INVARIANCE OF CYCLIC HOMOLOGY UNDER DERIVED EQUIVALENCE[1]

Bernhard Keller
ICRA VII, México, 1994

Abstract. We show that two algebras (projective over a commutative ground ring k) which are derived equivalent [23] share their cyclic homology. In particular, iterated tilting [9] [1] preserves cyclic homology. This completes results of Rickard's [23] and Happel's [7]. It also extends well known results on preservation of cyclic homology under Morita equivalence [21], [13], [22].

1. Hochschild homology and Cyclic homology

1.1 Definitions. We follow [21], [19], [20]. Let k be a commutative ring and A a k-algebra (associative with 1). The *Hochschild complex* $H(A)$ associated with A has the components

$$C_n = A^{\otimes(n+1)} \qquad (\text{where } \otimes = \otimes_k)$$

in degrees $n \geq 0$ and vanishing components in degrees $n < 0$. It is endowed with the differential

$$b := \sum_{i=0}^{n} (-1)^i d_i \,,$$

where $d_i : C_n \to C_{n-1}$ is defined by

$$d_i\,(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \begin{cases} a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n & \text{if } i < n \\ a_n a_0 \otimes \ldots \otimes a_{n-1} & \text{if } i = n. \end{cases}$$

By definition, *Hochschild homology of* A is $\mathrm{HH}_n\, A = \mathrm{H}_n H(A)$, $n \in \mathbf{Z}$. If A is projective over k, these groups admit the intrinsic interpretation

$$\mathrm{HH}_n\, A \xrightarrow{\sim} \mathrm{Tor}_n^{A^e}(A, A)\,, \quad \text{where } A^e = A \otimes A^{\mathrm{op}}.$$

To define *cyclic homology*, one introduces the operator

$$t : C_n \to C_n\,,\; t(a_0 \otimes \ldots \otimes a_n) = (-1)^n a_n \otimes a_1 \otimes \ldots \otimes a_{n-1}.$$

[1]1991 AMS Subject Classification: Primary 16E40; Secondary 18E30, 18G60.

Appeared in: Representation theory of algebras (Cocoyoc, 1994), 353–361, CMS Conf. Proc., 18, Amer. Math. Soc., Providence, RI, 1996.

Then the differential $b : C_n \to C_{n-1}$ induces a map in the coinvariants

$$\bar{b} : C_n/(1-t) \to C_{n-1}/(1-t) \,,$$

and, if k contains $\mathbf{Q}$, cyclic homology may be defined as the homology of the complex of coinvariants. If k is arbitrary, cyclic homology of A is the homology of the simple complex associated with the following double complex (the horizontal differential commutes with the vertical differential)

$$\begin{array}{ccccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
A^{\otimes 3} & \overset{1-t}{\leftarrow} & A^{\otimes 3} & \overset{N}{\leftarrow} & A^{\otimes 3} & \overset{1-t}{\leftarrow} & A^{\otimes 3} & \overset{N}{\leftarrow} \\
b\downarrow & & b'\downarrow & & b\downarrow & & b'\downarrow & \\
A^{\otimes 2} & \overset{1-t}{\leftarrow} & A^{\otimes 2} & \overset{N}{\leftarrow} & A^{\otimes 2} & \overset{1-t}{\leftarrow} & A^{\otimes 2} & \overset{N}{\leftarrow} \\
b\downarrow & & b'\downarrow & & b\downarrow & & b'\downarrow & \\
A & \overset{1-t}{\leftarrow} & A & \overset{N}{\leftarrow} & A & \overset{1-t}{\leftarrow} & A & \overset{N}{\leftarrow}
\end{array}$$

where $b' = \sum_{i=1}^{n-1}(-1)^i d_i$ and $N = 1 + t + t^2 + \ldots + t^n$. *There is no intrinsic interpretation of cyclic homology analogous to that of Hochschild homology.*

Remark. As the referee pointed out, in the notations of [12], one has the formula

$$\mathrm{HC}_* \, M = \mathrm{Tor}\,_*^{\Lambda}(k, M)$$

due to C. Kassel [12] for every mixed complex M. In particular, if $M(A)$ is the mixed complex associated with an algebra A, we find a formula bearing some resemblance to the intrinsic interpretation of Hochschild homology. However, this formula only gives an intrinsic interpretation of $\mathrm{HC}_* \, A$ in terms of $M(A)$ and one is then lead to ask for an intrinsic interpretation of the functor assigning $M(A)$ to A.

1.2 Properties.

1) We have $\mathrm{HC}_0 \, A = \mathrm{HH}_0 \, A = A/[A, A]$.

2) Cyclic homology and Hochschild homology are linked by Connes' long exact sequence
$$\mathrm{HH}_n \xrightarrow{I} \mathrm{HC}_n \xrightarrow{S} \mathrm{HC}_{n-2} \xrightarrow{B} \mathrm{HH}_{n-1} \,.$$
In particular, a morphism of algebras $A \to B$ inducing an isomorphism in Hochschild homology also induces an isomorphism in cyclic homology.

3) Cyclic homology and Hochschild homology are Morita invariants [21], [13], [22].

4) If A and B are k-algebras and M is an A-B-bimodule, then we have a canonical isomorphism
$$\mathrm{HC}_* \begin{pmatrix} A & M \\ 0 & B \end{pmatrix} \xrightarrow{\sim} \mathrm{HC}_*(A) \oplus \mathrm{HC}_*(B)$$
and similarly for Hochschild homology [11].

1.3 Examples.

1) We have an isomorphism of graded k -modules $\mathrm{HC}_*(k) \xrightarrow{\sim} k[u]$, where u is of degree 2.

2) If Q is a directed quiver and $I \subset kQ$ an admissible ideal, then we have an isomorphism
$$\mathrm{HC}_*(kQ/I) \xrightarrow{\sim} \bigoplus_{x \in Q_0} k[u]\,,$$
where Q_0 is the set of vertices of Q and $k[u]$ is defined as in example 1). This statement follows from property 4) by induction [3].

3) We have
$$\mathrm{HC}_n\,(k[\varepsilon]/(\varepsilon^2)) \xrightarrow{\sim} \mathrm{HC}_n\,(k) \oplus \bigoplus_{p=0}^{n} \mathrm{H}_{n-p}(\mathbf{Z}/(p+1), k).$$

4) More generally, C. Cibils has computed cyclic and Hochschild homology of all basic finite-dimensional algebras with vanishing radical square over a field [4].

5) D. Burghelea has computed cyclic homology of group algebras in terms of classifying spaces [2].

We refer to [20] and the references therein for numerous further examples.

2. Derived equivalences and cyclic homology

2.1 Derived equivalences. Let k be a commutative ring and let A and B be two k-algebras. Denote by $\mathcal{D}A$ the derived category of $\mathrm{Mod}\,A$, the category of (right) A-modules, and by $\mathcal{D}^b(\mathrm{Mod}\,A)$ the bounded derived category. Thus the objects of $\mathcal{D}A$ are all complexes of A-modules wheras the objects of $\mathcal{D}^b(\mathrm{Mod}\,A)$ are all bounded complexes. Denote by $\mathrm{per}\,A$ the full subcategory formed by complexes quasi-isomorphic to *perfect* complexes, i.e. bounded complexes of finitely generated projective A-modules. Suppose that B is flat as a k-module.

Theorem. *The following statements are equivalent*

(i) There is a complex of A-B-bimodules such that the total derived functor $? \otimes_A^{\mathbf{L}} X : \mathcal{D}A \to \mathcal{D}B$ *is an equivalence.*

(ii) There is a triangle equivalence $\mathcal{D}A \xrightarrow{\sim} \mathcal{D}B$.

(iii) There is a triangle equivalence $\mathcal{D}^b(\mathrm{Mod}\,A) \to \mathcal{D}^b(\mathrm{Mod}\,B)$.

(iv) There is a triangle equivalence $\mathrm{per}\,A \xrightarrow{\sim} \mathrm{per}\,B$.

(v) There is a complex $T \in \mathcal{D}B$ *such that*

  a) T *lies in* $\mathrm{per}\,B$, *and*

  b) we have $\mathrm{Hom}_{\mathcal{D}B}(T,T) \cong A$ *and* $\mathrm{Hom}_{\mathcal{D}B}(T,T[n]) = 0$ *for all* $n \neq 0$, *and*

  c) $\mathrm{per}\,B$ *equals the smallest strictly full triangulated subcategory of* $\mathcal{D}B$ *containing* T *and closed under forming direct summands.*

This theorem is due to J. Rickard [23], [24] (cf. also [14], [15], [16]). If B is right coherent, then the statements of the theorem are equivalent to

(vi) There is a triangle equivalence $\mathcal{D}^b(\mathrm{mod}\,A) \to \mathcal{D}^b(\mathrm{mod}\,B)$,

where $\operatorname{mod} A$ denotes the category of finitely presented A-modules. A complex T satisfying condition *v)* is called a *tilting complex* (note that we only assume here that T is *quasi-isomorphic* to a perfect complex; T is not assumed to be perfect). A complex of bimodules X satisfying condition *i)* is called a *bimodule tilting complex*. If X is a bimodule tilting complex then its restriction to B is a tilting complex. Derived equivalences between finite-dimensional algebras arise from tilting theory [9], [1], [8]. Each tilting module in the sense of [1] may be viewed as a bimodule tilting complex.

2.2 Invariance of Hochschild homology. Let k be a commutative ring and A and B two k-algebras which are projective over k. Suppose that there is a complex of A-B-bimodules X such that the functor

$$? \otimes_A^{\mathbf{L}} X : \mathcal{D}A \to \mathcal{D}B$$

is an equivalence, i.e. X is a bimodule tilting complex. The following theorem is due to D. Happel [7] and J. Rickard [24].

Theorem. *There is an isomorphism* $\mathrm{HH}_* A \cong \mathrm{HH}_* B$.

Sketch of proof. It is proved in [24] that there is a B-A-bimodule tilting complex Y and isomorphisms

$$X \otimes_B^{\mathbf{L}} Y \xrightarrow{\sim} A \text{ in } \mathcal{D}(A \otimes A^{\mathrm{op}}) \quad \text{and} \quad Y \otimes_A^{\mathbf{L}} X \xrightarrow{\sim} B \text{ in } \mathcal{D}(B \otimes B^{\mathrm{op}}).$$

Now using [24] again we have the following chain of isomorphisms in $\mathcal{D}k$

$$A \otimes_{A^e}^{\mathbf{L}} A \xrightarrow{\sim} (X \otimes_B^{\mathbf{L}} Y) \otimes_{A^e}^{\mathbf{L}} (X \otimes_B^{\mathbf{L}} Y) \xrightarrow{\sim} (Y \otimes_A^{\mathbf{L}} X) \otimes_{B^e}^{\mathbf{L}} (Y \otimes_A^{\mathbf{L}} X) \xrightarrow{\sim} B \otimes_{B^e}^{\mathbf{L}} B.$$

The claim follows since we have

$$\mathrm{HH}_n\, A \xrightarrow{\sim} \mathrm{Tor}_n^{A^e}(A, A) \xrightarrow{\sim} \mathrm{H}_n(A \otimes_{A^e}^{\mathbf{L}} A)$$

and similarly for B.

2.3 Invariance of Cyclic homology. Keep the hypotheses of (2.2). The absence of an intrinsic interpretation of cyclic homology makes the following theorem harder to prove than theorem (2.2).

Theorem. *There is an isomorphism* $\mathrm{HC}_* X : \mathrm{HC}_* A \xrightarrow{\sim} \mathrm{HC}_* B$.

Remarks. The morphism $\mathrm{HC}_* X$ is functorial in the following sense: Consider the full subcategory $\operatorname{rep}(A, B)$ of the derived category of A-B-bimodules formed by the bimodule complexes X which when restricted to B become quasi-isomorphic to perfect complexes (compare with [13]). One can show [17] that each such complex X gives rise to a morphism in cyclic homology

$$\mathrm{HC}_*(X) : \mathrm{HC}_*(A) \to \mathrm{HC}_*(B).$$

This morphism is functorial in the sense that if we view A as an A-A-bimodule complex, then $\mathrm{HC}_*(A) = \mathbf{1}$ and if $Y \in \operatorname{rep}(B, C)$ then $\mathrm{HC}_*(X \otimes_B^{\mathbf{L}} Y) = \mathrm{HC}_*(Y) \circ \mathrm{HC}_*(X)$.

Moreover, we show that $\mathrm{HC}_*(X)$ only depends on the class of X in the Grothendieck group of the triangulated category $\operatorname{rep}(A, B)$. These Grothendieck groups are naturally viewed as the morphism spaces of a category whose objects are all algebras.

A K*-theoretic equivalence* is an isomorphism of this category. Thus, cyclic homology is invariant under K-theoretic equivalence. For example, a finite-dimensional algebra over an algebraically closed field is K-theoretically equivalent to its largest semi-simple quotient (cf. [17]). Thus, if k is an algebraically closed field, the cyclic homology of a finite-dimensional algebra A of finite global dimension only depends on the number of isomorphism classes of simple A-modules. This yields the 'no loops conjecture' in the algebraically closed case, which was first proved by H. Lenzing [18]. We refer to K. Igusa's article [10] for a proof under more general hypotheses.

Proof. Consider the object $T = A \otimes_A^{\mathbf{L}} X$ of $\mathcal{D}B$. By section (2.1), it is quasi-isomorphic to a perfect complex over B. In particular, X lies in $\mathcal{D}^-(A^{\mathrm{op}} \otimes B)$. So there is a right bounded complex of projective $A^{\mathrm{op}} \otimes B$-modules P and a quasi-isomorphism $P \to X$ over $A^{\mathrm{op}} \otimes B$. Since A is projective over k, the components of P are projective over B. Thus, for any $n \in \mathbf{Z}$, the canonical map

$$\mathrm{Hom}_{\mathcal{H}B}(P, P[n]) \to \mathrm{Hom}_{\mathcal{D}B}(P, P[n])$$

is bijective. By the faithfulness of $? \otimes_A^{\mathbf{L}} X$, we have isomorphisms

$$\begin{array}{rcll} 0 & = & \mathrm{Hom}_{\mathcal{H}B}(P, P[n]) \,, & \text{for } n \neq 0 \\ A & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{H}B}(P, P). & \end{array} \qquad (1)$$

Now consider the endomorphism complex $C = \mathcal{H}om_B\,(P, P)$. Recall that C is a complex of k-modules whose nth component is formed by homogeneous maps of degree n

$$f : \bigoplus_{p \in \mathbf{Z}} P^p \to \bigoplus_{q \in \mathbf{Z}} P^q$$

of $\mathbf{Z}$-graded B-modules. It is endowed with the differential defined by

$$df = d \circ f - (-1)^n f \circ d$$

where $f \in C^n$. The composition of graded maps makes C into a differential graded k-algebra in the sense of [16]. We claim that C is flat as a DG k-module, i.e. that $C \otimes N$ is acyclic for each acyclic DG k-module N. Indeed, P is homotopy equivalent to a finite complex of finitely generated projective B-modules P'. Since $\mathcal{H}om_B\,(?, ?)$ induces a functor

$$(\mathcal{H}B)^{\mathrm{op}} \times (\mathcal{H}B) \to \mathcal{H}k,$$

this implies that C is homotopy equivalent to $\mathcal{H}om_B\,(P', P')$. In turn, $\mathcal{H}om_B\,(P', P')$ is obtained from $\mathcal{H}om_B\,(B, B) = B$ by forming shifts, mapping cones and direct summands. Since B is projective over k, we conclude that $B \otimes N$ and hence $C \otimes N$ are acyclic whenever N is acyclic.

Let us now consider the Hochschild complex $H(C)$ associated with C. It is the simple complex associated with the double complex whose n-th column is $C \otimes C^{\otimes n}$ (vanishing columns for $n < 0$) and whose horizontal differential is given by

$$\begin{array}{rcl} d(c_0 \otimes c_1 \otimes \ldots \otimes c_n) & = & \displaystyle\sum_{i=0}^{n-1} (-1)^i c_0 \otimes \ldots \otimes c_{i-1} \otimes c_i c_{i+1} \otimes c_{i+2} \otimes \ldots \otimes c_n \\ & & +(-1)^n c_n c_0 \otimes c_1 \otimes \ldots \otimes c_{n-1}. \end{array}$$

The column filtration of the double complex is complete and its subquotients are shifted copies of the complexes $C \otimes C^{\otimes n}$. Now consider the map

$$\lambda : A \to C = \mathcal{H}om_B\,(P, P)$$

given by the left action of A on the components of P. It is well known (and easy to check) that we have canonical isomorphisms

$$\mathcal{H}om_{\mathcal{H}B}(P, P[n]) \xrightarrow{\sim} \mathrm{H}^n C.$$

The formulae (1) therefore imply that λ is a quasi-isomorphism. Since $A\otimes ?$ and $C\otimes ?$ both preserve acyclicity, it follows that λ induces quasi-isomorphisms in all tensor powers $A \otimes A^{\otimes n} \to C \otimes C^{\otimes n}$ and hence a quasi-isomorphism between the Hochschild complexes associated with A and C. Now the cyclic complexes admit complete filtrations whose subquotients are homotopy equivalent to shifted copies of the Hochschild complexes. Thus λ also induces isomorphisms in the cyclic complexes.

We will now compare C to B. Consider the morphisms of DG algebras

$$\mathcal{H}om_B(P, P) \xrightarrow{\alpha} \mathcal{H}om_B(P \oplus Q, P \oplus Q) \xleftarrow{\beta} \mathcal{H}om_B(B, B) = B$$

given by

$$\alpha(f) = \begin{bmatrix} 0 & 0 \\ 0 & f \end{bmatrix}, \quad \beta(b) = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}.$$

We will show that α and β induce isomorphisms in Hochschild homology (and hence in cyclic homology). Now as a complex of B-modules, P is homotopy equivalent to a tilting complex. So B and P may be obtained from one another by forming shifts, mapping cones and direct summands. Our claim is therefore a consequence of the following

Lemma. *Let P and Q be complexes of B-modules which are homotopy equivalent to perfect complexes. Suppose that Q may be obtained from P by forming shifts, mapping cones and direct summands. Then the canonical map*

$$\mathcal{H}om_B(P, P) \to \mathcal{H}om_B(P \oplus Q, P \oplus Q), \ f \mapsto \begin{bmatrix} f & 0 \\ 0 & 0 \end{bmatrix}$$

induces isomorphisms in Hochschild homology and cyclic homology.

Proof. Put $C = \mathcal{H}om_B(P, P)$ and $D = \mathcal{H}om_B(P \oplus Q, P \oplus Q)$ and let

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in D.$$

The Hochschild resolution M associated with D is the total complex associated with the double complex whose n-th column is $D \otimes D^{\otimes n} \otimes D$ (vanishing columns for $n < 0$) and whose horizontal differential is given by

$$d(x_0 \otimes x_1 \otimes \ldots \otimes x_{n+1}) = \sum_{i=0}^{n} (-1)^i x_0 \otimes \ldots \otimes x_{i-1} \otimes x_i x_{i+1} \otimes x_{i+2} \otimes \ldots \otimes x_{n+1}.$$

We view it as a differential graded D-D-bimodule, respectively as a DG module over $D^e = D \otimes D^{\mathrm{op}}$, in the sense of [16]. It is easy to check that $M \otimes_{D^e} D$ is isomorphic to the Hochschild complex associated with D. Consider the DG submodule $M' \subset M$ given by the total complex of the double complex whose n-th column is $De \otimes C^{\otimes n} \otimes eD$ (vanishing columns for $n < 0$). Clearly M' is a D^e-submodule of M. It is easy to check that $M' \otimes_{D^e} D$ is isomorphic to the Hochschild complex $H(C)$ and that the inclusion $M' \subset M$ induces the same morphism $H(C) \to H(D)$ as the map

$C \to D$. To prove that $H(C) \to H(D)$ is an homotopy equivalence, we will prove that $M' \subset M$ is an homotopy equivalence of differential graded D^e-modules. Now both, M' and M are filtered by the column filtration with subquotients of the form $D \otimes L \otimes D$, where L is homotopy equivalent to a complex of projective k-modules. Moreover, the column filtrations split when considered as filtrations of graded D^e-modules. Thus M and M' have property (P) as D^e-modules in the sense of [16]. To prove that the inclusion $M' \subset M$ is an homotopy equivalence, it is therefore enough to prove that it is a quasi-isomorphism. Since the augmentation $\varepsilon : M \to D$ is a quasi-isomorphism (indeed, it is a homotopy equivalence of DG right D-modules), it is enough to show that the restriction of ε to M' induces a quasi-isomorphism $M' \to D$. For this we introduce the complex $R(X,Y)$, where X and Y are arbitrary complexes of B-modules. The complex $R(X,Y)$ is the total complex of the double complex whose n-th column is

$$\mathcal{H}om_B\,(P,Y) \otimes C^{\otimes n} \otimes \mathcal{H}om_B\,(X,P)$$

for $n \geq 0$ and whose column of index -1 is $\mathcal{H}om_B\,(X,Y)$. The other columns vanish. The differential is defined in analogy with that of the Hochschild resolution in degrees > 0 and via the augmentation in degree 0. Then $R(P \oplus Q, P \oplus Q)$ identifies with the mapping cone over $\varepsilon : M' \to D$ and we have to prove that $R(P \oplus Q, P \oplus Q)$ is acyclic. On the other hand, $R(P,P)$ is null-homotopic (it is the mapping cone over the augmentation of the Hochschild resolution for C). Now if we view R as a triangle functor

$$(\mathcal{H}B)^{\mathrm{op}} \times (\mathcal{H}B) \to \mathcal{H}k\,,$$

then this means that (P,P) is in the kernel of R. By the hypothesis on Q, we see that the objects $(P, P \oplus Q)$ and $(P \oplus Q, P \oplus Q)$ belong to the kernel as well. This means that $R(P \oplus Q, P \oplus Q)$ is null-homotopic (as a DG k-module) and hence acyclic.

References

[1] K. Bongartz, *Tilted algebras*, Representations of Algebras, Puebla 1980, Springer LNM **903** (1981), 26-38.

[2] D. Burghelea, *The cyclic homology of the group rings*, Comment. Math. Helv. **60** (1985), 354-365.

[3] C. Cibils, *Hochschild homology of a quiver whose algebra has no oriented cycles*, Representation Theory of finite-dimensional algebras, Proceedings, Ottawa 1984, Springer, 1986, 55–59.

[4] C. Cibils, *Cyclic and Hochschild homology of 2-nilpotent algebras*, K-theory **4** (1990), 131–141.

[5] P. Gabriel, A. Roiter, *Representations of finite-dimensional algebras*, Encyclopaedia of the Mathematical Sciences, Vol. 73, A. I. Kostrikin and I. V. Shafarevich (Eds.), Algebra VIII, Springer 1992.

[6] D. Happel, *On the derived Category of a finite-dimensional Algebra*, Comment. Math. Helv. **62**, 1987, 339–389.

[7] D. Happel, *Hochschild cohomology of finite-dimensional algebras*, Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin, 39^e année (Paris 1987/1988), Springer LNM **1404** (1989), 108–126.

[8] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Math. Soc. Lecture Note Series **119** (1988), Cambridge University Press.

[9] D. Happel, C. M. Ringel, *Tilted algebras*, Representations of algebras, Puebla 1980, Trans. Am. Math. Soc. **274** (1982), 399-443.

[10] K. Igusa, *Notes on the no loops conjecture*, J. Pure and Appl. Alg. **69** (1990), 161–176.

[11] L. Kadison, A relative cyclic cohomology theory useful for computation, C. R. Acad. Sci. Paris Sér. A-B **308** (1989), 569–573.

[12] Chr. Kassel, *Cyclic homology, comodules and mixed complexes*, J. Alg. **107** (1987), 195–216.

[13] Chr. Kassel, *K-théorie algébrique et cohomologie cyclique bivariantes*, C. R. Acad. Sci. Paris Sér. 1 **306** (1988), 799–802.

[14] B. Keller, *Chain complexes and stable categories*, Manus. math. **67** (1990), 379–417.

[15] B. Keller, *A remark on tilting theory and DG algebras*, Manus. Math. **79** (1993), 247–252.

[16] B. Keller, *Deriving DG categories*, Ann. scient. Ec. Norm. Sup., 4^e série **27** (1994), 63–102.

[17] B. Keller, *Invariance and localization for cyclic homology of DG algebras*, preprint, 1994.

[18] H. Lenzing, *Nilpotente Elemente in Ringen von endlicher globaler Dimension*, Math. Z. **108** (1969), 313-324.

[19] J.-L. Loday, *Cyclic homology: a survey*, Banach Center Publications (Warsaw) **18** (1986), 285–307.

[20] J.-L. Loday, *Cyclic Homology*, Grundlehren 301, Springer-Verlag, 1992.

[21] J.-L. Loday, D. Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv. **59** (1984), 565–591.

[22] R. McCarthy, *L'équivalence de Morita et l'homologie cyclique*, C. R. Acad. Sci. Paris Sér. A-B **307** (1988), 211–215.

[23] J. Rickard, *Morita theory for Derived Categories*, J. London Math. Soc. **39** (1989), 436–456.

[24] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. **43** (1991), 37–48.

[25] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Mathematica **65** (1988), 121–154.

[26] J.-L. Verdier, *Catégories dérivées, état 0*, SGA 4 1/2, Springer LNM **569**, 1977, 262–311.

Bernhard Keller
U.F.R. de Mathématiques
U.R.A. 748 du CNRS
Université Paris 7
2, place Jussieu,
75251 Paris Cedex 05, France
keller@mathp7.jussieu.fr