A BRIEF INTRODUCTION TO A-INFINITY ALGEBRAS

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Abstract. These are notes of a 90-minute talk given at the workshop on Derived categories, Quivers and Strings in Edinburgh, August 2004.

1. Introduction

A-infinity spaces and A-infinity algebras were invented at the beginning of the sixties by Stasheff [19]. In the seventies and the eighties, they were developped further by Smirnov [18], Kadeishvili [7], Prouté [14], Huebschmann [5], [6], ... especially with a view towards applications in topology. At the beginning of the nineties, the relevance of A-infinity structures in geometry and physics became apparent through the work of Getzler-Jones [4], Stasheff [20], Fukaya [3], Kontsevich [10], ..., and later Kontsevich-Soibelman [11], Seidel [17], ...

In this brief introduction, we will define A-infinity algebras and examine their basic properties. Then we will define A-infinity modules, the derived category and conclude with the description of triangulated categories via A-infinity algebras.

We refer to [8] [9] for a more detailed introduction with numerous references. K. Lefèvre's thesis [12] contains proofs of all the statements made in this brief introduction but we stress that most of the material we present is very classical.

2. A-infinity algebras

2.1. **Notations.** We will follow Fukaya's sign and degree conventions. For this, we need to introduce some notation:

Let k be a field. If V is a graded vector space, *i. e.*

$$V = \bigoplus_{p \in \mathbf{Z}} V^p ,$$

we denote by SV or $V[1]$ the graded space with $(SV)^p = V^{p+1}$ for all $p \in \mathbf{Z}$. We call SV the *suspension* or the *shift* of V.

If $f : V \to V'$ and $g : W \to W'$ are homogeneous maps between graded spaces, their *tensor product*

$$f \otimes g : V \otimes W \to V' \otimes W'$$

is defined by

$$(f \otimes g)(v \otimes w) = (-1)^{g\, v} f(v) \otimes g(w)$$

for all homogeneous elements $v \in V$ and $w \in W$.

If V and V' are complexes, *i. e.* endowed with differentials d homogeneous of degree 1 and square 0, we put $d_{SV} = -d_V$ and, for a homogeneous map $f : V \to V'$,

$$d(f) = d_{V'} \circ f - (-1)^f f \circ d_V .$$

Thus, f is a morphism of complexes iff $d(f) = 0$ and two morphisms of complexes f and f' are homotopic iff there is a morphism of graded spaces h such that $f' = f + d(h)$. We will use that homotopic morphisms induce the same map in homology.

Date: August 8, 2004, last modified on August 8, 2004.

Key words and phrases. A-infinity algebra, Derived category.

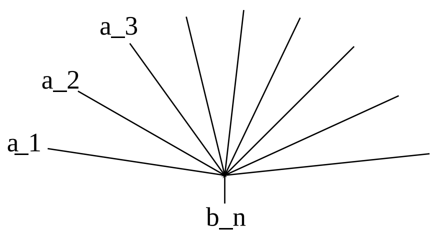
2.2. **A-infinity algebras.** An A_∞-*algebra* is a graded space A endowed with maps

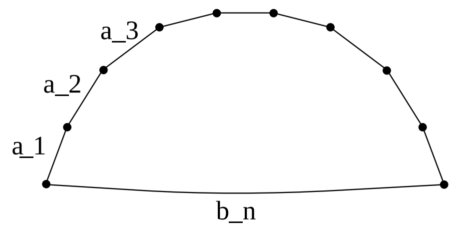
$$b_n : (SA)^{\otimes p} \to SA$$

defined for $n \geq 1$, homogeneous of degree 1 and such that, for all $n \geq 1$, we have

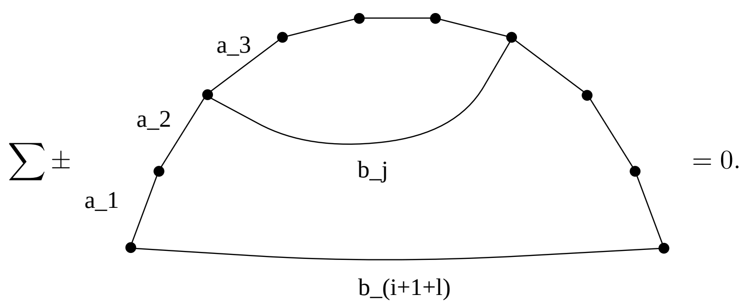
$$\sum_{i+j+l=n} b_{i+1+l} \circ (\mathbf{1}^{\otimes i} \otimes b_j \otimes \mathbf{1}^{\otimes l}) = 0 \tag{2.1}$$

as maps from $(SA)^{\otimes p}$ to SA. Here, the symbol $\mathbf{1}$ denotes the identity map of SA. We visualize b_n either as a planar tree with n leaves and one root or as a halfdisk whose upper arc is divided into segments, each of which symbolizes an 'input', and whose base segment symbolizes the 'output' of the operation.





Using this last representation, the defining identity (2.1) is depicted as follows:



Let us analyze the defining identity for small values of n: For $n = 1$, it states that $b_1^2 = 0$, so that (SA, b_1) is a complex. We also make A into a complex by endowing it with the shifted differential

$$m_1 = -b_1.$$

For $n = 2$, the defining identity becomes

$$b_1 b_2 + b_2(b_1 \otimes \mathbf{1} + \mathbf{1} \otimes b_1) = 0.$$

Note that $b_1 \otimes \mathbf{1} + \mathbf{1} \otimes b_1$ is the differential of $SA \otimes SA$ so that we obtain $d(b_2) = 0$, which means that $b_2 : SA \otimes SA \to SA$ is a morphism of complexes.

For $n = 3$, the identity (2.1) becomes

$$b_2\,(b_2 \otimes \mathbf{1} + \mathbf{1} \otimes b_2) + b_1 b_3 + b_3(b_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes b_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes b_1) = 0.$$

Here the second summand is $d(b_3)$ whereas the first is, up to a sign, the associator for the binary operation b_2. If we define $m_2 : A \otimes A \to A$ by

$$m_2(x, y) = (-1)^x b_2(x, y),$$

then we obtain that m_2 is associative up to a homotopy given by b_3.

For each $n > 3$, the identity (2.1) states that the maps $b_2, \ldots, b_{n-1}$ satisfy a certain quadratic identity up to a homotopy given by b_n. In this sense, an A_∞-algebra is an algebra associative up to a given system of higher homotopies.

It is a direct consequence of the definition, that if b_n vanishes for each $n \geq 1$, then (A, m_1, m_2) is a differential graded (=dg) algebra, *i. e.* m_2 is associative and m_1

a differential compatible with m_2 through the (graded) Leibniz rule. Conversely, each dg algebra gives rise to an A_∞-algebra with vanishing b_n, $n \geq 3$.

In particular, each ordinary associative algebra can be viewed as an A_∞-algebra concentrated in degree 0, and conversely.

2.3. **Examples via deformations.** Following an idea of Penkava-Schwarz [13], let us exhibit a large class of easily constructed but non trivial examples of A_∞-algebras. Let B be an ordinary algebra and $N \geq 1$ an integer. Let ε be an indeterminate of degree $2-N$. We first endow the graded space $A = B[\varepsilon]/(\varepsilon^2)$ with the trivial A_∞-structure given by the map b_2 induced by the multiplication of B and the maps $b_n = 0$ for all $n \neq 2$. Now let

$$c : B^{\otimes N} \to B$$

be any linear map. Define deformed multiplications

$$b'_n = \begin{cases} b_n & n \neq N \\ b_N + \varepsilon c & n = N. \end{cases}$$

Then it is easy to see that A *endowed with the* b'_n *is an* A_∞*-algebra iff* c *is a Hochschild cocycle for* B.

2.4. **Weak A-infinity algebras.** A *weak* A_∞*-algebra* is a graded space A endowed with maps $b_0 : k \to SA$ and b_n, $n \geq 0$, such that the identity 2.1 holds for all $n \geq 0$. The preceding example then naturally extends to the case where $N = 0$, where we start from a Hochschild 0-cocycle, *i. e.* a central element c of B. In general, in a weak A_∞-algebra, we have

$$b_1^2 = -b_2(b_0 \otimes \mathbf{1} + \mathbf{1} \otimes b_0) \neq 0$$

so that the homology with respect to b_1 is no longer defined and the above remarks no longer apply. Little is known about weak A_∞-algebras in general, but they do appear in nature as deformations.

2.5. **Morphisms and quasi-isomorphisms.** A *morphism of* A_∞*-algebras* $f : A \to B$ is given by maps

$$f_n : (SA)^{\otimes n} \to SA\,,\ n \geq 1\,,$$

homogeneous of degree 0 such that, for all $n \geq 1$, we have

$$\sum_{i+j+l=n} f_{i+1+l} \circ (\mathbf{1}^{\otimes i} \otimes b_j \otimes \mathbf{1}^{\otimes l}) = \sum_{i_1+\cdots i_s=n} b_s \circ (f_{i_1} \otimes \cdots \otimes f_{i_s}).$$

By looking at this equation for $n = 1$ and $n = 2$ we see that f_1 then induces a morphism of complexes from (A, m_1) to (B, m_1) which is compatible with m_2 up to an homotopy given by f_2. In particular, f_1 induces an *algebra morphism*

$$H^*A \to H^*B.$$

By definition, f is an A_∞-quasi-isomorphism if f_1 is a quasi-isomorphism (*i. e.* induces an isomorphism in homology).

The *composition* $f \circ g$ of two morphisms is given by

$$(f \circ g)_n = \sum_{i_1+\cdots+i_s=n} f_{i_s} \circ (g_{i_1} \otimes \cdots g_{i_s}).$$

The *identical morphism of* SA is given by $f_1 = \mathbf{1}$ and $f_n = 0$ for all $n \geq 2$.

It is easy to see that we do obtain a category. It contains the category of dg algebras and their morphisms as a non-full subcategory.

Proposition. *For each* A_∞*-algebra* A*, there is a universal* A_∞*-algebra morphism* $\varphi : A \to U(A)$ *to a dg algebra* $U(A)$*. Moreover,* φ *is an* A_∞*-quasi-isomorphism.*

The universal property means that for each dg algebra B, each A_∞-morphism $f : A \to B$ factors as $f = g \circ \varphi$ for a unique morphism of dg algebras $g : U(A) \to B$. The proposition tells us that, up to A_∞-quasi-isomorphism, A_∞-algebras are quite similar to dg algebras. However, in other respects, they are radically different from dg algebras, as the following proposition shows.

Proposition. *Let A be an A_∞-algebra, V a complex and $f_1 : A \to V$ a quasi-isomorphism of complexes. Then V admits a structure of A_∞-algebra such that f_1 extends to an A_∞-quasi-isomorphism $f : A \to V$.*

The analogous statement for dg algebras and their morphisms is of course completely wrong. For our complex V, we can take in particular the graded space H^*V with the zero differential (since we work over a field, the canonical surjection from the cycles to the homology of V splits and we obtain f_1 by composing a right inverse with the inclusion of the cycles). Then we obtain the first part of the

Theorem. *If A is an A_∞-algebra, then H^*A admits an A_∞-algebra structure such that*

(1) *$b_1 = 0$ and b_2 is induced from b_2^A, and*
(2) *there is an A_∞-quasi-isomorphism $A \to H^*A$ inducing the identity in homology.*

Moreover, this structure is unique up to (non unique) A_∞-isomorphism.

Note that uniqueness up to A_∞-quasi-isomorphism is trivial. The point is that here we can omit 'quasi'. An A_∞-algebra is *minimal* if $b_1 = 0$. The *minimal model* of an A_∞-algebra A is the space H^*A endowed with 'the' structure provided by the theorem. This structure can be computed as follows: Choose

$$h \circlearrowright A \underset{i}{\overset{p}{\rightleftarrows}} H^*A$$

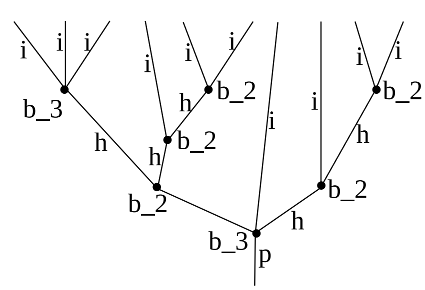
such that p and i are morphisms of complexes of degree 0 and h is a homogeneous map of degree -1 such that

$$pi = \mathbf{1}\ ,\ ip = \mathbf{1} + d(h)\ ,\ h^2 = 0.$$

Then the nth multiplication of the minimal model is constructed as

$$b_n^{min} = \sum_T b_n{}^T$$

where T ranges over the planar rooted trees T with n leaves and b_n^T is given by composing the tree-shaped diagram obtained by labelling each leaf by i, each branch point with m branches by b_m, each internal edge by h and the root by p.



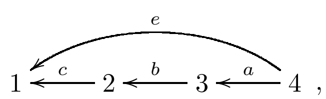
2.6. **Yoneda algebras.** Let B be a unital associative algebra, M a (right) B-module and $P \to M$ a projective resolution. Let $A = \mathsf{Hom}_B(P, P)$ be the differential graded endomorphism algebra of P (its nth component consists of the morphisms of graded objects of degree n and its differential is the supercommutator with the differential of P). Then A is in particular an A_∞-algebra and thus has a minimal model. Now the homology H^*A is isomorphic, as an algebra for m_2, to the Yoneda algebra

$$\mathsf{Ext}^*_B(M, M).$$

Thus we obtain higher multiplications on the Yoneda algebra. The simplest case where these are non-trivial is that of the algebra B given by the quiver

$$1 \xrightarrow{\gamma} 2 \xrightarrow{\beta} 3 \xrightarrow{\alpha} 4$$

with the relation $\alpha\beta\gamma = 0$ and the module M equal to the sum of the four simple B-modules. Then the Yoneda algebra is given by the quiver



where the arrows a, b, c are of degree 1, the arrow e is of degree 2, we have $m_2(c, b) = 0$, $m_2(b, a) = 0$ and $b_3(c, b, a) = e$.

2.7. **Units.** A *strict unit* for an A_∞-algebra A is an element $1 \in A^0$ which is a unit for m_2 and such that, for $n \neq 2$, the map b_n takes the value 0 as soon as one of its arguments equals 1. Unfortunately, strict unitality is not preserved by A_∞-quasi-isomorphisms. A *homological unit* for A is a unit for the associative algebra H^*A with the multiplication induced by m_2. Homological unitality is clearly preserved under A_∞-quasi-isomorphism but is not easy to handle in practical computations. Fortunately, it turns out that the two notions are not very different:

Proposition ([12, 3.2.1]). *Each (resp. minimal) homologically unital A_∞-algebra is A_∞-quasi-isomorphic (resp. A_∞-isomorphic) to a strictly unital A_∞-algebra.*

3. A-infinity modules

Let A be a homologically unital A_∞-algebra. An A_∞*-module* is a graded space M with maps

$$b_n : SM \otimes (SA)^{\otimes n-1} \to SM\ ,\ n \geq 1\ ,$$

homogeneous of degree 1 such that the identity 2.1 holds for all $n \geq 1$ (where we have to interpret b_n as b_n^A or b_n^M according to the type of its arguments) and that the induced action

$$H^*M \otimes H^*A \to H^*M$$

is unital. For example, the A_∞-algebra A can be viewed as a module over itself: the free module of rank one. The notions of *morphism* and *quasi-isomorphism* of A_∞-modules are defined in the natural way. The *derived category* $\mathsf{D}_\infty A$ is defined as the localization of the category of A_∞-modules (with degree 0 morphisms) with respect to the class of quasi-isomorphisms. Thus, its objects are all A_∞-modules and its morphisms are obtained from morphisms of A_∞-modules by formally inverting all quasi-isomorphisms. It turns out that the derived category is naturally a triangulated category. The *perfect derived category* $\mathsf{per}\, A$ is defined as the closure of the free A-module of rank one under shifts in both directions, extensions and passage to direct factors.

When A is an ordinary unital associative algebra, we have a natural functor from the category of complexes over the category $\mathsf{Mod}\, A$ of (right) A-modules to the category of A_∞-modules. This functor is faithful but neither full nor essentially surjective. Nevertheless, it induces an equivalence

$$\mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}_\infty A.$$

Under this equivalence, the perfect derived category corresponds to the full subcategory of complexes quasi-isomorphic to bounded complexes of finitely generated projective A-modules.

Let B be another homologically unital A_∞-algebra. An A_∞-*bimodule* is given by a graded space X with maps

$$b_{i,j} : (SA)^{\otimes i} \otimes SX \otimes (SB)^{\otimes j} \to SM \; , \; i+j \geq 0 \, ,$$

satisfying the identity 2.1 for all $i, j \geq 0$ and such that H^*X becomes a unital H^*A-H^*B-bimodule. For such a bimodule, one can define the *tensor product*

$$? \overset{\infty}{\otimes}_A X : \mathsf{D}_\infty A \to \mathsf{D}_\infty B \; , \; M \mapsto \bigoplus_{i=0}^{\infty} M \otimes (SA)^{\otimes i} \otimes X$$

(we only indicate the underlying graded space). The A_∞-algebras A and B are *derived equivalent* if there is an A_∞-bimodule X such that the associated tensor product is an equivalence.

This generalizes the now classical notion of derived equivalence for ordinary algebras: If A and B are ordinary algebras, then, by J. Rickard's theorem [15], they are derived equivalent iff B admits a tilting complex (*e. g.* a tilting module) with endomorphism ring A.

Let us call *algebraic* a triangulated category which is the homotopy category associated with a k-linear Quillen model category (recall that k is the ground field). The class of algebraic triangulated categories contains all homotopy categories of complexes over k-linear categories and is stable under passage to triangulated subcategories and Verdier localizations. Thus, it contains all triangulated categories 'of algebraic origin', *e. g.* derived categories of categories of coherent sheaves.

Recall that an additive category has *split idempotents* if every idempotent endomorphism admits a kernel (and thus gives rise to a direct sum decomposition). By a *generator* of a triangulated category $\mathcal{T}$, we mean an object G whose closure under shifts in both directions, extensions and passage to direct factors equals $\mathcal{T}$.

Theorem ([12, 7.6])**.** *Let $\mathcal{T}$ be a (k-linear) algebraic triangulated category with split idempotents and a generator G. Then there is a structure of A_∞-algebra on*

$$A = \bigoplus_{n \in \mathbf{Z}} \mathsf{Hom}_{\mathcal{T}}(G, G[n])$$

such that $m_1 = 0$, m_2 is given by composition and that the functor

$$\mathcal{T} \to \mathsf{Grmod}(A, m_2) \; , \; U \mapsto \bigoplus_{n \in \mathbf{Z}} \mathsf{Hom}_{\mathcal{T}}(G, U[n])$$

lifts to a triangle equivalence

$$\mathcal{T} \to \mathsf{per}(A).$$

Here Grmod denotes the category of graded right modules. For example, suppose that $\mathcal{T}$ is the bounded derived category of the category of coherent sheaves on projective n-space. Then, as Beilinson has shown [1] by 'resolving the diagonal', the object $G = \bigoplus_{i=0}^{n} O(-i)$ generates $\mathcal{T}$. Moreover, the algebra A is concentrated in degree 0 and of finite dimension and finite global dimension. Then we obtain equivalences

$$\mathsf{D}^b(\mathsf{coh}\, \mathbf{P}^n) \xrightarrow{\sim} \mathsf{per}(A) \xleftarrow{\sim} \mathsf{D}^b(\mathsf{mod}\, A) \, ,$$

where $\mathsf{mod}\, A$ denotes the category of finite-dimensional A-modules.

Beautiful theorems on the existence of generators in triangulated categories of geometric origin are due to Bondal-Van den Bergh [2] and to Rouquier [16].

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