# ERRATUM TO "DEFORMED CALABI–YAU COMPLETIONS"

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ABSTRACT. We correct an error and some inaccuracies that occurred in "Deformed Calabi–Yau completions". The most important point is that, as pointed out by W. K. Yeung, to show that the deformed Calabi-Yau completion has the Calabi–Yau property, one needs to assume that the deformation parameter comes from negative cyclic homology. Notice that this does hold in the case of Ginzburg dg algebras.

## 1. Calabi–Yau completions

We refer to [4] for unexplained notation and terminology. Let $k$ be a commutative ring, $n$ an integer and $\mathcal{A}$ a dg $k$-category. We may and will assume that $\mathcal{A}$ is cofibrant over $k$, *i.e.* each morphism complex $\mathcal{A}(X,Y)$ is cofibrant as a complex of $k$-modules. Moreover, we assume that $\mathcal{A}$ is homologically smooth [5], *i.e.* $\mathcal{A}$ is perfect in the derived category $\mathcal{D}(\mathcal{A}^e)$ of $\mathcal{A}$-bimodules. For a dg $\mathcal{A}$-bimodule $M$, put

$$M^\vee = \mathrm{RHom}_{\mathcal{A}^e}(M, \mathcal{A}^e).$$

Recall [2] that $\mathcal{A}$ is *(bimodule) $n$-Calabi–Yau* if there is an isomorphism

$$\varphi : \Sigma^n \mathcal{A}^\vee \xrightarrow{\sim} \mathcal{A}$$

in the derived category of $\mathcal{A}$-bimodules. The symmetry property originally imposed on $\varphi$ is automatic as shown in Appendix C of [8]. Following [9, 6, 11], we define an *$n$-Calabi-Yau structure on* $\mathcal{A}$ as the datum of a class $\eta$ in negative cyclic homology $HN_n(\mathcal{A})$ which is *non degenerate*, *i.e.* whose image under the canonical maps

$$HN_n(\mathcal{A}) \to HH_n(\mathcal{A}, \mathcal{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\mathcal{A}^e)}(\Sigma^n \mathcal{A}^\wedge, \mathcal{A})$$

is an isomorphism. Following [8], such a structure is called *exact* if $\eta$ is an image under Connes' map $B : HC_{n-1}(\mathcal{A}) \to HN_n(\mathcal{A})$.

Let $\theta$ be a cofibrant resolution of $\Sigma^{n-1}\mathcal{A}^\vee$. The *$n$-Calabi–Yau completion* $\Pi_n(\mathcal{A})$ was defined in [4] as the tensor category $T_{\mathcal{A}}(\theta)$. The following theorem is a more precise version of Theorem 4.8 of [4]. We include a proof since the statement is slightly stronger and the new proof more transparent.

**Theorem 1.1.** *The Calabi-Yau completion $\Pi_n(\mathcal{A})$ is homologically smooth and carries a canonical exact $n$-Calabi-Yau structure.*

**Remark 1.2.** *For finitely cellular dg categories $\mathcal{A}$, W. K. Yeung gives two proofs of this theorem: one in section 3.3 of [10] and a more geometric one in section 2.3 of [11]. The theorem generalizes his result to arbitrary homologically smooth dg categories $\mathcal{A}$.*

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**Remark 1.3.** *It is well-known that the Calabi-Yau completion, being a generalization of the* 2*-Calabi–Yau preprojective algebra* [1]*, should be viewed as a (shifted) non commutative cotangent bundle. This is very nicely explained in section 2.3 of* [11]*. Alternatively, one may view it as a (shifted) non commutative total space of the canonical bundle. This is made rigorous in section 3.5 of* [3].

*Proof of the Theorem.* Put $\mathcal{B} = \Pi_n(\mathcal{A})$. Notice that $\mathcal{B}$ is augmented over $\mathcal{A}$ in the sense that we have canonical dg functors $\mathcal{A} \to \mathcal{B} \to \mathcal{A}$ whose composition is the identity. Thus, Hochschild and cyclic homology of $\mathcal{A}$ are canonically direct summands of those of $\mathcal{B}$. We call the supplementary summands the *reduced* Hochschild respectively cyclic homology of $\mathcal{B}$. We would like to compute them. Since $\mathcal{B}$ is a tensor category (though over the non commutative ground category $\mathcal{A}$), it suffices to adapt the results of section 3.1 of [7]. The analogue of the small resolution $C^{\mathrm{sm}}(T(V))$ of remark 3.1.3 of [7] is the exact bimodule sequence

$$0 \longrightarrow \mathcal{B} \otimes_{\mathcal{A}} \theta \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{b'} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \longrightarrow \mathcal{B} \longrightarrow 0 \ , \tag{1.3.1}$$

where the map $b'$ takes $x \otimes t \otimes y$ to $xt \otimes y - x \otimes ty$ and the second map is composition. Notice that since $\mathcal{B}^e$ is cofibrant over $\mathcal{A}^e$, we have $\mathcal{B} \otimes_{\mathcal{A}} \otimes \mathcal{B} = \mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{B}^e = \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{B}^e$. Thus, this dg module is perfect over $\mathcal{B}^e$ and so is $\mathcal{B} \otimes_{\mathcal{A}} \theta \otimes_{\mathcal{A}} \mathcal{B} = \theta \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{B}^e$ so that $\mathcal{B}$ is indeed homologically smooth. By taking the derived tensor product of (1.3.1) over $\mathcal{B}^e$ with $\mathcal{B}$, we find that Hochschild homology of $\mathcal{B}$ is computed by the cone over the induced morphism

$$(\mathcal{B} \otimes_{\mathcal{A}} \theta \otimes_{\mathcal{A}} \mathcal{B}) \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B} \longrightarrow (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}) \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B}.$$

Notice that $\theta$ is cofibrant over $\mathcal{A}^e$ so that $\mathcal{B} \otimes_{\mathcal{A}} \theta \otimes_{\mathcal{A}} \mathcal{B} = \theta \otimes_{\mathcal{A}^e} \mathcal{B}^e$ is cofibrant over $\mathcal{B}^e$ and for the first term, we have

$$(\mathcal{B} \otimes_{\mathcal{A}} \theta \otimes_{\mathcal{A}} \mathcal{B}) \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B} = (\theta \otimes_{\mathcal{A}^e} \mathcal{B}^e) \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B} = \theta \otimes_{\mathcal{A}^e} \mathcal{B}.$$

As for the second term, notice that $\mathcal{B}$ is cofibrant over $\mathcal{A}$ so that we have

$$\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} = \mathcal{B} \overset{L}{\otimes}_{\mathcal{A}} \mathcal{B} = \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{B}^e$$

and therefore

$$(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}) \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B} = (\mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{B}^e) \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B} = \mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \mathcal{B}.$$

Now notice that in $\mathcal{B} = \mathcal{A} \oplus \theta \oplus (\theta \otimes_{\mathcal{A}} \theta) \oplus \cdots$, all the summands are cofibrant over $\mathcal{A}^e$ except the first one. Since we are interested in *reduced* Hochschild homology, the first term does not matter: Let us put $\overline{\mathcal{B}} = \mathcal{B}/\mathcal{A}$. Then $\overline{\mathcal{B}}$ is cofibrant over $\mathcal{A}^e$ and we have $\mathcal{A} \overset{L}{\otimes}_{\mathcal{A}^e} \overline{\mathcal{B}} = \mathcal{A} \otimes_{\mathcal{A}^e} \overline{\mathcal{B}}$. So the reduced Hochschild homology of $\mathcal{B}$ is computed by the cone over the morphism

$$\theta \otimes_{\mathcal{A}^e} \mathcal{B} \xrightarrow{b} \mathcal{A} \otimes_{\mathcal{A}^e} \overline{\mathcal{B}}.$$

It is not hard to check that the map $b$ is given by

$$t \otimes u \mapsto 1_y \otimes tu - (-1)^{|t||u|} 1_x \otimes ut \,,$$

where $t \in \theta(x, y)$ and $u \in \mathcal{B}(y, x)$. Notice that its kernel has the summand $\theta \otimes_{\mathcal{A}^e} \mathcal{A}$. The construction shows that the cone over $b$ is the analogue of the complex $C^{\mathrm{small}}(T(V)) = (V \otimes A \to A)$ of section 3.1.1 of [7], where $A = T(V)$. Now proceding further along this line, it is not hard to check that the analogue of the complex

$$\ldots \xrightarrow{\gamma} V \otimes A \xrightarrow{b} A \xrightarrow{\gamma} V \otimes A \xrightarrow{b} A \longrightarrow 0$$

computing the cyclic homology of $A = T(V)$ in Prop. 3.1.5 of [7] is the sum total dg module of the periodic complex

$$\dots \xrightarrow{\gamma} \theta \otimes_{\mathcal{A}^e} \mathcal{B} \xrightarrow{b} \mathcal{A} \otimes_{\mathcal{A}^e} \overline{\mathcal{B}} \xrightarrow{\gamma} \theta \otimes_{\mathcal{A}^e} \mathcal{B} \xrightarrow{b} \mathcal{A} \otimes_{\mathcal{A}^e} \overline{\mathcal{B}} \longrightarrow 0 \ ,$$

which computes the reduced cyclic homology of $\mathcal{B}$ and where $\gamma$ is given by

$$a \otimes (t_1 \dots t_n) \mapsto \sum_{i=1}^{n} \pm t_i \otimes (t_{i+1} \dots t_n a t_1 \dots t_{i-1}) \, ,$$

where the sign is given by the Koszul sign rule. Let us now exhibit a canonical element in the reduced $(n-1)$th cyclic homology of $\mathcal{B}$. We have canonical quasi-isomorphisms

$$\theta \otimes_{\mathcal{A}^e} \mathcal{A} \longleftarrow \theta \otimes_{\mathcal{A}^e} \mathbf{p}\mathcal{A} \longrightarrow \Sigma^{n-1}\mathrm{Hom}_{\mathcal{A}^e}(\mathbf{p}\mathcal{A}, \mathbf{p}\mathcal{A}) \ ,$$

where $\mathbf{p}\mathcal{A} \to \mathcal{A}$ is an $\mathcal{A}^e$-cofibrant resolution of $\mathcal{A}$. The identity on the right hand side corresponds to a Casimir element $c$ on the left hand side. This element yields a canonical class of homological degree $n-1$ in the summand $\mathcal{A} \otimes_{\mathcal{A}^e} \theta \cong \theta \otimes_{\mathcal{A}^e} \mathcal{A}$ of the last term of the complex

$$\dots \xrightarrow{\gamma} \theta \otimes_{\mathcal{A}^e} \mathcal{B} \xrightarrow{b} \mathcal{A} \otimes_{\mathcal{A}^e} \overline{\mathcal{B}} \xrightarrow{\gamma} \theta \otimes_{\mathcal{A}^e} \mathcal{B} \xrightarrow{b} \mathcal{A} \otimes_{\mathcal{A}^e} \overline{\mathcal{B}} \longrightarrow 0$$

and thus a canonical class in $HC^{\mathrm{red}}_{n-1}(\mathcal{B})$. Under Connes' map $HC^{\mathrm{red}}_{n-1}(\mathcal{B}) \to HH^{\mathrm{red}}_{n}(\mathcal{B})$, which is induced by $\gamma$, this class clearly corresponds to the class of the Casimir element $c$ in the summand $\theta \otimes_{\mathcal{A}^e} \mathcal{A}$ of $\theta \otimes_{\mathcal{A}^e} \mathcal{B}$. It remains to be checked that this class is non degenerate,*i.e.* that it is taken to an isomorphism by the canonical maps

$$\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B} \xrightarrow{\sim} \mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \Theta^\vee \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{B}^e}(\Theta, \mathcal{B}) \, ,$$

where $\Theta$ is the inverse dualizing complex $\mathcal{B}^\vee$. By the exact sequence (1.3.1), the $\mathcal{B}$-bimodule $\mathcal{B}$ is quasi-isomorphic to the cone over the map

$$\mathcal{B} \otimes_{\mathcal{A}} \theta \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{b'} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}.$$

As we have already noted, the left hand term is $\mathcal{B}^e$-cofibrant but the right hand term is not. We choose a surjective $\mathcal{A}^e$-cofibrant resolution $\mathbf{p}\mathcal{A} \to \mathcal{A}$. It yields a surjective quasi-isomorphism

$$\mathcal{B} \otimes_{\mathcal{A}} \mathbf{p}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{B} \longrightarrow \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \ ,$$

where now the left hand term is $\mathcal{B}^e$-cofibrant. We choose a lift $\widetilde{b'}$ of $b'$ along this quasi-isomorphism. We find that $\mathcal{B}$ has as $\mathcal{B}^e$-cofibrant resolution the cone over the map

$$\mathcal{B} \otimes_{\mathcal{A}} \theta \otimes_{\mathcal{A}} \mathcal{B} \xrightarrow{\widetilde{b'}} \mathcal{B} \otimes_{\mathcal{A}} \mathbf{p}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{B} \ ,$$

which we can rewrite as

$$\theta \otimes_{\mathcal{A}^e} \mathcal{B}^e \longrightarrow \mathbf{p}\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{B}^e \ .$$

By applying $\mathrm{Hom}_{\mathcal{B}^e}(?, \mathcal{B}^e)$ to this cone and using the adjunction, we find that $\Theta$ is the cylinder over

$$\mathrm{Hom}_{\mathcal{A}^e}(\mathbf{p}\mathcal{A}, \mathcal{B}^e) \longrightarrow \mathrm{Hom}_{\mathcal{A}^e}(\theta, \mathcal{B}^e).$$

We know that the class of $c$ in $\theta \otimes_{\mathcal{A}^e} \mathcal{A} \subset \theta \otimes_{\mathcal{A}^e} \mathcal{B}$ yields a morphism from this cylinder shifted by $n$ degrees to the $\mathcal{B}^e$-cofibrant resolution of $\mathcal{B}$, *i.e.* the cone over

$$\mathcal{B}^e \otimes_{\mathcal{A}^e} \theta \longrightarrow \mathcal{B}^e \otimes_{\mathcal{A}^e} \mathbf{p}\mathcal{A}.$$

To show that this morphism is invertible, we first choose a lift to the level of dg modules of the morphism

$$\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B} \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{B}^e}(\mathcal{B}^\vee, \mathcal{B}).$$

For this, we choose the $\mathcal{B}^e$-cofibrant resolution $\mathbf{p}\mathcal{B}$ of $\mathcal{B}$ given by the cone over

$$\theta \otimes_{\mathcal{A}^e} \mathcal{B}^e \longrightarrow \mathbf{p}\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{B}^e \ .$$

Notice that this is canonically isomorphic to the cone over

$$\mathcal{B}^e \otimes_{\mathcal{A}^e} \theta \longrightarrow \mathcal{B}^e \otimes_{\mathcal{A}^e} \mathbf{p}\mathcal{A} \ .$$

By taking the tensor product over $\mathcal{B}^e$ of the two preceding incarnations of $\mathbf{p}\mathcal{B}$ we find that the object $\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B}$ is lifted to the dg level by the total object of the square

(1.3.2)
$$\begin{array}{ccc} \theta \otimes_{\mathcal{A}^e} \mathcal{B}^e \otimes_{\mathcal{A}^e} \theta & \longrightarrow & \theta \otimes_{\mathcal{A}^e} \mathcal{B}^e \otimes_{\mathcal{A}^e} \mathbf{p}\mathcal{A} \\ \downarrow & & \downarrow \\ \mathbf{p}\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{B}^e \otimes_{\mathcal{A}^e} \theta & \longrightarrow & \mathbf{p}\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{B}^e \otimes_{\mathcal{A}^e} \mathbf{p}\mathcal{A}. \end{array}$$

On the other hand, we lift the object $\mathrm{RHom}_{\mathcal{B}^e}(\mathcal{B}^\vee, \mathcal{B})$ to the dg level by choosing for the first argument the cylinder over

$$\mathrm{Hom}_{\mathcal{A}^e}(\mathbf{p}\mathcal{A}, \mathcal{B}^e) \longrightarrow \mathrm{Hom}_{\mathcal{A}^e}(\theta, \mathcal{B}^e).$$

and for the second argument the cone over

$$\theta \otimes_{\mathcal{A}^e} \mathcal{B}^e \longrightarrow \mathbf{p}\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{B}^e \ .$$

so that $\mathrm{RHom}_{\mathcal{B}^e}(\mathcal{B}^\vee, \mathcal{B})$ is given by the total object of the square

(1.3.3)
$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{B}^e}(\mathrm{Hom}_{\mathcal{A}^e}(\theta, \mathcal{B}^e), \theta \otimes_{\mathcal{A}^e} \mathcal{B}^e) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}^e}(\mathrm{Hom}_{\mathcal{A}^e}(\mathbf{p}\mathcal{A}, \mathcal{B}^e), \theta \otimes_{\mathcal{A}^e} \mathcal{B}^e) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathcal{B}^e}(\mathrm{Hom}_{\mathcal{A}^e}(\theta, \mathcal{B}^e), \mathbf{p}\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{B}^e) & \longrightarrow & \mathrm{Hom}_{\mathcal{B}^e}(\mathrm{Hom}_{\mathcal{A}^e}(\mathbf{p}\mathcal{A}, \mathcal{B}^e), \mathbf{p}\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{B}^e) \end{array}$$

shifted by two degrees to the right. Now the morphism

$$\mathcal{B} \overset{L}{\otimes}_{\mathcal{B}^e} \mathcal{B} \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{B}^e}(\mathcal{B}^\vee, \mathcal{B}) \ ,$$

is lifted by the map induced by the natural morphism from square (1.3.2) to square (1.3.3) which, at each corner, is given by a natural map of the type

$$V \otimes_k W \longrightarrow \mathrm{Hom}_k(W^*, V) \ , \quad v \otimes w \mapsto (w^* \mapsto \langle w, w^* \rangle v) \ ,$$

where $V$ and $W$ are projective modules over $k$. We now need to lift the Casimir element $c$ to a cycle in the total object of square (1.3.2). First of all, $c$ lifts to a cycle corresponding to the identity under the quasi-isomorphism

$$\theta \otimes_{\mathcal{A}^e} \mathbf{p}\mathcal{A} \longrightarrow \Sigma^{n-1}\mathrm{Hom}_{\mathcal{A}^e}(\mathbf{p}\mathcal{A}, \mathbf{p}\mathcal{A}) \ \ ,$$

We can write this cycle as $\Sigma_i x_i^* \otimes x_i$. We lift it to the element $c_{ur} = \Sigma_i x_i^* \otimes 1 \otimes 1 \otimes x_i$ belonging to

$$\theta \otimes_{\mathcal{A}^e} \mathcal{A}^e \otimes_{\mathcal{A}^e} \mathbf{p}\mathcal{A} \subseteq \theta \otimes_{\mathcal{A}^e} \mathcal{B}^e \otimes_{\mathcal{A}^e} \mathbf{p}\mathcal{A}$$

in the upper right corner of the square 1.3.2 (the 1s are identities of objects which we omit from the notation). We also consider the element $c_{ll} = \Sigma_i x_i \otimes 1 \otimes 1 \otimes x_i^*$ belonging to

$$\mathbf{p}\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{A}^e \otimes_{\mathcal{A}^e} \otimes \theta \subseteq \mathbf{p}\mathcal{A} \otimes_{\mathcal{A}^e} \mathcal{B}^e \otimes_{\mathcal{A}^e} \otimes \theta$$

in the lower left corner of square (1.3.2). The image of $c_{ur}$ in the lower right corner of the square is

$$\Sigma_i(\widetilde{1}\otimes x_i^*\otimes 1\otimes x_i - \widetilde{1}\otimes 1\otimes x_i^*\otimes x_i)\,,$$

where $\widetilde{1}\in \mathbf{p}\mathcal{A}$ is a cycle lifting $1\in\mathcal{A}$, and the image of $c_{ll}$ in the lower right corner is

$$\Sigma_i(x_i\otimes x_i^*\otimes 1\otimes\widetilde{1} - x_i\otimes 1\otimes x_i^*\otimes\widetilde{1}).$$

These images are not equal but we will show that they are homologous. For this, it suffices to show that their images under a quasi-isomorphism are equal. Let us focus on the terms

$$x_i\otimes x_i^*\otimes 1\otimes\widetilde{1} \quad\text{and}\quad \widetilde{1}\otimes x_i^*\otimes 1\otimes x_i. \tag{1.3.4}$$

They lie in the summand

$$\mathbf{p}\mathcal{A}\otimes_{\mathcal{A}^e}(\theta\otimes\mathcal{A})\otimes_{\mathcal{A}^e}\mathbf{p}\mathcal{A}\subset \mathbf{p}\mathcal{A}\otimes_{\mathcal{A}^e}(\mathcal{B}\otimes\mathcal{B})\otimes_{\mathcal{A}^e}\mathbf{p}\mathcal{A}.$$

The natural morphism

$$\mathbf{p}\mathcal{A}\otimes_{\mathcal{A}^e}\mathcal{B}^e\otimes_{\mathcal{A}^e}\mathbf{p}\mathcal{A}\longrightarrow \mathcal{A}\otimes_{\mathcal{A}^e}\mathcal{B}^e\otimes_{\mathcal{A}^e}\mathcal{A} = \mathcal{A}\otimes_{\mathcal{A}^e}(\mathcal{B}\otimes_{\mathcal{A}}\mathcal{B})$$

may not be a quasi-isomorphism but it induces a quasi-isomorphism

$$\mathbf{p}\mathcal{A}\otimes_{\mathcal{A}^e}(\theta\otimes\mathcal{A})\otimes_{\mathcal{A}^e}\mathcal{A}\longrightarrow \mathcal{A}\otimes_{\mathcal{A}^e}(\theta\otimes_{\mathcal{A}}\mathcal{A}) = \mathcal{A}\otimes_{\mathcal{A}^e}\theta$$

because $\theta$ is cofibrant over $\mathcal{A}^e$. Clearly, the images of the terms (1.3.4) in $\mathcal{A}\otimes_{\mathcal{A}^e}\theta$ coincide. A similar argument works for the other terms in the sums. Thus, the images of $c_{ur}$ and $c_{ll}$ in the lower right corner are homologous, i.e. they differ by a boundary $d(c_{lr})$. Clearly, the element $c_{ur}+c_{ll}-c_{lr}$ forms a cycle in the total object of square (1.3.2) and this cycle lifts the chosen class $c$. Thus, the morphism induced by the class $c$ is given by a square commutative up to homotopy

$$\begin{array}{ccc}
\Sigma^n\mathrm{Hom}_{\mathcal{A}^e}(\mathbf{p}\mathcal{A},\mathcal{B}^e) & \longrightarrow & \Sigma^n\mathrm{Hom}_{\mathcal{A}^e}(\theta,\mathcal{B}^e)\\
{\scriptstyle c_{ur}}\downarrow & & \downarrow{\scriptstyle c_{ll}}\\
\theta\otimes_{\mathcal{A}^e}\mathcal{B}^e & \longrightarrow & \mathbf{p}\mathcal{A}\otimes_{\mathcal{A}^e}\mathcal{B}^e.
\end{array}$$

It remains to check that $c_{ur}$ and $c_{ll}$ map to the natural quasi-isomorphisms

$$\Sigma^n\mathrm{Hom}_{\mathcal{A}^e}(\mathbf{p}\mathcal{A},\mathcal{B}^e)\longrightarrow\theta\otimes_{\mathcal{A}^e}\mathcal{B}^e \quad\text{and}\quad \Sigma^n\mathrm{Hom}_{\mathcal{A}^e}(\theta,\mathcal{B}^e)\longrightarrow\mathbf{p}\mathcal{A}\otimes_{\mathcal{A}^e}\mathcal{B}^e\,.$$

These details are safely left to the reader. $\surd$

## 2. Deformed Calabi–Yau completions

We keep the notations and assumptions of section 1. In particular, $\mathcal{A}$ is assumed to be homologically smooth. Let $c$ be a class in the Hochschild homology $HH_{n-2}(\mathcal{A})$. Via the canonical isomorphism

$$HH_{n-2}(\mathcal{A})\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{D}(\mathcal{A}^e)}(\theta,\Sigma\mathcal{A})\,,$$

the class $c$ may be lifted to a closed degree 1 bimodule morphism $\delta:\theta\to\mathcal{A}$. The *deformed Calabi–Yau completion* $\Pi_n(\mathcal{A},c)$ was defined in section 5 of [4] as obtained from $\Pi_n(\mathcal{A})=T_{\mathcal{A}}(\theta)$ by replacing the differential $d$ with the unique derivation of the tensor category extending

$$\begin{bmatrix} d & 0\\ \delta & d\end{bmatrix}:\theta\oplus\mathcal{A}\to\theta\oplus\mathcal{A}.$$

Up to isomorphism, it only depends on $\mathcal{A}$ and the class $c$. It was claimed in Theorem 5.2 of [4], that the deformed Calabi–Yau completion is always $n$-Calabi–Yau. A counter-example to this claim is given by W. K. Yeung at the end of section 3.3 in [10]. As pointed out by

Yeung, a sufficient condition for the deformed $n$-Calabi–Yau completion to be $n$-Calabi–Yau is suggested by the work of Thanhoffer de Völcsey–Van den Bergh [9]: It suffices that the class $c$ lifts to the negative cyclic homology $HN_{n-2}(\mathcal{A})$. The following is Theorem 3.17 of [10].

**Theorem 2.1** (Yeung)**.** *Suppose that $\mathcal{A}$ is finitely cellular and $c$ is a class in $HH_{n-2}(\mathcal{A})$. Any lift $\widetilde{c}$ of $c$ to the negative cyclic homology $HN_{n-2}(\mathcal{A})$ determines an $n$-Calabi–Yau structure on the deformed Calabi–Yau completion $\Pi_n(\mathcal{A}, c)$.*

Notice that the assumptions do hold for Ginzburg dg categories. One would expect that, like theorem 1.1, this theorem should generalize to arbitrary homologically smooth dg categories.

## 3. Correction of some inaccuracies in "Deformed Calabi–Yau completions"

In section 3.6, one should point out in addition that the assumption that $Q$ is projective over $k$ implies that it is projective as an $\mathcal{R}$-bimodule (since $\mathcal{R}$ is discrete, so is $\mathcal{R}^e$), which in turn implies that the filtration (3.6.1) is split in the category of $\mathcal{R}$-bimodules.

At the end of the proof of Theorem 4.8, it is claimed that the transpose conjugate of $\widetilde{\lambda}$ maps to $\rho$. This is not true. However, the image of $\widetilde{\lambda}$ and $\rho$ define the same class in homology.

The statement of Theorem 6.1 is incorrect. One has to assume in addition that the set of 'minimal relations' $R$ generates the ideal $I$ (from the definition of $R$, it only follows that $R$ topologically generates the $J$-adic completion of $I$). Moreover, the condition 3) in the proof of the theorem should be replaced with

> for all $n \geq 1$, the differential $d$ maps $V^{-n-1}$ to $T_n$ and induces an isomorphism from $V^{-n-1}$ onto the head of the $H^0(T_n)$-bimodule $H^{-n}(T_n)$, where $T_n$ denotes the dg category $T_{\mathcal{R}}(V^0 \oplus \cdots \oplus V^{-n})$.

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