SINGULAR HOCHSCHILD COHOMOLOGY VIA THE SINGULARITY CATEGORY

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Abstract. We show that for a noetherian algebra A whose bounded dg derived category is smooth, the singular Hochschild cohomology (=Tate–Hochschild cohomology) is isomorphic, as a graded algebra, to the Hochschild cohomology of the dg singularity category of A. The existence of such an isomorphism is suggested by recent work of Zhengfang Wang.

1. Introduction

Let k be a commutative ring. We write $\otimes$ for $\otimes_k$. Let A be a right noetherian (non commutative) k-algebra projective over k. The *stable derived category* or *singularity category* of A is defined as the Verdier quotient

$$\mathsf{sg}(A) = \mathcal{D}^b(\mathsf{mod}\, A)/\,\mathsf{per}(A)$$

of the bounded derived category of finitely generated (right) A-modules by the *perfect derived category* $\mathsf{per}(A)$, *i.e.* the full subcategory of complexes quasi-isomorphic to bounded complexes of finitely generated projective modules. It was introduced by Buchweitz in an unpublished manuscript [4] in 1986 and rediscovered, in its scheme-theoretic variant, by Orlov in 2003 [28]. Notice that it vanishes when A is of finite global dimension and thus measures the degree to which A is 'singular', a view confirmed by the results of [28].

Let us suppose that the enveloping algebra $A^e = A \otimes A^{op}$ is also right noetherian. In analogy with Hochschild cohomology, in view of Buchweitz' theory, it is natural to define the *Tate–Hochschild cohomology* or *singular Hochschild cohomology* of A to be the graded algebra with components

$$HH^n_{sg}(A,A) = \mathsf{Hom}_{\mathsf{sg}(A^e)}(A, \Sigma^n A)\ ,\ n \in \mathbb{Z}\,,$$

where Σ denotes the suspension (=shift) functor. It was studied for example in [11, 2, 27] and more recently in [35, 36, 34, 37, 33, 5]. Wang showed in [35] that, like Hochschild cohomology [12], singular Hochschild cohomology carries a structure of Gerstenhaber algebra. Now recall that the Gerstenhaber algebra structure on Hochschild cohomology is a small part of much richer higher structure on the Hochschild cochain complex $C(A,A)$ itself, namely the structure of a B_∞-algebra in the sense of Getzler–Jones [14, 5.2] (whose definition was motivated by [1]) given by the Hochschild differential, the cup product and the brace operations [18, 13]. In [33], Wang improves on [35] by defining a singular Hochschild cochain complex $C_{sg}(A,A)$ and endowing it with a B_∞-structure which in particular yields the Gerstenhaber algebra structure on $HH^*_{sg}(A,A)$.

Using [19] Lowen–Van den Bergh showed in [24, Theorem 4.4.1] that the Hochschild cohomology of A is isomorphic to the Hochschild cohomology of the canonical differential graded (=dg) enhancement of the (bounded or unbounded) derived category of A and

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that the isomorphism lifts to the B_∞-level (cf. Corollary 7.6 of [30] for a related statement). Together with the complete structural analogy between Hochschild and singular Hochschild cohomology described above, this suggests the question whether the singular Hochschild cohomology of A is isomorphic to the Hochschild cohomology of the canonical dg enhancement $\mathsf{sg}_{dg}(A)$ of the singularity category $\mathsf{sg}(A)$ (note that such an enhancement exists by the construction of $\mathsf{sg}(A)$ as a Verdier quotient [21, 6]). Our main result is the following.

Theorem 1.1. a) *There is a canonical morphism of graded algebras Φ from the singular Hochschild cohomology of A to the Hochschild cohomology of the dg singularity category $\mathsf{sg}_{dg}(A)$.*

b) *If the bounded dg derived category $\mathcal{D}^b_{dg}(\mathsf{mod}\, A)$ is smooth, the morphism Φ is invertible.*

Recall that a dg category $\mathcal{A}$ is *smooth* if the identity bimodule

$$I_{\mathcal{A}} : (X, Y) \mapsto \mathcal{A}(X, Y)$$

is perfect in the derived category of $\mathcal{A}$-bimodules (i.e. the derived category of $\mathcal{A} \overset{L}{\otimes}_k \mathcal{A}^{op}$). According to Theorem A of Elagin–Lunts–Schnürer's [10], this holds for $\mathcal{D}^b_{dg}(\mathsf{mod}\, A)$ if A is a finite-dimensional algebra over any field k such that $A/\operatorname{\mathsf{rad}}(A)$ is separable over k (which is automatic if k is perfect). By Theorem B of [loc. cit.], it also holds if the algebra A is right noetherian and finitely generated over its center and the center is a finitely generated algebra over k.

Recall that localizations of smooth dg categories are smooth (Proposition 3.10 c) of [23]). So under the assumption of b), the dg singularity category is also smooth. It seems an interesting question to ask whether for arbitrary A, singular Hochschild cohomology is Hochschild cohomology of *some* dg category.

Example 1.2. *The author is grateful to Xiao-wu Chen for pointing out this example: Suppose that k is a field and $A \supset k$ a finite non separable field extension. Then the singularity category is quasi-equivalent to the zero dg category and so its Hochschild cohomology has to vanish. Notice that A and hence A^e are selfinjective so that the singularity category $\mathsf{sg}(A^e)$ is equivalent to the stable module category. Since A is a finite non separable extension of k, the bimodule A is non projective and hence non zero in the stable module category. Therefore, its endomorphism algebra, which is the zeroth singular Hochschild cohomology, is non zero. This shows that in general, the morphism Φ cannot be invertible. Notice that in this case, the dg category $\mathcal{D}^b_{dg}(\mathsf{mod}\, A)$ is derived Morita equivalent to the field A and thus is not smooth over the ground field k.*

Conjecture 1.3. *The morphism Φ of the theorem lifts to a morphism*

$$C_{sg}(A, A) \xrightarrow{\sim} C(\mathsf{sg}_{dg}(A), \mathsf{sg}_{dg}(A))$$

in the homotopy category of B_∞-algebras.

Notice that the B_∞-structure on Hochschild cohomology of dg categories is preserved (up to quasi-isomorphism) under Morita equivalences, cf. [19].

Theorem 1.4 (Chen–Li–Wang [5])**.** *The conjecture holds when k is a field and $A = kQ/(kQ_1)^2$ is the radical square zero algebra associated with a finite quiver Q without sources or sinks with arrow set Q_1.*

Let us mention an application of Theorem 1.1 obtained in joint work with Zheng Hua. Suppose that k is algebraically closed of characteristic 0 and let P the power series algebra $k[[x_1, \ldots, x_n]]$.

Theorem 1.5 ([17])**.** *Suppose that $Q \in P$ has an isolated singularity at the origin and $A = P/(Q)$. Then A is determined up to isomorphism by its dimension and the dg singularity category $\mathsf{sg}_{dg}(A)$.*

In [8, Theorem 8.1], Efimov proves a related but different reconstruction theorem: He shows that if Q is a polynomial, it is determined, up to a formal change of variables, by the differential $\mathbb{Z}/2$-graded endomorphism algebra E of the residue field in the differential $\mathbb{Z}/2$-graded singularity category together with a fixed isomorphism between H^*B and the exterior algebra $\Lambda(k^n)$.

In section 2, we generalize Theorem 1.1 to the non noetherian setting and prove the generalized statement. We comment on a possible lift of this proof to the B_∞-level in section 3. We prove Theorem 1.5 in section 4.

2. Generalization and proof

2.1. **Generalization to the non noetherian case.** We assume that A is an arbitrary k-algebra projective as a k-module. Its *pseudocoherent derived category* is the homotopy category of right bounded complexes of finitely generated projective modules with bounded homology:

$$\mathcal{D}^{pc}(A) = \mathcal{H}^{-,b}(\mathsf{proj}\, A).$$

Notice that when A is right noetherian, this category is canonically equivalent to $\mathcal{D}^b(\mathsf{mod}\, A)$. It has a canoncial dg enrichment

$$\mathcal{D}^{pc}_{dg}(A) = \mathcal{C}^{-,b}_{dg}(\mathsf{proj}\, A).$$

The singularity category $\mathsf{sg}(A)$ is defined as the Verdier quotient $\mathcal{D}^{pc}(A)/\mathcal{H}^b(\mathsf{proj}\, A)$. When A is right noetherian, this is equivalent to the definition given in the introduction.

The (partially) *completed singularity category* $\widehat{\mathsf{Sg}}(A)$ is defined as the Verdier quotient of the bounded derived category $\mathcal{D}^b(\mathsf{Mod}\, A)$ of all right A-modules by its full subcategory consisting of all complexes quasi-isomorphic to bounded complexes of arbitrary projective modules.

Lemma 2.2. *The canonical functor $\mathsf{sg}(A) \to \widehat{\mathsf{Sg}}(A)$ is fully faithful.*

Proof. Let M be a right bounded complex of finitely generated projective modules with bounded homology and P a bounded complex of arbitrary projective modules. Since the components of M are finitely generated, each morphism $M \to P$ in the derived category factors through a bounded complex P' with finitely generated projective components. This yields the claim. $\surd$

Since we do not assume that A^e is noetherian, the A-bimodule A will not, in general, belong to the singularity category $\mathsf{sg}(A^e)$. But it always belongs to the completed singularity category $\widehat{\mathsf{Sg}}(A^e)$. We define the singular Hochschild cohomology of A to be the graded algebra with components

$$HH^n_{sg}(A, A) = \mathsf{Hom}_{\widehat{\mathsf{Sg}}(A^e)}(A, \Sigma^n A)\ ,\ n \in \mathbb{Z}.$$

Theorem 2.3. *Even if A^e is non noetherian, there is a canonical morphism of graded algebras from the singular Hochschild cohomology of A to the Hochschild cohomology of the dg singularity category $\mathsf{sg}_{dg}(A)$. It is an isomorphism if the pseudocoherent dg derived category $\mathcal{D}^{pc}_{dg}(A)$ is smooth.*

Let P be a right bounded complex of projective A^e-modules. For $q \in \mathbb{Z}$, let $\sigma_{>q}P$ and $\sigma_{\leq q}P$ denote its stupid truncations:

$$\sigma_{>q}P : \ \ldots \longrightarrow 0 \longrightarrow P^{q+1} \longrightarrow P^{q+1} \longrightarrow \ldots$$
$$\sigma_{\leq q}P : \ \ldots \longrightarrow P^{q-1} \longrightarrow P^{q} \longrightarrow 0 \longrightarrow \ldots$$

so that we have a triangle

$$\sigma_{>q}P \longrightarrow P \longrightarrow \sigma_{\leq q}P \longrightarrow \Sigma\sigma_{>q}P.$$

We have a direct system

$$P = \sigma_{\leq 0}P \longrightarrow \sigma_{\leq -1}P \longrightarrow \sigma_{\leq -2}P \longrightarrow \ldots \longrightarrow P_{\leq q} \longrightarrow \ldots .$$

Lemma 2.4. *Let $L \in \mathcal{D}^b(\mathsf{Mod}\, A^e)$. We have a canonical isomorphism*

$$\mathsf{colim}\, \mathsf{Hom}_{\mathcal{D}A^e}(L, \sigma_{\leq q}P) \xrightarrow{\sim} \mathsf{Hom}_{\widehat{\mathsf{Sg}}(A^e)}(L, P).$$

In particular, if P is a projective resolution of A over A^e, we have

$$\mathsf{colim}\, \mathsf{Hom}_{\mathcal{D}A^e}(A, \Sigma^n\sigma_{\leq q}P) \xrightarrow{\sim} \mathsf{Hom}_{\widehat{\mathsf{Sg}}(A^e)}(A, \Sigma^n A)\ ,\ n \in \mathbb{Z}.$$

Proof. Clearly, if Q is a bounded complex of projective modules, each morphism $Q \to P$ in the derived category $\mathcal{D}A^e$ factors through $\sigma_{>q}P \to P$ for some $q \ll 0$. This shows that the morphisms $P \to \sigma_{\leq q}P$ form a cofinal subcategory in the category of morphisms $P \to P'$ whose cylinder is a bounded complex of projective modules. Whence the claim. $\surd$

2.5. **Proof of Theorem 2.3.** We refer to [20, 22, 31] for foundational material on dg categories. We will follow the terminology of [22] and use the model category structure on the category of dg categories constructed in [29]. For a dg category $\mathcal{A}$, denote by $X \mapsto Y(X)$ the dg Yoneda functor and by $\mathcal{D}\mathcal{A}$ the derived category. We write $\mathcal{A}^e$ for the enveloping dg category $\mathcal{A} \overset{L}{\otimes}_k \mathcal{A}^{op}$ and $I_{\mathcal{A}}$ for the *identity bimodule*

$$I_{\mathcal{A}} : (X, Y) \mapsto \mathcal{A}(X, Y).$$

By definition, the Hochschild cohomology of $\mathcal{A}$ is the graded endomorphism algebra of $I_{\mathcal{A}}$ in the derived category $D(\mathcal{A}^e)$. In the case of the algebra A, the identity bimodule is the A-bimodule A. Recall that if $F : \mathcal{A} \to \mathcal{B}$ is a fully faithful dg functor, the restriction $F_* : \mathcal{D}\mathcal{B} \to \mathcal{D}\mathcal{A}$ is a localization functor admitting fully faithful left and right adjoint functors F^* and $F^!$ given respectively by

$$F^* : M \mapsto M \overset{L}{\otimes}_{\mathcal{A}}\, {}_F\mathcal{B} \quad \text{and} \quad F^! : N \mapsto \mathsf{RHom}_{\mathcal{A}}(\mathcal{B}_F, N)\ ,$$

where ${}_F\mathcal{B} = \mathcal{B}(?, F-)$ and $\mathcal{B}_F = \mathcal{B}(F?, -)$.

Let $\mathcal{M}_0 = \mathcal{C}^{-,b}_{dg}(\mathsf{proj}\, A)$ denote the dg category of right bounded complexes of finitely generated projective A-modules with bounded homology. Notice that the morphism complexes of $\mathcal{M}_0$ have terms which involve infinite products of projective A-modules so that in general, the morphism complexes of $\mathcal{M}_0$ will not be cofibrant over k. Let $\mathcal{M} \to \mathcal{M}_0$ be a cofibrant resolution of $\mathcal{M}_0$. We assume, as we may, that the quasi-equivalence $\mathcal{M} \to \mathcal{M}_0$ is the identity on objects. Notice that the morphism complexes of $\mathcal{M}$ are cofibrant over k so that we have $\mathcal{M} \overset{L}{\otimes}_k \mathcal{M}^{op} \xrightarrow{\sim} \mathcal{M} \otimes \mathcal{M}^{op}$. Let $\mathcal{P} \subset \mathcal{M}$ be the full dg subcategory of $\mathcal{M}$ formed by the bounded complexes of finitely generated projective A-modules. Let $\mathcal{S}$ denote the dg quotient $\mathcal{M}/\mathcal{P}$. We assume, as we may, that $\mathcal{S}$ is cofibrant. In the homotopy category of dg categories, we have an isomorphism between $\mathsf{sg}_{dg}(A)$ and $\mathcal{S} = \mathcal{M}/\mathcal{P}$. Let B be the dg endomorphism algebra of A considered as an object of $\mathcal{P} \subset \mathcal{M}$. Notice that we have a quasi-isomorphism $B \to A$ and that both B and A are cofibrant over k. We view B as a dg category with one object whose endomorphism algebra is B. We have the obvious inclusion and projection dg functors

$$B \xrightarrow{i} \mathcal{M} \xrightarrow{p} \mathcal{S}.$$

Consider the fully faithful dg functors

$$B \otimes B^{op} \xrightarrow{\mathbf{1}\otimes i} B \otimes \mathcal{M}^{op} \xrightarrow{i\otimes \mathbf{1}} \mathcal{M} \otimes \mathcal{M}^{op} .$$

The restriction along $G = \mathbf{1} \otimes i$ admits the left adjoint G^* given by

$$G^* : X \mapsto \mathcal{M}_i \overset{L}{\otimes}_B X ,$$

and the restriction along $F = i \otimes \mathbf{1}$ admits the fully faithful left and right adjoints F^* and $F^!$ given by

$$F^* : Y \mapsto Y \overset{L}{\otimes}_B {}_i\mathcal{M} \quad \text{and} \quad F^! : Y \mapsto \mathsf{RHom}_B(\mathcal{M}_i, Y).$$

Since F^* and $F^!$ are the two adjoints of a localization functor, we have a canonical morphism $F^* \to F^!$.

Lemma 2.6. *If P is an arbitrary sum of copies of B^e, the morphism*

$$F^*G^*(P) \to F^!G^*(P)$$

is invertible.

Proof. Let P be the direct sum of copies of B^e indexed by a set J. Since F^* and G^* commute with (arbitrary) coproducts, the left hand side is the dg module

$$\bigoplus_J \mathcal{M}(i?, -) \overset{L}{\otimes}_B (B \otimes B) \overset{L}{\otimes}_B \mathcal{M}(?, i-) = \bigoplus_J \mathcal{M}(B, -) \otimes \mathcal{M}(?, B),$$

The right hand side is the dg module

$$\mathsf{RHom}_B(\mathcal{M}_i, \mathcal{M}_i \overset{L}{\otimes}_B (\bigoplus_J B \otimes B)) = \mathsf{RHom}_B(\mathcal{M}_i, \bigoplus_J \mathcal{M}(B, -) \otimes B).$$

Let us evaluate the canonical morphism at $(M, L) \in \mathcal{M} \otimes \mathcal{M}^{op}$. We find the canonical morphism

$$\bigoplus_J \mathcal{M}(B, L) \otimes \mathcal{M}(M, B) \to \mathsf{RHom}_B(\mathcal{M}(B, M), \bigoplus_j \mathcal{M}(B, L) \otimes B).$$

We have quasi-isomorphisms

$$\mathcal{M}(B, L) \otimes \mathcal{M}(M, B) \to \mathcal{M}_0(A, L) \otimes \mathcal{M}(M, B) \to L \otimes \mathcal{M}(M, B) \to L \otimes \mathsf{Hom}_A(M, A)$$

because $\mathcal{M}(M, B)$ and L are cofibrant over k. Now the equivalence $\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)$ takes $\mathcal{M}(B, L) \otimes B$ to $\mathcal{M}(B, L) \otimes A \xrightarrow{\sim} L \otimes A$. We have an quasi-isomorphism of dg B-modules $\mathcal{M}(B, M) \xrightarrow{\sim} \mathcal{M}_0(A, M) = M$ and so the equivalence $\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)$ takes $\mathcal{M}(B, M)$ to M. Whence an isomorphism

$$\mathsf{RHom}_B(\mathcal{M}(B, M), \bigoplus_J \mathcal{M}(B, L) \otimes B) \xrightarrow{\sim} \mathsf{RHom}_A(M, \bigoplus_J L \otimes A) = \mathsf{Hom}_A(M, \bigoplus_J L \otimes A).$$

Thus, we have to show that the canonical morphism

$$\bigoplus_J L \otimes \mathsf{Hom}_A(M, A) \to \mathsf{Hom}_A(M, \bigoplus_J L \otimes A)$$

is a quasi-isomorphism. Recall that L and M are right bounded complexes of finitely generated projective modules with bounded homology. We fix M and consider the morphism as a morphism of triangle functors with argument $L \in \mathcal{D}^b(\mathsf{Mod}\, A)$. Then we are reduced to the case where L is in $\mathsf{Mod}\, A$. In this case, the morphism becomes an isomorphism of complexes because the components of M are finitely generated projective. $\surd$

Let us put $H = F^! G^* : \mathcal{D}(B^e) \to \mathcal{D}(\mathcal{M}^e)$. Let us compute the image of the identity bimodule B under H. We have

$$H(B) = F^!(\mathcal{M}_i \overset{L}{\otimes}_B B) = F^!(\mathcal{M}_i) = \mathsf{RHom}_B(\mathcal{M}_i, \mathcal{M}_i)$$

and when we evaluate at L, M in $\mathcal{M}$, we find

$$H(B)(L, M) = \mathsf{RHom}_B(\mathcal{M}(i?, L), \mathcal{M}(i?, M)) = \mathsf{RHom}_B(\mathcal{M}(B, L), \mathcal{M}(B, M)).$$

We have seen in the above proof that the equivalence $\mathcal{D}(B) \xrightarrow{\sim} \mathcal{D}(A)$ takes $\mathcal{M}(B, L)$ to L. Whence quasi-isomorphisms

$$\begin{aligned} H(B)(L, M) = \mathsf{RHom}_B(\mathcal{M}(B, L), \mathcal{M}(B, M)) \xrightarrow{\sim} \mathsf{RHom}_A(L, M) &= \mathsf{Hom}(L, M) \\ &\xleftarrow{\sim} \mathcal{M}(L, M). \end{aligned}$$

Thus, the functor H takes the identity bimodule B to the identity bimodule $I_{\mathcal{M}}$. Since $F^!$ and G^* are fully faithful so is H. Denote by $\mathcal{N}$ the image under the composition of H with $\mathcal{D}(A^e) \xrightarrow{\sim} \mathcal{D}(B^e)$ of the closure of $\mathsf{Proj}\, A^e$ under finite extensions. Then H yields a fully faithful functor

$$\widehat{\mathsf{Sg}}(A^e) \to \mathcal{D}(\mathcal{M}^e)/\mathcal{N}$$

taking the bimodule A to the identity bimodule $I_{\mathcal{M}}$. Now notice that we have a Morita morphism of dg categories

$$\mathcal{S}^e \xleftarrow{\sim} \frac{\mathcal{M} \otimes \mathcal{M}^{op}}{\mathcal{P} \otimes \mathcal{M}^{op} + \mathcal{M} \otimes \mathcal{P}^{op}}.$$

The functor $p^* : \mathcal{D}(\mathcal{M}^e) \to \mathcal{D}(\mathcal{S}^e)$ induces the quotient functor

$$\frac{\mathcal{D}(\mathcal{M} \otimes \mathcal{M}^{op})}{\mathcal{N}} \longrightarrow \frac{\mathcal{D}(\mathcal{M} \otimes \mathcal{M}^{op})}{\mathcal{D}(\mathcal{P} \otimes \mathcal{M}^{op} + \mathcal{M} \otimes \mathcal{P}^{op})} = \mathcal{D}(\mathcal{S}^e)\ .$$

Since $p : \mathcal{M} \to \mathcal{S}$ is a localization, the image $p^*(I_{\mathcal{M}})$ is isomorphic to $I_{\mathcal{S}}$ (Proposition 3.10 a) of [23]). This yields the morphism Φ. Now let us assume that $\mathcal{M}$ is smooth. Since H is fully faithful, to show that Φ is an isomorphism, it suffices to show that p^* induces bijections

$$\mathsf{Hom}_{\mathcal{D}(\mathcal{M}^e)/\mathcal{N}}(I_{\mathcal{M}}, \Sigma^n I_{\mathcal{M}}) \longrightarrow \mathsf{Hom}_{\mathcal{D}(\mathcal{S}^e)}(p^*(I_{\mathcal{M}}), \Sigma^n p^*(I_{\mathcal{M}}))$$

for each $n \in \mathbb{Z}$. Now by our smoothness assumption, the object $I_{\mathcal{M}}$ lies in $\mathsf{per}(\mathcal{M}^e)$. The inclusion $\mathcal{P}^e \to \mathcal{M}^e$ induces an equivalence of $\mathsf{per}(\mathcal{P}^e)$ onto a full triangulated subcategory of $\mathcal{N}$.

Lemma 2.7. *The canonical functor*

$$\frac{\mathsf{per}(\mathcal{M}^e)}{\mathsf{per}(\mathcal{P}^e)} \longrightarrow \frac{\mathcal{D}(\mathcal{M}^e)}{\mathcal{N}}$$

is fully faithful.

Proof. This is a variation on Lemma 2.3 of [26], cf. also Lemma 5.3 of [20]. The claim follows from the following statement: Let $X \in \mathsf{per}(\mathcal{M}^e)$ and $Y \in \mathcal{D}(\mathcal{M}^e)$. Then each angle

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \xrightarrow[y]{} & Y', \end{array}$$

where the cone over y is in $\mathcal{N}$ may be completed to a commutative square

$$\begin{array}{ccc} X' & \xrightarrow{x} & X \\ \downarrow & & \downarrow \\ Y & \xrightarrow[y]{} & Y' \end{array}$$

where the cone over x is in $\mathsf{per}(\mathcal{P}^e)$. By construction, the cone N over y is an n-fold iterated extension of (infinite) sums of objects $P^\wedge$ for $P \in \mathcal{P}^e$ for some $n \geq 1$. We proceed by induction on n. If $n = 1$, then N itself is a sum of objects $P^\wedge$ and since X is compact, the composition $X \to Y' \to N$ factors through a finite subsum $N' \subset N$. We can then form a morphism of triangles

$$\begin{array}{ccccccc} X' & \xrightarrow{x} & X & \longrightarrow & N' & \longrightarrow & \Sigma X' \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow[y]{} & Y' & \longrightarrow & N & \longrightarrow & \Sigma Y. \end{array}$$

If $n > 1$, then N occurs in a triangle

$$N_0 \longrightarrow N \longrightarrow N_1 \longrightarrow \Sigma N_0$$

for objects N_0 and N_1 which are n'-fold extensions of sums of $P^\wedge$ for some $n' < n$. By forming an octahedron over the morphisms $Y' \to N \to N_1$, we obtain a commutative diagram

$$\begin{array}{ccccccc} & & & & N_0 & = & N_0 \\ & & & & \downarrow & & \downarrow \\ Y & \xrightarrow{y} & Y' & \longrightarrow & N & \longrightarrow & \Sigma Y \\ {\scriptstyle y_2}\downarrow & & \| & & \downarrow & & \downarrow{\scriptstyle \Sigma y_2} \\ Y'' & \xrightarrow[y_1]{} & Y' & \longrightarrow & N_1 & \longrightarrow & \Sigma Y'' \\ & & & & \downarrow & & \downarrow \\ & & & & \Sigma N_0 & = & \Sigma N_0 \end{array}$$

We see that y is the composition $y_1 \circ y_2$ of two morphisms whose mapping cones are n'-fold extensions of sums of objects $P^\wedge$, $P \in \mathcal{P}^e$. By the induction hypothesis, we can find a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{x_2} & X'' & \xrightarrow{x_1} & X \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xrightarrow[y_2]{} & Y'' & \xrightarrow[y_1]{} & Y', \end{array}$$

where the cones over x_1 and x_2 lie in $\mathsf{per}(\mathcal{P}^e)$. By the octahedral axiom, this also holds for the cone over $x = x_1 \circ x_2$. $\surd$

Thus, by the Lemma, it suffices to show that p^* induces bijections in the morphism spaces with target $I_\mathcal{M}$

$$\mathsf{Hom}_{\mathsf{per}(\mathcal{M}^e)/\,\mathsf{per}(\mathcal{P}^e)}(?, I_\mathcal{M}) \longrightarrow \mathsf{Hom}_{\mathcal{D}(\mathcal{S}^e)}(p^*(?), p^*(I_\mathcal{M}))\ .$$

Since the perfect derived category of $\mathcal{S}^e$ is equivalent to the quotient of $\mathsf{per}(\mathcal{M}^e)$ by the thick subcategory generated by the images under the Yoneda functor of objects in $\mathcal{P} \otimes \mathcal{M}^{op} + \mathcal{M} \otimes \mathcal{P}^{op}$, it suffices to show that $I_\mathcal{M}$ is right orthogonal in $\mathsf{per}(\mathcal{M}^e)/\,\mathsf{per}(\mathcal{P}^e)$ on the images under the Yoneda functor of the objects in $\mathcal{P}\otimes\mathcal{M}^{op}+\mathcal{M}\otimes\mathcal{P}^{op}$. To show that $I_{\mathcal{M}}$ is right orthogonal on $Y(\mathcal{M}\otimes\mathcal{P}^{op})$, it suffices to show that it is right orthogonal to an object $Y(M,B)$, $M\in\mathcal{M}$. Now a morphism in $\mathsf{per}(\mathcal{M}^e)/\mathsf{per}(\mathcal{P}^e)$ is given by a diagram of $\mathsf{per}(\mathcal{M}^e)$ representing a left fraction

$$Y(M,B)\longrightarrow I'_{\mathcal{M}}\longleftarrow I_{\mathcal{M}}$$

where the cone over $I_{\mathcal{M}}\to I'_{\mathcal{M}}$ lies in $\mathsf{per}(\mathcal{P}^e)$. For each object X of $\mathcal{D}(\mathcal{M}^e)$, we have canonical isomorphisms

$$\mathsf{Hom}_{\mathcal{D}\mathcal{M}^e}(Y(M,B),X)=H^0(X(M,B))=\mathsf{Hom}_{\mathcal{D}\mathcal{M}}(Y(M),X(?,B)).$$

Thus, the given fraction corresponds to a diagram in $\mathcal{D}(\mathcal{M})$ of the form

$$Y(M)\longrightarrow I'_{\mathcal{M}}(?,B)\longleftarrow I_{\mathcal{M}}(?,B)=\mathcal{M}(?,B)\ ,$$

where the cone over $I_{\mathcal{M}}(?,B)\to I'_{\mathcal{M}}(?,B)$ is the image under $\mathcal{D}A\xrightarrow{\sim}\mathcal{D}B\to\mathcal{D}\mathcal{M}$ of a bounded complex with projective components (which may be infinitely generated because as a right module, the tensor product $A\otimes A$ may be infinitely generated). Thus, the object $I'_{\mathcal{M}}(?,B)$ is a direct factor of a finite extension of shifts of arbitrary coproducts B. Since $Y(M)$ is compact, the given morphism $Y(M)\to I'_{\mathcal{M}}(?,M)$ must then factor through $Y(Q)$ for an object Q of $\mathcal{P}$. This means that the given morphism $Y(M,B)\to I'_{\mathcal{M}}$ factors through $Y(Q,B)$, which lies in $\mathcal{N}$. Thus, the given fraction represents the zero morphism of $\mathcal{D}(\mathcal{M}^e)/\mathcal{N}$, as was to be shown. The case of an object in $Y(\mathcal{P}\otimes\mathcal{M}^{op})$ is analogous. In summary, we have shown that the maps

$$\widehat{\mathsf{Sg}}(A^e)(A,\Sigma^n A)\xrightarrow{H}(\mathcal{D}(\mathcal{M}^e)/\mathcal{N})(I_{\mathcal{M}},\Sigma^n I_{\mathcal{M}})\xrightarrow{p^*}\mathcal{D}(\mathcal{S}^e)(I_{\mathcal{S}},\Sigma^n I_{\mathcal{S}})$$

are bijective, which implies the assertion on Hochschild cohomology.

3. Remark on a possible lift to the B_∞-level

Let $P\to A$ be a resolution of A by projective A-A-bimodules. Let us assume for simplicity that k is a field so that we can take $\mathcal{M}=\mathcal{M}_0$ and $B=A$. The proof in section 2 produces in fact isomorphisms in the derived category of k-modules

$$\begin{aligned}\mathsf{colim}\,\mathsf{RHom}_{A^e}(A,\sigma_{\leq q}P)&\to\mathsf{colim}\,\mathsf{RHom}_{\mathcal{M}^e}(I_{\mathcal{M}},H\sigma_{\leq q}P)\\&\to\mathsf{colim}\,\mathsf{RHom}_{\mathcal{S}^e}(I_{\mathcal{S}},p^*H\sigma_{\leq q}P)\\&=\mathsf{RHom}_{\mathcal{S}^e}(I_{\mathcal{S}},I_{\mathcal{S}}).\end{aligned}$$

For the bar resolution P, the truncation $\sigma_{\leq -q}P$ is canonically isomorphic to $\Sigma^q\Omega^q A$ so that the first complex carries a canonical B_∞-structure constructed by Wang [33]. As explained in the introduction, it is classical that the last complex carries a canonical B_∞-structure. It is not obvious to make the intermediate complexes explicit because the functor H, being a composition of a *right adjoint* with a left adjoint to a restriction functor, does not take cofibrant objects to cofibrant objects.

4. Proof of Theorem 1.5

By the Weierstrass preparation theorem, we may assume that Q is a polynomial. Let $P_0=k[x_1,\dots,x_n]$ and $S=P_0/(Q)$. Then S has isolated singularities but may have singularities other than the origin. Let $\mathfrak{m}$ be the maximal ideal of P_0 generated by the x_i and let R be the localization of S at $\mathfrak{m}$. Now R is local with an isolated singularity at $\mathfrak{m}$ and A is isomorphic to the completion $\widehat{R}$. By Theorem 3.2.7 of [16], in sufficiently high degrees r, the Hochschild cohomology of S is isomorphic to the homology in degree r of the complex

$$k[u] \otimes K(S, \partial_1 Q, \ldots, \partial_n Q)\ ,$$

where u is of degree 2 and K denotes the Koszul complex. Now S is isomorphic to $K(P_0, Q)$ and so $K(S, \partial_1 Q, \ldots, \partial_n Q)$ is isomorphic to

$$K(P_0, Q, \partial_1 Q, \ldots, \partial_n Q).$$

Since Q has isolated singularities, the $\partial_i Q$ form a regular sequence in P_0. So

$$K(P_0, Q, \partial_1 Q, \ldots, \partial_n Q)$$

is quasi-isomorphic to $K(M, Q)$, where $M = P_0/(\partial_1 Q, \ldots, \partial_n Q)$. Therefore, in high even degrees $2r$, the Hochschild cohomology of S is isomorphic to

$$T = k[x_1, \ldots, x_n]/(Q, \partial_1 Q, \ldots, \partial_n Q)$$

as an S-module. Since S and S^e are noetherian, this implies that the Hochschild cohomology of R in high even degrees is isomorphic to the localisation $T_{\mathfrak{m}}$. Since $R \otimes R$ is noetherian and Gorenstein (cf. Theorem 1.6 of [32]), by Theorem 6.3.4 of [4], the singular Hochschild cohomology of R coincides with Hochschild cohomology in sufficiently high degrees. Since $\mathbb{C}$ is a perfect field and S a finitely generated $\mathbb{C}$-algebra, the dg derived category $\mathcal{D}^b_{dg}(\mathsf{mod}\, S)$ is smooth, by Theorem B of [10]. The dg derived category $\mathcal{D}^b_{dg}(\mathsf{mod}\, R)$ is a dg localization of $\mathcal{D}^b_{dg}(\mathsf{mod}\, S)$ and hence is also smooth. Thus, by part b) of Theorem 1.1, the Hochschild cohomology of $\mathsf{sg}_{dg}(R)$ is isomorphic to the singular Hochschild cohomology of R and thus isomorphic to $T_{\mathfrak{m}}$ in high even degrees. Since R is a hypersurface, the dg category $\mathsf{sg}_{dg}(R)$ is isomorphic, in the homotopy category of dg categories, to the underlying differential $\mathbb{Z}$-graded category of the differential $\mathbb{Z}/2$-graded category of matrix factorizations of Q, cf. [9], [28] and Theorem 2.49 of [3]. Thus, it is 2-periodic and so is its Hochschild cohomology. It follows that the zeroth Hochschild cohomology of $\mathsf{sg}_{dg}(R)$ is isomorphic to $T_{\mathfrak{m}}$ as an algebra. The completion functor $? \otimes_R \widehat{R}$ yields an embedding $\mathsf{sg}(R) \to \mathsf{sg}(A)$ through which $\mathsf{sg}(A)$ identifies with the idempotent completion of the triangulated category $\mathsf{sg}(R)$, cf. Theorem 5.7 of [7]. Therefore, the corresponding dg functor $\mathsf{sg}_{dg}(R) \to \mathsf{sg}_{dg}(A)$ induces an equivalence in the derived categories and an isomorphism in Hochschild cohomology. So we find an isomorphism

$$HH^0(\mathsf{sg}_{dg}(A), \mathsf{sg}_{dg}(A)) \xrightarrow{\sim} T_{\mathfrak{m}}.$$

Since $Q \in k[x_1, \ldots, x_n]_{\mathfrak{m}}$ has an isolated singularity at the origin, we have an isomorphism

$$T_{\mathfrak{m}} \xrightarrow{\sim} k[[x_1, \ldots, x_n]]/(Q, \partial_1 Q, \ldots, \partial_n Q)$$

with the Tyurina algebra of $A = P/(Q)$. Now by the Mather–Yau theorem [25], more precisely by its formal version [15, Prop. 2.1], in a fixed dimension, the Tyurina algebra determines A up to isomorphism.

Notice that the Hochschild cohomology of the dg category of matrix factorizations considered as a differential $\mathbb{Z}/2$-graded category is different: As shown by Dyckerhoff [7], it is isomorphic to the Milnor algebra $P/(\partial_1 Q, \ldots, \partial_n Q)$ in even degree and vanishes in odd degree.

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References

[1] H. J. Baues, *The double bar and cobar constructions*, Compositio Math. **43** (1981), no. 3, 331–341.

[2] Petter Andreas Bergh and David A. Jorgensen, *Tate-Hochschild homology and cohomology of Frobenius algebras*, J. Noncommut. Geom. **7** (2013), no. 4, 907–937.

[3] Anthony Blanc, Marco Robalo, Bertrand Toën, and Gabriele Vezzosi, *Motivic realizations of singularity categories and vanishing cycles*, arXiv:1607.03012 [math.AG].

[4] Ragnar-Olaf Buchweitz, *Maximal Cohen–Macaulay modules and Tate cohomology over Gorenstein rings*, `http:hdl.handle.net/1807/16682` (1986), 155 pp.

[5] Xiao-Wu Chen, Huanhuan Li, and Zhengfang Wang, *The Hochschild cohomology of Leavitt path algebras and Tate–Hochschild cohomology*, in preparation.

[6] Vladimir Drinfeld, *DG quotients of DG categories*, J. Algebra **272** (2004), no. 2, 643–691.

[7] Tobias Dyckerhoff, *Compact generators in categories of matrix factorizations*, Duke Math. J. **159** (2011), no. 2, 223–274.

[8] Alexander I. Efimov, *Homological mirror symmetry for curves of higher genus*, Adv. Math. **230** (2012), no. 2, 493–530.

[9] David Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc. **260** (1980), no. 1, 35–64.

[10] Alexey Elagin, Valery A. Lunts, and Olaf M. Schnürer, *Smoothness of derived categories of algebras*, arXiv:1810.07626 [math.AG].

[11] Ching-Hwa Eu and Travis Schedler, *Calabi-Yau Frobenius algebras*, J. Algebra **321** (2009), no. 3, 774–815.

[12] Murray Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2) **78** (1963), 267–288.

[13] Ezra Getzler, *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology*, Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), Israel Math. Conf. Proc., vol. 7, Bar-Ilan Univ., Ramat Gan, 1993, pp. 65–78.

[14] Ezra Getzler and J. D. S. Jones, *Operads, homotopy algebra, and iterated integrals for double loop spaces*, hep-th/9403055.

[15] Gert-Martin Greuel and Thuy Huong Pham, *Mather-Yau theorem in positive characteristic*, J. Algebraic Geom. **26** (2017), no. 2, 347–355.

[16] Jorge Alberto Guccione, Jose Guccione, Maria Julia Redondo, and Orlando Eugenio Villamayor, *Hochschild and cyclic homology of hypersurfaces*, Adv. Math. **95** (1992), no. 1, 18–60.

[17] Zheng Hua and Bernhard Keller, *Cluster categories and rational curves*, arXiv:1810.00749 [math.AG].

[18] T. V. Kadeishvili, *A_∞-algebra structure in cohomology and the rational homotopy type*, Preprint 37, Forschungsschwerpunkt Geometrie, Universität Heidelberg, Mathematisches Institut, 1988.

[19] Bernhard Keller, *Derived invariance of higher structures on the Hochschild complex*, preprint, 2003, available at the author's home page.

[20] ———, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63–102.

[21] ———, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra **136** (1999), no. 1, 1–56.

[22] ———, *On differential graded categories*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.

[23] ———, *Deformed Calabi–Yau completions*, Journal für die reine und angewandte Mathematik (Crelles Journal) **654** (2011), 125–180, with an appendix by Michel Van den Bergh.

[24] Wendy Lowen and Michel Van den Bergh, *Hochschild cohomology of abelian categories and ringed spaces*, Adv. Math. **198** (2005), no. 1, 172–221.

[25] John N. Mather and Stephen S. T. Yau, *Classification of isolated hypersurface singularities by their moduli algebras*, Invent. Math. **69** (1982), no. 2, 243–251.

[26] Amnon Neeman, *The connection between the K–theory localisation theorem of Thomason, Trobaugh and Yao, and the smashing subcategories of Bousfield and Ravenel*, Ann. Sci. École Normale Supérieure **25** (1992), 547–566.

[27] Van C. Nguyen, *Tate and Tate-Hochschild cohomology for finite dimensional Hopf algebras*, J. Pure Appl. Algebra **217** (2013), no. 10, 1967–1979.

[28] D. O. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Tr. Mat. Inst. Steklova **246** (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 240–262.

[29] Goncalo Tabuada, *Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories*, C. R. Math. Acad. Sci. Paris **340** (2005), no. 1, 15–19. MR MR2112034 (2005h:18033)
[30] Bertrand Toën, *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math. **167** (2007), no. 3, 615–667.
[31] Bertrand Toën, *Lectures on dg-categories*, Topics in algebraic and topological K-theory, Lecture Notes in Math., vol. 2008, Springer, Berlin, 2011, pp. 243–302.
[32] Masoud Tousi and Siamak Yassemi, *Tensor products of some special rings*, J. Algebra **268** (2003), no. 2, 672–676.
[33] Zhengfang Wang, *Gerstenhaber algebra and Deligne's conjecture on Tate–Hochschild cohomology*, arXiv:1801.07990.
[34] ———, *Singular deformation theory and the invariance of the Gerstenhaber algebra structure on the singular Hochschild cohomology*, arXiv:1501.01641 [math.RT].
[35] ———, *Singular Hochschild cohomology and Gerstenhaber algebra structure*, arXiv:1508.00190 [math.RT].
[36] ———, *Singular Hochschild cohomology of radical square zero algebras*, arXiv:1511.08348 [math.RT].
[37] Zhengfang Wang and Manuel Rivera, *Singular Hochschild cohomology and algebraic string operations*, arXiv:1703.03899 [math.RT].

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