus_{i=1}^t \mathcal{O}_{C_i}, \bigoplus_{i=1}^t \mathcal{O}_{C_i}) \cong \operatorname{Ext}^*_A(\bigoplus_{i=1}^t S_i, \bigoplus_{i=1}^t S_i). \tag{2.3}$$

Let $\bar{l} = A/\operatorname{rad}(A)$ and $l = \bar{l}/ke_0$. Then there is a natural isomorphism of A-modules $l \cong \bigoplus_{i=0}^t S_i$, where S_0 is the simple A-module that corresponds to the summand R of $R \oplus \bigoplus_{i=1}^t N_i$. By Lemma 4.1 of [67], the vector space V in Theorem 2.8 can be chosen as $D\big(\Sigma \operatorname{Ext}_A^{\geq 1}(\bar{l}, \bar{l})\big)$. Therefore, $\Gamma := \widehat{T}_{\bar{l}} V / \widehat{T}_{\bar{l}} V e_0 \widehat{T}_{\bar{l}} V \cong \mathbb{D}B(\operatorname{Ext}^*_A(l, l))$ represents the noncommutative deformations of semi-simple collection $\{S_i\}_{i=1}^t \in \mathrm{D}^b(\operatorname{mod} A)$. Part (2) follows from the isomorphism 2.3. Denote by $\Gamma_{C_i}^{\widehat{Y}}$ the derived deformation algebra of $\mathcal{O}_{C_i}$. We have

$$\Gamma_{C_i}^{\widehat{Y}} \cong \Gamma / \sum_{j \neq i} \Gamma e_j \Gamma, \text{ and } H^0 \Gamma_{C_i}^{\widehat{Y}} \cong H^0 \Gamma / \sum_{j \neq i} H^0 \Gamma e_j H^0 \Gamma.$$

Then part (3) follows from (1) of Theorem 2.8. □

3. CALABI-YAU STRUCTURE AND CLUSTER CATEGORY

In this section, we first review several notions of Calabi–Yau property for triangulated categories, for homologically smooth dg algebras and for proper dg algebras. Then we recall geometric versions of the Calabi–Yau property and translate them into algebraic notions for endomorphism algebras of generators respectively for derived deformation algebras. Finally, we classify Calabi–Yau structures for 3-dimensional flopping contractions and review the cluster category.

3.1. **CY structures.**

3.1.1. *CY triangulated categories.* Let $\mathcal{T}$ be a Hom-finite k-linear triangulated category.

Definition 3.1. A *right Serre functor* for $\mathcal{T}$ is a triangle functor $S\colon \mathcal{T} \to \mathcal{T}$ such that there are bifunctorial isomorphisms

$$\mathcal{T}(Y, SX) \to D\mathcal{T}(X, Y)$$

for all $X, Y \in \mathcal{T}$. It is a *Serre functor* if it is an autoequivalence.

One can show that a right Serre functor exists if and only if for each object X of $\mathcal{T}$, the functor $D\mathcal{T}(X, ?)$ is representable and in this case, the right Serre functor is unique up to canonical isomorphism of triangle functors [8, 69]. Let d be an integer. The triangulated category $\mathcal{T}$ is *d-Calabi–Yau* if it admits a Serre functor isomorphic to Σ^d.

3.1.2. *CY smooth dg algebras.* A dg-k-algebra Γ is called *homologically smooth* if Γ is perfect in $\mathrm{D}(\Gamma^e)$. Then one checks that $\mathrm{D}_{fd}(\Gamma)$, the subcategory of $\mathrm{D}(\Gamma)$ consisting of the modules whose homology is of finite total dimension, is contained in the perfect derived category $\mathrm{per}(\Gamma)$. Put

$$\Theta_\Gamma = \mathbf{R}\mathrm{Hom}_{\Gamma^e}(\Gamma, \Gamma^e).$$

Then we have a canonical isomorphism

$$\mathrm{HH}_d(\Gamma) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{D}(\Gamma^e)}(\Theta_\Gamma, \Sigma^{-d}\Gamma).$$

Definition 3.2. The dg algebra Γ is called *bimodule* dCY if it is homologically smooth and there is an isomorphism in $\mathrm{D}(\Gamma^e)$

$$\eta : \Theta_\Gamma \xrightarrow{\sim} \Sigma^{-d}\Gamma.$$

A class $\eta \in \mathrm{HH}_d(\Gamma)$ is called a *dCY structure* if the corresponding morphism $\eta : \Theta_\Gamma \to \Sigma^{-d}\Gamma$ is an isomorphism in $\mathrm{D}(\Gamma^e)$. A dCY structure η is called *exact* if there exists a class $\xi \in \mathrm{HC}_{d-1}(\Gamma)$ such that $B\xi = \eta$, where B is the Connes morphism. We call a bimodule dCY algebra Γ an *exact dCY algebra* if the dCY structure is exact in addition.

Definition 3.3. The dg algebra Γ is said to satisfy the *relative dCY property* if for $L \in \mathrm{D}_{fd}(\Gamma)$ and $M \in \mathrm{per}(\Gamma)$, we have

$$D\mathbf{R}\mathrm{Hom}_{\mathrm{D}(\Gamma)}(L, M) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{D}(\Gamma)}(M, \Sigma^d L).$$

Remark 3.4. If Γ is a topologically homologically smooth pseudo-compact dg algebra in $\mathrm{PCAlgc}(l)$, we call Γ a bimodule dCY if η is an isomorphism in the pseudo-compact derived category of bimodules. The isomorphism η represents a class in the continuous Hochschild homology $\mathrm{HH}_d(\Gamma)$. Exactness is defined similarly by taking the continuous cyclic homology. We call a bimodule dCY pseudo-compact dg algebra Γ in $\mathrm{PCAlgc}(l)$ an *exact dCY algebra* if the dCY structure is exact in addition. Similarly, for a pseudo-compact algebra Γ in $\mathrm{PCAlgc}(l)$, the relative dCY property is defined by replacing $\mathrm{D}(\Gamma), \mathrm{per}(\Gamma)$ and $\mathrm{D}_{fd}(\Gamma)$ with their pseudo-dompact counter-parts.

Given a homologically smooth dg algebra Γ, it follows from Lemma 3.4 in [44] that we have the implications

$$\begin{aligned} \Gamma \text{ is bimodule dCY} &\Rightarrow \Gamma \text{ satisfies the relative dCY property} \\ &\Rightarrow \mathrm{D}_{fd}(\Gamma) \text{ is a Hom-finite dCY triangulated category.} \end{aligned}$$

A similar chain of implications holds in the pseudo-compact case.

3.1.3. *CY proper dg algebras.* Let A be a dg algebra. Suppose that A is *proper*, i.e. that its homology is of finite total dimension. Then the category $\mathrm{per}(A)$ is Hom-finite. The proper dg algebra A is called *perfectly dCY* if there is an isomorphism

$$DA \xrightarrow{\sim} \Sigma^d A$$

in $\mathrm{D}(A^e)$. By the following lemma, the triangulated category $\mathrm{per}(A)$ is then dCY.

Lemma 3.5. *Let A be an arbitrary dg algebra. Then for P perfect and M arbitrary in $\mathrm{D}(A)$, there is a canonical isomorphism*

$$\mathbf{R}\mathrm{Hom}_A(M, P\overset{\mathbb{L}}{\otimes}_A DA) \xrightarrow{\sim} D\mathbf{R}\mathrm{Hom}_A(P, M).$$

Proof. We have a canonical isomorphism

$$P\overset{\mathbb{L}}{\otimes}_A \mathrm{Hom}_k(A, k) \to \mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_A(P, A), k)$$

which takes a tensor $p\otimes f$ to the map taking $g : P \to A$ to $\pm f(g(p))$. It induces an isomorphism

$$\mathbf{R}\mathrm{Hom}_A(M, P\overset{\mathbb{L}}{\otimes}_A DA) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_A(M, \mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_A(P, A), k)).$$

The last term is canonically isomorphic to

$$\mathrm{Hom}_k(M\overset{\mathbb{L}}{\otimes}_A \mathbf{R}\mathrm{Hom}_A(P, A), k) = D\mathbf{R}\mathrm{Hom}_A(P, M).$$

□

Lemma 3.6. *Let A be a proper dg algebra which is also homologically smooth. Then DA and $\Theta_A = \mathbf{R}\mathrm{Hom}_{A^e}(A, A^e)$ are inverse to each other in $\mathrm{D}(A^e)$, i.e we have isomorphisms*

$$DA\overset{\mathbb{L}}{\otimes}_A\Theta_A \xrightarrow{\sim} A \quad \textit{and} \quad \Theta_A\overset{\mathbb{L}}{\otimes}_A DA \xrightarrow{\sim} A$$

in $\mathrm{D}(A^e)$.

Proof. Let us write $\otimes$ instead of $\overset{\mathbb{L}}{\otimes}$ and Hom instead of $\mathbf{R}$Hom. We have canonical isomorphisms in $\mathrm{D}(A^e)$

$$DA\otimes_A\Theta_A = DA\otimes_A\Theta_A\otimes_A A = (A\otimes_k DA)\otimes_{A^e}\Theta_A = \mathrm{Hom}_{A^e}(A, A\otimes_k DA)\ ,$$

where we have used that A is homologically smooth for the last isomorphism. Since A is proper, the last term is canonically isomorphic to

$$\mathrm{Hom}_{A^e}(A, \mathrm{Hom}_k(A, A)) = \mathrm{Hom}_A(A, A) = A.$$

The proof of the second isomorphism is analogous. □

Definition 3.7. Let k be a field of characteristic zero. Given a finite-dimensional A_∞-algebra A, a *cyclic A_∞-structure* of degree d on A is a non degenerate symmetric bilinear form

$$(?, ?) : A \times A \to \Sigma^d A$$

of degree d such that

$$(m_n(a_1, \dots, a_n), a_{n+1}) = (-1)^n(-1)^{|a_1|(|a_2|+\dots+|a_{n+1}|)}(m_n(a_2, \dots, a_{n+1}), a_1)$$

In this case, we have in particular an isomorphism $DA \xrightarrow{\sim} \Sigma^d A$ in the derived category of A-bimodules. Thus, if a dg algebra is quasi-isomorphic to an A_∞-algebra admitting a cyclic A_∞-structure of degree d, then it is perfectly dCY.

Lemma 3.8. *Let A be a proper augmented dg algebra and Γ the pseudo-compact dg algebra $\mathbb{D}BA$, where B denotes the bar construction. Then Γ is topologically homologically smooth. Moreover, if A is perfectly d-Calabi–Yau, then Γ is bimodule d-Calabi–Yau.*

Proof. Let C be the augmented dg coalgebra BA and $\tau : C \to A$ the canonical twisting cochain. Since τ is acyclic, the canonical morphism

$$C \to C \otimes_\tau A \otimes_\tau C$$

is a weak equivalence of dg C-bicomodules (cf. [38]). By applying the duality $\mathbb{D}$ we obtain that the morphism

$$\Gamma \otimes_\tau \mathbb{D}A \otimes_\tau \Gamma \to \Gamma \tag{3.1}$$

is a weak equivalence, where the tensor products are taken in the category of pseudo-compact vector spaces. Since A is proper, it follows that Γ is topologically homologically smooth. Now suppose that A is perfectly d-Calabi–Yau. Since A is proper, it is weakly equivalent to its pseudo-compact completion $\widehat{A}$. By the Calabi–Yau property, we have an isomorphism $\mathbb{D}A \xrightarrow{\sim} \Sigma^d \widehat{A}$ in the pseudo-compact derived category of dg $\widehat{A}$-bimodules. Now we compute the inverse dualizing complex of Γ using the resolution 3.1. We have isomorphisms in the pseudo-compact derived category of Γ-bimodules

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_{\Gamma^e}(\Gamma, \Gamma^e) &= \mathrm{Hom}_{\Gamma^e}(\Gamma \otimes_\tau \mathbb{D}A \otimes_\tau \Gamma, \Gamma^e) \\
&= \mathrm{Hom}^{\tau,\tau}_{psc}(\mathbb{D}A, \Gamma^e) \\
&= \mathrm{Hom}^{\tau,\tau}_{psc}(\Sigma^d \widehat{A}, \Gamma^e) \\
&= \Sigma^{-d} \Gamma \otimes_\tau \mathbb{D}A \otimes_\tau \Gamma \\
&= \Sigma^{-d} \Gamma.
\end{aligned}$$

Here $\mathrm{Hom}^{\tau,\tau}_{psc}$ denotes the space of morphisms in the category of pseudo-compact vector spaces twisted twice by τ. This shows that Γ is topologically homologically bimodule d-Calabi–Yau. □

3.1.4. *CY structures in geometry, algebraic consequences.* We let $k = \mathbb{C}$ be the field of complex numbers unless specified otherwise.

Definition 3.9. Let Y be a d-dimensional smooth quasi-projective $\mathbb{C}$-variety. We call Y a d-dimensional Calabi–Yau variety if there is an isomorphism $\omega_Y := \Omega^d_Y \cong \mathcal{O}_Y$, i.e. there exists a nowhere vanishing d-form. We call a nowhere vanishing section $\eta : \mathcal{O}_Y \to \omega_Y$ a *dCY structure* on Y. We call the dCY structure *exact* if the the d-form η is exact, i.e. there exists a $(d-1)$-form $\xi \in \Omega^{d-1}_Y$ such that $\mathrm{d}\xi = \eta$. If $\widehat{Y}$ is a smooth formal scheme, we may define dCY structure in a similar way by considering de Rham complex of formal scheme $\Omega^*_{\widehat{Y}}$. A quasi-projective Gorenstein d-fold Y is called *Gorenstein CY* if $\omega_Y \cong \mathcal{O}_Y$.

Recall from Theorem 3.1.1 of [9] that the derived category of quasi-coherent sheaves on a quasi-compact separated scheme admits a single perfect generator.

Proposition 3.10. *Let Y be a smooth quasi-projective Calabi–Yau d-fold.*

a) *Let G be a perfect generator of* $\mathrm{D}(\mathrm{Qcoh}Y)$ *and B its derived endomorphism dg algebra. Then B is bimodule d-Calabi–Yau.*

b) *Let L be a bounded complex of coherent sheaves with compact support on Y and A its derived endomorphism dg algebra. Then A is perfectly d-Calabi–Yau.*

Proof. a) In this proof, for simplicity, we write ordinary functors for derived functors. For a perfect object P, let $P^\vee$ denote its dual (the derived functor of the sheaf valued Hom-functor with values in $\mathcal{O}_Y$). The functor taking P to $P^\vee$ is an equivalence $\mathrm{per}(Y)^{op} \to \mathrm{per}(Y)$. Therefore, the object $G^\vee$ is a classical generator of $\mathrm{per}(Y)$ and thus another generator of $\mathrm{D}(\mathrm{Qcoh}Y)$. By Lemma 3.4.1 of [9], the object $G \boxtimes G^\vee$ is a perfect generator of $\mathrm{D}(\mathrm{Qcoh}(Y \times Y))$ and clearly its derived endomorphism dg algebra is $B \otimes B^{op}$. Let F^e denote the equivalence $\mathrm{Hom}(G \boxtimes G^\vee, ?)$ from $\mathrm{D}(\mathrm{Qcoh}Y)$ to $\mathrm{D}(B^e)$.

Let $\Delta : Y \to Y \times Y$ denote the diagonal embedding. By Lemma 3.10 of [25], the image under $F^e\Delta_*$ of the inverse of the canonical bundle of Y is $\Sigma^d\Theta_B$, where $\Theta_B = \mathrm{Hom}_{B^e}(B, B^e)$ is the inverse dualizing complex of B. Since the Lemma in [25] is stated and proved in the context of a smooth and *projective* variety, we repeat the argument. We first check that $F^e(\Delta_*\mathcal{O}_Y) = B$. By definition, we have $G \boxtimes G^\vee = p_1^*G \otimes p_2^*G^\vee$, where p_i is the projection from $Y \times Y$ to its ith component. We have

$$\begin{aligned} F^e(\Delta_*\mathcal{O}_Y) = \mathrm{Hom}(p_1^*G \otimes p_2^*G^\vee, \Delta_*\mathcal{O}_Y) &= \mathrm{Hom}(p_1^*, p_2^*G \otimes \Delta_*\mathcal{O}_Y) \\ &= \mathrm{Hom}(p_1^*G, \Delta_*(G \otimes \mathcal{O}_Y)) \\ &= \mathrm{Hom}(\Delta^*p_1^*G, G) = B. \end{aligned}$$

For an object M of $\mathrm{D}(B^e)$, we put $M^\vee = \mathrm{Hom}_{B^e}(M, B^e)$. We claim that for $P \in \mathrm{per}(Y \times Y)$, we have a canonical isomorphism

$$F^e(P^\vee) \xrightarrow{\sim} F^e(P)^\vee.$$

Indeed, the left hand side is

$$F^e(P^\vee) = \mathrm{Hom}(G \boxtimes G^\vee, P^\vee).$$

The right hand side is

$$\begin{aligned} F^e(P)^\vee &= \mathrm{Hom}_{B^e}(F^e(P), B^e) \\ &\cong \mathrm{Hom}_{B^e}(F^e(P), F^e(G \boxtimes G^\vee)) \\ &\cong \mathrm{Hom}(P, G \boxtimes G^\vee) \\ &\cong \mathrm{Hom}((G \boxtimes G^\vee)^\vee, P^\vee) \\ &\cong \mathrm{Hom}(G^\vee \boxtimes G, P^\vee). \end{aligned}$$

We see that $F^e(P^\vee)$ and $F^e(P)^\vee$ are isomorphic as complexes of vector spaces. Since $B \otimes B^{op}$ acts on the last object through the flip map $B \otimes B^{op} \xrightarrow{\sim} B^{op} \otimes B$, the B^e-module structures coincide. Let ω_Y denote the canonical sheaf of Y. To conclude that $\Sigma^{-d}\Delta_*(\omega_Y^{-1})$ corresponds to the inverse dualizing complex $B^\vee$ through the equivalence F^e, it now suffices to show that we have an isomorphism

$$(\Delta_*\mathcal{O}_Y)^\vee \cong \Sigma^{-d}\omega_Y^{-1}.$$

Now since Y is smooth, we can apply Grothendieck duality and find

$$(\Delta_*\mathcal{O}_Y)^\vee = \mathscr{H}om(\Delta_*\mathcal{O}_Y, \mathcal{O}_{Y \times Y}) \cong \Delta_*\mathscr{H}om(\mathcal{O}_Y, (\Delta^*\mathcal{O}_{Y \times Y}) \otimes \Sigma^{-d}\omega_\Delta),$$

where $d = \dim Y$ equals the relative dimension of Δ and

$$\omega_\Delta = \omega_Y \otimes \Delta^*\omega_{Y \times Y}^{-1}$$

is the relative dualizing bundle of Δ. Now $\mathscr{H}om(\mathcal{O}_Y, ?)$ is the identity functor and $\Delta^*\mathcal{O}_{Y \times Y}$ is $\mathcal{O}_Y$. Thus, we have

$$\Delta_*\mathscr{H}om(\mathcal{O}_Y, (\Delta^*\mathcal{O}_{Y \times Y}) \otimes \Sigma^{-d}\omega_\Delta) \cong \Delta_*(\omega_Y \otimes \Delta^*\Sigma^{-d}\omega_{Y \times Y}^{-1})$$

Finally, the adjunction formula for $\Delta : Y \to Y \times Y$ implies that ω_Y is isomorphic to the tensor product of $\Delta^*\omega_{Y \times Y}$ with the top exterior power of the normal bundle, which identifies with the canonical bundle. We deduce that $\Delta^*\omega_{Y \times Y}$ is isomorphic to $\omega_Y^{\otimes 2}$. This implies the result.

b) Let G be the functor $\mathbf{R}\mathrm{Hom}(L, ?) : \mathrm{D}(\mathrm{Qcoh}Y) \to \mathrm{D}(A)$. Since L is in particular perfect, it has a fully faithful left adjoint (given by the derived functor of $? \otimes_A L$). Therefore, it is a localization functor. Since $\mathrm{D}(\mathrm{Qcoh}\ Y)$ is equivalent to $\mathrm{D}(B)$ and B is bimodule d-Calabi–Yau, it follows from Proposition 3.10 d) of [44] that A is bimodule d-Calabi–Yau. Since L is a bounded complex of coherent sheaves with compact support, the derived endomorphism algebra A is proper. Now by Lemma 3.6, for a proper dg algebra, the dg bimodules Θ_A and DA are inverse to each other, i.e.

their derived tensor products in both directions are isomorphic to the identity bimodule A in $\mathrm{D}(A^e)$. Since Θ_A is isomorphic to $\Sigma^{-d}A$, it follows that DA is isomorphic of $\Sigma^d A$. ☐

From the relative Calabi–Yau property for the dg algebra B constructed in part a) of the Proposition, one can deduce a relative Calabi–Yau property for sheaves on Y. For completeness, we include a direct geometric proof via Grothendieck duality. Indeed the we prove a slightly more general result for Gorenstein CY d-folds.

Lemma 3.11. *Let Y be a quasi-projective Gorenstein CY d-fold. For $L \in \mathrm{D}^b_c(\mathrm{coh}\ Y)$ a bounded complex of coherent sheaves with compact support and $M \in \mathrm{D}(\mathrm{Qcoh}\ Y)$, there is a bifunctorial isomorphism*

$$D\mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ Y)}(L, M) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ Y)}(M, \Sigma^d L).$$

Proof. Since we work over characteristic zero, there exists a smooth projective variety $\overline{Y}$ compactifying Y. Denote the natural embedding by i. Because $\overline{Y}$ is Noetherian, we may identify $\mathrm{D}(\mathrm{coh}\ \overline{Y})$ with the subcategory $\mathrm{D}_{\mathrm{coh}}\ (\mathrm{Qcoh}\ \overline{Y}) \subset \mathrm{D}(\mathrm{Qcoh}\ \overline{Y})$ consisting of complexes with coherent cohomologies (adding isomorphic objects). There exists an extension $\overline{M} \in \mathrm{D}(\mathrm{Qcoh}\ \overline{Y})$ of M, i.e. $i^*\overline{M} = M$. We may set $\overline{M} = i_* M$. Note that $i^* = i^!$ because i is an open immersion and therefore is etale. Because the cohomology of L has compact support, $i_* L \in \mathrm{D}^b_{\mathrm{coh}}\ (\mathrm{Qcoh}\ \overline{Y})$ which is in particular perfect.

We recall the Grothendieck duality for unbounded derived category due to Neeman [54]. Let $f : X \to Y$ be a morphism of Noetherian separated schemes. The derived functor $f_* : \mathrm{D}(\mathrm{Qcoh}\ X) \to \mathrm{D}(\mathrm{Qcoh}\ Y)$ admits a left adjoint f^* and a right adjoint $f^!$. Consider the special case $f : \overline{Y} \to \mathrm{Spec}\ k$. Because $\overline{Y}$ is projective, conditions (a) and (b) of Fact 0.2 (ii) of Neeman ([54]) are satisfied (We can verify Conjecture 4.16 of Neeman, see Fact 0.8 (i) and (ii)). Then it follows from Fact 0.2 (ii) that $f^! k = \Sigma^d \omega_{\overline{Y}}$. Let $\overline{L}$ be a perfect complex on $\overline{Y}$ and $\overline{N}$ be an arbitrary object in $\mathrm{D}(\mathrm{Qcoh}\ \overline{Y})$. The Grothendieck duality implies that

$$\mathbf{R}\mathrm{Hom}_{\mathrm{D}(k)}(f_*(\overline{L}^\vee \otimes \overline{N}), k) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ \overline{Y})}(\overline{L}^\vee \otimes \overline{N}, f^! k).$$

The left hand side is equal to $D\mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ \overline{Y})}(\overline{L}, \overline{N})$ and the right hand side is equal to

$$\mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ \overline{Y})}(\overline{N}, \overline{L} \otimes \Sigma^d \omega_{\overline{Y}})$$

by Neeman's theorem (Fact 0.2 (ii) [54]). As a consequence, we get a bifunctorial isomorphism

$$D\mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ \overline{Y})}(\overline{L}, \overline{N}) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ \overline{Y})}(\overline{N}, \Sigma^d(\overline{L} \otimes \omega_{\overline{Y}})).$$

Now set $\overline{L} = i_* L$ and $\overline{N} = \overline{M}$. By adjunction,

$$\mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ \overline{Y})}(i_* L, \overline{M}) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ Y)}(L, i^! \overline{M}) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ Y)}(L, M),$$

and

$$\mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ \overline{Y})}(\overline{N}, \Sigma^d(i_* L \otimes \omega_{\overline{Y}})) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ Y)}(i^* \overline{N}, \Sigma^d(L \otimes i^* \omega_{\overline{Y}})) \simeq \mathbf{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Qcoh}\ Y)}(M, \Sigma^d L).$$

The last isomorphism follows from that fact that $i^* \omega_{\overline{Y}} = \omega_Y \cong \mathcal{O}_Y$ because i is an open immersion. The lemma is proved. ☐

Lemma 3.12. *Let $\{\mathcal{E}_i\}_{i=1}^t$ be a semi-simple collaction of sheaves in a smooth quasi-projective CY d-fold Y. Write $\Gamma := \Gamma^Y_{\mathcal{E}}$ for the derived deformation algebra of $\mathcal{E} := \bigoplus_{i=1}^t \mathcal{E}_i$. Then Γ is topologically homologically smooth and bimodule d-Calabi–Yau. If Y is only Gorenstein CY, then Γ satisfies the relative dCY property.*

Proof. Let A be the derived endomorphism algebra of $\mathcal{E}$. Since Y is smooth and $\mathcal{E}$ has compact support, it is proper. Moreover, it can be chosen augmented. We know that Γ is quasi-isomorphic to $\mathbb{D}BA$. Since Y is smooth and d-Calabi–Yau, A is perfectly d-Calabi–Yau by part b) of Proposition 3.10. Hence Γ is topologically smooth and bimodule d-Calabi–Yau by Lemma 3.8. The last statement follows from Lemma 3.11. $\square$

Remark 3.13. If we assume that Y is smooth and projective, then one can show that $\mathrm{Ext}^*_Y(\mathcal{E},\mathcal{E})$ is a cyclic A_∞-algebra. This can be proved by reducing to the analytic case and applying the holomorphic Chern-Simons theory (see Example 10.2.7 [50]). An algebraic proof can be found in Polishchuk's upcoming paper [63].

Proposition 3.14. *Let $\{\mathcal{E}_i\}_{i=1}^t$ be a semi-simple collection of sheaf in a smooth projective CY d-fold Y. Write $\Gamma := \Gamma^Y_{\mathcal{E}}$ for the derived deformation algebra of $\mathcal{E} := \bigoplus_{i=1}^t \mathcal{E}_i$. Then Γ is an exact 3CY-algebra.*

Proof. Homological smoothness of Γ follows from Lemma 3.8 because $\Gamma = \mathbb{D}BA$, where A is the derived endomorphism algebra of $\mathcal{E}$. The exactness follows from Theorem 12.1 of [70] and the remark above. $\square$

Theorem 3.15. *(Iyama-Reiten) Let R be an equi-codimensional Gorenstein normal domain of dimension d over an algebraically closed field k, and let A be an NCCR. Then A satisfies the relative dCY property. Moreover, if R is complete local then A is bimodule dCY.*

Proof. The fact that A satisfies the relative dCY property is proved in Theorem 4.23 of [72]. We have $A = \mathrm{End}_R(R \oplus N)$ for the Cohen-Macaulay module $N = \bigoplus_{i=1}^t N_i$ with $N_1, \ldots, N_t$ indecomposable. Denote by $\bar{l}$ the algebra k^{t+1}. By Theorem 1.1 of [67], the algebra A is quasi-isomorphic to a pseudo-compact dg algebra $\widetilde{\Gamma} := (\widehat{T_{\bar{l}}}(V), d)$ in $\mathrm{PCAlgc}(l)$ for a finite-dimensional graded $\bar{l}$-bimodule V concentrated in degrees ≤ 0 and a differential d taking V into the square of the kernel of the augmentation $\widehat{T_{\bar{l}}}(V) \to \bar{l}$. Since A is of finite global dimension, $\widetilde{\Gamma}$ is homologically smooth and by the first part, we have bifunctorial isomorphisms

$$\mathrm{Hom}_{\mathrm{D}(\widetilde{\Gamma})}(M, P) = D\mathrm{Hom}_{\mathrm{D}(\widetilde{\Gamma})}(P, \Sigma^d M)$$

for M in $\mathrm{D}_{fd}(\widetilde{\Gamma})$ and P in $\mathrm{per}(\widetilde{\Gamma})$. Let

$$\Theta = \mathbf{R}\mathrm{Hom}_{\widetilde{\Gamma}^e}(\widetilde{\Gamma}, \widetilde{\Gamma}^e)$$

be the inverse dualizing complex of $\widetilde{\Gamma}$. By Lemma 4.1 of [43], we have bifunctorial isomorphisms

$$\mathrm{Hom}_{\mathrm{D}(\widetilde{\Gamma})}(L \overset{\mathbb{L}}{\otimes}_{\widetilde{\Gamma}} \Theta, M) = D\mathrm{Hom}_{\mathrm{D}(\widetilde{\Gamma})}(M, L)$$

for M in $\mathrm{D}_{fd}(\widetilde{\Gamma})$ and an arbitrary object L of $\mathrm{D}(\widetilde{\Gamma})$. By combining these with the previous isomorphisms we find

$$\mathrm{Hom}_{\mathrm{D}(\widetilde{\Gamma})}(P \overset{\mathbb{L}}{\otimes}_{\widetilde{\Gamma}} \Theta, M) = \mathrm{Hom}_{D(\widetilde{\Gamma})}(\Sigma^{-d} P, M)$$

for P in $\mathrm{per}(\widetilde{\Gamma})$ and M in $\mathrm{D}_{fd}(\widetilde{\Gamma})$. Since an object L of $\mathrm{D}(\widetilde{\Gamma})$ is perfect if and only if $\mathrm{Hom}_{\mathrm{D}(\widetilde{\Gamma})}(L, M)$ is finite-dimensional for each M in $\mathrm{D}_{fd}(\widetilde{\Gamma})$, we see that $P \overset{\mathbb{L}}{\otimes}_{\widetilde{\Gamma}} \Theta$ is perfect for each perfect P. Now by taking $P = \widetilde{\Gamma}$ and $M = \Sigma^n S_i$, where $n \in \mathbb{Z}$ and S_i is one of the $t+1$ simple $\widetilde{\Gamma}$-modules, we see that as a right $\widetilde{\Gamma}$-module, Θ is isomorphic to $\Sigma^{-d}\widetilde{\Gamma}$. Since $\widetilde{\Gamma}$ has its homology concentrated in degree 0, there is an automorphism σ of $\widetilde{\Gamma}$ such that $\Sigma^d \Theta$ is isomorphic to ${}_\sigma\widetilde{\Gamma}$ as a bimodule. Since each object L of $\mathrm{D}_{fd}(\widetilde{\Gamma})$ is perfect, we have in particular

$$\mathrm{Hom}_{\mathrm{D}(\widetilde{\Gamma})}(L_\sigma, M) = \mathrm{Hom}_{\mathrm{D}(\widetilde{\Gamma})}(L, M)$$

for all L and M in $\mathrm{D}_{fd}(\widetilde{\Gamma})$, which shows that there is a functorial isomorphism $L \xrightarrow{\sim} L_\sigma$ for each L in $\mathrm{D}_{fd}(\widetilde{\Gamma})$. In particular, for L, we can take the finite-dimensional quotients of $\widetilde{\Gamma}$. We deduce that in each finite-dimensional quotient of $\widetilde{\Gamma}$, the automorphism σ induces an inner automorphism. Thus, σ itself is inner and ${}_\sigma\widetilde{\Gamma}$ is isomorphic to $\widetilde{\Gamma}$ as a bimodule. This shows that Θ is isomorphic to $\Sigma^{-d}\widetilde{\Gamma}$ as a bimodule. □

Corollary 3.16. *Let R be a local complete equi-codimensional Gorenstein normal domain of dimension d over an algebraically closed field k of characteristic zero, and let A be an NCCR. Let Γ be the dg algebra constructed in Theorem 2.8. Then Γ is topologically homologically smooth and bimodule dCY.*

3.2. **Classification of CY structures for 3-dimensional flopping contractions.** If $(\widehat{Y}, \widehat{f}, R)$ is a 3-dimensional formal flopping contraction, then R is a hypersurface. The natural isomorphism $\widehat{f}_*\omega_{\widehat{Y}} \cong \omega_R$ identifies a CY structure η on $\widehat{Y}$ with a nonzero section $\widehat{f}(\eta)$ of ω_R. By the Gorenstein property, $\widehat{f}(\eta)$ defines an isomorphism $R \cong \omega_R$.

Theorem 3.17. *Let $(\widehat{Y}, \widehat{f}, R)$ be a 3-dimensional formal flopping contraction. The space of 3CY structures can be identified with $R^\times$. Moreover, every Calabi-Yau structure on $\widehat{Y}$ is exact. The space of all exact liftings of a 3CY structure can be identified with the cohomology group $H^1(\widehat{Y}, \Omega^1_{\widehat{Y}})$.*

Proof. Assume that $C := \mathrm{Ex}(f)$ has t irreducible components $C_1, \ldots, C_t$. Because R has rational singularities, $H^0(\widehat{Y}, \Omega^3_{\widehat{Y}}) \cong H^0(\widehat{Y}, \mathcal{O}_{\widehat{Y}}) \cong R$. Then the first claim follows. The Hodge-to-de Rham spectral sequence with E_1 term

$$E_1^{pq} = H^q(\widehat{Y}, \Omega^p_{\widehat{Y}}),$$

converges to $H^{p+q}_{DR}(\widehat{Y}, \mathbb{C})$. We claim that $H^1(\Omega^1_{\widehat{Y}}) \cong \mathbb{C}^t$. Because the first Chern classes of $\widehat{\mathcal{L}}_i$ for $i = 1, \ldots, t$ are linearly independent, $\dim_{\mathbb{C}} H^1(\Omega^1_{\widehat{Y}}) \geq t$. We write $\widehat{X} := \operatorname{Spec} R$. By the Leray spectral sequence

$$H^p(\widehat{X}, \mathbf{R}^q\widehat{f}_*\Omega^1) \Rightarrow H^{p+q}(\widehat{Y}, \Omega^1),$$

we have an exact sequence

$$0 \to H^0(\widehat{X}, \mathbf{R}^1 f_*\Omega^1) \to H^1(\widehat{Y}, \Omega^1) \to H^1(\widehat{X}, f_*\Omega^1) \to 0$$

The right most term vanishes since $\widehat{X}$ is affine. Since $\widehat{Y}$ is a small resolution of 3-dimensional Gorenstein singularities, the normal bundle of C_i is $\mathcal{O}(a) \oplus \mathcal{O}(b)$ with $(a, b) = \{(-1,-1), (0,-2), (1,-3)\}$ ([59, Theorem 4]). By the exact sequence

$$0 \to \mathcal{O}(-a) \oplus \mathcal{O}(-b) \to \Omega^1|_{C_i} \to \mathcal{O}(-2) \to 0$$

we have $H^1(C_i, \Omega^1|_{C_i}) \cong H^1(C_i, \mathcal{O}(-2)) = \mathbb{C}$. By the normal crossing condition, there is a short exact sequence of sheaves

$$0 \longrightarrow \Omega^1|_C \to \Omega^1|_{C_1} \oplus \Omega^1|_{\cup_{i=2}^t C_i} \longrightarrow \Omega^1|p \longrightarrow 0$$

where $p = C_1 \cap (\cup_{i=2}^t C_i)$. Then we get a surjection

$$H^1(C, \Omega^1|_C) \to H^1(C_1, \Omega^1|_{C_1}) \oplus H^1(\cup_{i=2}^t C_i, \Omega^1_{\cup_{i=2}^t C_i}).$$

By induction, $\dim_{\mathbb{C}} H^1(C, \Omega^1|_C) \leq t$. Then the conclusion follows from the formal function theorem and the Lerray spectral sequence.

The term E_2^{30} of the Hodge-to-de Rham spectral sequence is the quotient $H^0(\widehat{Y}, \Omega^3)/\mathrm{d}H^0(\widehat{Y}, \Omega^2)$. Recall that $H^1(\widehat{Y}, \Omega^1)$ admits a $\mathbb{C}$-basis by c_1 of the line bundles $\widehat{\mathcal{L}}_i$ with $i = 1, \ldots, t$. Because $(1,1)$ classes are d-closed, the differential $H^1(\widehat{Y}, \Omega^1) \to H^0(\widehat{Y}, \Omega^3)/\mathrm{d}H^0(\widehat{Y}, \Omega^2)$ is zero. Moreover since

$H^2(\mathcal{O}_{\widehat{Y}}) = 0$, we have $E_r^{30} = E_2^{30}$ for $r \geq 2$. Because $H^3_{DR}(\widehat{Y}, \mathbb{C}) \cong H^3_{DR}(C, \mathbb{C}) = 0$, E_2^{30} must vanish. Therefore, all 3-forms are exact.

Denote by $\Omega^{\leq i}_{\widehat{Y}}$ the stupid truncation $\sigma_{\leq i}\Omega^*_{\widehat{Y}}$ of the de Rham complex. There is a long exact sequence of hypercohomology

$$\dots \xrightarrow{B} H^{i-n}(\widehat{Y}, \Omega^i_{\widehat{Y}}) \xrightarrow{I} \mathbb{H}^{2i-n}(\widehat{Y}, \Omega^{\leq i}_{\widehat{Y}}) \xrightarrow{S} \mathbb{H}^{2i-n}(\widehat{Y}, \Omega^{\leq i-1}_{\widehat{Y}}) \xrightarrow{B} H^{i-n+1}(\widehat{Y}, \Omega^i_{\widehat{Y}}) \longrightarrow \dots$$

Take $i = 3$ and $n = 4$. Let leftmost term vanishes and $\mathbb{H}^2(\widehat{Y}, \Omega^{\leq 3}_{\widehat{Y}}) = H^2_{DR}(\widehat{Y}) \cong H^1(\widehat{Y}, \Omega^1_{\widehat{Y}})$. So the last claim is proved. $\square$

Corollary 3.18. *Let R be a complete local equi-codimensional Gorenstein normal domain of dimension d over an algebraically closed field of characteristic zero, and let A be an NCCR. Then every dCY structure on A is exact.*

Proof. This is an immediate consequence of Theorem 3.15, Theorem 2.11 and Corollary 9.3 of [70]. $\square$

The following proposition follows immediately from the Hochschild-Kostant-Rosenberg theorem.

Proposition 3.19. *Let $(\widehat{Y}, \widehat{f}, R)$ be a 3-dimensional formal flopping contraction. Let $A = \mathrm{End}_R(R \oplus N)$ be the corresponding NCCR. Then there is a bijective correspondence between the space of 3CY structures (resp. exact 3CY structures) on $\widehat{Y}$ and that of A.*

3.3. **Cluster category.** Let Γ be a dg k-algebra. Suppose that Γ has the following properties:

(1) Γ is homologically smooth, i.e. Γ is a perfect Γ^e-module;
(2) for each $p > 0$, the space $H^p\Gamma$ vanishes;
(3) $H^0\Gamma$ is finite dimensional;
(4) Γ satisfies the relative 3CY property.

By property (1), $\mathrm{D}_{fd}(\Gamma)$ is a subcategory of the perfect derived category $\mathrm{per}(\Gamma)$. The *generalized cluster category* $\mathcal{C}_\Gamma$ is defined to be the triangle quotient $\mathrm{per}(\Gamma)/\mathrm{D}_{fd}(\Gamma)$. We denote by π the canonical projection functor $\pi : \mathrm{per}(\Gamma) \to \mathcal{C}_\Gamma$. For simplicity, we will omit the adjective 'generalized' and call $\mathcal{C}_\Gamma$ the cluster category associated to Γ. An object $T \in \mathcal{C}_\Gamma$ is called a *cluster-tilting object* if

(1) $\mathrm{Ext}^1_{\mathcal{C}_\Gamma}(T, T) = 0$;
(2) For any object X such that $\mathrm{Ext}^1_{\mathcal{C}_\Gamma}(T, X) = 0$, one has $X \in \mathrm{add}(T)$.

Amiot has proved that $\pi(\Gamma)$ is a cluster-tilting object in $\mathcal{C}_\Gamma$ (Theorem 2.1 [3]). We call $H^0\Gamma$ the *CY tilted algebra* associated to the cluster category $\mathcal{C}_\Gamma$, cf. [56].

Remark 3.20. If Γ is a pseudo-compact dg l-algebra in $\mathrm{PCAlgc}(l)$, we may define a continuous version of cluster category. Condition (1) is replaced by

(1′) Γ is topologically homologically smooth,

and the *topological* cluster category $\mathcal{C}_\Gamma$ is defined to be the triangle quotient $\mathrm{per}(\Gamma)/\mathrm{D}_{fd}(\Gamma)$ where $\mathrm{per}(\Gamma)$ and $\mathrm{D}_{fd}(\Gamma)$ are considered as subcategories of the pseudo-compact derived category. We refer to the Appendix of [48] for the details.

Remark 3.21. If we drop the assumption that $H^0\Gamma$ is finite dimensional, the quotient category $\mathcal{C}_\Gamma = \mathrm{per}(\Gamma)/\mathrm{D}_{fd}(\Gamma)$ is no longer Hom-finite. The Calabi–Yau property only holds when one restricts to suitable subcategories, cf. Proposition 2.16 of [61].

Theorem 3.22. [3, Theorem 2.1][2] *Let Γ be a dg k-algebra with the above properties. Then the cluster category $\mathcal{C}_\Gamma$ is Hom-finite and 2CY as a triangulated category. Moreover, the object $\pi(\Gamma)$ is a cluster tilting object. Its endomorphism algebra is isomorphic to $H^0\Gamma$.*

Definition 3.23. Let $\{C_i\}_{i=1}^t$ be a collection of smooth rational curves in a smooth quasi-projective CY 3-fold Y with fixed CY-structure $\eta : \mathcal{O}_Y \xrightarrow{\sim} \omega_Y$, such that $\{\mathcal{O}_{C_i}\}$ form a semi-simple collection. Denote by $\mathcal{C}(Y, \{C_i\}_{i=1}^t, \eta)$ for the (topological) cluster category associated to the derived deformation algebra of $\bigoplus_{i=1}^t \mathcal{O}_{C_i}$. We call $\mathcal{C}(Y, \{C_i\}_{i=1}^t, \eta)$ *the cluster category associated to the triple* $(Y, \eta, \{C_i\}_{i=1}^t)$.

Definition 3.24. Let R be a complete local equicodimensional Gorenstein normal domain of dimension 3 over an algebraically closed field k of characteristic zero, and let A be the NCCR associate to the collection of indecomposables $R, N_1, \ldots, N_t$. Fix a 3CY structure $\eta \in \mathrm{HH}_3(A, A)$. Denote by $\mathcal{C}(R, \{N_i\}_{i=1}^t, \eta)$ the cluster category associated to the dg algebra Γ constructed in Theorem 2.8, and call it *the cluster category associated to the triple* $(R, \{N_i\}_{i=1}^t, \eta)$.

A priori, the dg algebra Γ constructed in Theorem 2.8 is pseudo-compact. However, if $\mathrm{D}(\Gamma)$ denotes the ordinary derived category and $\mathrm{D}_{pc}(\Gamma)$ the pseudo-compact derived category, then the natural functor $\mathrm{D}_{pc}(\Gamma) \to \mathrm{D}(\Gamma)$ induces equivalences in the perfect derived categories and in the subcategories of objects with finite-dimensional total homology. Therefore, the two candidates for the cluster category are equivalent.

The following result is an immediate consequence of Corollary 2.12.

Corollary 3.25. *Let $(\widehat{Y}, \widehat{f}, R)$ be a 3-dimensional formal flopping contraction, and let A be the NCCR associated to the collection of indecomposables $R, N_1, \ldots, N_t$ constructed in Section 2.5. Fix a 3CY structure η on $\widehat{Y}$ and denote its counter part on A by the same symbol. Then there is a triangle equivalence bewteen the cluster categories*

$$\mathcal{C}(Y, \{C_i\}_{i=1}^t, \eta) \simeq \mathcal{C}(R, \{N_i\}_{i=1}^t, \eta).$$

4. Ginzburg algebras

In this section we introduce the notion of Ginzburg (dg) algebra and prove several properties of it. The cluster category can be defined via the Ginzburg algebra, which provides an effective tool to do computations.

4.1. Definitions.

Fix a commutative ring k. Let Q be a finite quiver, possibly with loops and 2-cycles. Denote by Q_0 and Q_1 the set of nodes and arrows of Q respectively. Denote by kQ the path algebra and by $\widehat{kQ}$ the complete path algebra with respect to the two-sided ideal generated by arrows. For each $a \in Q_1$, we define the cyclic derivative D_a with respect to a as the unique linear map

$$D_a : kQ/[kQ, kQ] \to kQ$$

which takes the class of a path p to the sum $\sum_{p=uav} vu$ taken over all decompositions of the path p. The definition can be extend to $\widehat{kQ}_{\mathrm{cyc}} := \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]^c$ where the superscript c stands for the completion with respect to the adic topology defined above. An element w in $\widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]^c$ is called a *potential* on Q. It is given by a (possibly infinite) linear combination of cycles in Q.

Definition 4.1. (Ginzburg) Let Q be a finite quiver and w a potential on Q. Let $\overline{Q}$ be the graded quiver with the same vertices as Q and whose arrows are

- the arrows of Q (of degree 0);

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[2] In the original statement of [3], the author assumed that Γ is bimodule 3CY. However, the proof is still valid under the weaker assumption that Γ satisfies the relative 3CY property.

- an arrow $a^*: j \to i$ of degree -1 for each arrow $a: i \to j$ of Q;
- a loop $t_i: i \to i$ of degree -2 for each vertex i of Q.

The (complete) Ginzburg (dg)-algebra $\mathfrak{D}(Q,w)$ is the dg k-algebra whose underlying graded algebra is the completion (in the category of graded vector spaces) of the graded path algebra $k\overline{Q}$ with respect to the two-sided ideal generated by the arrows of $\overline{Q}$. Its differential is the unique linear endomorphism homogeneous of degree 1 satisfying the Leibniz rule, and which takes the following values on the arrows of $\overline{Q}$:

- $da = 0$ for $a \in Q_1$;
- $d(a^*) = D_a w$ for $a \in Q_1$;
- $d(t_i) = e_i(\sum_{a\in Q_1}[a,a^*])e_i$ for $i \in Q_0$ where e_i is the idempotent associated to i.

Denote by l the product $\prod_{i\in Q_0} ke_i$. Then $\widehat{kQ}$ is isomorphic to the complete tensor algebra $\widehat{T}_l V$ with V being the vector space spanned by arrows of Q.

Remark 4.2. In most references, the above definition corresponds to the *complete Ginzburg algebra* while the algebra without taking the graded completion is called *Ginzburg algebra*. The complete Ginzburg algebra $\mathfrak{D}(Q,w)$ is considered as an object of PCAlgc(l). Because we only consider complete Ginzburg algebra in this paper, we will call it the Ginzburg algebra for simplicity.

Definition 4.3. Let Q be a finite quiver and w a potential on Q. The *Jacobi algebra* $\Lambda(Q,w)$ is the zeroth homology of $\mathfrak{D}(Q,w)$, which is the quotient algebra

$$\widehat{kQ}/((D_a w, a \in Q_1))^c$$

where $((D_a w, a\in Q_1))^c$ is the closed two-sided ideal generated by $D_a w$. A Ginzburg algebra $\mathfrak{D}(Q,w)$ is called *Jacobi-finite* if $\dim_k \Lambda(Q,w) < \infty$.

Van den Bergh showed the following result.

Theorem 4.4. *(Van den Bergh)*[44, Appendix] *Let Q be finite quiver and w be a potential. Then $\mathfrak{D}(Q,w)$ is topologically homologically smooth and bimodule 3CY.*

The above theorem was first proved by Van den Bergh in the algebraic setting in [44]. But the same proof can be adapted to the pseudo-compact case (cf. [70]). Given a Jacobi-finite Ginzburg algebra $\Gamma := \mathfrak{D}(Q,w)$, there is an associate cluster category $\mathcal{C}_\Gamma := \mathrm{per}(\Gamma)/\mathrm{D}_{fd}(\Gamma)$.

Remark 4.5. There exists a canonical exact CY structure on $\Gamma = \mathfrak{D}(Q,w)$. We follow the notation of [70] to write $M_l := M/[l,M]$ for a l-bimodule M. Because the reduced cyclic homology of Γ is equal to the homology of $(\Gamma/l + [\Gamma,\Gamma])_l$, a class of $\mathrm{HC}_2(\Gamma,\Gamma)$ is represented by a degree -2 element χ of Γ such that $d\chi \in l + [\Gamma,\Gamma]$. By the definition of d of Γ, $\chi := \sum_{i\in Q_0} t_i$ represents a class in $\mathrm{HC}_2(\Gamma,\Gamma)$. Because Γ is cofibrant, by Proposition 7.2.1 of [68] the Hochschild chain complex of Γ is quasi-isomorphic to the mapping cone of

$$\Omega^1_l\Gamma/[\Gamma,\Omega^1_l\Gamma] \xrightarrow{\partial_1} \Gamma/[l,\Gamma]$$

with differential defined by $\partial_1(aDb) = [a,b]$, where $Db = 1\otimes b - b\otimes 1$. In other words, a class in $\mathrm{HH}_3(\Gamma,\Gamma)$ is represented by a pair of elements (ω,a) of degree $(-2,-3)$ satisfying $\partial_1(\omega) = da$ and $d\omega = 0$. Because d and D commute, $(D\chi,0)$ represents a class in $\mathrm{HH}_3(\Gamma,\Gamma)$, which is the image of χ under the Connes map. Because $d\chi = \sum_a [a,a^*]$, $D\chi$ is nondegenerate. Clearly, given the class $[\chi] \in \mathrm{HC}_3(\Gamma,\Gamma)$ and a coordinate $\{a,a^*\}_{a\in Q_1}$ the potential w is defined as an element of $\Gamma^0/l + [\Gamma^0,\Gamma^0]$.

For a Ginzburg algebra $\Gamma = \mathfrak{D}(Q,w)$, denote $\Lambda(Q,w)$ by Λ for short. The image of w under the canonical map $\mathrm{HH}_0(\widehat{kQ},\widehat{kQ}) = \widehat{kQ}_{\mathrm{cyc}} \to \mathrm{HH}_0(\Lambda,\Lambda) = \Lambda_{\mathrm{cyc}}$, denoted by $[w]$, is a canonical class associated to the Ginzburg algebra $\mathfrak{D}(Q,w)$. Therefore, we see that starting from a Ginzburg algebra $\Gamma = \mathfrak{D}(Q,w)$ we get not only a triangulated category $\mathcal{C}_\Gamma$ but an additional piece of information that is a canonical class $[w]$ in the 0-th Hochschild homology of the CY tilted algebra. We will discuss the invariance property of this class in the next section.

4.2. **Existence and uniqueness of potential.** The definition of Ginzburg algebra is not homotopically invariant. It is important to know when a bimodule 3CY dg algebra admits a model given by a Ginzburg algebra. The following theorem is due to Van den Bergh.

Theorem 4.6. [70, Theorem 10.2.2] *Let k be a field and l be a finite dimensional commutative separable k-algebra. Assume that Γ is a pseudo-compact dg l-algebra in* $\mathrm{PCAlgc}(l)$ *concentrated in nonpositive degrees. Then the following are equivalent*

1. Γ *is exact 3CY.*
2. Γ *is weakly equivalent to a Ginzburg algebra $\mathfrak{D}(Q,w)$ for some finite quiver Q with w contains only cubic terms and higher.*

For a Ginzburg algebra $\Gamma := \mathfrak{D}(Q,w)$, there is a canonical class ξ in $\mathrm{HC}_2(\Gamma,\Gamma)$ represented by $\sum_i t_i$. Its image under the Connes map $B\xi$ is the class in $\mathrm{HH}_3(\Gamma,\Gamma)$ representing the 3CY structure. On the other hand, the following result of Van den Bergh provides a lot of examples of dg algebras whose 3CY structures can be lifted to exact ones.

Theorem 4.7. [70, Corollary 9.3] *Assume that k has characteristic zero and let Γ be a pseudo-compact dg algebra in* $\mathrm{PCAlgc}(l)$ *concentrated in degree zero. Then Γ is bimodule dCY if and only if it is exact dCY.*

By putting the above two theorems together, we see that if Γ is a pseudo-compact dg l-algebra in $\mathrm{PCAlgc}(l)$ concentrated in degree zero that is bimodule 3CY then it is quasi-isomorphic to a Ginzburg algebra $\mathfrak{D}(Q,w)$ for some finite quiver Q and potential w.

Now we assume that a bimodule 3CY dg algebra Γ in $\mathrm{PCAlgc}(l)$ is exact. So it admits a model given by $\mathfrak{D}(Q,w)$. Note that being bimodule CY and exact CY are homotopically invariant properties. The next proposition shows that the (right equivalent class of) the canonical class $[w]$ in $\mathrm{HH}_0(\Lambda(Q,w))$ for such a dg algebra is indeed a homotopy invariant. The proof is essentially contained already in Van den Bergh's proof of Theorem 4.6 (cf. proof of Theorem 11.2.1 of [70]). However, we recall it for completeness. See Remark 4.9 for a conceptional explanation of Van den Bergh's result.

Proposition 4.8. *(Van den Bergh) Let k be a field of characteristic zero and $l = ke_1 \times \ldots \times ke_n$. Let Γ be a pseudo-compact 3CY dg l-algebra in* $\mathrm{PCAlgc}(l)$ *with a fixed 3CY structure $\eta \in \mathrm{HH}_3(\Gamma,\Gamma)$ that is exact. Then there exists a canonical class $[w]$, defined up to right equivalence, in $H^0\Gamma/[H^0\Gamma, H^0\Gamma]^c$.*

Proof. First recall that for an algebra A and two classes c and c' in $A/[A,A]$, we say c is *right equivalent* to c' if there exists an algebra automorphism γ such that $\gamma_* c = c'$. The definition extends to the pseudo-compact case without difficulty.

There is an isomorphism $l \cong \Gamma/\mathrm{rad}(\Gamma)$. Since η is exact, we may choose $\xi \in \mathrm{HC}_2(\Gamma)$ such that $\eta = B\xi$. By the CY property, the algebra $\mathrm{Ext}^*_\Gamma(l,l)$ is finite dimensional with a symmetric invariant form of degree 3 which depends only on η (but not on the lifting ξ!). Denote by V the graded space $\bigoplus_{p=1}^3 \left(\Sigma^{1-p} D\mathrm{Ext}^p_\Gamma(l,l)\right)$. It is convenient to write

$$V = V_c + lz$$

with V_c concentrating in degree 0 and -1 and z being a l-central element of degree -2. By Koszul duality (Proposition A.5.4 of [70]), there exists a differential d such that $(\widehat{T_l} V, d)$ is weakly equivalent to Γ. By Corollary A.5.6 of [70], V is unique up to isomorphisms. The differential d on $\widehat{T}_l V$ is determined by the A_∞-structure on $\mathrm{Ext}^*_\Gamma(l,l)$, while equivalent A_∞-structures are related by a formal change of variables on $\mathrm{Ext}^*_\Gamma(l,l)$.

Now assume that $\Gamma = \widehat{T}_l V$. We claim that there exists a coordinate such that $dz \in V_c \otimes_l V_c$ and it is non-degenerate and anti-symmetric. The class $\xi \in \mathrm{HC}_2$ corresponds to an element $\overline{\chi}$ of degree -2 in $\Gamma/l + [\Gamma, \Gamma]$ such that its lift $\chi \in \Gamma$ satisfies that

$$d\chi = \sum_i [x_i, y_i] \ \mathrm{mod} \ \ [l, -] \tag{4.1}$$

for $x_i, y_i \in \Gamma$. By the nondegeneracy condition on η, $\chi = uz + v$ for some invertible element $u \in l$ and some degree -2 element v. Set $z' = \chi$. Then

$$dz' = \eta'_2 + \eta'_3 + \dots$$

such that $\eta'_n \in V_c^{\otimes_l n}$. Because $\widehat{T}(V_c + lz) = \widehat{T}(V_c + lz')$, η'_2 is again non-degenerate. On the other hand, η'_2 is skew symmetric by (4.1).

Now we may assume that $V = V_c + lz$ such that dz is a sum of commutators in $\widehat{T}_l V$

$$dz = \eta_2 + \eta_3 + \dots$$

and η_2 is non-degenerate. There exists a change of variables $q : \widehat{T}_l V \to \widehat{T}_l V$ such that $q(z) = z$ and $q(\eta) = \eta_2$ (see the proof of Theorem 11.2.1 (3) implies (2) of [70]). We may choose a basis x_i for $D\mathrm{Ext}^1_\Gamma(l,l)$ and the dual basis θ_i of $D\mathrm{Ext}^2_\Gamma(l,l)$ such that

$$\eta_2 = \sum_i [x_i, \theta_i].$$

Finally, we define

$$w := \sum_i \sum_{|\beta_{ij}| = n_{ij}} \frac{x_i \beta_{ij}}{n_{ij}}$$

where $d\theta_i = \sum_j \beta_{ij}$ is a decomposition of $d\theta_i$ into a sum of words and $|\beta_{ij}|$ denotes the length of the corresponding word. The cyclic invariance of w follows from equation 4.1.

The key observation is that the skew symmetric non-degenerate pairing on V_c or equivalently the symmetric non-degenerate pairing on $\mathrm{Ext}^*_\Gamma(l,l)$ depends only on the bimodule CY structure $\eta \in \mathrm{HH}_3(\Gamma, \Gamma)$ but not on the choice of the exact lifting ξ (see the proof of Lemma 11.1.2 of [70]). By an abuse of notations, we denote by $\eta \in V_c \otimes_l V_c$ the non-degenerate anti-symmetric form. Define

$$\omega_\eta := \frac{1}{2} D\eta' D\eta''$$

to be the bisymplectic form of degree -1 on $\widehat{T}_l V_c$ (see Section 10.1 of [70]). By Lemma 11.3.1 of [70], for any $f \in T_l V_c$ we have

$$df = \{w, f\}_{\omega_\eta}$$

where $\{?, ?\}_{\omega_\eta}$ is the Poisson bracket of degree -1 associated to the bisymplectic form ω_η. For degree reason, w has degree 0. Because w has no constant term, such w is unique. Therefore, different choices of potentials will only differ by a formal (nonlinear) change of variables in x_i. In other words, any two potentials w and w' in $\widehat{\mathbb{C}Q}_{\mathrm{cyc}}$ that we obtain are right equivalent. As a consequence $[w]$ and $[w']$ are right equivalent in $H^0\Gamma/[H^0\Gamma, H^0\Gamma]$. ☐

Remark 4.9. Theorem 11.2.1 of [69], can be viewed as a Darboux-Weinstein theorem in noncommutative formal symplectic geometry. We refer to Appendix of [44] for an introduction. Working on $\mathrm{Ext}^*_\Gamma(l,l)$, the cyclic A_∞-structure can be interpreted as a symplectic structure. The symplectic structure restricts to the truncation $\mathrm{Ext}^1_\Gamma(l,l) \oplus \mathrm{Ext}^2_\Gamma(l,l)$ so that $\mathrm{Ext}^1_\Gamma(l,l)$ is a (graded) Lagrangian.

Then Theorem 11.2.1 of [69] says that there exists a coordinate on $\mathrm{Ext}^1_\Gamma(l,l)$ under which the symplectic form can be normalized so that it has constant coefficients, which is in particular exact. The differential d of Γ can be interpreted as a homological vector field of degree 1. Then the contraction of the normalized symplectic form by d is the exterior derivation of a potential (hamiltonian) w of degree 0. Note that a different choice of Darboux coordinates can only differ by a change of variables on $\mathrm{Ext}^1_\Gamma(l,l)$, which leads to the above proposition.

Corollary 4.10. *Let $(\widehat{Y},\widehat{f},R)$ be a 3-dimensional formal flopping contraction with reduced exceptional fiber $\mathrm{Ex}(f)=\bigcup_{i=1}^t C_i$, and let $A=\mathrm{End}_R(\bigoplus_{i=1}^t N_i\oplus R)$ be the NCCR associated to it. Fix a 3CY structure η on $\widehat{Y}$, and therefore on A. Denote the CY tilted algebra of $\mathcal{C}(Y,\{C_i\}_{i=1}^t,\eta)\simeq\mathcal{C}(R,\{N_i\}_{i=1}^t,\eta)$ by Λ. Then there exists a canonical class, defined up to right equivalence, on $\mathrm{HH}_0(\Lambda)=\Lambda/[\Lambda,\Lambda]^c$ represented by a potential.*

The canonical class $[w]$ in the 0-th Hochschild homology of $H^0\Gamma$ is part of the “classical shadow” of the CY structure. The class plays a crucial role in the geometric applications. When Γ is weakly equivalent to a Jacobi-finite Ginzburg algebra $\mathfrak{D}(\widehat{F},w)$ for a complete free algebra $\widehat{F}=k\langle\langle x_1,\dots,x_n\rangle\rangle$, then this class vanishes if and only if w is right equivalent to a weighted homogeneous noncommutative polynomial (see Theorem 4.16). Therefore, the quasi-homogeneity of a potential is indeed a homotopy invariant of the CY algebra. This motivates the following definition.

Definition 4.11. Let k be a field and l be a finite dimensional commutative separable k-algebra, and let Γ be a pseudo-compact dg l-algebra in $\mathrm{PCAlgc}(l)$ concentrated in nonpositive degrees. Assume that Γ is exact 3CY. Then Γ is called *quasi-homogeneous* if the canonical class $[w]$ is (right equivalent to) zero.

The notion of quasi-homogeneity is expected to be independent of choices of CY structure. In the case of simple flopping contractions, the first author and Gui-song Zhou have conjectured that this notion of quasi-homogeneity is indeed equivalent to the quasi-homogeneity of the underlying hypersurface singularity R (see Conjecture 4.18 [29]).

4.3. **Properties of Jacobi-finite Ginzburg algebras.** In this section, we collect several results about Jacobi-finite Ginzburg algebras. We take k to be the field of complex numbers, though some of the results are valid more generally.

Theorem 4.12. [29, Theorem 3.16] *Let Q be a finite quiver and w be a potential in $\widehat{\mathbb{C}Q}_{\mathrm{cyc}}$. Assume that the Jacobi algebra $\Lambda(Q,w)$ is finite dimensional. Then w is right equivalent to a formal series with only finitely many nonzero terms.*

As a consequence, we may assume the potential is a noncommutative polynomial to begin with if the Jacobi algebra is known to be finite dimensional.

Theorem 4.13. *(Noncommutative Mather-Yau theorem)*[29, Theorem 3.6] *Let Q be a finite quiver and let $w,w'\in\widehat{\mathbb{C}Q}_{\mathrm{cyc}}$ be two potentials with only cubic terms and higher. Suppose that the Jacobi algebras $\Lambda(Q,w)$ and $\Lambda(Q,w')$ are both finite dimensional. Then the following two statements are equivalent:*

(1) *There is an algebra isomorphism $\gamma:\Lambda(Q,w)\cong\Lambda(Q,w')$ so that $\gamma_*([w])=[w']$ in $\Lambda(Q,w')_{\mathrm{cyc}}$.*
(2) *w and w' are right equivalent in $\widehat{\mathbb{C}Q}_{\mathrm{cyc}}$.*

The noncommutative Mather–Yau theorem has an immediate application to Ginzburg algebras.

Corollary 4.14. [29, Theorem 4.2] *Fix a finite quiver Q. Let $w,w'\in\widehat{\mathbb{C}Q}_{\mathrm{cyc}}$ be two potentials with only cubic terms and higher, such that the Jacobi algebras $\Lambda(Q,w)$ and $\Lambda(Q,w')$ are both finite* *dimensional. Assume there is an algebra isomorphism* $\gamma : \Lambda(Q,w) \to \Lambda(Q,w')$ *so that* $\gamma_*([w]) = [w']$. *Then there exists a dg algebra isomorphism*

$$\beta : \mathfrak{D}(Q,w) \xrightarrow{\cong} \mathfrak{D}(Q,w')$$

such that $\beta(t_i) = t_i$ *for any* $i \in Q_0$.

Definition 4.15. Fix $\widehat{F}$ to be the complete free associative algebra $\mathbb{C}\langle\langle x_1, \dots, x_n \rangle\rangle$. Let $(r_1, \dots, r_n)$ be a tuple of rational numbers. A potential $w \in \widehat{F}_{\mathrm{cyc}} := \widehat{F}/[\widehat{F}, \widehat{F}]^c$ is said to be *weighted-homogeneous of type* $(r_1, \dots, r_n)$ if it has a representative which is a linear combination of monomials $x_{i_1}x_{i_2}\cdots x_{i_p}$ such that $r_{i_1} + r_{i_2} + \dots r_{i_p} = 1$.

Theorem 4.16. *(Noncommutative Saito theorem) Let* $w \in \widehat{F}_{\mathrm{cyc}}$ *be a potential with only cubic terms and higher such that the Jacobi algebra associated to* w *is finite dimensional. Then* $[w] = 0$ *if and only if* w *is right equivalent to a weighted-homogenous potential of type* $(r_1, \dots, r_n)$ *for some rational numbers* $r_1, \dots, r_n$ *which lie strictly between* 0 *and* $1/2$. *Moreover, in this case, all such types* $(r_1, \dots, r_n)$ *agree with each other up to permutations on the indices* $1, \dots, n$.

Theorem 4.17. *Let* Q *be a finite quiver and* w *a Jacobi-finite potential on* Q. *Let* Γ *be the complete Ginzburg algebra associated with* (Q,w). *Denote by* Λ_{dg} *the dg endomorphism algebra of* Γ *in the canonical dg enhancement of the cluster category* $\mathcal{C}_\Gamma$. *Then there is a canonical isomorphism in the homotopy category of dg algebras*

$$\Gamma \xrightarrow{\sim} \tau_{\leq 0}\Lambda_{dg}.$$

Proof. There is a canonical morphism

$$\Gamma = \mathbf{R}\mathrm{Hom}_\Gamma(\Gamma,\Gamma) \to (\mathcal{C}_\Gamma)_{dg}(\Gamma,\Gamma) = \Lambda_{dg}$$

where the right hand side denotes the dg endomorphism algebra of Γ in the canonical dg enhancement $(\mathcal{C}_\Gamma)_{dg}$ of $\mathcal{C}_\Gamma$. It suffices to show that the canonical map

$$H^{-p}(\Gamma) = \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{-p}\Gamma) \to \mathrm{Hom}_{\mathcal{C}_\Gamma}(\Gamma, \Sigma^{-p}\Gamma)$$

is invertible for $p \geq 0$. By Proposition 2.8 of [3], we have

$$\mathrm{Hom}_{\mathcal{C}_\Gamma}(\Gamma, \Sigma^{-p}\Gamma) = \mathrm{colim}_n \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \tau_{\leq n}(\Sigma^{-p}\Gamma)).$$

We have

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \tau_{\leq n}(\Sigma^{-p}\Gamma)) = \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \Sigma^{-p}\Gamma).$$

Consider the canonical triangle

$$\tau_{\leq n}\Gamma \longrightarrow \Gamma \longrightarrow \tau_{>n}\Gamma \longrightarrow \Sigma(\tau_{\leq n}\Gamma).$$

The object $\tau_{>n}\Gamma$ belongs to $\mathrm{D}_{fd}(\Gamma)$ and $\Sigma^{-p}\Gamma$ belongs to $\mathrm{per}(\Gamma)$. By the 3-Calabi-Yau property, we have

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Sigma^{-1}\tau_{>n}\Gamma, \Sigma^{-p}\Gamma) = D\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{p+2}\tau_{>n}\Gamma))$$

which vanishes because $\tau_{>n}\Gamma$ has no homology in degrees > 0. Similarly, we have

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{>n}\Gamma, \Sigma^{-p}\Gamma) = D\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{p+3}(\tau_{>n}\Gamma))$$

which vanishes for the same reason. Thus we have

$$\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\tau_{\leq n}\Gamma, \Sigma^{-p}\Gamma) = \mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Gamma, \Sigma^{-p}\Gamma).$$

□

Corollary 4.18. *Let Q be a quiver with one node and arbitrary number of loops and $\Gamma = \mathfrak{D}(Q,w)$ a Jacobi-finite Ginzburg algebra. Then $H^0\Gamma$ is self-injective and there is an isomorphism*

$$\Sigma^2\Gamma \xrightarrow{\sim} \tau_{\leq -1}\Gamma$$

in the derived category of dg Γ-modules. In particular, we have $H^i(\Gamma) = 0$ for odd i and $H^i(\Gamma) \cong H^0(\Gamma)$ for even $i \leq 0$.

Proof. By [3], the cluster category $\mathcal{C}_\Gamma$ is a Hom-finite 2-Calabi-Yau category and the image T of Γ in $\mathcal{C}_\Gamma$ is a cluster-tilting object in $\mathcal{C}_\Gamma$. By Theorem 4.1 of [1], the cluster-tilting objects of $\mathcal{C}_\Gamma$ are in bijection with the support τ-tilting modules over $\mathrm{End}(T)$. Since $\mathrm{End}(T) = H^0(\Gamma)$ is local, the only support τ-tilting modules over $\mathrm{End}(T)$ are 0 and $\mathrm{End}(T)$, by example 6.1 of [loc. cit.]. Thus, the only cluster-tilting objects of $\mathcal{C}_\Gamma$ are T and ΣT. In particular, $\Sigma^2 T$ has to be isomorphic to T (since $\mathrm{Hom}(\Sigma T, \Sigma^2 T) = 0$ and $\Sigma^2 T$ must be a cluster-tilting object). This implies that $H^0\Gamma = \mathrm{End}(T)$ is self-injective, since, by the 2-Calabi-Yau property, we have an isomorphism of right $\mathrm{End}(T)$-modules

$$D\mathrm{Hom}(T,T) = \mathrm{Hom}(T, \Sigma^2 T) = \mathrm{Hom}(T,T).$$

Let $\phi : \Sigma^2\Gamma \to \Gamma$ be a lift of an isomorphism $\Sigma^2 T \to T$ in $\mathcal{C}_\Gamma$. Let $p \geq 2$. In the commutative square

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Sigma^p\Gamma, \Sigma^2\Gamma) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}_\Gamma}(\Sigma^p T, \Sigma^2 T) \arrow[d, "\phi_*"] \\
\mathrm{Hom}_{\mathrm{per}(\Gamma)}(\Sigma^p\Gamma, \Gamma) \arrow[r] & \mathrm{Hom}_{\mathcal{C}_\Gamma}(\Sigma^p T, T)
\end{tikzcd}$$

the horizontal arrows are isomorphisms by Theorem 4.17 and the right vertical arrow ϕ_* is an isomorphism. Thus, the morphism $\phi : \Sigma^2\Gamma \to \Gamma$ induces isomorphisms in H^i for $i \leq 2$. Moreover, we have $H^{-1}(\Gamma) = \mathrm{Hom}(T, \Sigma^{-1}T) = \mathrm{Hom}(T, \Sigma T) = 0$ since $\Sigma^{-1}T$ is isomorphic to ΣT. It follows that ϕ induces an isomorphism

$$\Sigma^2\Gamma \xrightarrow{\sim} \tau_{\leq -1}\Gamma.$$

□

Remark 4.19. For the pair (Q,w) associate to 3-dimensional flopping contractions, one can show that $H^0\Gamma$ is indeed symmetric (see Theorem 6.5). In the context of general contraction with one dimensional fiber, Kawamata has proved that the classical (multi-pointed) deformation algebra of the reduced exceptional fiber is always self-injective (see Proposition 6.3 [34]). So in particular it is Gorenstein. This result overlaps with the above corollary in the case of simple flopping contractions. For a general finite quiver Q, the 0-th homology of a Jacobi-finite Ginzburg algebra $\mathfrak{D}(Q,w)$ is not self-injective. Moreover, Kawamata proves that the deformation algebra is always isomorphic to its opposite algebra (see Corollary 6.3 of [35]).

Corollary 4.20. *Let Γ be the Ginzburg algebra of a Jacobi-finite quiver with potential. Let T be the image of Γ in the cluster category $\mathcal{C}_\Gamma$.*

a) *$H^0\Gamma$ is selfinjective if and only if $H^{-1}\Gamma$ vanishes if and only if T is isomorphic to $\Sigma^2 T$ in $\mathcal{C}_\Gamma$.*
b) *If the identity functor of $\mathcal{C}_\Gamma$ is isomorphic to Σ^2, then $H^0\Gamma$ is symmetric and there is an isomorphism of graded algebras $H^0(\Gamma)\otimes k[u^{-1}] \xrightarrow{\sim} H^*(\Gamma)$, where u is of degree 2.*

Proof. a) By Theorem 4.17, the space $H^{-1}\Gamma$ is isomorphic to $\mathcal{C}_\Gamma(T, \Sigma^{-1}T)$ and $H^0\Gamma$ is isomorphic to the endomorphism algebra of T. By Proposition 3.6 of [32], the endomorphism algebra is selfinjective if and only if $\mathcal{C}_\Gamma(T, \Sigma^{-1}T)$ vanishes if and only if T is isomorphic to $\Sigma^2 T$ in $\mathcal{C}_\Gamma$.

b) By combining the functorial isomorphism from T to $\Sigma^2 T$ with the Calabi-Yau property we get an isomorphism of bimodules over the endomorphism algebra of T

$$\mathcal{C}_\Gamma(T,T) \xrightarrow{\sim} \mathcal{C}_\Gamma(T,\Sigma^2 T) \xrightarrow{\sim} D\mathcal{C}_\Gamma(T,T).$$

Since $H^0\Gamma$ is in particular selfinjective, the space $H^{-1}\Gamma$ vanishes by a). We get an isomorphism of graded algebras

$$k[u,u^{-1}]\otimes_k \mathcal{C}_\Gamma(T,T) \xrightarrow{\sim} \bigoplus_{p\in\mathbb{Z}} \mathcal{C}_\Gamma(T,\Sigma^p T)$$

where u is of degree 2 by sending u to the functorial isomorphism $T \xrightarrow{\sim} \Sigma^2 T$. Thanks to Theorem 4.17, by truncation, we get an isomorphism of graded algebras

$$k[u^{-1}]\otimes_k H^0\Gamma \xrightarrow{\sim} H^*\Gamma.$$

□

4.4. **Silting theory for a non positive dg algebra and its zeroth homology.** Let $\mathcal{T}$ be a triangulated category. Recall that a *tilting object* for $\mathcal{T}$ is a classical generator T of $\mathcal{T}$ such that $\mathcal{T}(T,\Sigma^p T)$ vanishes for all $p \neq 0$. A *silting object* [47] for $\mathcal{T}$ is a classical generator T of $\mathcal{T}$ such that $\mathcal{T}(T,\Sigma^p T)$ vanishes for all $p > 0$. The advantage of silting objects over tilting objects is that (under suitable finiteness assumptions) they are stable under mutation [2].

We recall fundamental definitions and results from [2]. Assume from now on that $\mathcal{T}$ is k-linear, Hom-finite and has split idempotents. In particular, it is a Krull–Schmidt category, i.e. indecomposables have local endomorphism rings and each object is a finite direct sum of indecomposables (which are then unique up to isomorphism and permutation). An object of $\mathcal{T}$ is *basic* if it is a direct sum of pairwise non isomorphic indecomposables. If X is an object of $\mathcal{T}$ and $\mathcal{U}$ a full subcategory, a *left $\mathcal{U}$-approximation of X* is a morphism $f : X \to U$ to an object of $\mathcal{U}$ such that each morphism $X \to V$ to an object of $\mathcal{U}$ factors through $f : X \to U$. It is *minimal* if each endomorphism $g : U \to U$ such that $g \circ f = f$ is an isomorphism. Notice that the morphism $f : X \to U$ is a minimal left $\mathcal{U}$-approximation iff the morphism $f^* : \mathcal{U}(U,?) \to \mathcal{T}(X,?)|_{\mathcal{U}}$ is a projective cover in the category of left $\mathcal{U}$-modules. In particular, minimal left approximations are unique up to non unique isomorphism when they exist. Existence is automatic if $\mathcal{U}$ has finitely many indecomposables U_1, …, U_n (which is the case in our applications) because then the functor $\mathcal{T}(X,?)|_{\mathcal{U}}$ corresponds to a finite-dimensional left module over the finite-dimensional endomorphism algebra of the sum of the U_i. A *(minimal) right $\mathcal{U}$-approximation* is defined dually. For an object X of $\mathcal{T}$, we denote by $\mathrm{add}X$ the full subcategory formed by all direct factors of finite direct sums of copies of X.

Let M be a basic silting object of $\mathcal{T}$ and X an indecomposable direct summand of M. Denote by M/X the object such that $M \cong X \oplus M/X$. By definition, the *left mutation $\mu_X(M)$ of M at X* is the silting object $M/X \oplus Y$, where Y is defined by a triangle

$$X \longrightarrow E \longrightarrow Y \longrightarrow \Sigma X$$

and $X \to E$ is a minimal left $\mathrm{add}(M/X)$-approximation. It is not hard to show that then $E \to Y$ is a minimal right $\mathrm{add}(M/X)$-approximation which implies that Y is indeed indecomposable.The right mutation $\mu^-_X(M)$ is defined dually. The right mutation of $\mu_X(M)$ at Y is isomorphic to M. The *silting quiver* $\mathrm{silt}(\mathcal{T})$ has as vertices the isomorphism classes of basic silting objects of $\mathcal{T}$ and an arrow $M \to \mu_X(M)$ for each left mutation.

There is a partial order on the set of isomorphism classes of basic silting objects: By definition, we have $M \geq N$ if $\mathrm{Hom}(M,\Sigma^p N) = 0$ for all $p > 0$.

Now let A be a dg k-algebra whose homologies H^pA are finite-dimensional and vanish in all degrees $p > 0$. Then $\mathrm{per}(A)$ is a Hom-finite k-linear triangulated category with split idempotents and A is a silting object of $\mathrm{per}(A)$. We write $\mathrm{silt}(A)$ for $\mathrm{silt}(\mathrm{per}(A))$ and $\mathrm{silt}(A)^0$ for the connected component containing the direct sum of the indecomposable summands of A (notice that we have not assumed A to be basic as an object of $\mathrm{per}(A)$). An example of a Jacobi-finite Ginzburg algebra A where $\mathrm{silt}(A)$ is not connected is given in example 4.3 of [62]. Our aim is to compare $\mathrm{silt}(A)$ with $\mathrm{silt}(H^0A)$. Note that by our assumption on A, we have a canonical morphism $A \to H^0A$ in the homotopy category of dg algebras. We write $F : \mathrm{per}(A) \to \mathrm{per}(H^0A)$ for the derived tensor product over A with H^0A and $G : \mathrm{D}(H^0A) \to \mathrm{D}(A)$ for the restriction along $A \to H^0A$.

Theorem 4.21. a) *For each basic silting object M of $\mathrm{per}(A)$, the object FM is a basic silting object of $\mathrm{per}(H^0A)$ and the map*

$$\mathrm{Hom}_{\mathrm{D}A}(M, M) \to \mathrm{Hom}_{\mathrm{D}H^0A}(FM, FM)$$

is an isomorphism. If we have $M \geq N$ for a basic silting object N, then $FM \geq FN$.

b) *If M is a basic silting object of $\mathrm{per}(A)$ and X an indecomposable summand of M, then $F\mu_X(M)$ is isomorphic to $\mu_{FX}(FM)$.*

c) *The functor F induces a morphism of quivers $\mathrm{silt}(A) \to \mathrm{silt}(H^0A)$ and a surjective morphism $\mathrm{silt}(A)^0 \to \mathrm{silt}(H^0A)^0$.*

d) *Suppose we have an isomorphism $H^0(A) \otimes k[u^{-1}] \cong H^*(A)$ restricting to the identity on H^0A, where u is of degree 2. Then for two basic silting objects M and N, we have $M \geq N$ if and only if $FM \geq FN$. As a consequence, the morphism $\mathrm{silt}(A) \to \mathrm{silt}(H^0A)$ is injective on vertices and the morphism $\mathrm{silt}(A)^0 \to \mathrm{silt}(H^0A)^0$ is an isomorphism.*

Proof. As a first step, let us prove that if X is an object of $\mathrm{D}(A^e)$ with vanishing homology in degrees > 0, then $M \overset{\mathbb{L}}{\otimes}_A X$ is in the closure in $\mathrm{D}A$ of M under suspensions, extensions, and arbitrary coproducts. Indeed, since A^e has its homology concentrated in degrees ≤ 0, the object X lies in the closure of A^e under suspensions, arbitrary coproducts and extensions. Thus, the object $M \overset{\mathbb{L}}{\otimes}_A X$ lies in the closure of $M \otimes_A A^e = M \otimes A$ under suspensions, arbitrary coproducts and extensions.

Let us now prove a). For $p > 0$, we have

$$\mathrm{Hom}_{\mathrm{D}H^0A}(FM, \Sigma^p FM) = \mathrm{Hom}_{\mathrm{D}A}(M, \Sigma^p GFM) = \mathrm{Hom}_{\mathrm{D}A}(M, \Sigma^p M \overset{\mathbb{L}}{\otimes}_A H^0A).$$

By the first step, the object $M \overset{\mathbb{L}}{\otimes}_A H^0A$ lies in the closure of M under suspensions, extensions and arbitrary coproducts. Since M is compact and there are no non zero morphisms from M to $\Sigma^p M$ for $p > 0$, it follows that we have no non zero morphisms from FM to $\Sigma^p FM$ for $p > 0$. Similarly, one shows that $M \geq N$ implies $FM \geq FN$. Moreover, since the object A belongs to the triangulated subcategory of $\mathrm{per}(A)$ generated by M, the object $H^0A = FA$ belongs to the triangulated subcategory of $\mathrm{per}(H^0A)$ generated by FM. Thus, FM is indeed a silting object in $\mathrm{per}(H^0A)$. Let us prove that F induces an isomorphism from the endomorphism ring of M to that of FM. The truncation triangle

$$\begin{tikzcd} \tau_{\leq -1}A \arrow[r] & A \arrow[r] & H^0A \arrow[r] & \Sigma\tau_{\leq -1}A \end{tikzcd}$$

yields a triangle

$$\begin{tikzcd} M \overset{\mathbb{L}}{\otimes}_A (\tau_{\leq -1}A) \arrow[r] & M \arrow[r] & GFM \arrow[r] & \Sigma M \overset{\mathbb{L}}{\otimes}_A (\tau_{\leq -1}A) \end{tikzcd} .$$

By the first step, the object $M \overset{\mathbb{L}}{\otimes}_A (\tau_{\leq -1}A)$ and its suspension belong to the closure in $\mathrm{D}A$ under suspensions, extensions and arbitrary coproducts of the object ΣM. Since M is silting and compact, it sends no non zero morphisms to the first and the last terms of the triangle and the map

$$\mathrm{Hom}(M, M) \to \mathrm{Hom}(M, GFM) = \mathrm{Hom}(FM, FM)$$

is indeed bijective, as was to be shown. It follows that F induces an equivalence $\operatorname{add}(M) \to \operatorname{add}(FM)$ which shows in particular that FM is basic. Part b) follows from a) because $\mu_X(M)$ is defined entirely in terms of the category $\operatorname{add}(M)$ and triangles. Part c) follows from b).

Let us show d). By a), we only have to show that if $FM \geq FN$, then $M \geq N$. Consider the triangle

$$N \overset{\mathbb{L}}{\otimes}_A (\tau_{\leq -1} A) \longrightarrow N \longrightarrow GFN \longrightarrow \Sigma N \overset{\mathbb{L}}{\otimes}_A (\tau_{\leq -1} A) .$$

By assumption, we have

$$0 = \operatorname{Hom}(FM, \Sigma^p FN) = \operatorname{Hom}(M, \Sigma^p GFN)$$

for $p > 0$. It suffices to show that $\operatorname{Hom}(M, \Sigma^p N \overset{\mathbb{L}}{\otimes}_A (\tau_{\leq -1} A)) = 0$ for $p > 0$. Put $X = \tau_{\leq -1} A$ and $X_q = \tau_{\geq -q} \tau_{\leq -1} A$, $q \geq 0$. As q tends to infinity, the X_q form an inverse system with Milnor limit X. Since N is perfect, the Milnor limit of the system $N \overset{\mathbb{L}}{\otimes}_A X_q$ identifies with $N \overset{\mathbb{L}}{\otimes}_A X$. Thus, we have an exact sequence

$$0 \longrightarrow \lim{}^1 (M, \Sigma^{p-1} N \overset{\mathbb{L}}{\otimes}_A X_q) \longrightarrow (M, \Sigma^p N \overset{\mathbb{L}}{\otimes}_A X) \longrightarrow \lim (M, \Sigma^p N \overset{\mathbb{L}}{\otimes}_A X_q) \longrightarrow 0 ,$$

where we abbreviate $\operatorname{Hom}(?, -)$ to $(?, -)$. Since M is perfect and the $N \overset{\mathbb{L}}{\otimes}_A X_q$ have finite-dimensional homologies, the terms in $\lim^1$ are finite-dimensional so that $\lim^1$ vanishes by the Mittag–Leffler condition. It remains to be shown that the $\operatorname{Hom}(M, \Sigma^p N \overset{\mathbb{L}}{\otimes}_A X_q)$ vanish. Now by our assumption, X_q is an iterated extension of objects $\Sigma^{2i} H^0 A$, $i \geq 1$, so that $N \overset{\mathbb{L}}{\otimes}_A X_q$ is an iterated extension of objects $N \overset{\mathbb{L}}{\otimes}_A \Sigma^{2i} H^0 A$, $i \geq 1$. This implies the assertion since $\operatorname{Hom}(M, \Sigma^p N \overset{\mathbb{L}}{\otimes}_A H^0)$ vanishes for $p > 0$ by assumption. ☐

Example 4.22. Suppose that Γ is a Jacobi-finite Ginzburg algebra associated with a finite quiver and a potential not containing cycles of length ≤ 2. Then $A = \Gamma$ satisfies our assumptions and Γ is a basic silting object in $\operatorname{per}(\Gamma)$. Let M be a silting object in $\operatorname{per}(\Gamma)$ and Γ' the derived endomorphism algebra of M. Then the homologies $H^p(\Gamma')$ are finite-dimensional and vanish in degrees $p > 0$. Since M generates $\operatorname{per}(\Gamma)$, the Γ'–Γ–bimodule M yields an algebraic triangle equivalence $\mathrm{D}(\Gamma') \xrightarrow{\sim} \mathrm{D}(\Gamma)$. Conversely, if we start from a dg algebra Γ' whose homologies are finite-dimensional and vanish in degrees > 0 and from an algebraic triangle equivalence $\mathrm{D}(\Gamma') \xrightarrow{\sim} \mathrm{D}(\Gamma)$, then the image M of Γ' in $\operatorname{per}(\Gamma)$ is a silting object. In any case, the dg algebra Γ' is exactly bimodule 3-Calabi-Yau has its homology concentrated in degrees ≤ 0. By Van den Bergh's theorem [70], the dg algebra Γ' is again a Jacobi-finite Ginzburg algebra (up to weak equivalence). In particular, for M we can take the mutation $M' = \mu_X \Gamma$, where $X = e_i \Gamma$ for a vertex i of the quiver of Γ. We define the associated Ginzburg algebra Γ' to be the *left mutation of* Γ *at* i. Notice that by construction, we have a canonical derived equivalence from Γ' to Γ. In the same way, we can define the right mutation Γ'' of Γ at i using the right mutation $M'' = \mu_X^-(\Gamma)$ of Γ at X. The right mutation Γ'' turns out to be quasi-isomorphic to the left mutation Γ'. Indeed, by Theorem 4.17, these algebras are the $\tau_{\leq 0}$–truncations of the derived endomorphism algebras of the images $\pi(M')$ and $\pi(M'')$ in the cluster category $\mathcal{C}_\Gamma$. Now we have $\pi(M') \cong \pi(M'')$ because they are the left resp. right mutation in the sense of Iyama-Yoshino [31] of the cluster-tilting object $\pi(\Gamma)$ at $\pi(X)$ and for cluster-tilting objects in 2-Calabi-Yau triangulated categories, right and left mutation coincide up to isomorphism.

Now let A be a pseudocompact dg algebra in $\operatorname{PCAlgc}(l)$ strictly concentrated in degrees ≤ 0. Let e be an idempotent of $H^0 A$ and A' the derived endomorphism algebra of the image of A in the Verdier quotient of $\operatorname{per}(A)$ by the thick subcategory generated by eA. Then A' is concentrated in degrees

≤ 0 and we have a canonical morphism $A \to A'$ in the homotopy category of $\mathrm{PCAlgc}(l)$. If A is of the form $(\widehat{T_l}(V), d)$ for a pseudocompact l-bimodule V concentrated in degrees ≤ 0, where $\widehat{T_l}(V)$ is the completed tensor algebra, then A' is quasi-isomorphic to the quotient of A by the twosided closed ideal generated by e (cf. [12]). Put $A_0 = H^0 A'$ so that we have a canonical morphism $p : A \to A_0$. Let B and B_0 be pseudocompact dg algebras in $\mathrm{PCAlgc}(l)$, $X \in \mathrm{D}(A^{op} \otimes B)$ such that X_B is perfect and $Q \in \mathrm{D}(B^{op} \otimes B_0)$ such that Q_{B_0} is perfect.

Proposition 4.23. *Suppose that* $eX \overset{\mathbb{L}}{\otimes}_B Q$ *vanishes and that the object* $X \overset{\mathbb{L}}{\otimes}_B Q$ *of* $\mathrm{D}(B_0)$ *has no selfextensions in degrees* $p < 0$. *Then there is an object* Y *of* $\mathrm{D}(A_0^{op} \otimes B_0)$, *unique up to isomorphism, such that we have an isomorphism*

$$X \overset{\mathbb{L}}{\otimes}_B Q \xrightarrow{\sim} A_0 \overset{\mathbb{L}}{\otimes}_{A_0} Y$$

in $\mathrm{D}(A^{op} \otimes B_0)$. *Thus, the square*

$$\begin{tikzcd}
\mathrm{D}(A) \arrow[r, "X"] \arrow[d, "A_0"'] & \mathrm{D}(B) \arrow[d, "Q"] \\
\mathrm{D}(A_0) \arrow[r, "Y"'] & \mathrm{D}(B_0)
\end{tikzcd}$$

is commutative up to isomorphism, where we write dg bimodules instead of derived tensor products by dg bimodules.

Remark 4.24. In our applications in this article, the idempotent e will be 0. We state and prove the proposition in the general case because it provides an alternative approach to the problem of relating the tilting theory of maximal modification algebras [73] to that of the associated contraction algebras as treated by August in [5]. Let R be a complete local cDV singularity and M a maximal basic rigid object in the category of Cohen-Macaulay modules over R containing R as a direct summand. We can take $A = \mathrm{End}_R(M)$ and e the idempotent corresponding to the projection on R. Then $A_0 = H^0 A'$ is isomorphic to the stable endomorphism algebra of M, i.e. the contraction algebra associated with M. Let N be another maximal basic rigid object containing R as a direct summand, B its endomorphism algebra and B_0 the associated contraction algebra. Then $X = \mathrm{Hom}_R(N, M)$ yields a derived equivalence $? \overset{\mathbb{L}}{\otimes}_A X : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(B)$ taking eA to eB. Moreover, the complex $X \overset{\mathbb{L}}{\otimes}_B B_0$ is a silting object of $\mathrm{per}(B_0)$ (as it follows from silting reduction [2] combined with part a) of Theorem 4.21) and hence a tilting object since B_0 is symmetric. Thus, the hypotheses of the proposition hold and there is a canonical two-sided tilting complex Y in $\mathrm{D}(A_0^{op} \otimes B_0)$. Clearly, the construction is compatible with compositions via derived tensor products.

Proof of the Proposition. Put $U = X \overset{\mathbb{L}}{\otimes}_B Q$ viewed as an object in $\mathrm{D}(A^{op} \otimes B_0)$. The morphism $A \to A'$ induces the Verdier quotient

$$\mathrm{per}(A) \to \mathrm{per}(A') = \mathrm{per}(A)/\mathbf{thick}(eA)$$

and is therefore a dg quotient. By the universal property of the dg quotient, there is an object Z in $\mathrm{D}(A'^{op} \otimes B_0)$ unique up to isomorphism such that the restriction of Z along $A^{op} \otimes B_0 \to A'^{op} \otimes B_0$ is isomorphic to U in $\mathrm{D}(A^{op} \otimes B_0)$. We have a canonical morphism in the homotopy category of dg algebras

$$A' \longrightarrow \mathbf{R}\mathrm{Hom}_{B_0}(Z, Z).$$

Since by assumption $\mathbf{R}\mathrm{Hom}_{B_0}(Z, Z)$ is concentrated in degrees ≥ 0 and A' in degrees ≤ 0, this morphism factors uniquely through a morphism

$$A_0 = H^0 A' \to \mathbf{R}\mathrm{Hom}_{B_0}(Z, Z).$$

We thus obtain an A_0-B_0-bimodule structure on Z. Write Y for this dg bimodule. By construction, the image of Y under the restriction $\mathrm{D}(A_0^{op}\otimes B_0) \to \mathrm{D}(A'^{op}\otimes B_0)$ is isomorphic to Z. Now suppose we have a second object Y' in $\mathrm{D}(A_0^{op}\otimes B_0)$ which becomes isomorphic to Z in $\mathrm{D}(A'^{op}\otimes H^0B)$. We have a chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}(A'^{op}\otimes B_0)}(Y,Y') &= \mathrm{Hom}_{\mathrm{D}(A'^e)}(A', \mathbf{R}\mathrm{Hom}_{B_0}(Y,Y')) \\
&= \mathrm{Hom}_{\mathrm{D}(A'^e)}(H^0A', H^0\mathbf{R}\mathrm{Hom}_{B_0}(Y,Y')) \\
&= \mathrm{Hom}_{H^0(A'^e)}(H^0A', H^0\mathbf{R}\mathrm{Hom}_{H^0B}(Y,Y')) \\
&= \mathrm{Hom}_{\mathrm{D}(H^0(A')^e)}(H^0A', H^0\mathbf{R}\mathrm{Hom}_{H^0B}(Y,Y')) \\
&= \mathrm{Hom}_{\mathrm{D}(H^0(A')^e)}(H^0A', \mathbf{R}\mathrm{Hom}_{H^0B}(Y,Y')) \\
&= \mathrm{Hom}_{\mathrm{D}(H^0(A')^{op}\otimes H^0B)}(Y,Y').
\end{aligned}$$

Clearly, isomorphisms correspond to isomorphisms under this bijection. This shows the uniqueness. □

Let A and B be dg k-algebras whose homologies are finite-dimensional and vanish in degrees > 0. Let C be a finite-dimensional basic k-algebra (i.e. C is basic as a right module over itself) and Z an object of $\mathrm{D}(C^{op}\otimes H^0B)$ such that

$$?\overset{\mathbb{L}}{\otimes}_C Z : \mathrm{D}(C) \to \mathrm{D}(H^0B)$$

is an equivalence. Notice that Z_{H^0B} is a tilting object in $\mathrm{per}(H^0B)$ and in particular a silting object, which is basic by our assumption on C.

Theorem 4.25. *Assume that Z_{H^0B} belongs to the connected component $\mathrm{silt}(H^0B)^0$. Then there is a dg algebra A whose homologies H^pA are finite-dimensional and vanish in degrees $p > 0$, a derived equivalence $?\overset{\mathbb{L}}{\otimes}_A X : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(B)$, an isomorphism of algebras $\phi : H^0A \xrightarrow{\sim} C$ and an isomorphism*

$$_{\phi}Z \xrightarrow{\sim} X\overset{\mathbb{L}}{\otimes}_B H^0B$$

in $\mathrm{D}(A^{op}\otimes H^0B)$, where the left A-module structure on $_{\phi}Z$ is defined via the composition $A \to H^0A \to C$. In particular, we have a diagram, commutative up to isomorphism

$$\begin{tikzcd}
\mathrm{D}(A) \arrow[r, "X"] \arrow[d, "H^0A"'] & \mathrm{D}(B) \arrow[d, "H^0B"] \\
\mathrm{D}(H^0A) \arrow[r, "_{\phi}Z"'] & \mathrm{D}(H^0B)
\end{tikzcd}$$

where we write dg bimodules instead of derived tensor products by dg bimodules.

Proof. By part c) of Theorem 4.21, there is a silting object M of $\mathrm{per}(B)$ such that $M\overset{\mathbb{L}}{\otimes}_B H^0B$ is isomorphic to Z_{H^0B}. We let A be the derived endomorphism algebra of M and $X \in \mathrm{D}(A^{op}\otimes B)$ the dg bimodule given by M with its canonical left A-action. Since $X\overset{\mathbb{L}}{\otimes}_B H^0B$ is isomorphic to the tilting object Z_{H^0B}, it has no self-extensions in degree < 0. Therefore, Proposition 4.23 yields an object Y of $\mathrm{D}(H^0(A)^{op}\otimes H^0(B))$ and an isomorphism

$$\psi : X\overset{\mathbb{L}}{\otimes}_B H^0B \xrightarrow{\sim} A\overset{\mathbb{L}}{\otimes}_{H^0A} Y$$

in $\mathrm{D}(A^{op}\otimes H^0B)$. By part a) of Theorem 4.21, we have an isomorphism $\mathrm{End}(M) \xrightarrow{\sim} \mathrm{End}(M\overset{\mathbb{L}}{\otimes}_B H^0B)$. By construction, we have an isomorphism $H^0A \xrightarrow{\sim} \mathrm{End}(M)$ or equivalently $H^0A \xrightarrow{\sim} \mathrm{End}(X_B)$. Thus, the composition of $?\overset{\mathbb{L}}{\otimes}_B H^0B$ with $?\overset{\mathbb{L}}{\otimes}_A X$ induces an isomorphism

$$H^0A \xrightarrow{\sim} \mathrm{End}_{\mathrm{D}(H^0B)}(X\overset{\mathbb{L}}{\otimes}_B H^0B).$$

Via ψ, we get an isomorphism

$$H^0A \xrightarrow{\sim} \mathrm{End}_{\mathrm{D}(H^0B)}(Y)$$

given by the left action of H^0A on Y. We choose an isomorphism $Y_{H^0B} \xrightarrow{\sim} Z_{H^0B}$ in $\mathrm{D}(H^0B)$ and define $\phi : H^0A \xrightarrow{\sim} C$ so as to make the following square commutative

$$\begin{tikzcd}
H^0A \arrow[r] \arrow[d, "\phi"'] & \mathrm{End}(Y_{H^0B}) \arrow[d] \\
C \arrow[r] & \mathrm{End}(Z_{H^0B}).
\end{tikzcd}$$

By Lemma 2.6, the chosen isomorphism $Y_{H^0B} \xrightarrow{\sim} Z_{H^0B}$ lifts to an isomorphism $Y \xrightarrow{\sim} {}_{\phi}Z$ in $\mathrm{D}(H^0(A)^{op}\otimes H^0(B))$. Whence a composed isomorphism

$$X\overset{\mathbb{L}}{\otimes}_B H^0B \xrightarrow{\sim} A\overset{\mathbb{L}}{\otimes}_{H^0A} Y \xrightarrow{\sim} A\overset{\mathbb{L}}{\otimes}_{H^0A}{}_{\phi}Z.$$

□

4.5. **Cyclic homology and preservation of the canonical class.** Let k be a field of characteristic 0 and l a finite product of copies of k. Let V be a pseudocompact l-bimodule and d a continous differential on the completed tensor algebra $\widehat{T}_l(V)$. Put $A = (\widehat{T}_l(V), d)$. We define $\Omega_l A$ by the short exact sequence

$$0 \longrightarrow \Omega^1 A \longrightarrow A\otimes_l A \xrightarrow{\mu} A \longrightarrow 0\ ,$$

where μ is the multiplication of A. Then the morphism

$$A\otimes_l V\otimes_l A \to \Omega^1 A$$

taking $a\otimes v\otimes b$ to $av\otimes b - a\otimes vb$ is an isomorphism of graded l-bimodules, cf. Example 3.10 of [64]. We can describe the induced differential on $A\otimes_l V\otimes_l A$ as follows (cf. Proposition 3.7 of [44]): Let $D : A \to A\otimes_l V\otimes_l A$ be the unique continuous bimodule derivation which restricts to the map $v \mapsto 1\otimes v\otimes 1$ on V. We have

$$D(v_1 \dots v_n) = 1\otimes v_1\otimes(v_2 \dots v_n) + \sum_{i=2}^{n-1} v_1 \dots v_{i-1}\otimes v_i\otimes v_{i+1} \dots v_n + (v_1 \dots v_{n-1})\otimes v_n\otimes 1.$$

Then the induced differential on $A\otimes_l V\otimes_l A$ sends $a\otimes v\otimes b$ to

$$(-1)^{|a|} aD(dv)b + (da)\otimes v\otimes b + (-1)^{(|v|+|a|)} a\otimes v\otimes(db).$$

For an l-bimodule M, we write M_l for the coinvariant module $M/[l, M]$. For an A-bimodule M, we let $M_\natural$ be the coinvariant module $M/[A, M]$. We have an isomorphism of graded modules

$$(A\otimes_l V\otimes A)_\natural \xrightarrow{\sim} (V\otimes A)_l$$

taking $a\otimes v\otimes b$ to $(-1)^{|a|(|v|+|b|)} v\otimes ba$. The induced differential on the right hand side is given as follows: If $D(dv) = \sum_i a_i\otimes v_i\otimes b_i$, then

$$d(v\otimes a) = (-1)^{|v|} v\otimes(da) + \sum_i (-1)^{|a_i|(|v_i|+|b_i|+|a|)} v_i\otimes b_i a a_i.$$

Following section 3 of [64] we define morphisms of complexes

$$\partial_1 : (V\otimes_l A)_l \to A_l \quad \text{and} \quad \partial_0 : A_l \to (V\otimes_l A)_l$$

as follows: ∂_1 sends $v \otimes a$ to $va - (-1)^{|v||a|} a \otimes v$ and ∂_0 sends $v_1 \ldots v_n$ to

$$\sum_i \pm v_i \otimes v_{i+1} \ldots v_n v_1 \ldots v_{i-1},$$

where the sign is determined by the Koszul sign rule. We then have $\partial_0 \partial_1 = 0 = \partial_1 \partial_0$.

The (continuous) Hochschild homology of A is computed by the total complex of

$$(V \otimes A)_l \xrightarrow{\partial_1} A_l$$

and the (continuous) cyclic homology of A is computed by the product total complex of

$$\ldots \longrightarrow (V \otimes_l A)_l \xrightarrow{\partial_1} A_l \xrightarrow{\partial_0} (V \otimes_l A)_l \xrightarrow{\partial_1} A_l \ .$$

Since k is of characteristic 0, the morphism $A_l \to A/([A,A]+l)$ induces a quasi-isomorphism from this complex to $A/([A,A]+l)$. The ISB-sequence

$$\ldots \longrightarrow \mathrm{HH}_n \xrightarrow{I} \mathrm{HC}_n \xrightarrow{S} \mathrm{HC}_{n-2} \xrightarrow{B} \mathrm{HH}_{n-1} \longrightarrow \ldots$$

is induced by the following sequence

$$\begin{tikzcd}
\ldots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & (V \otimes A)_l \arrow[r, "\partial_1"] \arrow[d, equal] & A_l \arrow[d, equal] \\
\ldots \arrow[r] & (V \otimes_l A)_l \arrow[r, "\partial_1"] \arrow[d, equal] & A_l \arrow[r, "\partial_0"] \arrow[d, equal] & (V \otimes_l A)_l \arrow[r, "\partial_1"] \arrow[d] & A_l \arrow[d] \\
\ldots \arrow[r] & (V \otimes_l A)_l \arrow[r, "\partial_1"] \arrow[d] & A_l \arrow[r] \arrow[d, "\partial_0"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
\ldots \arrow[r] & 0 \arrow[r] & (V \otimes_l A)_l \arrow[r, "\partial_1"] & A_l \arrow[r] & 0
\end{tikzcd}$$

Notice that the first three rows form a short exact sequence and that the composition of the last vertical morphism with the second last vertical morphism is only homotopic to zero.

Now let Q be a finite quiver, l the product over the ke_i, where i runs through the vertices of Q, w a potential on Q and $A = \Gamma$ the associated complete Ginzburg algebra with generators the arrows α of Q in degree 0, the reversed arrows α^* in degree -1 and the loops t_i in degree -2. Then A is of the form $(\widehat{T_l}(V), d)$, where V is the l-bimodule with basis given by the arrows α, α^* and t_i. Let t be the sum of the t_i. By definition, we have

$$d(t) = \sum_\alpha [\alpha, \alpha^*].$$

Thus, t defines an element in $HC_2(A)$.

Lemma 4.26. *The image of the class of t under $S : HC_2(A) \to HC_0(A)$ is the canonical class $[w]$, i.e. the image of w under the projection $HC_0(T_l V) \to HC_0(A)$.*

Proof. We compute $S(t)$ using the above description of S. We need to lift t to an element of the total complex computing cyclic homology. We have

$$d(t) = \partial_1 (\sum_\alpha \alpha \otimes \alpha^*).$$

We have

$$d(\sum_{\alpha} \alpha\otimes\alpha^*) = \sum_{\alpha} \alpha\otimes d(\alpha^*) = \sum_{\alpha} \alpha\otimes D_\alpha(w).$$

Thus, we have

$$d(\sum_{\alpha} \alpha\otimes\alpha^*) = \partial_0(w)$$

and $S(t)$ is the image of w in $H^0(A/([A,A]+l) = HC_0(\Gamma)$. Notice that $BS(t) = B(w)$ is indeed a boundary in the Hochschild complex: It is the differential of

$$\sum_{\alpha} \alpha\otimes\alpha^* - t.$$

$\square$

Corollary 4.27. *Let $\Gamma' = \Gamma(Q', w')$ be a Ginzburg algebra and A a pseudo-compact dg algebra in* $\mathrm{PCAlgc}(l)$ *concentrated in degrees ≤ 0. Let X be a dg A-Γ'-bimodule such that $?\overset{\mathbb{L}}{\otimes}_A X : \mathrm{D}(A) \to \mathrm{D}(\Gamma')$ is an equivalence. Then there is a quiver with potential (Q'', w'') and a weak equivalence $s : \Gamma(Q'', w'') \to A$ such that for the restriction ${}_sX$ along s, the isomorphism $HC_0({}_sX)$ takes the class $[w'']$ to $[w']$.*

Proof. We know that the class $[t'] \in HC_2(\Gamma')$ is non degenerate in the sense that $B[t'] \in HH_3(\Gamma')$ defines an isomorphism $\Sigma^3\Theta_{\Gamma'} \xrightarrow{\sim} \Gamma'$ in $\mathrm{D}(\Gamma'^e)$, where $\Theta_{\Gamma'}$ is the inverse dualizing complex. Thus the image τ of $[t']$ under $HC_2(X)^{-1}$ is a non degenerate element of $HC_2(A)$. The proof of Theorem 10.2.2 in [70] then shows that there is a quiver Q'', a potential w'' and a weak equivalence $s : \Gamma(Q'', w'') \to A$ which takes $[t'']$ to τ. Thus, the composition $HC_3(X) \circ HC_3(s) = HC_3({}_sX)$ takes $[t'']$ to $[t']$ and the isomorphism $HC_0({}_sX)$ takes $[w''] = S[t']$ to $[w'] = S[t']$. $\square$

5. CY TILTED ALGEBRAS AND SINGULARITIES

5.1. **Basics on Hochschild cohomology.** Let k be a commutative ring and A be a unital k-algebra projective over k. Denote by $\overline{A}$ the quotient $A/k\cdot 1$. Define the *normalized bar complex* associated to A to be the complex $B_kA := A\otimes_k T\Sigma\overline{A}\otimes_k A$ with differential $\sum_{i=0}^{n-1}(-1)^i b_i : A\otimes\overline{A}^{\otimes n-1}\otimes A \to A\otimes\overline{A}^{\otimes n-2}\otimes A$

$$b_i(a_0, \ldots, a_n) = (a_0, \ldots, a_i a_{i+1}, \ldots, a_n).$$

It is a projective bimodule resolution of A. Let M be an A-bimodule. The *Hochschild cochain complex* with coefficients in the bimodule M is defined to be the complex $C^*(A, M) := \mathrm{Hom}_{A^e}(B_k(A), M)$ with differential

$$\delta(f) = -(-1)^n f \circ b$$

for $f : A\otimes\overline{A}^{\otimes n}\otimes A \to M$. The i-th *Hochschild cohomology of the algebra A with coefficients in the bimodule* M is defined to be $\mathrm{HH}^i(A, M) := H^i(C^*(A, M), \delta)$.

Let A be an augmented dg k-algebra. Denote by $\overline{A}$ the kernel of the augmentation. Then the bar complex B_kA is equipped with a second differential induced from the differential d_A on A. Given an A-bimodule M, the Hochschild cochain complex $C^*(A, M)$ is equipped with a second differential d induced by d_A and the internal differential d_M on M. The ith *Hochschild cohomology of the dg algebra A with coefficients in the bimodule* M is defined to be $\mathrm{HH}^i(A, M) := H^i(C^*(A, M), d + \delta)$.

It is a well-known result that $\mathrm{HH}^i(A, M)$ is isomorphic to $\mathrm{Ext}^i_{A^e}(A, M)$. When A is a smooth commutative k-algebra, $\mathrm{HH}^*(A, A)$ is isomorphic to the polyvector fields on Spec A by the Hochschild-Kostant-Rosenberg theorem. For non-smooth algebras, there exist different variants of Hochschild cohomology.

Let A be an associative k-algebra projective over k. Define the module of *Kähler differentials* Ω_A to be the kernel of the multiplication map $\mu : A\otimes A \to A$. Clearly, Ω_A inherits a bimodule structure from $A\otimes A$. It is easy to show that Ω_A is generated as a bimodule by the elements of the form $xdy := xy\otimes 1 - x\otimes y$. The left and right module structure are given by

$$a(xdy) = (ax)dy,\ (xdy)a = xd(ya) - xyda.$$

Define the module of n-forms to be the n-fold tensor product

$$\Omega_A^n := \Omega_A\otimes_A\Omega_A\otimes \ldots \otimes_A\Omega_A.$$

Using the above identities, one can check that Ω_A^n is generated as a bimodule by the elements of the form $a_0da_1da_2\ldots da_n$. There is an isomorphism of bimodules $\Omega_A^n \cong A\otimes_k\overline{A}^{\otimes n}$ defined by

$$a_0da_1da_2\ldots da_n \mapsto a_0\otimes a_1\otimes \ldots \otimes a_n.$$

Set $\Omega_A^0 = A$ and $\Omega_A^1 = \Omega_A$. Write Ω_A^* for $\bigoplus_{n\geq 0}\Omega_A^n$. The bimodule structure on Ω_A naturally extends to an associative algebra structure on Ω_A^*. The obvious differential

$$D : a_0da_1da_2\ldots da_n \mapsto da_0da_1da_2\ldots da_n$$

makes Ω_A^* into a differential graded algebra.

Consider the chain maps $\theta_n : C^*(A, \Sigma^n\Omega_A^n) \to C^*(A, \Sigma^{n+1}\Omega_A^{n+1})$ between the Hochschild cochain complexes defined by $f \mapsto f\otimes \mathrm{Id}_{\Sigma A}$.

Definition 5.1. Let A be an associative k-algebra. Then the *singular Hochschild cochain complex* of A, denoted by $C^*_{sg}(A, A)$, is defined as the colimit of the inductive system in the category of cochain complexes of k-modules,

$$0 \longrightarrow C^*(A, A) \xrightarrow{\theta_0} C^*(A, \Sigma\Omega_A^1) \xrightarrow{\theta_1} \ldots \longrightarrow C^*(A, \Sigma^n\Omega_A^n) \xrightarrow{\theta_n} \ldots .$$

Namely, $C^*_{sg}(A, A) := \mathrm{colim}_n C^*(A, \Sigma^n\Omega_A^n)$. Its cohomology groups are denoted by $\mathrm{HH}^*_{sg}(A, A)$.

By construction, we have a natural chain morphism from $C^*(A, A)$ to $C^*_{sg}(A, A)$, which induces a natural morphism from $\mathrm{HH}^*(A, A)$ to $\mathrm{HH}^*_{sg}(A, A)$.

Let A be a Noetherian k-algebra. Define $\mathrm{D}_{sg}(A)$ to be the Verdier quotient of $\mathrm{D}^b(A)$ by the subcategory $\mathrm{per}(A)$. We denote the extension group in $\mathrm{D}_{sg}(A)$ by $\underline{\mathrm{Ext}}^i_A(?, ?)$. The singular Hochschild cohomology groups are related to the extension groups in $\mathrm{D}_{sg}(A^e)$.

Proposition 5.2. *(Theorem 3.6 [71]) Let A be a Noetherian k-algebra. Then there exists a natural isomorphism*

$$\mathrm{HH}^*_{sg}(A, A) \xrightarrow{\sim} \underline{\mathrm{Ext}}^*_{A^e}(A, A),$$

such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Ext}^*_{A^e}(A, A) \arrow[r] & \underline{\mathrm{Ext}}^*_{A^e}(A, A) \\
\mathrm{HH}^*(A, A) \arrow[r] \arrow[u, "\cong"] & \mathrm{HH}^*_{sg}(A, A) \arrow[u, "\cong"']
\end{tikzcd}$$

From Wang's result, we see that the singular Hochschild cohomology admits a structure of graded commutative algebra.

5.2. **Hochschild cohomology of Gorenstein algebras.** A (not necessarily commutative) Noetherian ring A is called *Gorenstein* if it has finite injective dimension both as a left and right A-module. As in the commutative case, we denote by CM_A the category of maximal Cohen-Macaulay (left) A-modules and denote by $\underline{\mathrm{CM}}_A$ its stable category. Buchweitz proved that if A is Gorenstein, then $\underline{\mathrm{CM}}_A$ is equipped with a structure of triangulated category and $\underline{\mathrm{CM}}_A \cong \mathrm{D}_{sg}(A)$.

We recall a fundamental result on extension groups in the stable category over Gorenstein rings due to Buchweitz.

Proposition 5.3. *(Corollary 6.3.4 of* [13]*) Let A be a Gorenstein ring and let X and Y be objects in $\mathrm{D}^b(A)$. There exists a positive integer m depending on A, X and Y such that the natural morphism $\mathrm{Ext}^i_A(X,Y) \to \underline{\mathrm{Ext}}^i_A(X,Y)$ is surjective for $i = m$ and is an isomorphism for $i > m$.*

Combining Proposition 5.2 and Proposition 5.3, we obtain the following result.

Corollary 5.4. *Let R be a commutative noetherian Gorenstein k-algebra. If $R \otimes R$ is noetherian, there exists a positive integer m such that for $i > m$, the natural morphism*

$$\mathrm{HH}^i(R,R) \to \mathrm{HH}^i_{sg}(R,R)$$

is an isomorphism.

Proof. By Proposition 5.2, we get a morphism $\mathrm{HH}^i(R,R) \to \mathrm{HH}^i_{sg}(R,R)$ for all i. In order to apply Proposition 5.3, we need to check that $R \otimes R$ is Gorenstein. This follows from Theorem 1.6 of [66]. ☐

A commutative local complete Gorenstein k-algebra $\hat{R}$ is called a hypersurface algebra if $\hat{R} \cong k[[x_1,\ldots,x_n]]/(g)$. We say that $\hat{R}$ is a hypersurface algebra with isolated singularities if g has an isolated critical point.

Theorem 5.5. *(Theorem 3.2.7* [23]*) Let $R = k[x_1,\ldots,x_n]/(g)$ be a hypersurface algebra with isolated singularities. Denote by M_g the Milnor algebra $k[x_1,\ldots,x_n]/(\frac{\partial g}{\partial x_1},\ldots,\frac{\partial g}{\partial x_n})$, and by K_g and T_g the kernel and cokernel of the endomorphism of M_g defined by multiplication with g. Then for $r \geq n$, there is an isomorphism of R-modules*

$$\begin{cases} \mathrm{HH}^r(R,R) \cong T_g & r \text{ is even,} \\ \mathrm{HH}^r(R,R) \cong K_g & r \text{ is odd.} \end{cases}$$

Proof. The proof in [23] shows that in degrees $r \geq n$, the Hochschild cohomology $\mathrm{HH}^r(R,R)$ is isomorphic to the homology in degree r of the complex

$$k[u] \otimes K(R, \frac{\partial g}{\partial x_1}, \ldots, \frac{\partial g}{\partial x_n}),$$

where u is of degree 2 and K denotes the Koszul complex. Put $P = k[x_1,\ldots,x_n]$. Since R is quasi-isomorphic to $K(P,g)$ and the $(\partial g)/(\partial x_i)$ form a regular sequence in P, the Koszul complex is quasi-isomorphic to $K(M_g, g)$. ☐

Note that T_g is the *Tyurina algebra* $k[x_1,\ldots,x_n]/(g, \frac{\partial g}{\partial x_1},\ldots,\frac{\partial g}{\partial x_n})$. Clearly K_g is a module over T_g.

Lemma 5.6. *Let A be a commutative k-algebra such that A and A^e are noetherian. Let $S \subset A$ be a multiplicative subset. If M is a finitely generated A-module and L an A-module, we have a canonical isomorphism*

$$\mathbf{R}\mathrm{Hom}_{A^e}(L, M)_S \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{A^e_S}(L_S, M_S).$$

Proof. Since L is finitely generated over A, it is finitely generated over A^e. Since A^e is noetherian, we have a projective resolution $P \to L$ with finitely generated components. This implies that we have isomorphisms

$$\mathbf{R}\mathrm{Hom}_{A^e}(L, M)_S = \mathrm{Hom}_{A^e}(P, M)_S = \mathrm{Hom}_{A^e}(P, M_S).$$

Since $A_S \otimes_A P \otimes_A A_S \to L_S$ is a projective resolution over A_S^e, we find

$$\mathrm{Hom}_{A^e}(P, M_S) = \mathrm{Hom}_{A_S^e}(A_S \otimes_A P \otimes_A A_S, M_S) = \mathbf{R}\mathrm{Hom}_{A_S^e}(L_S, M_S).$$

□

Remark 5.7. In the setting of Theorem 5.5, assume that g has isolated singularities and that the origin is a singular point of the vanishing locus of g. If we denote by $\mathfrak{m}$ the maximal ideal $(x_1, \ldots, x_n) \subset k[x_1, \ldots, x_n]$, then $g \in \mathfrak{m}$. Denote by $M_{g,\mathfrak{m}}$, $T_{g,\mathfrak{m}}$ and $K_{g,\mathfrak{m}}$ the localizations of M_g, T_g and K_g. It follows from the lemma that Theorem 5.5 still holds if one replaces R by $R_\mathfrak{m} := k[x_1, \ldots, x_n]_\mathfrak{m}/(g)$ and replaces T_g and K_g by $T_{g,\mathfrak{m}}$ and $K_{g,\mathfrak{m}}$.

For a Noetherian k-algebra A, the derived category of singularities $\mathrm{D}_{sg}(A)$ is equipped with a canonical dg enhancement, obtained from its construction as a Verdier quotient of two canonically enhanced triangulated categories [40, 17]. Instead of $\mathrm{HH}^*_{sg}(A, A)$, one may also consider the Hochschild cohomology of the dg category $\mathrm{D}_{sg}(A)$, which we will denote by $\mathrm{HH}^*(\mathrm{D}_{sg}(A))$.

Theorem 5.8. *(Keller [45]) There is a canonical isomorphism of graded algebras*

$$\mathrm{HH}^*(\mathrm{D}_{sg}(A)) \xrightarrow{\sim} \mathrm{HH}^*_{sg}(A, A).$$

Now we establish the main result of the subsection.

Theorem 5.9. *Let $\hat{R} = k[[x_1, \ldots, x_n]]/(g)$ be a hypersurface algebra with isolated singularity. Denote by $\hat{T}_g$ the Milnor algebra of g. Then there is an isomorphism of k-algebras*

$$\mathrm{HH}^0(\mathrm{D}_{sg}(\hat{R})) \cong \hat{T}_g.$$

Moreover, if $\hat{R}' = k[[x_1, \ldots, x_n]]/(g')$ is another hypersurface algebra with isolated singularity such that $\mathrm{D}_{sg}(\hat{R}')$ is equivalent with $\mathrm{D}_{sg}(\hat{R})$, then $\hat{R}'$ is isomorphic to $\hat{R}$.

Proof. Because g has an isolated critical point, we may assume that g is a polynomial without loss of generality. Denote by R the algebra $k[x_1, \ldots, x_n]_\mathfrak{m}/(g)$. Notice that R has an isolated singularity at the origin and that its completion identifies with $\hat{R}$. By Theorem 5.7 of [19], the trianglated category $\mathrm{D}_{sg}(\hat{R})$ is the Karoubi envelope of $\mathrm{D}_{sg}(R)$. Therefore, the two dg categories have equivalent derived categories and there is a natural isomorphism $\mathrm{HH}^*(\mathrm{D}_{sg}(R)) \xrightarrow{\sim} \mathrm{HH}^*(\mathrm{D}_{sg}(\hat{R}))$. Orlov proved in [57] that $\mathrm{D}_{sg}(R)$ is triangle equivalent with the homotopy category of matrix factorizations $\mathrm{MF}(k[x_1, \ldots, x_n]_\mathfrak{m}, g)$. The triangle equivalence is lifted to an equivalence of dg categories by the work of Blanc, Robalo, Toën and Vezzosi [7]. Therefore, the dg category $\mathrm{D}_{sg}(R)$ is 2-periodic and so is its Hochschild cohomology. So there exists a natural isomorphism of $\hat{R}$-modules

$$\mathrm{HH}^0(\mathrm{D}_{sg}(R)) \xrightarrow{\sim} \mathrm{HH}^{2r}(\mathrm{D}_{sg}(R))$$

for all $r \in \mathbb{Z}$. By Theorem 5.8, we have

$$\mathrm{HH}^{2r}(\mathrm{D}_{sg}(R)) \xrightarrow{\sim} \mathrm{HH}^{2r}_{sg}(R, R).$$

By Corollary 5.4, for $r \gg 0$, we have

$$\mathrm{HH}^{2r}(R, R) \xrightarrow{\sim} \mathrm{HH}^{2r}_{sg}(R, R)$$

and by Theorem 5.5 and Remark 5.7, we have

$$\mathrm{HH}^{2r}(R, R) \xrightarrow{\sim} T_g.$$

Because g has an isolated critical point, there is an isomorphism

$$\hat{T}_g \cong T_g.$$

Then the first claim follows. The second claim folllows from the formal version of the Mather-Yau theorem (see Theorem 1.2 [24]). ☐

5.3. **Classification of 3-dimensional smooth flops.** Let $(\widehat{Y}, \widehat{f}, R)$ be a 3-dimensional formal flopping contraction with $\mathrm{Ex}(f) = \bigcup_{i=1}^{t} C_i$, and let $A = \mathrm{End}_R(\bigoplus_{i=1}^{t} N_i \oplus R)$ be the NCCR associated to it. We have associated to it an exact 3CY algebra: the derived deformation algebra Γ of the semi-simple collection $\mathcal{O}_{C_1}, \ldots, \mathcal{O}_{C_t}$, and the cluster category $\mathcal{C}_\Gamma$. By the theorem of Van den Bergh, Γ is weakly equivalent to a Ginzburg algebra $\mathfrak{D}(Q, w)$ with t nodes. Recall that the CY tilted algebra $\Lambda := H^0\Gamma$ is isomorphic to $\mathrm{End}_{\mathcal{C}_\Gamma}(\pi(\Gamma))$. By [49], R is isomorphic to $k[[x, y, u, v]]/(g)$ for some g with isolated critical point. It is natural to ask whether these algebraic and categorical invariants can determine the flop analytically. One conjecture was formulated by Donovan and Wemyss for the CY tilted algebra.

Conjecture 5.10. *Let $(\widehat{Y}, \widehat{f}, R)$ and $(\widehat{Y}', \widehat{f}', R')$ be two 3-dimensional simple formal flopping contractions with associated CY tilted algebras Λ and Λ'. Then the following are equivalent*

(1) *R is isomorphic to R'.*
(2) *Λ is isomorphic to Λ'.*

Donovan and Wemyss have extended this conjecture to the case of not necessarily simple formal flopping contractions by replacing (2) with

(2') Λ is derived equivalent to Λ',

cf. Conjecture 1.3 in [4]. In this situation, the implication from (1) to (2) is known to be true by iterating a construction of Dugas [18]. The implication from (2') to (1) is one of the main open problems in the homological minimal model program for 3-folds. In this section, we will prove a slightly weaker version of this implication.

Theorem 5.11. *Let $(\widehat{Y}, \eta, \widehat{f}, R)$ and $(\widehat{Y}', \eta', \widehat{f}', R')$ be formal flopping contractions and denote by $(\Lambda, [w])$ and $(\Lambda', [w'])$ the associated CY tilted algebras and the canonical classes of potentials. If there exists a derived equivalence from Λ to Λ' given by a bimodule complex Z such that the induced map $\mathrm{HH}_0(Z)$ takes $[w]$ to $[w']$, then R is isomorphic to R'.*

The proof of this theorem will take up the rest of the section. The key idea of the proof is to reconstruct the Tyurina algebra of R from the CY tilted algebra Λ, and then apply the Mather–Yau theorem to some hypersurface singularities. To achieve this, we first prove that the Tyurina algebra of R can be recovered from the (dg-enhanced) cluster category $\mathcal{C}_\Gamma$, which is clearly determined by the derived deformation algebra Γ. In the last step, we prove that Γ can be reconstructed from the CY tilted algebra Λ together with a certain structure of classical shadow of the CY structure on Γ. This is where $[w]$ enters the picture.

Note that when R admits a NCCR, then $\mathrm{D}_{sg}(R)$ has another dg model via the triangle equivalence $\mathrm{D}_{sg}(R) \simeq \mathcal{C}_\Gamma$. We first prove that these two models are dg equivalent.

Lemma 5.12. *In the homotopy category of dg categories, there is an isomorphism between $\mathcal{C}_\Gamma = \mathrm{per}(\Gamma)/\mathrm{D}_{fd}(\Gamma)$ and the category of singularities $\mathrm{D}_{sg}(R) = \mathrm{D}^b(R)/K^b(\mathrm{proj}(R))$, both equipped with their canonical dg enhancements.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two pretriangulated dg categories. We call a triangle functor $F : H^0(\mathcal{A}) \to H^0(\mathcal{B})$ algebraic if there is a dg $\mathcal{A}$-$\mathcal{B}$-bimodule X such that we have a square of triangle functors, commutative up to isomorphism

$$\begin{tikzcd}
H^0(\mathcal{A}) \arrow[r, "F"] \arrow[d] & H^0(\mathcal{B}) \arrow[d] \\
\mathrm{D}(\mathcal{A}) \arrow[r, "?\overset{\mathbb{L}}{\otimes}_{\mathcal{A}} X"'] & \mathrm{D}(\mathcal{B}),
\end{tikzcd}$$

where the vertical arrows are induced by the Yoneda functors. We have to show that the triangle equivalence $\mathcal{C}_\Gamma \xrightarrow{\sim} \mathrm{D}_{sg}(R)$ is algebraic. We use the notations of subsection 2.4 and put $N = N_1 \oplus \ldots \oplus N_t$. Let $\mathcal{F}$ denote the thick triangulated subcategory of $\mathrm{per}(A)$ generated by the simples $S_1, \ldots, S_t$. Let us recall from Proposition 3 of [58] that we have a diagram of triangle functors, commutative up to isomorphism and whose rows and columns are exact sequences of triangulated categories

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathrm{per}(A)/\mathrm{per}(R) \arrow[r] \arrow[u] & \underline{\mathrm{CM}}_R \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[u, equal] & \mathrm{per}(A) \arrow[r] \arrow[u] & \mathrm{D}^b(\mathrm{CM}_R) \arrow[r] \arrow[u] & 0 \\
 & & \mathrm{per}(R) \arrow[r, equal] \arrow[u] & \mathrm{per}(R) \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Here the category $\mathrm{D}^b(\mathrm{CM}_R)$ is the bounded derived category of the exact category CM_R, the functor $\mathrm{per}(R) \to \mathrm{D}^b(\mathrm{CM}_R)$ is induced by the inclusion $\mathrm{proj}(R) \to \mathrm{CM}_R$, the functor $\mathrm{per}(A) \to \mathrm{D}^b(\mathrm{CM}_R)$ is induced by $?\overset{\mathbb{L}}{\otimes}_A (R \oplus N)$ and $\mathrm{per}(R) \to \mathrm{per}(A)$ is induced by the inclusion $\mathrm{add}(R) \to \mathrm{add}(R \oplus N) = \mathrm{proj}(A)$. We endow $\mathrm{per}(A)/\mathrm{per}(R)$ with the dg enhancement given by the dg quotient [40, 17]. It is then clear that the triangle functors of the middle row and of the middle column are algebraic. Let us show that the functor $\mathrm{D}^b(\mathrm{CM}_R) \to \underline{\mathrm{CM}}_R$ is algebraic. The canonical dg enhancement of $\underline{\mathrm{CM}}_R$ is given by the triangle equivalence from the homotopy category $H^0(\mathcal{A})$ of the dg category $\mathcal{A}$ of acyclic complexes over $\mathrm{proj}(R)$ to $\underline{\mathrm{CM}}_R$ taking an acyclic complex P to its zero cycles $Z^0(P)$. Let $\mathcal{B}$ be the dg enhanced derived category $\mathrm{D}^b_{dg}(\mathrm{CM}_R)$. We define a $\mathcal{B}$-$\mathcal{A}$-bimodule X by putting

$$X(P, M) = \mathrm{Hom}(P, M)$$

where P is an acyclic complex of finitely generated projective R-modules and M a bounded complex of finitely generated Cohen-Macaulay R-modules. One checks that the derived tensor product by X induces the canonical triangle functor $\mathrm{D}^b(\mathrm{CM}_R) \to \underline{\mathrm{CM}}_R$. It follows that, at the level of dg categories, $\underline{\mathrm{CM}}_R$ identifies with the dg quotient of $\mathrm{D}^b(\mathrm{CM}_R)$ by $\mathrm{per}(R)$. It follows that the induced functor $\mathrm{per}(A)/\mathrm{per}(R) \to \underline{\mathrm{CM}}_R$ is algebraic. Thus, the whole diagram is made up of algebraic functors. Now using the notations of Theorem 2.8 put $\widetilde{\Gamma} = (\widehat{T_l} V, d)$ so that we have a quasi-isomorphism $\widetilde{\Gamma} \xrightarrow{\sim} A$. It induces an algebraic equivalence $\mathrm{per}(\widetilde{\Gamma}) \xrightarrow{\sim} \mathrm{per}(A)$. This equivalence induces an algebraic equivalence $\mathrm{tria}(e_0 \widetilde{\Gamma}) \to \mathrm{per}(R)$, where $\mathrm{tria}(e_0 \widetilde{\Gamma})$ is the triangulated subcategory generated by the $\widetilde{\Gamma}$-module $e_0 \widetilde{\Gamma}$. The quotient map $\widetilde{\Gamma} \to \Gamma$ induces an algebraic triangle functor $\mathrm{per}(\widetilde{\Gamma}) \to \mathrm{per}(\Gamma)$ and we know from Lemma 7.2 of [36] that it is a localization with kernel $\mathrm{tria}(e_0 \widetilde{\Gamma})$. We obtain a diagram of triangle functors, commutative up to isomorphism, whose vertical arrows are equivalences and whose rows are exact.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{tria}(e_0\widetilde{\Gamma}) \arrow[r] \arrow[d] & \mathrm{per}(\widetilde{\Gamma}) \arrow[r] \arrow[d] & \mathrm{per}(\Gamma) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{per}(R) \arrow[r] & \mathrm{per}(A) \arrow[r] & \mathrm{per}(A)/\mathrm{per}(R) \arrow[r] & 0.
\end{tikzcd}$$

By passage to the dg quotient, the rightmost vertical arrow is an algebraic triangle equivalence. Similarly, we obtain a diagram with exact rows and whose vertical arrows are equivalences.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathrm{per}(\Gamma) \arrow[r] \arrow[d] & \mathcal{C}_\Gamma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F} \arrow[r] & \mathrm{per}(A)/\mathrm{per}(R) \arrow[r] & \underline{\mathrm{CM}}_R \arrow[r] & 0.
\end{tikzcd}$$

Again by passage to the dg quotient, the rightmost vertical arrow is an algebraic triangle equivalence. □

Proof of Theorem 5.11. Let (Q, w) and (Q', w') be the quivers with potential constructed from the formal flopping contractions and Γ and Γ' the associated Ginzburg dg algebras. By part (2) of Theorem 7.2 of [5], the silting quiver $\mathrm{silt}(\Lambda')$ is connected (notice that since Λ' is symmetric, tilting objects coincide with silting objects in $\mathrm{per}(\Lambda')$. Let A be the dg algebra obtained from Theorem 4.25. Its homologies $H^p(A)$ are finite-dimensional and vanish for $p > 0$. Thus, it is quasi-isomorphic to a dg algebra of the form $(\widehat{T}_l V, d)$, where V is a graded bimodule whose components vanish in degrees > 0 and are finite-dimensional in all degrees ≤ 0. So we may assume that A is in $\mathrm{PCAlgc}(l)$. By Theorem 4.25, there is moreover a dg A-Γ'-bimodule X yielding an equivalence $?\overset{\mathbb{L}}{\otimes}_A X : \mathrm{D}(A) \xrightarrow{\sim} \mathrm{D}(\Gamma')$ and an isomorphism $\phi : H^0(A) \to \Lambda$ such that we have an isomorphism

$$ {}_\phi Z \xrightarrow{\sim} X \overset{\mathbb{L}}{\otimes}_{\Gamma'} \Lambda' $$

in the derived category $\mathrm{D}(A^{op} \otimes \Lambda')$ and in particular a square, commutative up to isomorphism

$$\begin{tikzcd}
A \arrow[r, "X"] \arrow[d, "(H^0A)_\phi"'] & \Gamma' \arrow[d, "\Lambda'"] \\
\Lambda \arrow[r, "Z"'] & \Lambda',
\end{tikzcd}$$

where we write dg bimodules instead of derived tensor products and the top and bottom arrows are equivalences. By Corollary 4.27, there is a quiver with potential (Q'', w'') and a weak equivalence $s : \Gamma'' \to A$ from the associated Ginzburg algebra Γ'' to A such that the isomorphism $HC_0({}_sX)$ takes the class $[w'']$ to $[w']$. Thus the isomorphism $HH_0({}_{s\phi}(Z))$ takes $[w'']$ to $[w']$ and the isomorphism $\phi \circ H^0(s) : H^0(\Gamma'') \to \Lambda$ takes $[w'']$ to $[w]$. We may assume all potentials to contain no cycles of length ≤ 2 and then it follows that $\phi \circ H^0(s)$ induces an isomorphism of quivers $Q'' \xrightarrow{\sim} Q$. By Corollary 4.14, there is an isomorphism $\beta : \Gamma'' \to \Gamma$. The dg bimodule ${}_{s\beta^{-1}}X$ yields an algebraic triangle equivalence $\mathrm{per}(\Gamma) \xrightarrow{\sim} \mathrm{per}(\Gamma')$. By Lemma 5.12, it induces an algebraic triangle equivalence $\mathrm{D}_{sg}(R) \xrightarrow{\sim} \mathrm{D}_{sg}(R')$ and therefore an algebra isomorphism $\mathrm{HH}^0(\mathrm{D}_{sg}(R)) \cong \mathrm{HH}^0(\mathrm{D}_{sg}(R'))$. By Theorem 5.9, we get an isomorphism $R \cong R'$. □

6. CONTRACTIBILITY OF RATIONAL CURVE

6.1. **dg $k[u^{-1}]$-algebras.** In this section, we define dg-$k[u^{-1}]$-algebras and study their properties. All the definitions and results can be adapted to the pseudo-compact case, with notations appropriately replaced by their pseudo-compact counterparts.

Definition 6.1. Let S be a commutative dg algebra. Denote by $\mathscr{C}(S)$ the category of complexes of S-modules with the monoidal structure given by the tensor product over S. A dg S-algebra is an algebra in $\mathscr{C}(S)$, i.e. a dg S-module with an S-bilinear multiplication and a unit.

Let k be a field, S a commutative dg k-algebra and Γ a dg S-algebra. By restriction, each dg Γ-module becomes a dg S-module and the morphism complexes between dg Γ-modules are naturally dg S-modules. Thus, the derived category $\mathrm{D}(\Gamma)$ is naturally enriched over $\mathrm{D}(S)$.

Definition 6.2. Let $S = k[u^{-1}]$ be the commutative dg algebra with $\deg(u) = 2, \deg(u^{-1}) = -2$ and zero differential. We call a dg k-algebra A $k[u^{-1}]$-*enhanced* if A is isomorphic to a dg $k[u^{-1}]$-algebra in the homotopy category of dg k-algebras.

Let us put $S = k[u^{-1}]$ and $K = k[u, u^{-1}]$. For a dg S-module M, we have a canonical isomorphism

$$H^*(M \otimes_S K) = H^*(M) \otimes_S K.$$

We call M a *torsion module* if $M \otimes_S K$ is acyclic. This happens if and only if $H^*(M)$ is a torsion module, i.e. for each m in $H^*(M)$, there exists a $p \gg 0$ such that $mu^{-p} = 0$.

Let A be a dg S-algebra. The functor taking a dg A-module M to the dg $A \otimes_S K$-module $M \otimes_S K$ preserves quasi-isomorphisms. Thus, it induces a functor $? \otimes_S K : \mathrm{D}(A) \to \mathrm{D}(A \otimes_S K)$. The kernel of this functor consists of the dg A-modules which are torsion as dg S-modules. We write $\mathrm{D}(A)_{u^{-1}\text{-}tor}$ for the kernel and $\mathrm{per}(A)_{u^{-1}\text{-}tor}$ for its intersection with the perfect derived category $\mathrm{per}(A)$.

Lemma 6.3. *We have exact sequences of triangulated categories*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{D}(A)_{u^{-1}\text{-}tor} \arrow[r] & \mathrm{D}(A) \arrow[r] & \mathrm{D}(A \otimes_S K) \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{per}(A)_{u^{-1}\text{-}tor} \arrow[r] & \mathrm{per}(A) \arrow[r] & \mathrm{per}(A \otimes_S K) \arrow[r] & 0. \end{tikzcd}$$

Proof. The restriction along $A \to A \otimes_S K$ induces a fully faithful right adjoint to $? \otimes_S K : \mathrm{D}(A) \to \mathrm{D}(A \otimes_S K)$. Thus, the latter functor is a localization functor. By definition, its kernel is $\mathrm{D}(A)_{u^{-1}\text{-}tor}$ so that we obtain the first sequence. To deduce the second one, it suffices to show that the kernel of $\mathrm{D}(A) \to \mathrm{D}(A \otimes_S K)$ is compactly generated. Indeed, let P be the cone over the morphism

$$A \to \Sigma^{-2} A$$

given by the multiplication by u^{-1}. Clearly P is compact and we claim that it generates the kernel. For this, it suffices to show that the right orthogonal of P in the kernel vanishes. Indeed, let M be in the kernel. If $\mathbf{R}\mathrm{Hom}_A(C, M)$ vanishes, then the morphism $\Sigma^2 M \to M$ given by the multiplication by u^{-1} is a quasi-isomorphism. Thus, u^{-1} acts in $H^*(M)$ by an isomorphism. But on the other hand, $H^*(M)$ is torsion. So $H^*(M)$ vanishes and M is acyclic as was to be shown. □

A dg S-algebra A concentrated in non positive degrees is *non degenerate* if the morphism $A \to \Sigma^{-2} A$ given by the multiplication by u^{-1} induces isomorphisms $H^n(A) \xrightarrow{\sim} H^{n-2}(A)$ for all $n \leq 0$.

Lemma 6.4. *Let A be a dg k-algebra concentrated in nonpositive degrees. Assume that A is homologically smooth, that $H^n(A)$ is finite-dimensional for each $n \in \mathbb{Z}$ and that A admits a non degenerate $k[u^{-1}]$-enhancement. Then the subcategory $\mathrm{D}_{fd}(A) \subset \mathrm{per}(A)$ coincides with $\mathrm{per}(A)_{u^{-1}\text{-}tor}$.*

Proof. Since A is homologically smooth, $\mathrm{D}_{fd}(A)$ is contained in $\mathrm{per}(A)$ and clearly it consists of torsion modules. Conversely, we know from the proof of Lemma 6.3 that $\mathrm{per}(A)_{u^{-1}\text{-}tor}$ is the thick subcategory of $\mathrm{D}(A)$ generated by the cone C over the morphism $A \to \Sigma^{-2}A$ given by the multiplication by u^{-1}. Since A is non degenerate, the object C lies in $\mathrm{D}_{fd}(A)$. □

Proposition 6.5. *Let Q be a finite quiver with potential w such that the associated Ginzburg algebra $\Gamma = \mathfrak{D}(Q,w)$ has finite-dimensional Jacobi algebra. Assume that Γ is equipped with a nondegenerate $k[u^{-1}]$-enhancement. Then $\mathcal{C}_\Gamma$ is a $\mathbb{Z}/2$-graded 0CY triangulated category, equivalent with the category of perfect modules over $A \otimes_{k[u^{-1}]} k[u,u^{-1}]$ as $k[u,u^{-1}]$-enhanced triangulated categories. In particular, the Jacobi algebra $H^0(\Gamma)$ is a symmetric Frobenius algebra.*

Proof. We may assume that Γ itself is a differential graded $k[u^{-1}]$-algebra. The multiplication with u^{-1} yields a functorial morphism $\mathrm{Id} \to \Sigma^{-2}$ of triangle functors $\mathrm{D}(\Gamma) \to \mathrm{D}(\Gamma)$ and $\mathrm{per}(\Gamma) \to \mathrm{per}(\Gamma)$. This induces a functorial morphism of triangle functors $\mathrm{Id} \to \Sigma^{-2}$ in the cluster category $\mathcal{C}_\Gamma$. To check that it is invertible, it is enough to check that its action on the cluster-tilting object $\Gamma \in \mathcal{C}_\Gamma$ is invertible (since $\mathcal{C}_\Gamma$ equals its thick subcategory generated by Γ). Now by our assumption, the morphism $u^{-1} : \Sigma^2\Gamma \to \Gamma$ induces isomorphisms in H^n for $n \leq -2$ and the 0-map for $n \geq -1$. Thus, it induces a quasi-isomorphism $\Sigma^2\Gamma \to \tau_{\leq -2}\Gamma$. Since the canonical morphism $\tau_{\leq -2}\Gamma \to \Gamma$ becomes invertible in the cluster category, it follows that u induces an isomorphism $\Sigma^2\Gamma \xrightarrow{\sim} \Gamma$ in $\mathcal{C}_\Gamma$. The rest follows because we have isomorphisms of $H^0(\Gamma)$-bimodules

$$H^0(\Gamma) \xrightarrow{\sim} \mathcal{C}_\Gamma(\Gamma,\Gamma) \xrightarrow{\sim} \mathcal{C}_\Gamma(\Gamma,\Sigma^2\Gamma) \xrightarrow{\sim} D\mathcal{C}_\Gamma(\Gamma,\Gamma).$$

□

6.2. **$k[u^{-1}]$-structures on Ginzburg algebras associated to contractible curves.** Recall that for a Jacobi-finite Ginzburg algebra $\Gamma := \mathfrak{D}(Q,w)$ associate to quiver Q with only one vertex (and multiple loops), $H^0(\Gamma)$ is self-injective. On the other hand, if Γ is $k[u^{-1}]$-enhanced then $H^0(\Gamma)$ is symmetric.

The simplest case is when $\Gamma = k[t]$ with zero differential. It is $k[u^{-1}]$-enhanced by setting $u^{-1} = t$. This is the derived deformation algebra for $\mathcal{O}_C$ of a $(-1,-1)$-curve, which is always contractible.

Proposition 6.6. *Let $F = k\langle\langle x\rangle\rangle$ be the complete free algebra of one generator and $w \in F$ be an element with no constant term. Then $\Gamma := \mathfrak{D}(F,w)$ is $k[u^{-1}]$-enhanced.*

Proof. A general element $w \in F$ is of the form:

$$w = x^{n+1} + \text{higher order terms}.$$

We assume that $n \geq 2$. The Jacobi algebra of $\mathfrak{D}(F,w)$ is isomorphic to $k[[x]]/(x^n)$. It is always finite dimensional. Because $w = x^{n+1} \cdot u$ for some unit $u \in k[[x]]$, $[w] = 0$ in $k[[x]]/(x^n)$. By Theorem 4.16, w is right equivalent to x^{n+1}. Without loss of generality, we may assume $w = \frac{x^{n+1}}{n+1}$ to begin with. Then the Ginzburg algebra $\mathfrak{D}(F,w)$ is isomorphic to $k\langle\langle x,\theta,t\rangle\rangle$ with $dt = [x,\theta]$ and $d\theta = x^n$. It is easy to check that the two-sided differential ideal $(t,[x,\theta])$ is acyclic. As a consequence, the quotient morphism

$$\Gamma = (k\langle\langle x,\theta,t\rangle\rangle, d) \to \Gamma' := (k\langle\langle x,\theta,t\rangle\rangle/(t,[x,\theta]), d)$$

is a quasi-isomorphism of dg algebras. Note that Γ' is isomorphic to the complex

$$\ldots k[[x]]\theta^3 \xrightarrow{d} k[[x]]\theta^2 \xrightarrow{d} k[[x]]\theta \xrightarrow{d} k[[x]] \longrightarrow 0$$

where

$$d(\theta^{2k}) = 0,\ d(\theta^{2k+1}) = x^n\theta^{2k}.$$

Define the action of u^{-1} on Γ' by multiplication by θ^2. It is easy to check it makes Γ' a dg $k[u^{-1}]$-algebra. □

From the above proposition, we see that the Ginzburg algebras associated to the "one-loop quiver" are essentially classified by the dimension of their Jacobi algebras. Moreover, they all admit $k[u^{-1}]$-enhancements. If $\dim_k(H^0\Gamma) = n$ for $n > 1$, then Γ is equivalent to the derived deformation algebra of a floppable $(0,-2)$ curve of width n (see [55] for the geometric definition of width). The following corollary can be viewed as a noncommutative counter part of the classification theorem of Reid [55].

Corollary 6.7. *Let C be a rational curve in a quasi-projective smooth CY 3-fold Y of normal bundle $\mathcal{O}_C \oplus \mathcal{O}_C(-2)$. Denote its derived deformation algebra by Γ. Then*

1. *C is movable if and only if Γ has infinite dimensional Jacobi algebra;*
2. *If C is rigid then it is contractible. The dimension of $H^0\Gamma$ is equal to n for $n > 1$ if and only if the underlying singularity is isomorphic to the germ of hypersurface $x^2+y^2+u^2+v^{2n}=0$ at the origin.*

It is proved by Laufer (c.f. [59]) that a contractible rational curve in a CY 3-fold must have normal bundle of types $(-1,-1)$, $(0,-2)$ or $(1,-3)$. Donovan and Wemyss give an example of a rigid rational curve of type $(1,-3)$ that is not nc rigid (see Example 6.4 in [16]). In their example, there exists a birational morphism that contracts a divisor containing the $(1,-3)$ curve. Kawamata asked whether it is true that C is contractible if it is nc rigid (see Question 6.6 of [35]). We formulate a conjecture in terms of the derived deformation algebra.

Conjecture 6.8. *Let $C \subset Y$ be a nc rigid rational curve in a smooth quasi-projective CY 3-fold. Denote its associated derived deformation algebra by Γ_C^Y. Then C is contractible if and only if Γ_C^Y is $k[u^{-1}]$-enhanced.*

Note that one direction of the conjecture follows from our Theorem 4.17.

Proposition 6.9. *Let $C \subset Y$ be a contractible rational curve in a smooth quasi-projective CY 3-fold. Then Γ_C^Y is $k[u^{-1}]$-enhanced.*

Proof. Denote R for the ring of formal functions on the singularity underlying the contraction. For simplicity, we denote the derived deformation algebra Γ_C^Y by Γ. By Proposition 5.12, $\mathcal{C}_\Gamma$ is quasi-equivalent to $\mathrm{D}_{sg}(R)$ as dg categories. Under the equivalence, the projection image of Γ is identified with the Cohen-Macaulay module $N \in \mathrm{D}_{sg}(R) \cong \underline{\mathrm{CM}}_R$. By Theorem 4.17, Γ is isomorphic to $\tau_{\le 0}\Lambda_{dg}$ where Λ_{dg} is the dg endomorphism algebra of N in $\underline{\mathrm{CM}}_R$. Because R is a hypersurface ring, the dg category $\underline{\mathrm{CM}}_R$ carries a canonical $\mathbb{Z}/2$-graded structure (equivalently $k[u,u^{-1}]$-structure) by Eisenbud's theorem [20]. Therefore, Γ is $k[u^{-1}]$-enhanced. $\square$

We have already seen that in $(-1,-1)$ and $(0,-2)$ cases, the $k[u^{-1}]$-structure on Γ can be computed explicitly. However, we don't have any explicit construction for the $(1,-3)$ case even though we know it must exist. We do have an explicit formula for the symmetric Frobenius structure on the CY tilted algebra $H^0\Gamma$ in term of the residue map of matrix factorizations (see [27]).

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