

THE DONOVAN–WEMYSS CONJECTURE VIA THE DERIVED AUSLANDER–IYAMA CORRESPONDENCE

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ABSTRACT. We provide an outline of the proof of the Donovan–Wemyss Conjecture in the context of the Homological Minimal Model Program for threefolds. The proof relies on results of August, of Hua and the second-named author, Wemyss, and on the Derived Auslander–Iyama Correspondence—a recent result by the first- and third-named authors.

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INTRODUCTION

We work over the field \mathbb{C} of complex numbers. A *compound Du Val (=cDV) singularity* is a complete local hypersurface

$$R \cong \frac{\mathbb{C}[[x, y, z, t]]}{(f + tg)},$$

where $\mathbb{C}[[x, y, z]]/(f)$ is a Kleinian surface singularity and $g \in \mathbb{C}[[x, y, z, t]]$ is arbitrary. Introduced by Reid in the early 1980s [Rei83], cDV singularities form an important class of three-dimensional singularities in birational geometry and play a significant role in the Minimal Model Program (MMP) for threefolds [KM98, Sec. 5.3] as well as in the Homological MMP [Wem18]. We refer the reader to [Aug19, Ch. 1] and [Wem23] for introductions to the subject.

This note is concerned with the following geometric situation: Let R be an isolated cDV singularity and $p: X \rightarrow \mathbf{Spec}(R)$ a crepant resolution, that is p is a proper birational map with smooth source such that the pullback of the dualising sheaf of $\mathbf{Spec}(R)$ along p is the dualising sheaf of X . It follows that $\mathbf{Spec}(R)$ has a unique singular point \mathfrak{m} and the (reduced) exceptional fibre $p^{-1}(\mathfrak{m}) = \bigcup_{i=1}^n C_i$ is a union of curves, with $C_i \cong \mathbb{P}_{\mathbb{C}}^1$ [VdB04, Lemma 3.4.1]. To these data, Donovan and Wemyss [DW16, DW19] associate a (basic, connected) finite-dimensional algebra $\Lambda_{\text{con}} = \Lambda_{\text{con}}(p)$, the *contraction algebra of p* , which represents the functor of ‘simultaneous non-commutative deformations’ of the reduced exceptional fibre. By construction, Λ_{con} is a \mathbb{C}^n -augmented algebra, and hence in particular determines the number n of irreducible components of the exceptional fibre. The contraction

2020 *Mathematics Subject Classification*. Primary 14E30; Secondary 13D03.

Key words and phrases. Minimal model program; compound Du Val singularity; crepant resolution; contraction algebra; Hochschild cohomology; Auslander correspondence.

algebra encodes a surprising amount of information stemming from the given geometric setup. For example, when p contracts a single curve, the contraction algebra recovers known invariants such as Reid’s width [Rei83] and the Gopakumar–Vafa invariants [Kat08], see [Tod15]. Neither the dimension nor the Gabriel quiver of contraction algebras suffice for differentiating cDV singularities [DW16, Table 2]. In fact, it is well-known that there are continuous families of pairwise non isomorphic cDV singularities (that is ‘cDV singularities have moduli’). Notwithstanding, at the risk of stating the obvious, let us point out that the contraction algebra is equipped with crucial data in the form of the multiplication law and that this law is essential in recovering the above mentioned invariants. Equipped with their algebra structure, contraction algebras distinguish between non-isomorphic isolated cDV singularities that admit a crepant resolution in all known examples. These considerations motivate the following remarkable conjecture.

Conjecture A (Donovan and Wemyss [DW16]). Let R_1 and R_2 be isolated cDV singularities with crepant resolutions

$$p_1: X_1 \rightarrow \mathrm{Spec}(R_1) \quad \text{and} \quad p_2: X_2 \rightarrow \mathrm{Spec}(R_2).$$

Then, the contraction algebras $\Lambda_{\mathrm{con}}(p_1)$ and $\Lambda_{\mathrm{con}}(p_2)$ are derived equivalent if and only if there is an isomorphism of algebras $R_1 \cong R_2$.

The original conjecture was formulated only in the case of single-curve contractions; algebraically, this corresponds to the case where the contraction algebras are local and thus derived equivalence reduces to mere isomorphism of algebras since contraction algebras are basic, see [Zim14, Prop. 6.7.4] for example. In the above form, which allows for contracting multiple curves, the conjecture appeared in print in [Aug20, Conj. 1.3].

That the contraction algebras of a given isolated cDV singularity are derived equivalent follows by combining results from Wemyss [Wem18] and Dugas [Dug15]. In this note we provide an outline of the proof of the remaining part of **Conjecture A**. This proof first appeared in the appendix to [JM22] written by the second-named author where it is explained how the conjecture follows by combining previous results of August [Aug20] and [HK18] with the Derived Auslander–Iyama Correspondence—the main result in [JM22]. For the sake of concreteness, we restrict ourselves to the specific context of the conjecture, with the understanding that most concepts and results that are presented here are but special cases of a much more general theory that the reader can find in the original sources. We hope that this sacrifice in generality makes the proof of the conjecture accessible to a broader readership.

Acknowledgements. The authors thank the organisers of the Online Workshop on Higher Dimensional Homological Algebra at the IPM-Isfahan in May 2022, where the application of the Derived Auslander–Iyama Correspondence to the Donovan–Wemyss Conjecture was first found. The authors thank Michael Wemyss and Zheng Hua for interesting conversations and email exchanges; in addition, they thank M. W. for suggesting the terminology ‘ $2\mathbb{Z}$ -derived contraction algebra’ used in these proceedings. The third-named author is also grateful to Matt Booth for answering questions.

Financial support. F. M. was partially supported by grants PID2020-117971GB-C21 funded by MCIN/AEI/10.13039/501100011033 and P20_01109 (JUNTA/FEDER, UE).

1. PRELIMINARIES

In this section we collect preliminary definitions and results that are needed in our proof of [Conjecture A](#). We use freely the theories of differential graded categories [\[Kel94, Kel06\]](#) and A_∞ -categories [\[LH03\]](#). We denote the derived category of an algebra or, more generally, a DG algebra A by $D(A)$; the perfect derived category of A , that is the full subcategory of $D(A)$ spanned by its compact objects, is denoted by $D^c(A)$. All (DG) modules are right (DG) modules.

1.1. $2\mathbb{Z}$ -cluster tilting objects. Let \mathcal{T} be a triangulated category whose underlying additive category is Krull–Schmidt and has finite-dimensional morphism spaces.

Definition 1.1.1 ([\[IY08, GKO13\]](#)). A basic¹ object $T \in \mathcal{T}$ is *2-cluster tilting* if the following conditions hold:

- (1) The object T is *rigid*: $\mathcal{T}(T, T[1]) = 0$.
- (2) For each object $X \in \mathcal{T}$ there exists a triangle $T_1 \rightarrow T_0 \rightarrow X \rightarrow T_1[1]$ with $T_0, T_1 \in \text{add}(T)$, where $\text{add}(T)$ is the smallest additive subcategory of \mathcal{T} containing T that is closed under direct summands.

We say that T is *$2\mathbb{Z}$ -cluster tilting* if it is 2-cluster tilting and $T \cong T[2]$.

Remark 1.1.2. Clearly, if $T \in \mathcal{T}$ is a 2- or $2\mathbb{Z}$ -cluster tilting object, then T generates \mathcal{T} as a triangulated category with split idempotents (which is to say that T is a classical generator of \mathcal{T}). In particular, if the triangulated category \mathcal{T} is algebraic² then there exists a DG algebra A and an equivalence of triangulated categories

$$\mathcal{T} \xrightarrow{\sim} D^c(A), \quad T \mapsto A.$$

Remark 1.1.3. Given a basic 2-cluster tilting object $T \in \mathcal{T}$, one can produce a new such object by a procedure called *mutation* that, in a nutshell, replaces a single indecomposable direct summand of T by a new one, see [\[IY08\]](#) for a precise definition. This process, which can be iterated, is an important reason for introducing 2-cluster tilting objects into the framework of the Homological MMP, see [\[Wem18\]](#) and compare with [Section 1.3](#).

Remark 1.1.4. In general, $2\mathbb{Z}$ -cluster tilting objects are *not* invariant under mutation (however, see [\[HI11, Sec. 4\]](#)). In the context of the Homological MMP this problem does not occur since the notions of 2- and $2\mathbb{Z}$ -cluster tilting object coincide, see [Section 1.2](#).

1.2. Maximal Cohen–Macaulay modules and singularity categories. Let R be an isolated cDV singularity and

$$\text{CM}(R) := \{M \in \text{mod}(R) \mid \text{depth}(M) = \dim(R)\}$$

be the category of *maximal Cohen–Macaulay R -modules* [\[Yos90, LW12\]](#). The category $\text{CM}(R)$ is a Frobenius exact category and, therefore, the stable category $\underline{\text{CM}}(R)$ has an induced structure of a triangulated category; moreover, there is a canonical equivalence of triangulated categories

$$\underline{\text{CM}}(R) \xrightarrow{\sim} D_{\text{sg}}(R) := D^b(\text{mod}(R))/K^b(\text{proj}(R)),$$

where $D_{\text{sg}}(R)$ is the *singularity category of R* , see [\[Buc21\]](#) for details. We record the following facts for later use:

- [\[Yos90, Prop. 1.18\]](#) Since R is complete local, $\underline{\text{CM}}(R) \simeq D_{\text{sg}}(R)$ is a Krull–Schmidt category with finite-dimensional morphism spaces.

¹An object in a Krull–Schmidt additive category is basic if all of its indecomposable direct summands have multiplicity one.

²A triangulated category is *algebraic* if it is equivalent—as a triangulated category—to the stable category of a Frobenius exact category [\[Kel94\]](#).

- [Aus78] Since R is 3-dimensional, $\underline{\mathbf{CM}}(R) \simeq \mathbf{D}_{\text{sg}}(R)$ is a 2-Calabi–Yau triangulated category [Kon98, Kel08], that is there is a natural isomorphism

$$D\text{Hom}(X, Y) \xrightarrow{\simeq} \text{Hom}(Y, X[2]), \quad X, Y \in \underline{\mathbf{CM}}(R) \simeq \mathbf{D}_{\text{sg}}(R),$$

where $V \mapsto DV$ denotes the passage to the \mathbb{C} -linear dual.

- [Eis80] Since R is a hypersurface, $\underline{\mathbf{CM}}(R) \simeq \mathbf{D}_{\text{sg}}(R)$ is a 2-periodic triangulated category, that is there is an isomorphism of exact functors $[2] \cong \mathbf{1}$. In particular, the notions of 2- and $2\mathbb{Z}$ -cluster tilting object coincide in this context.
- The endomorphism algebra of any basic object X in $\underline{\mathbf{CM}}(R) \simeq \mathbf{D}_{\text{sg}}(R)$ is a symmetric algebra. This is an immediate consequence of the natural isomorphisms

$$\text{Hom}(X, X) \cong D\text{Hom}(X, X[2]) \cong D\text{Hom}(X, X).$$

We also consider the DG category $\mathbf{D}_{\text{sg}}(R)_{\text{dg}}$, which is defined as the DG quotient [Kel99, Dri04] of the canonical DG enhancements of the triangulated categories $\mathbf{D}^b(\text{mod}(R))$ and $\mathbf{K}^b(\text{proj}(R))$. By construction,

$$H^0(\mathbf{D}_{\text{sg}}(R)_{\text{dg}}) = \mathbf{D}_{\text{sg}}(R).$$

1.3. Contraction algebras via $2\mathbb{Z}$ -cluster tilting objects. Let R be an isolated cDV singularity and $p: X \rightarrow \text{Spec}(R)$ a crepant resolution. As explained in the introduction, to this geometric setup Donovan and Wemyss associate a basic finite-dimensional algebra $\Lambda_{\text{con}} = \Lambda_{\text{con}}(p)$. We recall an alternative construction of the algebra Λ_{con} that is more adapted to the methods we utilise in this note. Given p as above, a theorem of Van den Bergh [VdB04] furnishes a tilting bundle

$$\mathcal{O}_X \oplus \mathcal{N} = \mathcal{O}_X \oplus \mathcal{N}(p) \in \text{coh}(X)$$

and Wemyss proves [Wem18] that there is an isomorphism of algebras

$$\Lambda_{\text{con}} \cong \underline{\text{End}}_R(N)$$

between the contraction algebra of p and the stable endomorphism algebra of $N := H^0(\mathcal{N}) \in \underline{\mathbf{CM}}(R)$. Remarkably, when viewed as an object of the triangulated category $\underline{\mathbf{CM}}(R)$, the R -module N is a $2\mathbb{Z}$ -cluster tilting object. Conversely, given a $2\mathbb{Z}$ -cluster tilting object $T \in \underline{\mathbf{CM}}(R)$, there exists a crepant resolution of $\text{Spec}(R)$ whose associated contraction algebra is isomorphic to $\underline{\text{End}}_R(T)$. We summarise the previous discussion in the following theorem.³

Theorem 1.3.1 ([Wem18]). *Let R be an isolated cDV singularity and assume that $\text{Spec}(R)$ admits a crepant resolution. Then, the contraction algebras of R are precisely the endomorphism algebras of $2\mathbb{Z}$ -cluster tilting objects in the triangulated category $\underline{\mathbf{CM}}(R) \simeq \mathbf{D}_{\text{sg}}(R)$.*

The following theorem of August reduces **Conjecture A** from a derived equivalence to an isomorphism problem. The proof leverages the characterisation of contraction algebras provided by **Theorem 1.3.1**.

Theorem 1.3.2 ([Aug20, Thm. 1.4]). *Let R be an isolated cDV singularity and assume that $\text{Spec}(R)$ admits a crepant resolution. The contraction algebras of R form a single and complete derived equivalence class of basic algebras.*

³Wemyss proves even more: Up to isomorphism on both sides, crepant resolutions of R correspond bijectively to (basic) $2\mathbb{Z}$ -cluster tilting objects in $\mathbf{D}_{\text{sg}}(R)$. In particular, the number of isomorphism classes of $2\mathbb{Z}$ -cluster tilting objects in $\mathbf{D}_{\text{sg}}(R)$ if finite, for the number of minimal models of $\text{Spec}(R)$ is finite [KM87].

Corollary 1.3.3. *Let R_1 and R_2 be isolated cDV singularities with crepant resolutions*

$$p_1: X_1 \rightarrow \mathrm{Spec}(R_1) \quad \text{and} \quad p_2: X_2 \rightarrow \mathrm{Spec}(R_2)$$

and corresponding contraction algebras $\Lambda_1 = \Lambda_{\mathrm{con}}(p_1)$ and $\Lambda_2 = \Lambda_{\mathrm{con}}(p_2)$. If the algebras Λ_1 and Λ_2 are derived equivalent, then there exists a contraction algebra Λ of R_2 such that $\Lambda \cong \Lambda_1$.

1.4. Hochschild cohomology. Let A be a graded algebra. The bigraded Hochschild cochain complex has components

$$\mathbf{C}^{p,q}(A, A) := \mathrm{Hom}_{\mathbb{C}}(A^{\otimes p}, A(q)), \quad p \geq 0, q \in \mathbb{Z},$$

where $V \mapsto V(1)$ is the (vertical) degree shift of graded vector spaces, equipped with the Hochschild differential $x \mapsto \partial(x)$ of bidegree $(1, 0)$, see [Mur20] for the precise definition, related structure (described below) and sign conventions. The first degree is called *horizontal* or *Hochschild degree* and the second is the *vertical* or *internal degree*; the sum of both is the *total degree* and we denote it by $|x|$ (we also use this notation for the degree of an element in a singly-graded vector space). The component of total degree n of the Hochschild complex is

$$\prod_{p+q=n} \mathbf{C}^{p,q}(A, A).$$

The Hochschild complex is equipped with a *brace algebra* structure, consisting of operations called *braces* (which we do not describe explicitly here)

$$\begin{aligned} \mathbf{C}^{\bullet,*}(A, A)^{\otimes^{n+1}} &\longrightarrow \mathbf{C}^{\bullet,*}(A, A), \\ x_0 \otimes x_1 \otimes \cdots \otimes x_n &\longmapsto x_0\{x_1, \dots, x_n\}, \end{aligned}$$

that are defined for all $n \geq 1$, and satisfy the *brace relation*

$$\begin{aligned} &x\{y_1, \dots, y_p\}\{z_1, \dots, z_q\} \\ &= \sum_{0 \leq i_1 \leq j_1 \leq \cdots \leq i_p \leq j_p \leq q} (-1)^\epsilon \{z_1, \dots, z_{i_1}, y_1\{z_{i_1+1}, \dots, z_{j_1}\}, z_{j_1+1}, \dots \\ &\quad \dots, z_{i_p}, y_p\{z_{i_p+1}, \dots, z_{j_p}\}, z_{j_p+1}, \dots, z_q\}. \end{aligned}$$

Above, ϵ reflects the Koszul sign rule with respect to the total degree shifted by -1 . Brace operations have horizontal degree $-n$ and vertical degree 0 and the n -th brace operation vanishes when x_0 has horizontal degree $< n$. The Hochschild complex is a bigraded associative algebra equipped with the *cup-product*

$$x \cdot y = (-1)^{|x|-1} m_2\{x, y\},$$

where $m_2 \in \mathbf{C}^{2,0}(A, A)$ is, up to a sign, the multiplication in the graded algebra A . The Hochschild complex also has the structure of a (horizontally-shifted) graded Lie algebra, with Lie bracket

$$[x, y] = x\{y\} - (-1)^{(|x|-1)(|y|-1)} y\{x\}$$

of horizontal degree -1 and vertical degree 0 . The Hochschild differential is given by

$$\partial(x) = [m_2, x]$$

and satisfies the corresponding Leibniz rules with respect to the previous associative product and Lie bracket.

The relation between brace operations, the Hochschild differential and the cup product are encoded in the following straightforward consequences of the brace relation.

Lemma 1.4.1. *The following formula holds for all $n \geq 1$:*

$$\begin{aligned} \partial(x_0\{x_1, \dots, x_n\}) &= \partial(x_0)\{x_1, \dots, x_n\} \\ &+ \sum_{i=1}^n (-1)^{\sum_{j=0}^{i-1} |x_j| - i} x_0\{x_1, \dots, \partial(x_i), \dots, x_n\} \\ &+ (-1)^{|x_0| - 1 + |x_0||x_1|} x_1 \cdot x_0\{x_2, \dots, x_{n-1}\} \\ &+ \sum_{i=1}^{n-1} (-1)^{\sum_{j=0}^i |x_j| - i - 1} x_0\{x_1, \dots, x_i \cdot x_{i+1}, \dots, x_n\} \\ &+ (-1)^{\sum_{j=0}^{n-1} |x_j| - n - 1} x_0\{x_1, \dots, x_{n-1}\} \cdot x_n. \end{aligned}$$

Lemma 1.4.2. *The following formula holds for all $n \geq 1$:*

$$(x \cdot y)\{z_1, \dots, z_n\} = \sum_{i=0}^n (-1)^{|y| \sum_{j=1}^i (|z_j| - 1)} x\{z_1, \dots, z_i\} \cdot y\{z_{i+1}, \dots, z_n\}.$$

Lemma 1.4.1 for $n = 1$ proves that the induced associative product in *Hochschild cohomology*

$$\mathrm{HH}^{\bullet,*}(A, A) = H^{\bullet,*}(\mathbf{C}^{\bullet,*}(A, A))$$

(the cohomology of the Hochschild complex) is graded commutative with respect to the total degree. This products satisfies the following compatibility relation with the (horizontally shifted) Lie algebra structure,

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{(|x|-1)|y|} y \cdot [x, z],$$

and hence Hochschild cohomology is a *Gernstenhaber algebra*. For this we use both **Lemmas 1.4.1** and **1.4.2**, for $n = 2$ and $n = 1$ respectively. The Hochschild complex $\mathbf{C}^{\bullet,*}(A, M)$ and Hochschild cohomology $\mathrm{HH}^{\bullet,*}(A, M)$ are defined, more generally, for M an A -bimodule, but it does not have any multiplicative structure in this general case. For the existence of a graded associative algebra structure it suffices that M be an associative algebra in A -bimodules, see also [Mur22, Sec. 1].

1.5. Minimal A_∞ -algebras. We now describe minimal A_∞ -algebras and their morphisms in terms of the Hochschild complex. A *minimal A_∞ -algebra structure* on a graded algebra A is a Hochschild cochain

$$m = (0, 0, 0, m_3, \dots, m_n, \dots) \in \mathbf{C}^{\bullet,*}(A, A)$$

of total degree 2 such that the Maurer–Cartan equation

$$(1.5.1) \quad \partial(m) + m\{m\} = \partial(m) + \frac{1}{2}[m, m] = 0$$

is satisfied. The pair (A, m) is also denoted

$$(A, m_3, \dots, m_n, \dots).$$

If $g: A' \rightarrow A$ is a graded algebra isomorphism, then

$$m * g = (0, 0, 0, g^{-1}m_3g^{\otimes 3}, \dots, g^{-1}m_n g^{\otimes n}, \dots)$$

is a minimal A_∞ -algebra structure on A' . If

$$m' = (0, 0, 0, m'_3, \dots, m'_n, \dots) \in \mathbf{C}^{\bullet,*}(A, A)$$

is another minimal A_∞ -algebra structure on A , an *A_∞ -isomorphism with identity linear part*

$$f: (A, m) \longrightarrow (A, m')$$

is a Hochschild cochain

$$f = (0, 0, f_2, f_3, \dots, f_n, \dots) \in \mathbf{C}^{\bullet,*}(A, A)$$

of total degree 1 such that the following Hochschild cochain vanishes

$$(1.5.2) \quad \partial(f) + f \cdot f + \sum_{r \geq 0} m' \{f, \dots, f\} - m - f\{m\}.$$

More generally, an A_∞ -*isomorphism* between minimal A_∞ -algebras

$$f: (A, m) \longrightarrow (A', m')$$

consists of an isomorphism of graded algebras

$$f_1: A \longrightarrow A'$$

and a Hochschild cochain

$$(0, 0, f_2, f_3, \dots, f_n, \dots) \in \mathbf{C}^{\bullet,*}(A, A')$$

of total degree 1 such that

$$(0, 0, f_1^{-1} f_2, f_1^{-1} f_3, \dots, f_1^{-1} f_n, \dots): (A, m) \longrightarrow (A, m' * f_1)$$

is an A_∞ -isomorphism with identity linear part.

2. THE DERIVED DONOVAN–WEMYSS CONJECTURE

In this section we discuss one of the main results in [HK18]—crucial to our proof of the Donovan–Wemyss conjecture—and explain how it implies a *derived version* of the conjecture (Corollary 2.2.4).

2.1. $2\mathbb{Z}$ -derived contraction algebras. By means of the equivalence of triangulated categories $\underline{\mathbf{CM}}(R) \simeq \mathbf{D}_{\text{sg}}(R)$, the contraction algebra associated to a crepant resolution p of an isolated cDV singularity can be promoted to the DG algebra

$$\mathbf{\Lambda}_{\text{con}} = \mathbf{\Lambda}_{\text{con}}(p) := \mathbf{R}\text{End}(N)$$

given by the derived endomorphism algebra of the corresponding $2\mathbb{Z}$ -cluster tilting object $N = N(p) \in \mathbf{D}_{\text{sg}}(R)_{\text{dg}}$. By construction $H^0(\mathbf{\Lambda}_{\text{con}}) \cong \Lambda_{\text{con}}$ and, as a consequence of the 2-periodicity of the singularity category of R ,

$$H^\bullet(\mathbf{\Lambda}_{\text{con}}) \cong \Lambda_{\text{con}}[i^{\pm 1}] = \Lambda_{\text{con}} \otimes_{\mathbb{C}} \mathbb{C}[i^{\pm 1}], \quad |i| = -2.$$

We refer to the DG algebra $\mathbf{\Lambda}_{\text{con}}$ as the *$2\mathbb{Z}$ -derived contraction algebra of p* . The (soft) non-positive truncation $\mathbf{\Lambda}_{\text{con}}^{\leq 0} = \tau^{\leq 0} \mathbf{\Lambda}_{\text{con}}$ of $\mathbf{\Lambda}_{\text{con}}$ is quasi-isomorphic to the *derived contraction algebra of p* considered for example in [Boo19, Boo21, HK18], and there is an isomorphism of graded algebras

$$H^\bullet(\mathbf{\Lambda}_{\text{con}}^{\leq 0}) \cong \Lambda_{\text{con}}[i] = \Lambda_{\text{con}} \otimes_{\mathbb{C}} \mathbb{C}[i], \quad |i| = -2.$$

The $2\mathbb{Z}$ -derived contraction algebra $\mathbf{\Lambda}_{\text{con}}$ is a localisation of $\mathbf{\Lambda}_{\text{con}}^{\leq 0}$ (see [Boo21, Thms. 6.4.6 and 7.2.3] and [HK18, Thm. 4.17]) and $\mathbf{\Lambda}_{\text{con}}$ can also be interpreted as a *non-connective* variant of $\mathbf{\Lambda}_{\text{con}}^{\leq 0}$.

Notice also that, since $2\mathbb{Z}$ -cluster tilting objects are in particular classical generators, there is a canonical quasi-equivalence of DG categories

$$\mathbf{D}^c(\mathbf{\Lambda}_{\text{con}})_{\text{dg}} \xrightarrow{\sim} \mathbf{D}_{\text{sg}}(R)_{\text{dg}}$$

that induces an equivalence of triangulated categories

$$\mathbf{D}^c(\mathbf{\Lambda}_{\text{con}}) \xrightarrow{\sim} \mathbf{D}_{\text{sg}}(R).$$

Although we do not need this fact in the sequel, we mention that there is an equivalence of triangulated categories⁴ [HK18, Theorem 4.17 and Lemma 5.12]

$$\mathcal{C}(\mathbf{\Lambda}_{\text{con}}^{\leq 0}) := \mathbf{D}^c(\mathbf{\Lambda}_{\text{con}}^{\leq 0}) / \mathbf{D}^{\text{fd}}(\mathbf{\Lambda}_{\text{con}}^{\leq 0}) \simeq \mathbf{D}_{\text{sg}}(R),$$

⁴For a DG algebra A , we denote by $\mathbf{D}^{\text{fd}}(A)$ the full subcategory of $\mathbf{D}(A)$ spanned by the DG A -modules with finite-dimensional total cohomology.

that is compatible with the canonical DG enhancements on either side. The category $\mathcal{C}(\Lambda_{\text{con}}^{\leq 0})$ is known as the *Amiot cluster category of $\Lambda_{\text{con}}^{\leq 0}$* [Ami07] and, indeed, establishing a link between birational geometry and the theory of cluster categories was one of the objectives in [HK18].

2.2. The Derived Donovan–Wemyss Conjecture. The following theorem of Hua and the second-named author settles a derived version of [Conjecture A](#).

Theorem 2.2.1 ([HK18, Thm. 5.9]). *Let $R = \mathbb{C}\langle x, y, z, t \rangle / (f)$ be an isolated cDV singularity. Then, there is an isomorphism of algebras*

$$\mathrm{HH}^0(\mathrm{D}_{\mathrm{sg}}(R)_{\mathrm{dg}}) \cong \frac{\mathbb{C}\langle x, y, z, t \rangle}{(f, \partial_x f, \partial_y f, \partial_z f, \partial_t f)}$$

between the 0-th Hochschild cohomology of the DG category $\mathrm{D}_{\mathrm{sg}}(R)_{\mathrm{dg}}$ and the Tyurina algebra of R . In particular, if R' is a further isolated cDV singularity such that the DG categories $\mathrm{D}_{\mathrm{sg}}(R)_{\mathrm{dg}}$ and $\mathrm{D}_{\mathrm{sg}}(R')_{\mathrm{dg}}$ are quasi-equivalent, then there is an isomorphism of algebras $R \cong R'$.

Remark 2.2.2. The proof of [Theorem 2.2.1](#) relies on a comparison result [Kel18, Kel19] between the singular Hochschild cohomology (=Hochschild–Tate cohomology) of R and the Hochschild cohomology of the DG category $\mathrm{D}_{\mathrm{sg}}(R)_{\mathrm{dg}}$. The appearance of the Tyurina algebra stems from an earlier result of the Buenos Aires Cyclic Homology Group [GGRV92, Thm. 3.2.7]. That the Tyurina algebra, together with the dimension of R , determines the isomorphism type of the singularity is a theorem of Mather and Yau [MY82].

Remark 2.2.3. In [Dyc11], Dyckerhoff shows that the 0-th Hochschild cohomology of $\mathrm{D}_{\mathrm{sg}}(R)_{\mathrm{dg}}$ —viewed as a $\mathbb{Z}/2$ -graded DG category—is isomorphic to the Milnor algebra

$$\frac{\mathbb{C}\langle x, y, z, t \rangle}{(\partial_x f, \partial_y f, \partial_z f, \partial_t f)}$$

of the singularity (which does not determine the isomorphism type of the singularity, even if one knows the dimension). Thus, in [Theorem 2.2.1](#) it is crucial to consider $\mathrm{D}_{\mathrm{sg}}(R)_{\mathrm{dg}}$ as a \mathbb{Z} -graded DG category.

Corollary 2.2.4 (Derived Donovan–Wemyss Conjecture). *Let R_1 and R_2 be isolated cDV singularities with crepant resolutions*

$$p_1: X_1 \rightarrow \mathrm{Spec}(R_1) \quad \text{and} \quad p_2: X_2 \rightarrow \mathrm{Spec}(R_2).$$

If the $2\mathbb{Z}$ -derived contraction algebras $\Lambda_{\text{con}}(p_1)$ and $\Lambda_{\text{con}}(p_2)$ are quasi-isomorphic, then there is an isomorphism of algebras $R_1 \cong R_2$.

Proof. Indeed, if the DG algebras $\Lambda_{\text{con}}(p_1)$ and $\Lambda_{\text{con}}(p_2)$ are quasi-isomorphic, then the DG categories

$$\mathrm{D}^c(\Lambda_{\text{con}}(p_1))_{\mathrm{dg}} \simeq \mathrm{D}_{\mathrm{sg}}(R_1)_{\mathrm{dg}} \quad \text{and} \quad \mathrm{D}^c(\Lambda_{\text{con}}(p_2))_{\mathrm{dg}} \simeq \mathrm{D}_{\mathrm{sg}}(R_2)_{\mathrm{dg}}$$

are quasi-equivalent. [Theorem 2.2.1](#) then implies the existence of an isomorphism of algebras $R_1 \cong R_2$. \square

3. UNIQUENESS OF THE $2\mathbb{Z}$ -DERIVED CONTRACTION ALGEBRA

In this section we prove that $2\mathbb{Z}$ -derived contraction algebras are determined up to quasi-isomorphism by their zeroth cohomology plus a minimal amount of additional algebraic data (see [Corollary 3.4.6](#) for the precise statement). Before that, we formulate a closely related result ([Theorem 3.1.1](#)) that states that two $2\mathbb{Z}$ -derived contraction algebras whose zeroth cohomologies are isomorphic as algebras must be quasi-isomorphic, and use this result to prove [Conjecture A](#).

3.1. Proof of the Donovan–Wemyss Conjecture. In view of [Corollaries 1.3.3](#) and [2.2.4](#), [Conjecture A](#) is an immediate consequence of the following theorem, the proof of which is given in [Section 3.4](#).

Theorem 3.1.1. *Let R_1 and R_2 be isolated cDV singularities with crepant resolutions*

$$p_1: X_1 \rightarrow \mathrm{Spec}(R_1) \quad \text{and} \quad p_2: X_2 \rightarrow \mathrm{Spec}(R_2).$$

If the contraction algebras $\Lambda(p_1)$ and $\Lambda(p_2)$ are isomorphic, then the $2\mathbb{Z}$ -derived contraction algebras $\mathbf{\Lambda}_{\mathrm{con}}(p_1)$ and $\mathbf{\Lambda}_{\mathrm{con}}(p_2)$ are quasi-isomorphic.

Remark 3.1.2. [Theorem 3.1.1](#) is a special case of [[JM22](#), Thm. 5.1.10], see [Section 4.2](#).

Proof of [Conjecture A](#) using [Theorem 3.1.1](#). Let R_1 and R_2 be isolated cDV singularities with crepant resolutions

$$p_1: X_1 \rightarrow \mathrm{Spec}(R_1) \quad \text{and} \quad p_2: X_2 \rightarrow \mathrm{Spec}(R_2)$$

whose corresponding contraction algebras $\Lambda_{\mathrm{con}}(p_1)$ and $\Lambda_{\mathrm{con}}(p_2)$ are derived equivalent. In view of [Corollaries 1.3.3](#) and [2.2.4](#), we may and we will assume that $\Lambda_{\mathrm{con}}(p_1)$ and $\Lambda_{\mathrm{con}}(p_2)$ are isomorphic and hence, by [Theorem 3.1.1](#), the $2\mathbb{Z}$ -derived contraction algebras $\mathbf{\Lambda}_{\mathrm{con}}(p_1)$ and $\mathbf{\Lambda}_{\mathrm{con}}(p_2)$ are quasi-isomorphic. Finally, [Corollary 2.2.4](#) yields the desired algebra isomorphism $R_1 \cong R_2$. \square

3.2. The restricted universal Massey product. The proof of [Theorem 3.1.1](#) makes use of an invariant of the $2\mathbb{Z}$ -derived contraction algebra, a certain Hochschild cohomology class of bidegree $(4, -2)$ that we call the restricted universal Massey product. As we explain below, this invariant is induced by the first possibly non-trivial higher operation on a minimal A_∞ -algebra model of the $2\mathbb{Z}$ -derived contraction algebra.

Setting 3.2.1. Fix an isolated cDV singularity R that admits a crepant resolution $p: X \rightarrow \mathrm{Spec}(R)$, and let $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathrm{con}}(p)$ be the corresponding $2\mathbb{Z}$ -derived contraction algebra so that $H^0(\mathbf{\Lambda}) \cong \Lambda = \Lambda_{\mathrm{con}}(p)$ is the contraction algebra defined by Donovan and Wemyss. For simplicity, we treat the isomorphism of graded algebras

$$H^\bullet(\mathbf{\Lambda}) \cong \Lambda[\mathfrak{l}^{\pm 1}] = \Lambda \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{l}^{\pm 1}], \quad |\mathfrak{l}| = -2,$$

as an identification.

Kadeishvili’s Homotopy Transfer Theorem [[Kad82](#)] provides us with a minimal A_∞ -algebra structure, unique up to A_∞ -isomorphism with identity linear part,

$$B = (\Lambda[\mathfrak{l}^{\pm 1}], m_3, m_4, m_5, \dots)$$

on the cohomology algebra $\Lambda[\mathfrak{l}^{\pm 1}]$. Since $\Lambda[\mathfrak{l}^{\pm 1}]$ is concentrated in even degrees and, by definition,

$$m_n: \Lambda[\mathfrak{l}^{\pm 1}]^{\otimes n} \longrightarrow \Lambda[\mathfrak{l}^{\pm 1}]$$

is a morphism of degree $2 - n$, we conclude that $m_n = 0$ whenever n is odd. We write

$$B = (\Lambda[\mathfrak{l}^{\pm 1}], m_4, m_6, m_8, \dots)$$

as a way to record this observation. We refer to B as a *minimal (A_∞ -algebra) model* of the DG algebra $\mathbf{\Lambda}$ and fix it for the rest of the section.

Remark 3.2.2. The passage from DG to A_∞ -algebras is a matter of technical convenience: The homotopy theories of *non-unital* DG and of A_∞ -algebras are equivalent, [[LV12](#), Thm. 11.4.8]. In particular, two *non-unital* DG algebras are quasi-isomorphic if and only if their minimal models are A_∞ -isomorphic [[LV12](#), Thms. 11.4.9 and 10.3.10]. Here, we are exclusively interested in *unital* DG algebras,

but this is not a problem since by [Mur14, Prop. 6.2] two *unital* DG algebras are quasi-isomorphic as *non-unital* DG algebras if and only if they are quasi-isomorphic as *unital* DG algebras.

Consider now the bigraded Hochschild cochain complex

$$\mathbb{C}^{p,q}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]) := \text{Hom}_{\mathbb{C}}(\Lambda[\iota^{\pm 1}]^{\otimes p}, \Lambda[\iota^{\pm 1}](q)), \quad p \geq 0, q \in \mathbb{Z},$$

recalled in Section 1.4. Since $m_3 = 0$, the A_{∞} -equations imply that $\partial(m_4) = 0$ ([LH03, Lemme B.4.1]); hence we obtain a class

$$(3.2.3) \quad \{m_4\} = \{m_4^{\Lambda}\} \in \text{HH}^{4,-2}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}])$$

that we call the *universal Massey product (of length 4)*. It follows from the definition of A_{∞} -morphism ([LH03, Lemme B.4.2]) that the class $\{m_4\}$ does not depend on the choice of minimal model for $\mathbf{\Lambda}$ and hence the universal Massey product can and will be regarded as an invariant of the latter DG algebra.

Consider now the graded-algebra morphism $j: \Lambda \hookrightarrow \Lambda[\iota^{\pm 1}]$ given by the inclusion of the degree 0 part. The morphism j induces a restriction morphism on Hochschild cohomology⁵

$$j^*: \text{HH}^{\bullet,*}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]) \longrightarrow \text{HH}^{\bullet,*}(\Lambda, \Lambda[\iota^{\pm 1}]),$$

where the space on the right is the Hochschild cohomology of Λ , viewed as a graded algebra concentrated in degree 0, with coefficients in the graded Λ -bimodule $\Lambda[\iota^{\pm 1}]$. In particular, since the degree -2 component $\Lambda \cdot \iota$ of $\Lambda[\iota^{\pm 1}]$ is isomorphic to the diagonal Λ -bimodule, we obtain a class

$$(3.2.4) \quad j^*\{m_4\} = j^*\{m_4^{\Lambda}\} \in \text{HH}^{4,-2}(\Lambda, \Lambda[\iota^{\pm 1}]) = \text{HH}^4(\Lambda, \Lambda \cdot \iota) = \text{Ext}_{\Lambda^e}^4(\Lambda, \Lambda),$$

where $\Lambda^e = \Lambda \otimes_{\mathbb{C}} \Lambda^{\text{op}}$ is the enveloping algebra of Λ ; we call the class $j^*\{m_4\}$ the *restricted universal Massey product (of length 4)* and, as with the unrestricted version, we regard it as an invariant of the $2\mathbb{Z}$ -derived contraction algebra $\mathbf{\Lambda}$. Notice also that the previous discussion applies verbatim to any DG algebra A whose cohomology is isomorphic to the graded algebra $\Lambda[\iota^{\pm 1}]$, so that we may associate to A its restricted universal Massey product $j^*\{m_4^A\}$.

The following theorem is the first main step towards the proof of Theorem 3.1.1.

Theorem 3.2.5. *The restricted universal Massey product $j^*\{m_4\}$, when viewed as an element of the space $\text{Ext}_{\Lambda^e}^4(\Lambda, \Lambda)$ of Yoneda extensions of Λ -bimodules, can be represented by an exact sequence*

$$0 \rightarrow \Lambda \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \Lambda \rightarrow 0$$

with projective middle terms. In particular, $\Omega_{\Lambda^e}^4(\Lambda) \cong \Lambda$ in the stable category of Λ -bimodules.

Proof. The first claim is a special case of [JM22, Cor. 4.5.17]. Indeed, by definition, the $2\mathbb{Z}$ -derived contraction algebra $\mathbf{\Lambda}$ is the derived endomorphism algebra of a $2\mathbb{Z}$ -cluster tilting object in $\text{D}^c(\mathbf{\Lambda}) \simeq \text{D}_{\text{sg}}(R)$, which is one of the equivalent conditions in *loc. cit.* The second claim follows immediately from the first. \square

Remark 3.2.6. The proof of [JM22, Cor. 4.5.17], and hence that of Theorem 3.2.5, is non-trivial. In the special case of the contraction algebra, it is possible that detailed knowledge of the first non-trivial higher operation m_4 of some minimal model of the $2\mathbb{Z}$ -derived contraction algebra allows for establishing the desired property of the restricted universal Massey product $j^*\{m_4\}$ directly. The approach taken in [JM22], which deals with an abstract and more general situation, rather leverages the fact that Λ is the endomorphism algebra of a $2\mathbb{Z}$ -cluster tilting object $T \in \text{D}_{\text{sg}}(R)$.

⁵In fact, the morphism j^* is surjective, see [JM22, Prop. 4.6.9] and take $\sigma = \mathbf{1}$ and $d = 2$ (which is even and hence no signs occur in the formulas therein).

The upshot is that the additive closure $\text{add}(T)$ of T has an induced structure of a so-called *4-angulated category*, that is $\text{add}(T)$ is equipped with a natural class of diagrams \square_{GKO} , called *4-angles*, of the form⁶

$$T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow T_4 \rightarrow T_1[2]$$

that satisfies axioms analogous to those of triangulated categories [GKO13]. On the other hand, an extension of Λ -bimodules

$$0 \rightarrow \Lambda \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow \Lambda \rightarrow 0$$

with P_0, P_1, P_2 projective-injective (but perhaps not P_3) that represents the class $j^*\{m_4\} \in \text{Ext}_{\Lambda^e}^4(\Lambda, \Lambda)$ yields a class of 4-angles $\square_{j^*\{m_4\}}$ defined in terms of certain exactness properties [Ami07, Lin19]; the class $\square_{j^*\{m_4\}}$ is *a priori* not known to form a 4-angulation of $\text{add}(T)$. The crux of the argument is then to prove that

$$\square_{\text{GKO}} = \square_{j^*\{m_4\}}$$

so that the class $\square_{j^*\{m_4\}}$ is indeed a 4-angulation of $\text{add}(T)$; this agreement relies on a delicate analysis of the relationship between Toda brackets, Massey products and the classes \square_{GKO} and $\square_{j^*\{m_4\}}$. Finally, in view of the exactness properties defining the class $\square_{j^*\{m_4\}}$ (now known to be 4-angulation), a theorem of Auslander and Reiten [AR91] for detecting projective bimodules implies that P_3 must be a projective Λ -bimodule, which is what [Theorem 3.2.5](#) claims. The reader is referred to [JM22] for details.

Recall that the contraction algebra is Frobenius (in fact, symmetric). Consequently, its enveloping algebra is also a Frobenius algebra and we may consider the *Hochschild–Tate cohomology*

$$\underline{\text{HH}}^{\bullet,*}(\Lambda, \Lambda[l^{\pm 1}]) := \underline{\text{Ext}}_{\Lambda^e}^{\bullet,*}(\Lambda, \Lambda[l^{\pm 1}])$$

defined in terms of the extension spaces in the stable category of graded Λ -bimodules; thus,

$$\text{HH}^{>0,*}(\Lambda, \Lambda[l^{\pm 1}]) = \underline{\text{HH}}^{>0,*}(\Lambda, \Lambda[l^{\pm 1}])$$

and there is a surjection

$$\text{HH}^{0,*}(\Lambda, \Lambda[l^{\pm 1}]) \twoheadrightarrow \underline{\text{HH}}^{0,*}(\Lambda, \Lambda[l^{\pm 1}]).$$

The multiplication on $\Lambda[l^{\pm 1}]$ endows $\underline{\text{HH}}^{\bullet,*}(\Lambda, \Lambda[l^{\pm 1}])$ with the structure of a bi-graded algebra, see [Mur22, Sec. 5] for details.

Corollary 3.2.7. *The restricted universal Massey product $j^*\{m_4\}$, when viewed as an element of the Hochschild–Tate cohomology $\underline{\text{HH}}^{\bullet,*}(\Lambda, \Lambda[l^{\pm 1}])$, is a unit.*

Proof. Immediate from [Theorem 3.2.5](#) and [Mur22, Prop. 5.7 and Rmk. 5.8], which characterises the units in $\underline{\text{HH}}^{\bullet,*}(\Lambda, \Lambda[l^{\pm 1}])$ of positive Hochschild (=horizontal) degree. \square

Remark 3.2.8. In [Corollary 3.2.7](#) it is essential to pass from Hochschild to Hochschild–Tate cohomology in order to have units of positive Hochschild degree.

⁶Recall that $[2] \cong \mathbf{1}$ in $\text{D}_{\text{sg}}(R)$.

3.3. Hochschild cohomology of the graded contraction algebra. In this section we compute the Hochschild cohomology of the graded algebra

$$\Lambda_{\text{con}}[\iota^{\pm 1}] = \Lambda[\iota^{\pm 1}] = \Lambda \otimes_{\mathbb{C}} \mathbb{C}[\iota^{\pm 1}], \quad |\iota| = -2,$$

that we call *graded contraction algebra*, in terms of the Hochschild cohomology of the Donovan–Wemyss contraction algebra $\Lambda_{\text{con}} = \Lambda$.

First, notice that ι lies in the (graded) centre of $\Lambda[\iota^{\pm 1}]$, which is

$$Z(\Lambda[\iota^{\pm 1}]) = \text{HH}^{0,*}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]);$$

hence

$$\iota \in \text{HH}^{0,-2}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]).$$

We introduce the *fractional Euler derivation*

$$\bar{\delta} \in \mathbb{C}^{1,0}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]),$$

which acts by the formula

$$\bar{\delta}: a \mapsto \frac{|a|}{2}a,$$

where we observe that $\frac{|a|}{2}$ is an integer since $\Lambda[\iota^{\pm 1}]$ is concentrated in even degrees. It is a cocycle with cohomology class

$$\delta \in \text{HH}^{1,0}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]).$$

Proposition 3.3.1. *The following statements hold:*

- (1) *There is an isomorphism of graded commutative algebras*

$$\text{HH}^{\bullet,*}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]) \cong \text{HH}^{\bullet}(\Lambda, \Lambda)[\iota^{\pm 1}, \delta].$$

The graded Lie algebra structure on the right hand side is induced by the (usual) Lie algebra structure on $\text{HH}^{\bullet,}(\Lambda, \Lambda)$ by setting*

$$\begin{aligned} [\iota, \text{HH}^{\bullet}(\Lambda, \Lambda)] &= 0, & [\iota, \delta] &= 0, \\ [\delta, \text{HH}^{\bullet}(\Lambda, \Lambda)] &= 0, & [\delta, \delta] &= -\iota. \end{aligned}$$

- (2) *There is an isomorphism of graded algebras*

$$\text{HH}^{\bullet,*}(\Lambda, \Lambda[\iota^{\pm 1}]) \cong \text{HH}^{\bullet}(\Lambda, \Lambda)[\iota^{\pm 1}].$$

Moreover, the morphism

$$j^*: \text{HH}^{\bullet,*}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]) \longrightarrow \text{HH}^{\bullet,*}(\Lambda, \Lambda[\iota^{\pm 1}])$$

induced by the inclusion $j: \Lambda \hookrightarrow \Lambda[\iota^{\pm 1}]$ of the degree 0 part is the apparent natural projection with kernel the graded ideal generated by δ .

- (3) *There is an isomorphism of graded algebras*

$$\underline{\text{HH}}^{\bullet,*}(\Lambda, \Lambda[\iota^{\pm 1}]) \cong \underline{\text{HH}}^{\bullet}(\Lambda, \Lambda)[\iota^{\pm 1}].$$

Furthermore, the comparison map

$$\text{HH}^{\bullet,*}(\Lambda, \Lambda[\iota^{\pm 1}]) \longrightarrow \underline{\text{HH}}^{\bullet,*}(\Lambda, \Lambda[\iota^{\pm 1}])$$

is the apparent extension of the comparison map $\text{HH}^{\bullet}(\Lambda, \Lambda) \rightarrow \underline{\text{HH}}^{\bullet}(\Lambda, \Lambda)$.

Proof. All of the forthcoming claims follow from the proof of [JM22, Prop. 4.6.9] for $\sigma = \mathbf{1}_{\Lambda}$ and $d = 2$.

- (1) The Hochschild complex $\mathbb{C}^{\bullet,*}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}])$ contains the subcomplex

$$\mathbb{C}_{\mathbb{C}[\iota^{\pm 1}]}^{\bullet,*}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}])$$

of $\mathbb{C}[\iota^{\pm 1}]$ -linear cochains; this subcomplex is also an associative subalgebra and a Lie subalgebra of the \mathbb{C} -linear Hochschild complex.

The composite

$$\mathbb{C}_{\mathbb{C}[\iota^{\pm 1}]}^{\bullet,*}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]) \xrightarrow{i} \mathbb{C}^{\bullet,*}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]) \xrightarrow{j^*} \mathbb{C}^{\bullet,*}(\Lambda, \Lambda[\iota^{\pm 1}]),$$

of the inclusion of the $\mathbb{C}[\iota^{\pm 1}]$ -linear Hochschild cochains into the \mathbb{C} -linear ones with the restriction of scalars along the inclusion $j: \Lambda \hookrightarrow \Lambda[\iota^{\pm 1}]$ of the degree 0 part is an *isomorphism* of DG algebras. The target, unlike the source, does not *a priori* carry any Lie algebra structure. Nevertheless, there is an obvious isomorphism of DG algebras

$$\mathbf{C}^{\bullet,*}(\Lambda, \Lambda[\iota^{\pm 1}]) \cong \mathbf{C}^{\bullet}(\Lambda, \Lambda)[\iota^{\pm 1}]$$

that we regard as an identification, and the composite isomorphism

$$\mathbf{C}_{\mathbb{C}[\iota^{\pm 1}]}^{\bullet,*}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]) \cong \mathbf{C}^{\bullet}(\Lambda, \Lambda)[\iota^{\pm 1}]$$

is also a Lie algebra map when we regard the target as a Lie algebra extension of $\mathbf{C}^{\bullet}(\Lambda, \Lambda)$ with ι a central element (in the Lie-algebra sense).

The morphism

$$\begin{aligned} \mathbf{C}^{\bullet,*}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]) &\longrightarrow \mathbf{C}^{\bullet}(\Lambda, \Lambda)[\iota^{\pm 1}, \bar{\delta}], \\ x &\longmapsto j^*(x) - \iota^{-1} \cdot j^*([x, \iota]) \cdot \bar{\delta}, \end{aligned}$$

is a quasi-isomorphism of DG algebras with quasi-inverse

$$\begin{aligned} \mathbf{C}^{\bullet}(\Lambda, \Lambda)[\iota^{\pm 1}] \oplus \mathbf{C}^{\bullet}(\Lambda, \Lambda)[\iota^{\pm 1}] \cdot \bar{\delta} &\longrightarrow \mathbf{C}^{\bullet,*}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}]), \\ x + y \cdot \bar{\delta} &\longmapsto i(x) + i(y) \cdot \bar{\delta}. \end{aligned}$$

(The latter is just a morphism of complexes since $\bar{\delta}^2 \neq 0$ in the target, it only vanishes in cohomology.) In fact, the relevant composite equals the identity of $\mathbf{C}^{\bullet}(\Lambda, \Lambda)[\iota^{\pm 1}, \bar{\delta}]$. The Lie bracket formulas in the statement of the proposition follow from the definition of the fractional Euler class, $\mathbb{C}[\iota^{\pm 1}]$ -linear cochains and degree considerations.

(2) The statement follows easily from the previous computations.

(3) The statement is consequence of the fact that $\underline{\mathbf{H}\mathbf{H}}^{\bullet,*}(\Lambda, \Lambda)$ is obtained from $\mathbf{H}\mathbf{H}^{\bullet,*}(\Lambda, \Lambda)$ by inverting any element of

$$\mathbf{H}\mathbf{H}^4(\Lambda, \Lambda) = \text{Ext}_{\Lambda^e}^4(\Lambda, \Lambda)$$

representing the 4-periodicity of Λ , and similarly when the coefficients lie in $\Lambda[\iota^{\pm 1}]$. \square

Below, we use the isomorphisms in [Proposition 3.3.1](#) as identifications.

Corollary 3.3.2. *Let $u \in \underline{\mathbf{H}\mathbf{H}}^4(\Lambda, \Lambda)$ be a unit. There exists a unique Hochschild class*

$$m \in \mathbf{H}\mathbf{H}^{4,-2}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}])$$

such that

$$j^*(m) = u \cdot \iota, \quad \frac{1}{2}[m, m] = 0.$$

Proof. The first equation in the statement is equivalent to m being of the form

$$(3.3.3) \quad m = (u + x \cdot \delta) \cdot \iota$$

for some $x \in \mathbf{H}\mathbf{H}^3(\Lambda, \Lambda)$. Using the relations in a Gerstenhaber algebra, the second equation is equivalent to

$$0 = ([u, u] - 2u \cdot x) \cdot \iota^2 - 2[x, u] \cdot \iota^2 \cdot \delta.$$

This means that both summands must vanish. For the first one, this is equivalent to

$$x = \frac{1}{2}u^{-1}[u, u].$$

This takes place in the piece of $\mathbf{H}\mathbf{H}^{\bullet}(\Lambda, \Lambda)$ that agrees with $\underline{\mathbf{H}\mathbf{H}}^{\bullet}(\Lambda, \Lambda)$, and is compatible with the second summand since

$$0 = \frac{1}{2}[[u, u], u] = [ux, u] = u[x, u] - [u, u]x = u[x, u] - u^{-1}[u, u]^2 = u[x, u],$$

so $[x, u] = 0$. The first step follows from the graded Jacobi identity and we also use that $[u, u]^2 = 0$ since $[u, u]$ has odd total degree and Hochschild cohomology is graded commutative. \square

The following result should be compared with equation (3.3.3); its proof is similar to that of Corollary 3.3.2.

Corollary 3.3.4. *Let $u \in \mathbf{HH}^4(\Lambda, \Lambda)$ be a unit such that $[u, u] = 0$. Given*

$$(x + y \cdot \delta) \cdot \iota^q \in \mathbf{HH}^{p, -2q}(\Lambda[\iota^{\pm 1}], \Lambda[\iota^{\pm 1}])$$

with $p \geq 2$, $x \in \mathbf{HH}^p(\Lambda, \Lambda)$ and $y \in \mathbf{HH}^{p-1}(\Lambda, \Lambda)$, if $[u \cdot \iota, (x + y \cdot \delta) \cdot \iota^q] = 0$ then

$$(x + y \cdot \delta) \cdot \iota^q = [u \cdot \iota, u^{-1} \cdot \delta \cdot x \cdot \iota^{q-1}].$$

We obtain the following more precise information on a minimal A_∞ -model of the $2\mathbb{Z}$ -derived contraction algebra.

Proposition 3.3.5. *The $2\mathbb{Z}$ -derived contraction algebra has a minimal A_∞ -model*

$$(\Lambda[\iota^{\pm 1}], m_4, m_6, \dots)$$

such that m_n is $\mathbb{C}[\iota^{\pm 1}]$ -linear for all $n \geq 4$. In particular, $\{m_4\} = u \cdot \iota$ for some unit $u \in \mathbf{HH}^4(\Lambda, \Lambda)$ satisfying $[u, u] = 0$.

Proof. The first part follows from [HK18]. The rest is a direct consequence of Proposition 3.3.1 and the fact that $[\{m_4\}, \{m_4\}] = 0$, which follows from (1.5.1). \square

3.4. Proof of Theorem 3.1.1. The introduction of the restricted universal Massey product of \mathbf{A} is justified by the following result and the subsequent corollary. Theorem 3.4.1 is an immediate consequence of [JM22, Thm. B], and the latter theorem is obtained an application of the obstruction theory for the existence of A_∞ -structures developed by the third-named author in [Mur20]. In this note we give a direct proof of Theorem 3.4.1 that leverages our detailed knowledge of the relationship between the Hochschild cohomology of the contraction algebra and that of its graded variant (see Section 3.3), although part of the techniques used to prove [JM22, Thm. B] are utilised in some guise.

Theorem 3.4.1. *Let A be a DG algebra such that $H^\bullet(A) = \Lambda[\iota^{\pm 1}]$ as graded algebras. If*

$$j^* \{m_4^A\} = j^* \{m_4^\Lambda\} \in \mathbf{HH}^{\bullet, *}(A, \Lambda[\iota^{\pm 1}]),$$

then A is quasi-isomorphic to the $2\mathbb{Z}$ -derived contraction algebra \mathbf{A} via a quasi-isomorphism that induces the identity in cohomology.

Proof. Let

$$(\Lambda[\iota^{\pm 1}], m_4, m_6, \dots)$$

be a minimal model for the $2\mathbb{Z}$ -derived contraction algebra as in Proposition 3.3.5, and

$$(\Lambda[\iota^{\pm 1}], m'_4, m'_6, \dots)$$

a minimal model for A . Inductively, we will construct an A_∞ -isomorphism with identity linear part

$$f = (0, 0, 0, f_3, 0, f_5, \dots): (\Lambda[\iota^{\pm 1}], m_4, m_6, \dots) \longrightarrow (\Lambda[\iota^{\pm 1}], m'_4, m'_6, \dots),$$

and this clearly suffices to prove the claim. Notice that, necessarily, $f_{2n} = 0$ for all $n \geq 0$ since $\Lambda[\iota^{\pm 1}]$ is concentrated in even degrees.

We proceed as follows. For all $n \geq 0$ we define a Hochschild cochain of total degree 1

$$f^{(n)} = (0, 0, 0, f_3^{(n)}, 0, \dots, f_{2n+1}^{(n)}, 0, \dots)$$

such that $f^{(n)}$ coincides with $f^{(n-1)}$ up to Hochschild degree $2n - 2$ and

$$(3.4.2)^n \quad \partial(f^{(n)}) + f^{(n)} \cdot f^{(n)} + \sum_{r \geq 0} m' \{f^{(n)}, \dots, f^{(n)}\} - m - f^{(n)} \{m\}$$

vanishes up to Hochschild degree $2n + 2$. If we achieve this goal, then we can take

$$f = (0, 0, 0, f_3^{(3)}, 0, \dots, f_{2n-3}^{(n)}, 0, \dots).$$

Indeed, f coincides with $f^{(n)}$ up to Hochschild degree $2n - 2$, so (1.5.2) coincides with (3.4.2)ⁿ up to Hochschild degree $2n - 1$. In particular (1.5.2) vanishes up to Hochschild degree $2n - 1$ for all $n \geq 0$. Therefore (1.5.2) fully vanishes, so f is indeed an A_∞ -isomorphism with identity linear part.

We start with $f^{(0)} = 0$. With this choice, (3.4.2)⁰ clearly vanishes up to Hochschild degree 2.

Below, when defining $f^{(n)}$ we will only specify $f_{2n-1}^{(n)}$ and $f_{2n+1}^{(n)}$ since in smaller Hochschild degrees they are determined by $f^{(n-1)}$ and in higher Hochschild degrees they are irrelevant. Moreover, we will also use that (3.4.2)ⁿ⁻¹ and (3.4.2)ⁿ agree (and hence both vanish) up to Hochschild degree $2n - 1$.

Since $j^* \{m_4\} = j^* \{m'_4\}$, then $\{m_4\} = \{m'_4\}$ by Corollary 3.3.2, so there exists $f_3^{(1)}$ such that

$$(3.4.3) \quad \partial(f_3^{(1)}) + m'_4 - m_4 = 0.$$

This proves that (3.4.2)¹ vanishes up to Hochschild degree 4.

Assume we have constructed up to $f^{(n-1)}$ for some $n \geq 2$. Let us see how to construct $f^{(n)}$. We know by [LH03, Lemme B.4.2] that the Hochschild degree $2n + 2$ part of (3.4.2)ⁿ⁻¹, that we simply denote by a , is an obstruction cocycle ($\partial(a) = 0$) which vanishes in cohomology if and only if there exists $f_{2n+1}^{(n)}$ such that, taking $f_{2n-1}^{(n)} = f_{2n-1}^{(n-1)}$, (3.4.2)ⁿ vanishes up to Hochschild degree $2n + 2$. Indeed, any $f_{2n+1}^{(n)}$ such that $a + \partial(f_{2n+1}^{(n)}) = 0$ would do. We claim that

$$(3.4.4) \quad [m_4, a] + \partial(b - f_3^{n-1} \{a\})$$

vanishes, where b is the Hochschild degree $2n + 4$ part of (3.4.2)ⁿ⁻¹. We prove this claim below. Now, we deduce from Proposition 3.3.5 and Corollary 3.3.4 that there exist Hochschild cochains c_{2n-1} and c_{2n+1} such that

$$a + \partial(c_{2n+1}) + [m_4, c_{2n-1}] = 0, \quad \partial(c_{2n-1}) = 0.$$

If we set

$$f_{2n-1}^{(n)} = f_{2n-1}^{(n-1)} + c_{2n-1}, \quad f_{2n+1}^{(n)} = \begin{cases} c_5 + f_3^{(1)} \{c_3\} - \frac{1}{2} c_3 \{c_3\}, & n = 2; \\ c_{2n+1} + f_3 \{c_{2n-1}\}, & n > 2; \end{cases}$$

we complete the induction step since the Hochschild degree $2n$ part of (3.4.2)ⁿ is,

$$\partial(c_{2n-1}) = 0,$$

and its Hochschild degree $2n + 2$ part is, for $n = 2$,

$$\begin{aligned} a + \partial\left(c_5 + f_3^{(1)} \{c_3\} - \frac{1}{2} c_3 \{c_3\}\right) + c_3 \cdot c_3 + m'_4 \{c_3\} - c_3 \{m_4\} \\ = a + \partial(c_5) + \partial(f_3^{(1)}) \{c_3\} + f_3^{(1)} \{\partial(c_3)\} + m'_4 \{c_3\} - c_3 \{m_4\} \\ = a + \partial(c_5) + (m_4 - m'_4) \{c_3\} + m'_4 \{c_3\} - c_3 \{m_4\} \\ = a + \partial(c_5) + [m_4, c_3] = 0, \end{aligned}$$

where we use that $\partial(c_3\{c_3\}) = 2c_3 \cdot c_3$ by [Lemma 1.4.1](#), and for $n > 2$,

$$\begin{aligned} & a + \partial\left(c_{2n+1} + f_3\{c_{2n-1}\}\right) + m'_4\{c_{2n-1}\} - c_{2n-1}\{m_4\} \\ &= a + \partial(c_{2n+1}) + \partial(f_3^{(n-1)})\{c_{2n-1}\} + f_3^{(n-1)}\{\partial(c_{2n-1})\} + m'_4\{c_{2n-1}\} - c_{2n-1}\{m_4\} \\ &= a + \partial(c_{2n+1}) + (m_4 - m'_4)\{c_{2n-1}\} + m'_4\{c_{2n-1}\} - c_{2n-1}\{m_4\} \\ &= a + \partial(c_{2n+1}) + [m_4, c_{2n-1}] = 0. \end{aligned}$$

We finish the proof with the vanishing of [\(3.4.4\)](#). In what follows, let us write $\Xi = \text{(3.4.2)}^{n-1}$ and $f = f^{(n-1)}$, so as not to overload notation. Note that [\(3.4.4\)](#) is the Hochschild degree $2n + 5$ part of

$$(3.4.5) \quad [m, \Xi] + \partial(\Xi - f\{\Xi\}).$$

This cochain vanishes in Hochschild degrees $< 2n + 5$.

We now start a series of computations. We number most terms for bookkeeping purposes. In the first equation we use [Lemma 1.4.1](#),

$$\begin{aligned} \partial(\Xi) &= \overbrace{\partial(f) \cdot f}^{\textcircled{1}} - \overbrace{f \cdot \partial f}^{\textcircled{2}} + \overbrace{\sum_{r \geq 0} \partial(m')\{f, \cdot, \cdot, f\}}^{\textcircled{3}} \\ &\quad - \overbrace{\sum_{r \geq 1} \sum_{i=1}^r m'\{f, \cdot, \cdot, \partial(f), \cdot, \cdot, f\}}^{\textcircled{4}} \\ &\quad - \overbrace{\sum_{r \geq 1} f \cdot m'\{f, \cdot, \cdot, f\}}^{\textcircled{5}} - \overbrace{\sum_{r \geq 2} \sum_{i=1}^{r-1} m'\{f, \cdot, \cdot, f^2, \cdot, \cdot, f\}}^{\textcircled{6}} \\ &\quad + \overbrace{\sum_{r \geq 1} m'\{f, \cdot, \cdot, f\} \cdot f}^{\textcircled{7}} - \overbrace{\partial(m)}^{\textcircled{8}} - \overbrace{\partial(f)\{m\}}^{\textcircled{9}} - \overbrace{f\{\partial(m)\}}^{\textcircled{10}} + \overbrace{f \cdot m}^{\textcircled{11}} - \overbrace{m \cdot f}^{\textcircled{12}} \end{aligned}$$

Since m and m' are A_∞ -algebra structures,

$$\begin{aligned} \textcircled{8} &= -m\{m\}, \\ \textcircled{10} &= -f\{m\{m\}\}, \\ \textcircled{3} &= -\sum_{r \geq 0} m'\{m'\}\{f, \cdot, \cdot, f\} \\ &= -\sum_{r \geq 0} \sum_{0 \leq i \leq j \leq r} m'\{f, \cdot, \cdot, m'\{f, \cdot, \cdot, f\}, \cdot, \cdot, f\}, \\ &= -\overbrace{\sum_{r \geq 0} m'\{m'\}\{f, \cdot, \cdot, f\}}^{\textcircled{13}} - \overbrace{\sum_{r \geq 1} \sum_{\substack{0 \leq i \leq j \leq r \\ j-i < r}} m'\{f, \cdot, \cdot, m'\{f, \cdot, \cdot, f\}, \cdot, \cdot, f\}}^{\textcircled{14}} \end{aligned}$$

Here we also use the brace relation. We also split the following summations in two parts,

$$\begin{aligned} \textcircled{4} &= \overbrace{m'\{\partial(f)\}}^{\textcircled{15}} + \overbrace{\sum_{r \geq 2} \sum_{i=1}^r m'\{f, i-1, \partial(f), r-i, f\}}^{\textcircled{16}}, \\ \textcircled{6} &= \overbrace{m'\{f^2\}}^{\textcircled{17}} + \overbrace{\sum_{r \geq 3} \sum_{i=1}^{r-1} m'\{f, i-1, f^2, r-i-1, f\}}^{\textcircled{18}}. \end{aligned}$$

Consider the following cochain, that we decompose using the brace relation,

$$\begin{aligned} \overbrace{\sum_{r \geq 0} m'\{f, r, f\}\{m\}}^{\textcircled{19}} &= \sum_{r \geq 0} \sum_{i=0}^r m'\{f, i, m, r-i, f\} \\ &\quad + \sum_{r \geq 1} \sum_{i=0}^r m'\{f, i-1, f\{m\}, r-i, f\} \\ &= \overbrace{m'\{m\}}^{\textcircled{20}} + \overbrace{\sum_{r \geq 1} \sum_{i=0}^r m'\{f, i, m, r-i, f\}}^{\textcircled{21}} + \overbrace{m'\{f\{m\}\}}^{\textcircled{22}} \\ &\quad + \overbrace{\sum_{r \geq 2} \sum_{i=0}^r m'\{f, i-1, f\{m\}, r-i, f\}}^{\textcircled{23}}. \end{aligned}$$

Consider also the following cochain, which is computed by using [Lemma 1.4.2](#),

$$\overbrace{f^2\{m\}}^{\textcircled{24}} = \overbrace{f \cdot f\{m\}}^{\textcircled{25}} - \overbrace{f\{m\} \cdot f}^{\textcircled{26}}.$$

Since Ξ vanishes up to Hochschild degree $2n+1$ and its Hochschild degree $2n+2$ part is a cocycle, the following cochains vanish up to Hochschild degree $2n+5$,

$$\begin{aligned} \sum_{r \geq 2} \sum_{i=1}^r m'\{f, i-1, \Xi, r-i, f\} &= \textcircled{16} + \textcircled{18} + \textcircled{14} - \textcircled{21} - \textcircled{23}, \\ \partial(f)\{\Xi\} + m'\{\Xi\} - m\{\Xi\}, &\quad f\{\partial(\Xi)\}. \end{aligned}$$

Notice that

$$\begin{aligned} \Xi \cdot f &= \textcircled{1} + f^3 + \textcircled{7} - \textcircled{12} - \textcircled{26}, & f \cdot \Xi &= \textcircled{2} + f^3 + \textcircled{5} - \textcircled{11} - \textcircled{25}, \\ m'\{\Xi\} &= \textcircled{15} + \textcircled{17} + \textcircled{13} - \textcircled{20} - \textcircled{22}, & \Xi\{m\} &= \textcircled{9} + \textcircled{24} + \textcircled{19} + \textcircled{8} + \textcircled{10}. \end{aligned}$$

Using all the above, we obtain the following relations, where \equiv stands for congruence modulo cochains vanishing in Hochschild degrees $< 2n + 5$,

$$\begin{aligned}
(3.4.5) &= m\{\Xi\} + \Xi\{m\} + \partial(\Xi) - \partial(f)\{\Xi\} - f\{\partial(\Xi)\} + f \cdot \Xi - \Xi \cdot f \\
&\equiv m\{\Xi\} - (\textcircled{9} + \textcircled{24} + \textcircled{19} + \textcircled{8} + \textcircled{10}) \\
&\quad + (\textcircled{1} - \textcircled{2} + \textcircled{3} - \textcircled{4} - \textcircled{5} - \textcircled{6} + \textcircled{7} - \textcircled{8} - \textcircled{9} - \textcircled{10} + \textcircled{11} - \textcircled{12}) \\
&\quad + (\textcircled{15} + \textcircled{17} + \textcircled{13} - \textcircled{20} - \textcircled{22}) - m\{\Xi\} - 0 \\
&\quad + (\textcircled{2} + f^3 + \textcircled{5} - \textcircled{11} - \textcircled{25}) - (\textcircled{1} + f^3 + \textcircled{7} - \textcircled{12} - \textcircled{26}) \\
&\quad + (\textcircled{16} + \textcircled{18} + \textcircled{14} - \textcircled{21} - \textcircled{23}) = 0.
\end{aligned}$$

This finally concludes the proof. \square

Corollary 3.4.6. *Let A be a DG algebra such that $H^\bullet(A) = \Lambda[l^{\pm 1}]$ as graded algebras. If the restricted universal Massey product*

$$j^*\{m_4^A\} \in \underline{\mathrm{HH}}^{\bullet,*}(\Lambda, \Lambda[l^{\pm 1}])$$

is a unit, then A is quasi-isomorphic to the $2\mathbb{Z}$ -derived contraction algebra Λ .

Proof. Firstly, we observe that the group of graded-algebra automorphisms of $\Lambda[l^{\pm 1}]$ acts on the right of $\underline{\mathrm{HH}}^{\bullet,*}(\Lambda, \Lambda[l^{\pm 1}])$ by conjugation. In particular, the group $Z(\Lambda)^\times$ of units of the centre of Λ acts on $\underline{\mathrm{HH}}^{\bullet,*}(\Lambda, \Lambda[l^{\pm 1}])$ via the graded-algebra automorphisms

$$g_u : x \mapsto xu^i, \quad |x| = 2i,$$

where $u \in Z(\Lambda)^\times$. The induced action on

$$\underline{\mathrm{HH}}^{4,-2}(\Lambda, \Lambda[l^{\pm 1}]) \cong \mathrm{Ext}_{\Lambda^e}^4(\Lambda, \Lambda)$$

has the following explicit description: Given a unit $u \in Z(\Lambda)^\times$ and an exact sequence of Λ -bimodules

$$\eta : 0 \rightarrow \Lambda \xrightarrow{f} X_3 \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow \Lambda \rightarrow 0,$$

we let $[\eta] \cdot u$ be the class of the exact sequence

$$0 \rightarrow \Lambda \xrightarrow{f'} X_3 \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow \Lambda \rightarrow 0,$$

where $f' := u^{-1}f$. The above action clearly restricts to the set of units in $\underline{\mathrm{HH}}^{\bullet,*}(\Lambda, \Lambda[l^{\pm 1}])$ of bidegree $(4, -2)$, since these are precisely the classes that can be represented by an exact sequence with projective(-injective) middle terms [Mur22, Rmk. 5.8], and is in fact transitive on this set. To prove the latter claim on the transitivity of the action we appeal to [Che21, Cor. 2.3], which allows us to lift stable bimodule isomorphisms $\Lambda \simeq \Lambda$ to honest bimodule isomorphisms $\Lambda \cong \Lambda$, see the proof of [JM22, Prop. 2.2.16].

Let $u \in Z(\Lambda)^\times$ be a unit such that

$$j^*\{m_4^A\} \cdot u = j^*\{m_4^\Lambda\} \in \underline{\mathrm{HH}}^{4,-2}(\Lambda, \Lambda[l^{\pm 1}]).$$

Given a minimal model

$$(\Lambda[l^{\pm 1}], m_4^A, m_6^A, m_8^A, \dots)$$

of the DG algebra A we define a new minimal A_∞ -algebra structure

$$(\Lambda[l^{\pm 1}], \underline{m}_4^A, \underline{m}_6^A, \underline{m}_8^A, \dots) = (\Lambda[l^{\pm 1}], m_4^A, m_6^A, m_8^A, \dots) * g_u$$

with n -ary operations $\underline{m}_n^A := g_u^{-1}m_4^A g_u^{\otimes n}$ (notice that $\underline{m}_4^A \neq m_4^A$ as soon as $u \neq 1$). It is easy to see that

$$j^*\{\underline{m}_4^A\} = j^*\{m_4^A\} \cdot u = j^*\{m_4^\Lambda\}$$

and that there is an isomorphism of A_∞ -algebras

$$(\Lambda[l^{\pm 1}], m_4^A, m_6^A, m_8^A, \dots) \rightsquigarrow (\Lambda[l^{\pm 1}], \underline{m}_4^A, \underline{m}_6^A, \underline{m}_8^A, \dots).$$

Thus, A is quasi-isomorphic to any DG algebra model B of the minimal A_∞ -algebra on the right-hand side, see [Remark 3.2.2](#), and by [Theorem 3.4.1](#) the DG algebras B and $\mathbf{\Lambda}$ are quasi-isomorphic. Consequently, the DG algebras A and $\mathbf{\Lambda}$ are quasi-isomorphic, which is what we needed to prove. \square

We are ready to prove [Theorem 3.1.1](#).

Proof of [Theorem 3.1.1](#). Let R_1 and R_2 be isolated cDV singularities with crepant resolutions

$$p_1: X_1 \rightarrow \mathrm{Spec}(R_1) \quad \text{and} \quad p_2: X_2 \rightarrow \mathrm{Spec}(R_2)$$

whose corresponding contraction algebras $\Lambda_1 = \Lambda(p_1)$ and $\Lambda_2 = \Lambda(p_2)$ are isomorphic. We need to prove that the $2\mathbb{Z}$ -derived contraction algebras $\mathbf{\Lambda}_1 = \mathbf{\Lambda}_{\mathrm{con}}(p_1)$ and $\mathbf{\Lambda}_2 = \mathbf{\Lambda}_{\mathrm{con}}(p_2)$ are quasi-isomorphic. Recall that

$$H^\bullet(\mathbf{\Lambda}_1) \cong \Lambda_1[l^{\pm 1}] \quad \text{and} \quad H^\bullet(\mathbf{\Lambda}_2) \cong \Lambda_2[l^{\pm 1}], \quad |l| = -2.$$

In view of the assumption that $\Lambda_1 \cong \Lambda_2$, we obtain a chain of isomorphisms of graded algebras

$$H^\bullet(\mathbf{\Lambda}_1) \cong \Lambda_1[l^{\pm 1}] \cong \Lambda_2[l^{\pm 1}] \cong H^\bullet(\mathbf{\Lambda}_2).$$

In particular, we may and we will identify minimal models of $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ via the above isomorphism. For simplicity, let $\Lambda = \Lambda_1 \cong \Lambda_2$. [Corollary 3.2.7](#) shows that the restricted universal Massey products

$$j^* \{m_4^{\mathbf{\Lambda}_1}\} \in \mathrm{HH}^{4,-2}(\Lambda, \Lambda[l^{\pm 1}]) \quad \text{and} \quad j^* \{m_4^{\mathbf{\Lambda}_2}\} \in \mathrm{HH}^{4,-2}(\Lambda, \Lambda[l^{\pm 1}])$$

are units in the Hochschild–Tate cohomology $\mathrm{HH}^{\bullet,*}(\Lambda, \Lambda[l^{\pm 1}])$. [Corollary 3.4.6](#) implies that the DG algebras $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are quasi-isomorphic, which is what we needed to prove. \square

We also have the following important corollary.

Corollary 3.4.7. *Let R be an isolated cDV singularity that admits a crepant resolution. Then, the singularity category $\mathrm{D}_{\mathrm{sg}}(R)$ admits a unique DG enhancement in the sense of [\[BK90\]](#).*

Proof. Let \mathcal{A} be a DG enhancement of $\mathrm{D}_{\mathrm{sg}}(R)$. By definition, this means that \mathcal{A} is a pre-triangulated DG category and there exists an equivalence of triangulated categories

$$H^0(\mathcal{A}) \simeq \mathrm{D}_{\mathrm{sg}}(R).$$

Let $T \in \mathrm{D}_{\mathrm{sg}}(R)$ be a $2\mathbb{Z}$ -cluster tilting object and A the derived endomorphism algebra of T computed by means of the DG enhancement \mathcal{A} . In particular $\mathrm{D}^c(A)_{\mathrm{dg}} \simeq \mathcal{A}$ since T is a classical generator. By definition, $H^\bullet(A) \cong H^\bullet(\mathbf{\Lambda})$ where $\mathbf{\Lambda}$ is the derived endomorphism algebra of T computed by means of the canonical DG enhancement of $\mathrm{D}_{\mathrm{sg}}(R)$. The proofs of [Theorem 3.2.5](#) and [Corollary 3.2.7](#) only rely on the fact that $T \in \mathrm{D}_{\mathrm{sg}}(R)$ is a $2\mathbb{Z}$ -cluster tilting object, see [Remark 3.2.6](#). Consequently, the restricted universal Massey product $j^* \{m_4^A\}$ is a unit in $\mathrm{HH}^{\bullet,*}(\Lambda, \Lambda[l^{\pm 1}])$ and, by [Corollary 3.4.6](#), the DG algebras A and $\mathbf{\Lambda}$ are quasi-isomorphic. Therefore, the DG categories

$$\mathcal{A} \simeq \mathrm{D}^c(A)_{\mathrm{dg}} \quad \text{and} \quad \mathrm{D}_{\mathrm{sg}}(R)_{\mathrm{dg}}$$

are quasi-equivalent. This shows that every DG enhancement of $\mathrm{D}_{\mathrm{sg}}(R)$ is equivalent to the canonical DG enhancement and the claim follows. \square

Remark 3.4.8. **Corollary 3.4.7** is stronger than **Conjecture A**, as it shows that the DG category $\mathbf{D}_{\text{sg}}(R)_{\text{dg}}$ is determined by Λ_{con} up to isomorphism in \mathbf{Hmo} , the Morita category of small DG categories [Tab05].

4. CONCLUDING REMARKS

In this section we collect some observations, some of which follow easily from the results in the previous sections.

4.1. Formality of contraction algebras. Recall that a DG algebra A is *formal* if there is a quasi-isomorphism $A \simeq H^\bullet(A)$, where the graded algebra $H^\bullet(A)$ is viewed as a DG algebra with vanishing differential. We observe that $2\mathbb{Z}$ -derived contraction algebras are almost never formal.⁷

Theorem 4.1.1. *Let R be an isolated cDV singularity with a crepant resolution and Λ_{con} a $2\mathbb{Z}$ -derived contraction algebra for R . The following statements are equivalent:*

- (1) *The $2\mathbb{Z}$ -derived contraction algebra Λ_{con} is formal.*
- (2) *There is an isomorphism of algebras $\Lambda_{\text{con}} \cong \mathbb{C}$.*
- (3) *There is an isomorphism of algebras*

$$R \cong \mathbb{C}[[x, y, z, t]]/(xy - zt),$$

so that $\text{Spec}(R)$ is the base of the Atiyah flop [Ati58].

If the above equivalent conditions hold, then there is a quasi-isomorphism

$$\Lambda_{\text{con}} \simeq \mathbb{C}[t^{\pm 1}], \quad |t| = -2,$$

where the graded algebra $\mathbb{C}[t^{\pm 1}]$ is equipped with the trivial differential.

Proof. (1) \Rightarrow (2) If the $2\mathbb{Z}$ -derived contraction algebra Λ_{con} is formal, then its restricted universal Massey product $j^*\{m_4\}$ is represented by the trivial sequence in $\mathbf{HH}^{4,-2}(\Lambda_{\text{con}}, \Lambda_{\text{con}}[t^{\pm 1}]) = \text{Ext}_{\Lambda_{\text{con}}^e}^4(\Lambda_{\text{con}}, \Lambda_{\text{con}})$. In view of **Theorem 3.2.5**, this can only happen if Λ_{con} is projective as a Λ_{con} -bimodule or, equivalently, if the algebra Λ_{con} is semisimple. Since contraction algebras are basic and connected, we must have an isomorphism of algebras $\Lambda_{\text{con}} \cong \mathbb{C}$, which is what we needed to prove.

(2) \Rightarrow (1) If $\Lambda_{\text{con}} \cong \mathbb{C}$, then there is an isomorphism of graded algebras

$$H^\bullet(\Lambda_{\text{con}}) \cong \Lambda_{\text{con}}[t^{\pm 1}] \cong \mathbb{C}[t^{\pm 1}], \quad |t| = -2.$$

It is well-known (and easy to prove using Kadeishvili's Theorem [Kad88], for example) that the latter graded algebra is intrinsically formal, that is every DG algebra with cohomology algebra $\mathbb{C}[t^{\pm 1}]$ is formal. In particular, Λ_{con} is formal.⁸

(2) \Leftrightarrow (3) In view of the validity of **Conjecture A**, it is enough to observe that \mathbb{C} is indeed isomorphic to the contraction algebra of the Atiyah flop [DW16, Table 2]. \square

Remark 4.1.2. The (non-)formality of the derived contraction algebra $\Lambda_{\text{con}}^{\leq 0}$ is investigated in [Boo19, Ch. 9] where minimal models of this DG algebra are computed in various examples, see also [Boo18, Boo21, Boo22].

⁷This fact will come as no surprise to the experts, but we did not find a proof of it in the literature.

⁸Alternatively, one can prove this fact using the results in this note as follows: Since the enveloping algebra

$$\Lambda_{\text{con}}^e \cong \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$$

is semisimple, the Hochschild–Tate cohomology $\mathbf{HH}^{\bullet,*}(\Lambda_{\text{con}}, \Lambda_{\text{con}}[t^{\pm 1}])$ vanishes in positive Hochschild degrees. Then, by **Theorem 3.4.1**, the $2\mathbb{Z}$ -derived contraction algebra Λ_{con} is quasi-isomorphic to its cohomology algebra $H^\bullet(\Lambda_{\text{con}})$ for the condition on the agreement of the corresponding universal Massey products is trivially satisfied. Of course, this is essentially the same proof as the one using Kadeishvili's Theorem, which is indeed a special case of [JM22, Thm. B].

Remark 4.1.3. More generally, Kadeishvili’s Theorem [Kad88] can be used to prove that the Laurent polynomial algebra $K[t^{\pm 1}] = K \otimes_{\mathbb{C}} \mathbb{C}[t^{\pm 1}]$ is intrinsically formal if K is a finite-dimensional algebra of projective dimension at most 2 as a K -bimodule [Sai20, Cor. 4.2]. Such an algebra K , however, cannot be a contraction algebra unless $K = \mathbb{C}$ since contraction algebras are basic and connected, and a symmetric \mathbb{C} -algebra has finite projective dimension as a bimodule over itself if and only if it is separable (semisimple).

4.2. Relationship to the Derived Auslander–Iyama Correspondence. Let \mathcal{T} be a triangulated category whose underlying additive category is Krull–Schmidt and has finite-dimensional morphism spaces. A basic object $T \in \mathcal{T}$ is *d-cluster tilting*, $d \geq 1$, if the following conditions are satisfied [IY08, Bel15]:

- The object T is *d-rigid*: $\mathcal{T}(T, T[i]) = 0$ for all $0 < i < d$.
- For each object $X \in \mathcal{T}$ there exists a diagram

$$\begin{array}{ccccccc} & & T_{d-2} & \cdots & & T_1 & \longrightarrow & T_0 \\ & \nearrow & & \searrow & & \nearrow & & \searrow \\ T_{d-1} & \longleftarrow & +1 & \longrightarrow & X_{d-2} & \cdots & X_2 & \longleftarrow & +1 & \longrightarrow & X_1 & \longleftarrow & +1 & \longrightarrow & X \end{array}$$

in which $T_i \in \text{add}(T)$, $0 \leq i < d$, the oriented triangles denote exact triangles in \mathcal{T} and the unoriented triangles commute. The object T is *d \mathbb{Z} -cluster tilting* if it is *d-cluster tilting* and $T \cong T[d]$. **Theorem 3.1.1** is a special case of the theorem below. For a finite-dimensional algebra Λ , we let $\text{proj}(\Lambda)$ be the category of finite-dimensional projective Λ -modules. For example, if $\Lambda = \mathcal{T}(T, T)$, then the Yoneda functor

$$\mathcal{T} \supseteq \text{add}(T) \xrightarrow{\sim} \text{proj}(\Lambda), \quad X \mapsto \mathcal{T}(T, X),$$

is an equivalence (of additive categories).

Theorem 4.2.1 (Derived Auslander–Iyama Correspondence [JM22, Thm. 5.1.10]). *Let $d \geq 1$. There is a bijective correspondence between the following:*

- (1) *Quasi-isomorphism classes of DG algebras A that satisfy the following:*
 - *The algebra $H^0(A)$ is a basic finite-dimensional algebra.*
 - *The free DG A -module $A \in \text{D}^c(A)$ is a $d\mathbb{Z}$ -cluster tilting object.*
- (2) *Equivalence classes of pairs (Λ, I) consisting of*
 - *a basic finite-dimensional self-injective algebra Λ and*
 - *an invertible Λ -bimodule I such that $\Omega_{\Lambda^e}^{d+2}(\Lambda) \cong I$ in the stable category of Λ -bimodules.*

The correspondence is given by the formula $A \mapsto (H^0(A), H^{-d}(A))$.

Remark 4.2.2. The case $d = 1$ of **Theorem 4.2.1** is one way to formulate the main result in [Mur22]. Indeed, an object $T \in \mathcal{T}$ is 1-cluster tilting if and only if it is $1\mathbb{Z}$ -cluster tilting if and only if $\text{add}(T) = \mathcal{T}$; the latter condition means that \mathcal{T} is a triangulated category of finite type in the terminology of [Mur22].

Remark 4.2.3. Let (Λ, I) be a pair as in **Theorem 4.2.1(2)**. Since the algebra Λ is assumed to be basic, the map

$$\text{Aut}(\Lambda) \longrightarrow \text{Pic}(\Lambda), \quad \sigma \longmapsto [{}_1\Lambda_\sigma]$$

from the group of algebra automorphisms of Λ to the Picard group of invertible Λ -bimodules induces an isomorphism of groups [Bol84, Prop. 3.8]

$$\text{Out}(\Lambda) \xrightarrow{\sim} \text{Pic}(\Lambda), \quad [\sigma] \longmapsto [{}_1\Lambda_\sigma],$$

where $\text{Out}(\Lambda) = \text{Aut}(\Lambda)/\text{Inn}(\Lambda)$ is the group of outer automorphisms of Λ . In particular, there exists $\sigma \in \text{Aut}(\Lambda)$ such that $I \cong {}_1\Lambda_\sigma$ as Λ -bimodules. The condition $\Omega_{\Lambda^e}^{d+2}(\Lambda) \simeq {}_1\Lambda_\sigma$ expresses the fact that the algebra Λ is *twisted $(d+2)$ -periodic*

with respect to σ . When $\sigma = \mathbf{1}$ or, equivalently, $I \cong \Lambda$, the algebra Λ is said to be $(d+2)$ -periodic. For example, contraction algebras are known to be 4-periodic. We refer the reader to [ES08] and the references therein for information on (twisted) periodic algebras.

Theorem 3.1.1 follows from the *injectivity* of the correspondence in **Theorem 4.2.1** with $d = 2$; the proof of **Theorem 3.1.1** outlined in this note effectively goes through the proof of the latter in the special case of $2\mathbb{Z}$ -derived contraction algebras. We record the resulting characterisation of contraction algebras for the sake of completeness.

Theorem 4.2.4. *Let R be an isolated cDV singularity with a crepant resolution $p: X \rightarrow \mathrm{Spec}(R)$. Up to quasi-isomorphism, the $2\mathbb{Z}$ -derived contraction algebra $\mathbf{\Lambda} = \mathbf{\Lambda}_{\mathrm{con}}(p)$ is the unique DG algebra with the following properties:*

- (1) $\mathbf{\Lambda} \in \mathrm{D}^c(\mathbf{\Lambda})$ is a $2\mathbb{Z}$ -cluster tilting object.
- (2) There is an isomorphism of algebras $H^0(\mathbf{\Lambda}) \cong \Lambda = \Lambda_{\mathrm{con}}(p)$.
- (3) There is an isomorphism of Λ -bimodules $H^{-2}(\mathbf{\Lambda}) \cong \Lambda$.

In other words, $\mathbf{\Lambda}$ is determined up to quasi-isomorphism by its image (Λ, Λ) under the Derived Auslander–Iyama Correspondence.

Proof. That the $2\mathbb{Z}$ -derived contraction algebra satisfies the first two properties follows from **Theorem 1.3.1** since, by definition, $\mathbf{\Lambda}$ is the derived endomorphism algebra of a $2\mathbb{Z}$ -cluster tilting object in $\mathrm{D}_{\mathrm{sg}}(R) \simeq \mathrm{D}^c(\mathbf{\Lambda})$. Therefore $\mathbf{\Lambda}$ belongs to the class of DG algebras in **Theorem 4.2.1(1)** with $d = 2$. Given that Λ is 4-periodic, the third property follows. Moreover, $\mathbf{\Lambda}$ is determined up to quasi-isomorphism by its image (Λ, Λ) under the Derived Auslander–Iyama Correspondence, which is what we needed to prove. \square

4.3. Isolated cDV singularities with non-smooth minimal models. Contraction algebras are defined for an arbitrary cDV singularity R that is neither isolated nor admits a crepant resolution. However, contraction algebras are finite-dimensional if and only if R defines an isolated singularity [DW19, Summary 5.6], and this finite-dimensionality is crucial to our approach. On the other hand, R admits a crepant resolution if and only if the singularity category $\mathrm{D}_{\mathrm{sg}}(R)$ admits a $2\mathbb{Z}$ -cluster tilting object, see [BIKR08, Thm. 5.4] and the references therein. If, on the other hand, the minimal models⁹ of R are singular, then the contraction algebras of R are the endomorphism algebras of *maximal rigid objects* in $\mathrm{D}_{\mathrm{sg}}(R)$ [Wem18], that is objects $T \in \mathrm{D}_{\mathrm{sg}}(R)$ such that

$$\mathrm{add}(T) = \{X \in \mathrm{D}_{\mathrm{sg}}(R) \mid \mathrm{Hom}(T \oplus X, (T \oplus X)[1]) = 0\}.$$

It is easy to verify that 2-cluster tilting objects are maximal rigid, but the converse is false in general [BIKR08, BMV10]; moreover, if there exists a 2-cluster tilting object then every maximal rigid object is also 2-cluster tilting [BIRS09, Thm. II.1.8].¹⁰ In any case, one may still define the $2\mathbb{Z}$ -derived contraction algebras of R as the derived endomorphism algebras of maximal rigid objects in $\mathrm{D}_{\mathrm{sg}}(R)$, computed in terms of the canonical DG enhancement of the latter triangulated category. Note, however, that **Theorem 4.2.1** does not cover the case of maximal rigid objects that are not $2\mathbb{Z}$ -cluster tilting and hence it cannot be applied to prove a version of **Theorem 4.2.4**

⁹In the context of the MMP in dimension three and higher, minimal models play the role of minimal resolutions of surfaces. We do not recall the technical definition in this note, but only mention that minimal models are permitted to have ‘mild’ singularities as long as they remain ‘closer’ to the original space than a smooth resolution (which always exists by a famous theorem of Hironaka [Hir64].)

¹⁰The reader should compare this statement with the following geometric fact: If one minimal model of $\mathrm{Spec}(R)$ is smooth, then all of its minimal models are also smooth [Kol89, Cor. 4.11].

for isolated cDV singularities that do not admit a crepant resolution. Finally, we mention that the apparent variant of [Conjecture A](#) does not hold for isolated cDV singularities whose minimal models are not smooth, see [[Boo21](#), Ex 8.4.2] for an explicit counterexample.

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