A REMARK ON A THEOREM BY C. AMIOT

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Abstract. C. Amiot has classified the connected triangulated k-categories with finitely many isoclasses of indecomposables satisfying suitable hypotheses. We remark that her proof shows that these triangulated categories are determined by their underlying k-linear categories. We observe that if the connectedness assumption is dropped, the triangulated categories are still determined by their underlying k-categories together with the action of the suspension functor on the set of isoclasses of indecomposables.

1. The connected case

We refer to [1] for unexplained notation and terminology. Let k be an algebraically closed field and $\mathcal{T}$ a k-linear Hom-finite triangulated category with split idempotents. Recall Theorem 7.2 of [1]:

Theorem 1 (Amiot). *Suppose $\mathcal{T}$ is connected, algebraic, standard and has only finitely many isoclasses of indecomposables. Then there exists a Dynkin quiver Q and a triangle autoequivalence Φ of $\mathcal{D}^b(\mathsf{mod}\, kQ)$ such that $\mathcal{T}$ is triangle equivalent to the triangulated* [6] *orbit category $\mathcal{D}^b(\mathsf{mod}\, kQ)/\Phi$.*

Our aim is to show that the proof of this theorem in [1] actually shows that a given k-linear equivalence $\mathcal{D}^b(\mathsf{mod}\, kQ)/F \xrightarrow{\sim} \mathcal{T}$, where F is a k-linear autoequivalence, lifts to a triangle equivalence $\mathcal{D}^b(\mathsf{mod}\, kQ)/\Phi \xrightarrow{\sim} \mathcal{T}$, where Φ is a triangle autoequivalence lifting F. Thus, we obtain the

Corollary 2. *Under the hypotheses of the theorem, the k-linear structure of $\mathcal{T}$ determines its triangulated structure up to triangle equivalence.*

Proof. The facts that $\mathcal{T}$ is connected, standard and has only finitely many isoclasses of indecomposables imply that there is a Dynkin quiver Q, a k-linear autoequivalence F of $\mathcal{D}^b(\mathsf{mod}\, kQ)$ and a k-linear equivalence

$$G : \mathcal{D}^b(\mathsf{mod}\, kQ)/F \xrightarrow{\sim} \mathcal{T}.$$

This follows from the work of Riedtmann [7], cf. section 6.1 of [1]. We will show that F lifts to an (algebraic) triangle autoequivalence of $\mathcal{D}^b(\mathsf{mod}\, kQ)$ and G to an (algebraic) triangle equivalence Γ. We give the details in the case of F which were omitted in [1]. Put $\mathcal{D} = \mathcal{D}^b(\mathsf{mod}\, kQ)$. Since $\mathcal{D}$ is triangulated, the category $\mathsf{mod}\,\mathcal{D}$ of finitely presented functors $\mathcal{D}^{op} \to \mathsf{Mod}\, k$ is an exact Frobenius category and we have a canonical isomorphism of functors $\underline{\mathsf{mod}}\,\mathcal{D} \to \underline{\mathsf{mod}}\,\mathcal{D}$

$$\Sigma_m^3 \xrightarrow{\sim} \Sigma_{\mathcal{D}}\ ,$$

where $\Sigma_{\mathcal{D}} : \mathsf{mod}\,\mathcal{D} \to \mathsf{mod}\,\mathcal{D}$ denotes the functor $\mathsf{mod}\,\mathcal{D} \to \mathsf{mod}\,\mathcal{D}$ induced by Σ and Σ_m is the suspension functor of the stable category $\underline{\mathsf{mod}}\,\mathcal{D}$, cf. [4, 16.4]. Notice that Σ_m

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only depends on the underlying k-category of $\mathcal{D}$. Now if S_U denotes the simple $\mathcal{D}$-module associated with an indecomposable object U of $\mathcal{D}$, we have

$$S_{\Sigma U} = \Sigma_{\mathcal{D}} S_U \xrightarrow{\sim} \Sigma_m^3 S_U$$

in the stable category $\underline{\mathsf{mod}}\,\mathcal{D}$. Since F is a k-linear autoequivalence, the functor it induces in $\underline{\mathsf{mod}}\,\mathcal{D}$ commutes with Σ_m and we have $S_{F\Sigma U} \cong S_{\Sigma F U}$ in $\underline{\mathsf{mod}}\,\mathcal{D}$ for each indecomposable U of $\mathcal{D}$. It follows that if S_U is not zero in $\underline{\mathsf{mod}}\,\mathcal{D}$, then we have an isomorphism $F\Sigma U \cong \Sigma F U$ in $\mathcal{D}$. Now S_U is zero in $\underline{\mathsf{mod}}\,\mathcal{D}$ only if S_U is projective, which happens if and only if the canonical map $P_U \to S_U$ is an isomorphism, where $P_U = \mathcal{D}(?, U)$ is the projective module associated with U. This is the case only if no arrows arrive at U in the Auslander–Reiten quiver of $\mathcal{D}$ and this happens if and only if no arrows start or arrive at U. The same then holds for the suspensions $\Sigma^n U$, $n \in \mathbb{Z}$. Since we have assumed that $\mathcal{T}$ and hence $\mathcal{D}$ is connected, this case is impossible. Therefore, we have an isomorphism $\Sigma F U \cong F\Sigma U$ for each indecomposable U of $\mathcal{D}$. It follows that $T = F(kQ)$ is a tilting object of $\mathcal{D}$. By [5], we can lift T to a kQ-bimodule complex Y which is even unique in the derived category of bimodules if we take the isomorphism $kQ \xrightarrow{\sim} \mathsf{End}_{\mathcal{D}}(F(kQ))$ into account. Since the k-linear functors F and $\Phi = ? \overset{L}{\otimes}_{kQ} Y$ are isomorphic when restricted to $\mathsf{add}(kQ)$, they are isomorphic as k-linear functors by Riedtmann's knitting argument [7]. Since the triangulated category $\mathcal{T}$ is algebraic, we may assume that it equals the perfect derived category $\mathsf{per}\,\mathcal{A}$ of a small dg k-category $\mathcal{A}$. Using Riedtmann's knitting argument again, it follows from the proof of Theorem 7.2 in [1] that the composition

$$\mathcal{D} \xrightarrow{\pi} \mathcal{D}/\Phi \xrightarrow{G} \mathcal{T} = \mathsf{per}\,\mathcal{A}$$

lifts to a triangle functor $? \overset{L}{\otimes}_{kQ} X$ for a kQ-$\mathcal{A}$-bimodule X. Moreover, it is shown there that this composition factors through an algebraic triangle equivalence

$$\Gamma : \mathcal{D}/\Phi \xrightarrow{\sim} \mathcal{T} = \mathsf{per}\,\mathcal{A}.$$

Since the compositions $\Gamma \circ \pi$ and $G \circ \pi$ are isomorphic as k-linear functors, the functors Γ and G are isomorphic as k-linear functors. $\surd$

2. The non connected case

Let k be an algebraically closed field and $\mathcal{T}$ a k-linear Hom-finite triangulated category with split idempotents and finitely many isomorphism classes of indecomposables. We assume that $\mathcal{T}$ is algebraic and standard but possibly non connected.

Assume first that $\mathcal{T}$ is Σ-connected, i.e. that the k-linear orbit category $\mathcal{T}/\Sigma$ is connected. Then the argument at the beginning of the above proof shows that either $\mathcal{T}$ is connected or $\mathcal{T}$ is k-linearly equivalent to $\mathcal{D}^b(\mathsf{mod}\,kA_1)/F$ for a k-linear equivalence F of $\mathcal{D}^b(\mathsf{mod}\,kA_1)$. Clearly F lifts to a triangle autoequivalence, namely a power Σ^N, of the suspension functor of $\mathcal{D}^b(\mathsf{mod}\,kA_1)$. We may assume $N > 0$ equals the number of isoclasses of indecomposables of $\mathcal{T}$. Since the underlying k-category of $\mathcal{T}$ is abelian and semi-simple, all triangles of $\mathcal{T}$ split and $\mathcal{T}$ is triangle equivalent to $\mathcal{D}^b(\mathsf{mod}\,kA_1)/\Sigma^N$.

Let us now drop the Σ-connectedness assumption on $\mathcal{T}$. Then clearly $\mathcal{T}$ decomposes, as a triangulated category, into finitely many Σ-connected components (the pre-images of the connected components of $\mathcal{T}/\Sigma$). Each of these is either connected or triangle equivalent to $\mathcal{D}^b(\mathsf{mod}\,kA_1)/\Sigma^N$ for some $N > 0$. Thus, the indecomposables of $\mathcal{T}$ either lie in connected components or in Σ-connected non connected components and the triangle equivalence class of the latter is determined by the action of Σ on the isomorphism classes of indecomposables. We obtain the

Corollary 3. *$\mathcal{T}$ is determined up to triangle equivalence by its underlying k-category and the action of Σ on the set of isomorphism classes of indecomposables.*

We refer to Theorem 6.5 of [3] for an analogous result concerning the Σ-finite triangulated categories $\mathcal{T}$ and to [2] for an application.

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