RELATIVE CLUSTER CATEGORIES AND HIGGS CATEGORIES WITH INFINITE-DIMENSIONAL MORPHISM SPACES

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With an appendix by Chris Fraser and Bernhard Keller

Dedicated to Professor Henning Krause on the occasion of his 60th birthday

Abstract. Cluster algebras *with coefficients* are important since they appear in nature as coordinate algebras of varieties like Grassmannians, double Bruhat cells, unipotent cells, $\cdots$. The approach of Geiss–Leclerc–Schröer often yields Frobenius exact categories which allow to categorify such cluster algebras. In previous work, the second-named author has constructed Higgs categories and relative cluster categories in the relative Jacobi-finite setting (arXiv:2109.03707). Higgs categories generalize the Frobenius categories used by Geiss–Leclerc–Schröer.

In this article, we construct the Higgs category and the relative cluster category in the relative Jacobi-infinite setting under suitable hypotheses. This covers for example the case of Jensen–King–Su's Grassmannian cluster category. As in the relative Jacobi-finite case, the Higgs category is no longer exact but still extriangulated in the sense of Nakaoka-Palu. We also construct a cluster character refining Plamondon's.

In the appendix, Chris Fraser and the first-named author categorify quasi-cluster morphisms using Frobenius categories. A recent application of this result is due to Matthew Pressland, who uses it to prove a conjecture by Muller–Speyer.

Contents

1. Introduction

Cluster categories were introduced in 2006 by Buan-Marsh-Reineke-Reiten-Todorov [2] in order to categorify acyclic cluster algebras [11, 12, 13] without coefficients. Caldero and Chapoton used the geometry of quiver Grassmannians to define the cluster character [5], i.e. a decategorification map which yields a bijection from the set of isomorphism classes of indecomposable objects of the cluster category of a Dynkin quiver to the set of cluster variables in the associated cluster algebra. More generally, for (antisymmetric) cluster algebras associated with acyclic quivers, Caldero-Keller [7] showed that the Caldero-Chapoton map induces a bijection between the set of isomorphism classes of indecomposable rigid objects and the set of cluster variables.

Key words and phrases. Ice quiver with potential, relative Ginzburg algebra, Higgs category, cluster character, quasi-cluster homomorphism.

In order generalize the representation-theoretic approach to cluster algebras from acyclic quivers to quivers with oriented cycles, Derksen–Weyman–Zelevinsky [8, 9] extended the mutation operation from quivers to quivers with potential and their representations. In the case where the quiver with potential is Jacobi-finite, Amiot [1] generalized the construction of the cluster category [1]. The cluster character constructed by Palu in [33] induces a bijection [6] from the isoclasses of the reachable rigid indecomposables of the (generalized) cluster category to the cluster variables of the associated cluster algebra. Plamondon [36] generalized Amiot's and Palu's constructions to arbitrary quivers with potential.

Cluster algebras with coefficients are of great importance in geometric examples. They appear in nature as coordinate algebras of varieties like Grassmannians, double Bruhat cells, unipotent cells, ... cf. for example [3, 18, 41]. The work of Geiss–Leclerc–Schröer provides Frobenius exact categories which allow to categorify such cluster algebras [17, 18] in many cases. In their approach, the Frobenius exact category is a full subcategory of the category of modules over a certain preprojective algebra. Geiss–Leclerc–Schröer's setting was axiomatized by Fu–Keller [15] using stably 2-Calabi–Yau Frobenius categories. They also observed that not all cluster algebras with coefficients admit such a categorification (for example, acyclic cluster algebras with principal coefficients do not, cf. Remark 5.7 of [15]).

In order to extend Geiss–Leclerc–Schröer's approach to larger classes of cluster algebras with coefficients, the second-named author introduced relative cluster categories and Higgs categories in [43]. However, the setting he considered was restricted to the case of ice quivers with potential (Q, F, W) whose associated relative Jacobi algebra is finite-dimensional. In this article, our aim is to construct the associated Higgs category $\mathcal{H}(Q, F, W)$ and the relative cluster category $\mathcal{C}(Q, F, W)$ under a much weaker assumption (cf. Assumption 1). We also construct a canonical cluster character in this setting. Higgs categories generalize the Frobenius categories used by Geiss-Leclerc-Schröer. Our cluster character generalizes Plamondon's [36] to the relative context.

Let us state our main results more precisely: Let (Q, F, W) be an ice quiver with potential and $\mathbf{\Gamma} = \mathbf{\Gamma}(Q, F, W)$ the associated relative Ginzburg algebra (cf. section 2.3). Let $e = \sum_{i\in F} e_i$ be the idempotent associated with the set of frozen vertices. Let $J(Q, F, W) = H^0(\mathbf{\Gamma})$ be the corresponding relative Jacobian algebra. We denote by $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$ the closure under finite direct sums and summands of $e\mathbf{\Gamma}$ in the perfect derived category $\mathrm{per}\mathbf{\Gamma}$. The *relative cluster category* $\mathcal{C}(Q, F, W)$ is defined as the idempotent completion of the quotient of $\mathrm{per}\mathbf{\Gamma}$ by the thick subcategory generated by all simple $H^0(\mathbf{\Gamma})$-modules associated with non frozen vertices of Q. The *Higgs category* $\mathcal{H}(Q, F, W)$ is a certain extension closed full subcategory of $\mathcal{C}(Q, F, W)$ (cf. Definition 3.21). Let $(\overline{Q}, \overline{W})$ be the quiver with potential obtained from (Q, F, W) by deleting the frozen part F and $\mathbf{\Gamma}(\overline{Q}, \overline{W})$ the Ginzburg algebra of $(\overline{Q}, \overline{W})$. Then we have a dg quotient morphism

$$p\colon \mathbf{\Gamma}(Q, F, W) \to \mathbf{\Gamma}(\overline{Q}, \overline{W}).$$

It induces a triangulated quotient functor $p^*\colon \mathcal{C}(Q, F, W) \to \mathcal{C}(\overline{Q}, \overline{W})$, where $\mathcal{C}(\overline{Q}, \overline{W})$ (or $\overline{\mathcal{C}}$) is the associated generalized cluster category. Let $\mathcal{D}(Q, F, W) \subseteq \mathcal{C}(Q, F, W)$ be the full subcategory of $\mathcal{C}(Q, F, W)$ whose objects are the M in $\mathcal{C}(Q, F, W)$ whose image $p^*(M)$ lies in Plamondon's category $\mathcal{D}(\overline{Q}, \overline{W})$ (see Subsection 4.1). In the following theorem, we abbreviate $\mathbf{\Gamma} = \mathbf{\Gamma}(Q, F, W)$ and $\mathcal{H} = \mathcal{H}(Q, F, W)$.

Theorem 1.1. *(Theorem 4.17) Let (Q, F, W) be an ice quiver with potential such that $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$ is functorially finite in $\mathrm{add}(\mathbf{\Gamma})$.*

1) *We have an equivalence of k-categories*
$$\mathcal{H}(Q, F, W)/[\mathcal{P}] \xrightarrow{\sim} \mathcal{D}(\overline{Q}, \overline{W}).$$
2) *If $(\overline{Q}, \overline{W})$ is Jacobi-finite, then $\mathcal{H}(Q, F, W)$ is a Frobenius extriangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$ and the equivalence in (1) preserves the extriangulated structure. Moreover, we have equalities*
$$\mathcal{D}(\overline{Q}, \overline{W}) = \mathcal{C}(\overline{Q}, \overline{W})$$
and
$$\mathcal{D}(Q, F, W) = \mathcal{C}(Q, F, W).$$
3) *If moreover $\mathbf{\Gamma}$ is concentrated in degree 0, then the* boundary algebra $B = eH^0(\mathbf{\Gamma})e$ *is* fp_∞-*Gorenstein of injective dimension at most e with respect to $\mathbf{\Gamma}$ and the Higgs category $\mathcal{H}$ is equivalent to the category $\mathrm{gpr}_\infty^{\leqslant 3}B$ (cf. section 3.6). Moreover, $\mathbf{\Gamma}$ is a canonical cluster-tilting object of $\mathcal{H}$ with endomorphism algebra $\mathrm{End}_{\mathcal{H}}(\mathbf{\Gamma}) = H^0(\mathbf{\Gamma}) = J(Q, F, W)$.*

4) *Let* $\mathcal{M} = \mathrm{add}(\mathbf{\Gamma}) \subseteq \mathcal{H}$. *Under the assumptions of* 3)*, the exact sequence of triangulated categories*

$$0 \to \mathrm{pvd}_e(\mathbf{\Gamma}) \to \mathrm{per}\mathbf{\Gamma} \to \mathcal{C}(Q,F,W) \to 0$$

is equivalent to

$$0 \to \mathcal{K}^b_{\mathcal{H}-ac}(\mathcal{M}) \to \mathcal{K}^b(\mathcal{M}) \to \mathcal{D}^b(\mathcal{H}) \to 0.$$

In particular, the relative cluster category $\mathcal{C}(Q,F,W)$ *is equivalent to the bounded derived category* $\mathcal{D}^b(\mathcal{H})$ *of* $\mathcal{H}$*.*

In section 5, we generalize Fu-Keller's cluster character to a cluster character

$$CC = X_? \colon \mathrm{obj}(\mathcal{H}) \to \mathbb{Q}[x_{r+1}, \ldots, x_n][x_1^{\pm1}, x_2^{\pm1}, \ldots, x_r^{\pm1}]$$

defined for Hom-infinite Higgs categories $\mathcal{H}$. Thus, the Higgs category yields an additive categorification for cluster algebras with *non-invertible* coefficients and the cluster character is a decategorification map. Let us point out that Grabowski and Pressland will construct a cluster character for certain extriangulated categories in their upcoming paper [20].

Now we assume that $(\overline{Q}, \overline{W})$ is Jacobi-finite. By definition, the Higgs category $\mathcal{H}(Q,F,W)$ is a full subcategory of $\mathcal{C}(Q,F,W)$. The following theorem shows that the map CC defined on $\mathcal{H}(Q,F,W)$ canonically extends to a map

$$CC_{loc} \colon \mathcal{C}(Q,F,W) \to \mathbb{Q}[x_1^{\pm1}, \ldots, x_r^{\pm1}, x_{r+1}^{\pm1}, \ldots, x_n^{\pm1}]$$

defined on the *whole* relative cluster category. Thus, we can consider the triangulated category $\mathcal{C}(Q,F,W)$ as an additive categorification of a cluster algebra with *invertible* coefficients and the map CC_{loc} as a decategorification map.

Theorem 1.2. *(Theorem 5.11) Let* (Q,F,W) *be an ice quiver with potential such that* $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$ *is functorially finite in* $\mathrm{add}(\mathbf{\Gamma})$*. We assume moreover that* $(\overline{Q}, \overline{W})$ *is Jacobi-finite. Consider the following diagram, where* $\overline{CC}$ *is the cluster character constructed by Plamondon in* [36]*,*

$$\begin{array}{ccccc}
\mathcal{H} & \hookrightarrow & \mathcal{C}(Q,F,W) & \overset{CC_{loc}}{\dashrightarrow} & \mathbb{Q}[x_1^{\pm1}, \ldots, x_r^{\pm1}, x_{r+1}^{\pm1}, \ldots, x_n^{\pm1}] \\
\downarrow{\scriptstyle p^*} & & & & \downarrow{\scriptstyle x_i \mapsto 1,\ \forall i>r} \\
\underline{\mathcal{H}} & \xrightarrow{\sim} & \mathcal{C}(\overline{Q}, \overline{W}) & \xrightarrow{\overline{CC}} & \mathbb{Q}[x_1^{\pm1}, \ldots, x_r^{\pm1}].
\end{array}$$

There is a unique map

$$CC_{loc} \colon \mathcal{C}(Q,F,W) \to \mathbb{Q}[x_1^{\pm1}, \ldots, x_r^{\pm1}, x_{r+1}^{\pm1}, \ldots, x_n^{\pm1}]$$

such that the above diagram commutes and

1) *for each triangle in* $\mathcal{C}(Q,F,W)$

$$P \to X \to M \to \Sigma P$$

with $P \in \mathrm{thick}_{\mathcal{C}}(\mathcal{P})$*, we have*

$$CC_{loc}(X) = CC_{loc}(P) \cdot CC_{loc}(M)\,;$$

2) *the restriction* $CC_{loc}|_{\mathcal{H}}$ *is the cluster character* $X_?$ *defined in Section 5.*
3) *for each object* P *in* $\mathrm{thick}_{\mathcal{C}}(\mathcal{P})$*, we have* $CC_{loc}(P) = x^{[P]}$*, where* $[P] \in K_0(\mathrm{per}\mathbf{\Gamma}) \simeq \mathbb{Z}^n$*.*

This article is organized as follows. In Section 2 we first recall the definitions of ice quivers with potential and the mutation operations. Then we give the construction of (complete) relative Ginzburg algebras. In Section 3, we use Plamondon's technique to define the Higgs category. The relationship between the Higgs category and Plamondon's category is explained in Section 4.1.

Let (Q,F,W) be an ice quiver with potential and $\mu_v(Q,F,W) = (Q',F',W')$ its mutation at a vertex v. If v is an unfrozen vertex, we show that mutation at v yields an equivalence between the relative cluster categories of (Q,F,W) and (Q',F',W') (see Proposition 3.29). If v is a frozen source or frozen sink, the mutation at v yields an equivalence between $\mathcal{D}(Q,F,W)$ and $\mathcal{D}(Q',F',W')$ (see Proposition 3.31).

Section 5 is devoted to the construction of the cluster character $CC = X_?$ (with respect to $\mathbf{\Gamma}$) on $\mathcal{H}(Q, F, W)$. We use an argument similar to Plamondon's to show the multiplication formula, cf. Section 5.2. Then we extend our cluster character to a map

$$CC_{loc} \colon \mathcal{C}(Q, F, W) \to \mathbb{Q}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}].$$

In Section 6, we show that the decategorification of the equivalence associated with the mutation at a frozen source (or sink) is a quasi-cluster isomorphism. In the final section 7, we explain how the class of ice quivers with potential which come from Postnikov diagrams fits into the theory developed in this article.

In appendix A, Chris Fraser and the first-named author use Frobenius categorifications (which can often be constructed using the main results of this paper) to construct quasi-cluster isomorphisms. In [40], Matthew Pressland has recently applied these results to prove a conjecture by Muller–Speyer [31, Rem. 4.7] linking the two canonical cluster structures on a positroid variety by a quasi-cluster isomorphism.

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Notation and conventions

Throughout this paper, k will denote an algebraically closed field of characteristic zero. We denote by $D = \mathrm{Hom}_k(-, k)$ the k-dual. All modules are right modules unless stated otherwise.

Let $\mathcal{T}$ be any triangulated category. For any subcategory $\mathcal{T}'$ of $\mathcal{T}$, we denote by $\mathrm{ind}\mathcal{T}'$ the set of isomorphism classes of indecomposable objects of $\mathcal{T}$ contained in $\mathcal{T}'$. Denote by $\mathrm{add}\mathcal{T}'$ the full subcategory of $\mathcal{T}$ whose objects are all direct summands of finite direct sums of objects in $\mathcal{T}'$. The subcategory $\mathcal{T}'$ is *rigid* if for any two objects X and Y of $\mathcal{T}'$, we have $\mathrm{Hom}_{\mathcal{T}}(X, \Sigma Y) = 0$.

Let $\mathcal{P}$ be a subcategory of $\mathcal{T}$. We denote by $[\mathcal{P}]$ the ideal of morphisms in $\mathcal{T}$ factoring through an object of $\mathcal{P}$. Then the corresponding additive quotient category with respect to $\mathcal{P}$ is denoted by $\mathcal{T}/[\mathcal{P}]$. Denote by $\mathrm{tri}_{\mathcal{T}}(\mathcal{P})$ the triangulated subcategory of $\mathcal{T}$ gennerated by $\mathcal{P}$, i.e. the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{P}$.

For collections $\mathcal{X}$ and $\mathcal{Y}$ of objects in $\mathcal{T}$. we denote by $\mathcal{X} * \mathcal{Y}$ the collection of objects $Z \in \mathcal{T}$ appearing in a triangle $X \to Z \to Y \to \Sigma X$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

2. Preliminaries

2.1. Ice quivers with potential.

Definition 2.1. A *quiver* is a tuple $Q = (Q_0, Q_1, s, t)$, where Q_0 and Q_1 are sets, and $s, t \colon Q_1 \to Q_0$ are functions. We think of the elements of Q_0 as vertices and of those of Q_1 as arrows, so that each $\alpha \in Q_1$ is realised as an arrow $\alpha \colon s(\alpha) \to t(\alpha)$. We call Q *finite* if Q_0 and Q_1 are finite sets.

Definition 2.2. Let Q be a quiver. A quiver $F = (F_0, F_1, s', t')$ is a *subquiver* of Q if it is a quiver such that $F_0 \subseteq Q_0$, $F_1 \subseteq Q_1$ and the functions s' and t' are the restrictions of s and t to F_1 . We say F is a *full subquiver* if $F_1 = \{\alpha \in Q_1 \colon s(\alpha), t(\alpha) \in F_0\}$, so that a full subquiver of Q is completely determined by its set of vertices.

Definition 2.3. An *ice quiver* is a pair (Q, F), where Q is a finite quiver and F is a (not necessarily full) subquiver of Q. We call F_0, F_1 and F the frozen vertices, arrows and subquiver respectively. We also call $Q_0 \setminus F_0$ and $Q_1 \setminus F_1$ the unfrozen vertices and arrows respectively.

Let k be a field. Let Q be a finite quiver.

Definition 2.4. Let S be the semisimple k-algebra $\prod_{i\in Q_0} ke_i$. The vector space kQ_1 naturally becomes an S-bimodule. Then the *complete path algebra* of Q is the completed tensor algebra

$$\widehat{kQ} = \widehat{T_S}(kQ_1).$$

It has underlying vector space

$$\prod_{d=0}^{\infty} (kQ_1)^{\otimes_S^d}$$

and multiplication given by concatenation. The algebra $\widehat{kQ}$ becomes a graded pseudocompact S-algebra in the sense of [4].

Definition 2.5. [39, Definition 2.8] A *potential* on Q is an element W in $HH_0(\widehat{kQ}) = \widehat{kQ}/[\widehat{kQ}, \widehat{kQ}]$ expressible as a (possibly infinite) linear combination of homogeneous elements of degree at least 2, such that any term involving a loop has degree at least 3. An *ice quiver with potential* is a tuple (Q, F, W) in which (Q, F) is a finite ice quiver and W is a potential on Q. If $F = \emptyset$ is the empty quiver, then $(Q, \emptyset, W) = (Q, W)$ is simply called a *quiver with potential.* We say that W is *irredundant* if each term of W includes at least one unfrozen arrow.

A potential can be thought of as an infinite formal linear combination of cyclic paths in Q (of length at least 2), considered up to the equivalence relation on such cycles induced by

$$\alpha_n \cdots \alpha_1 \sim \alpha_{n-1} \cdots \alpha_1 \alpha_n.$$

Definition 2.6. Let $p = \alpha_n \cdots \alpha_1$ be a cyclic path, with each $\alpha_i \in Q_1$, and let $\alpha \in Q_1$ be any arrow. Then the *cyclic derivative* of p with respect to α is

$$\partial_\alpha p = \Sigma_{\alpha_i=\alpha} \alpha_{i-1} \cdots \alpha_1 \cdots \alpha_{i+1}.$$

We extend ∂_α by linearity and continuity. Then it determines a map $HH_0(\widehat{kQ}) \to \widehat{kQ}$. For an ice quiver with potential (Q, F, W), we define the *relative Jacobian algebra* $J(Q, F, W)$ (or J_{rel}) as

$$\widehat{kQ}/\overline{\langle \partial_\alpha W \colon \alpha \in Q_1 \setminus F_1 \rangle}.$$

If $F = \emptyset$, we call $J(Q, W) = J(Q, \emptyset, W)$ the *Jacobian algebra* of the quiver with potential (Q, W). An ice quiver with potential (Q, F, W) is called *Jacobi-finite* if the relative Jacobian algebra $J(Q, F, W)$ is finite-dimensional. Otherwise, we say it is *Jacobi-infinite.*

Definition 2.7. Let Q be a quiver. An ideal of $\widehat{kQ}$ is called *admissible* if it is contained in the square of the closed ideal generated by the arrows of Q. We call an ice quiver with potential (Q, F, W) *reduced* if W is irredundant and the Jacobian ideal of $\widehat{kQ}$ determined by F and W is admissible. An ice quiver with potential (Q, F, W) is *trivial* if its relative Jacobian algebra $J(Q, F, W)$ is a product of copies of the base field k.

2.2. **Mutation of ice quivers with potentials.**

Two ice quivers with potential (Q, F, W) and (Q', F', W') are *right equivalent* if $Q_0 = Q'_0$, $F_0 = F'_0$ and there exists an S-algebra isomorphism $\varphi \colon \widehat{kQ} \to \widehat{kQ'}$ such that

(1) $\varphi|_S = \mathbf{1}_S$,
(2) $\varphi(\widehat{kF}) = \widehat{kF'}$, where $\widehat{kF}$ and $\widehat{kF'}$ are treated in the the natural way as subalgebras of $\widehat{kQ}$ and $\widehat{kQ'}$ respectively, and
(3) $\varphi(W)$ equals W' in $HH_0(\widehat{kQ'})$.

In that case, the relative Jacobian algebras of the two ice quivers with potential are isomorphic (see [39, Proposition 3.10]). Let (Q, F, W) be an ice quiver with potential. By [39, Theorem 3.6], there exists a reduced ice quiver with potential $(Q_{red}, F_{red}, W_{red})$ such that $J(Q, F, W) \cong J(Q_{red}, F_{red}, W_{red})$. And $(Q_{red}, F_{red}, W_{red})$ is uniquely determined up to right equivalence by the right equivalence class of (Q, F, W). We call $(Q_{red}, F_{red}, W_{red})$ the *reduction* of (Q, F, W).

2.2.1. *Mutation at unfrozen vertices.* Let v be an unfrozen vertex of Q such that no loops or 2-cycles of Q are incident with v. The mutation at the vertex v is the new ice quiver with potential $\mu_v(Q,F,W)$ obtained from (Q,F,W) (see [39]) in the following way:

(1) For each pair of arrows $\alpha\colon u\to v$ and $\beta\colon v\to w$, add an unfrozen 'composite' arrow $[\beta\alpha]\colon u\to w$ to Q.
(2) Reverse each arrow incident with v. This gives a new ice quiver (Q',F).
(3) Pick a representative $\widetilde{W}$ of W in kQ such that no term of W begins at v (which is possible since there are no loops at v). For each pair of arrows α,β as in (1), replace each occurrence of $\beta\alpha$ in $\widetilde{W}$ by $[\beta\alpha]$, and add the term $[\beta\alpha]\alpha^*\beta^*$. This gives a new potential W'.

The mutation $\mu_v(Q,F,W)$ of (Q,F,W) at v is then defined to be the reduction of (Q',F,W').

2.2.2. *Mutation at frozen vertices.* Let (Q,F,W) be an ice quiver with potential. Let v be a frozen vertex.

Definition 2.8. We say that v is a *frozen source* of Q if v is a source vertex of F and there are no unfrozen arrows with source v. Similarly, we say that v is a *frozen sink* of Q if v is a sink vertex of F and there are no unfrozen arrows with target v. For two vertices i and j, we say that they have the *same state* if they are both in F_0 or $Q_0\setminus F_0$. Otherwise, we say that they have *different states.* Similarly, for two arrows in Q, we say that they have the *same state* if they are both in F_1 or $Q_1\setminus F_1$. Otherwise, we say that they have *different states.*

Let v be a frozen source or a frozen sink. The mutation at the vertex v is the new ice quiver with potential $\mu_v(Q,F,W)$ obtained from (Q,F,W) (see [39, 42]) in the following way:

(1) For each pair of arrows $\alpha\colon u\to v$ and $\beta\colon v\to w$, add an unfrozen 'composite' arrow $[\beta\alpha]\colon u\to w$ to Q.
(2) Replace each arrow $\alpha\colon u\to v$ by an arrow $\alpha^*\colon v\to u$ of the same state as α and each arrow $\beta\colon v\to w$ by an arrow $\beta^*\colon w\to v$ of the same state as β. This gives a new ice quiver (Q',F').
(3) Pick a representative $\widetilde{W}$ of W in kQ such that no term of W begins at v (which is possible since there are no loops at v). For each pair of arrows α,β as in (1), replace each occurrence of $\beta\alpha$ in $\widetilde{W}$ by $[\beta\alpha]$, and add the term $[\beta\alpha]\alpha^*\beta^*$. This gives a new potential W'.

The mutation $\mu_v(Q,F,W)$ of (Q,F,W) at v is then defined to be the reduction of (Q',F',W').

2.3. Complete relative Ginzburg dg algebras.

Definition 2.9. Let (Q,F,W) be a finite ice quiver with potential. Let $\widetilde{Q}$ be the graded quiver with the same vertices as Q and whose arrows are

- the arrows of Q,
- an arrow $a^*\colon j\to i$ of degree -1 for each arrow a of Q not belonging to F,
- a loop $t_i\colon i\to i$ of degree -2 for each vertex i of Q not belonging to F.

Define the *completed relative Ginzburg dg algebra* $\mathbf{\Gamma}(Q,F,W)$ as the dg algebra whose underlying graded space is the completed graded path algebra $\widehat{k\widetilde{Q}}$. Its differential is the unique k-linear continuous endomorphism of degree 1 which satisfies the Leibniz rule

$$d(uv)=d(u)v+(-1)^p u d(v)$$

for all homogeneous u of degree p and all v, and takes the following values on the arrows of $\widetilde{Q}$:

- $d(a)=0$ for each arrow a of Q,
- $d(a^*)=\partial_a W$ for each arrow a of Q not belonging to F,
- $d(t_i)=e_i(\sum_{a\in Q_1}[a,a^*])e_i$ for each vertex i of Q not belonging to F where e_i is the lazy path corresponding to the vertex i.

Definition 2.10. Let F be any finite quiver. Let $\widetilde{F}$ be the graded quiver with the same vertices as F and whose arrows are

- the arrows of F,
- an arrow $\widetilde{a}: j\to i$ of degree 0 for each arrow a of F,
- a loop $r_i: i\to i$ of degree -1 for each vertex i of F.

Define *complete derived preprojective algebra* $\mathbf{\Pi}_2(F)$ as the dg algebra whose underlying graded space is the completed graded path algebra $\widehat{k\widetilde{F}}$. Its differential is the unique k-linear continuous endomorphism of degree 1 which satisfies the Leibniz rule

$$d(u \circ v) = d(u) \circ v + (-1)^p u \circ d(v)$$

for all homogeneous u of degree p and all v, and takes the following values on the arrows of $\widetilde{F}$:

- $d(a) = 0$ for each arrow a of F,
- $d(\widetilde{a}) = 0$ for each arrow a in F,
- $d(r_i) = e_i(\sum_{a \in F_1}[a, \widetilde{a}])e_i$ for each vertex i of F, where e_i is the lazy path corresponding to the vertex i.

Lemma 2.11. With the above notations, $J(Q, F, W)$ is isomorphic to $\mathrm{H}^0(\mathbf{\Gamma}(Q, F, W))$.

Let (Q, F, W) be a finite ice quiver with potential. Since W can be viewed as an element in $HC_0(\widehat{kQ})$, the class $c = B(W)$ is an element in $HH_1(\widehat{kQ})$, where

$$B\colon HC_0(\widehat{kQ}) \to HH_1(\widehat{kQ})$$

is Connes' connecting map (see [28, Section 6.1]).

We denote by $G\colon \widehat{kF} \hookrightarrow \widehat{kQ}$ the canonical dg inclusion. Let $HH_0(G)$ be the 0-th Hochschild homology of G (see [43, Section 2.4]). Then $\xi = (0, c)$ is an element of $HH_0(G)$. Applying the relative deformed 3-Calabi–Yau completion of $G\colon kF \hookrightarrow kQ$ with respect to ξ, we get the following dg functor (see [43, Section 7.2] and [42, Section 4])

$$\boldsymbol{G}_{rel}\colon \mathbf{\Pi}_2(F) \to \mathbf{\Gamma}(Q, F, W).$$

We call it *Ginzburg functor* (see [42, Section 4]). In the notations of [42, Section 4], it is given explicitly as follows:

- $\boldsymbol{G}_{rel}(i) = i$ for each frozen vertex $i \in F_0$,
- $\boldsymbol{G}_{rel}(a) = a$ for each arrow $a \in F_1$,
- $\boldsymbol{G}_{rel}(\widetilde{a}) = -\partial_a W$ for each arrow $a \in F_1$,
- $\boldsymbol{G}_{rel}(r_i) = e_i(\sum_{a \in Q_1 \setminus F_1}[a, a^*])e_i$ for each frozen vertex $i \in F_0$.

Let $e = \sum_{i \in F} e_i$ be the idempotent associated with the set of frozen vertices and $\mathrm{pvd}_e(\mathbf{\Gamma})$ the full subcategory of $\mathrm{pvd}(\mathbf{\Gamma})$ whose objects are dg modules Mwhose restriction to $e\mathbf{\Gamma}e$ is acyclic. In another words, $\mathrm{pvd}_e(\mathbf{\Gamma})$ is equal to

$$\mathrm{thick}_{\mathcal{D}(\mathbf{\Gamma})}\langle S_i \,|\, i \in Q_0 \setminus F_0 \rangle,$$

i.e. the thick subcategory of $\mathcal{D}(\mathbf{\Gamma})$ generated by all unfrozen simple modules.

Proposition 2.12. [43, Corollary 3.13] *For any dg module N and any dg module M in $\mathrm{pvd}_e(\mathbf{\Gamma})$, there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathcal{D}(\mathbf{\Gamma})}(M, N) \cong D\mathrm{Hom}_{\mathcal{D}(\mathbf{\Gamma})}(N, \Sigma^3 M).$$

Proposition 2.13. [43, Proposition 7.7] *Let (Q, F, W) be a finite ice quiver with potential. Let $\overline{Q}$ be the quiver obtained from Q by deleting all vertices in F and all arrows incident with vertices in F. Let $\overline{W}$ be the potential on $\overline{Q}$ obtained by deleting all cycles passing through vertices of F in W. Then*

$$\mathbf{\Pi}_2(F) \xrightarrow{\boldsymbol{G}_{rel}} \mathbf{\Gamma}(Q, F, W) \to \mathbf{\Gamma}(\overline{Q}, \overline{W})$$

is a homotopy cofiber sequence of dg categories, where $\mathbf{\Gamma}(\overline{Q}, \overline{W})$ is the Ginzburg algebra (see [28, Section 6]*) associated with the quiver with potential $(\overline{Q}, \overline{W})$.*

For simplicity of notation, we write $\mathbf{\Gamma}$ instead of $\mathbf{\Gamma}(Q, F, W)$ and use $\overline{\mathbf{\Gamma}}$ for $\mathbf{\Gamma}(\overline{Q}, \overline{W})$.

Denote by $p : \mathbf{\Gamma} \to \overline{\mathbf{\Gamma}}$ the canonical quotient functor. Then p induces the usual triple of adjoint functors $(p^*, p_*, p^!)$ between $\mathcal{D}(\overline{\mathbf{\Gamma}})$ and $\mathcal{D}(\mathbf{\Gamma})$, where p_* is the restriction functor, $p^* = ? \overset{\mathbf{L}}{\otimes}_{\mathbf{\Gamma}} \overline{\mathbf{\Gamma}}$ and $p^! = \mathbf{R}\mathrm{Hom}_{\mathbf{\Gamma}}(\overline{\mathbf{\Gamma}}, ?)$.

Proposition 2.14. *Let $e = \sum_{i \in F} e_i$ be the idempotent associated with the set of frozen vertices. We have an exact sequence of triangulated categories*

$$\mathrm{per}(e\mathbf{\Gamma}_{rel}e) \to \mathrm{per}(\mathbf{\Gamma}_{rel}) \xrightarrow{p^*} \mathrm{per}(\mathbf{\Gamma}(\overline{Q}, \overline{W})).$$

Proof. This follows from the lemma below based on [10, Theorem 3.1]. $\surd$

Lemma 2.15. *Let $\mathcal{A}$ be a dg category and $\mathcal{B}$ a full dg subcategory. Suppose that the functor*

$$p^*\colon \mathrm{add}(H^0(\mathcal{A})) \to \mathrm{add}(H^0(\mathcal{A}/\mathcal{B}))$$

is essentially surjective. Then we have an equivalence of triangulated categories

$$\mathrm{per}\mathcal{A}/\mathrm{per}\mathcal{B} \xrightarrow{\sim} \mathrm{per}\mathcal{A}/\mathcal{B},$$

where $\mathcal{A}/\mathcal{B}$ is the Drinfeld dg quotient (see [10]*) of $\mathcal{A}$ by $\mathcal{B}$.*

Proof. The perfect derived category $\mathrm{per}(\mathcal{A}/\mathcal{B})$ is generated, as triangulated category, by the retracts of the representable functor $(\mathcal{A}/\mathcal{B})(?, x)$, $x \in \mathcal{A}$. By [10, Theorem 3.1], the triangulated functor $p^*\colon \mathrm{per}\mathcal{A}/\mathrm{per}\mathcal{B} \to \mathrm{per}\mathcal{A}/\mathcal{B}$ is an equivalence up to direct summands. To show that it is dense, it suffices to check that its image contains the retract of the representable functor $(\mathcal{A}/\mathcal{B})(?, x)$, $x \in \mathcal{A}$. Then it follows from our assumption. $\surd$

3. Relative cluster categories and Higgs Categories

Let (Q, F, W) be an ice quiver with potential. Denote by $\mathbf{\Gamma}$ the associated complete relative Ginzburg dg algebra $\mathbf{\Gamma}(Q, F, W)$. Let $e = \sum_{i \in F} e_i$ be the idempotent associated with the set of frozen vertices and $\mathcal{P}$ the additive subcategory $\mathrm{add}(e\mathbf{\Gamma})$ of $\mathrm{per}\mathbf{\Gamma}$.

Definition 3.1. The *relative cluster category* $\mathcal{C}(Q, F, W)$ (or $\mathcal{C}$) of (Q, F, W) is defined as the idempotent completion of the Verdier quotient of triangulated categories

$$\mathrm{per}(\mathbf{\Gamma})/\mathrm{pvd}_e(\mathbf{\Gamma}).$$

If $F = \emptyset$, the *cluster category* associated with (Q, W) is defined as $\mathcal{C}(Q, \emptyset, W)$ and we denote it by $\mathcal{C}(Q, W)$.

Remark 3.2. If (Q, F, W) is Jacobi-finite, then the Verdier quotient $\mathrm{per}(\mathbf{\Gamma})/\mathrm{pvd}_e(\mathbf{\Gamma})$ is idempotent complete (see [43, Corollary 4.15]) and $(\overline{Q}, \overline{W})$ is also Jacobi-finite (see [43, Proposition 4.20]). The Verdier quotient $\mathrm{per}(\mathbf{\Gamma}(\overline{Q}, \overline{W}))/\mathrm{pvd}(\mathbf{\Gamma}(\overline{Q}, \overline{W}))$ is also idempotent complete (see [1]).

We denote by $\overline{\mathcal{D}}$ the unbounded derived category of $\mathbf{\Gamma}(\overline{Q}, \overline{W})$ and by $\overline{\mathcal{C}}$ the cluster category $\mathcal{C}(\overline{Q}, \overline{W})$ associated with $(\overline{Q}, \overline{W})$.

Proposition 3.3. [43, Corollary 4.22] *We have the following commutative diagram*

$$\begin{array}{ccccc}
 & & \mathrm{thick}(\mathcal{P}) & = & \mathrm{thick}(\mathcal{P}) \\
 & & \downarrow & & \downarrow \\
\mathrm{pvd}_e(\mathbf{\Gamma}) & \hookrightarrow & \mathrm{per}\mathbf{\Gamma} & \xrightarrow{\pi^{rel}} & \mathcal{C}(Q, F, W) \\
\downarrow{\cong} & & \downarrow{p^*} & & \downarrow{p^*} \\
\mathrm{pvd}(\overline{\mathbf{\Gamma}}) & \hookrightarrow & \mathrm{per}\overline{\mathbf{\Gamma}} & \xrightarrow{\pi} & \mathcal{C}(\overline{Q}, \overline{W}),
\end{array}$$

where the columns and rows are exact sequences of triangulated categories.

Proposition 3.4. [29, Lemma 2.17] *The perfect derived categories* $\mathrm{per}\mathbf{\Gamma}$ *and* $\mathrm{per}\overline{\mathbf{\Gamma}}$ *are Krull–Schmidt categories.*

Let $\mathcal{D}^{\leqslant 0}$ (and $\mathcal{D}^{\geqslant 0}$ respectively) be the full subcategory of $\mathcal{D}(\mathbf{\Gamma})$ whose objects are those X whose homology is concentrated in non-positive (and non-negative, respectively) degrees. Since $\mathbf{\Gamma}$ is connective, the pair $(\mathcal{D}^{\leqslant 0}, \mathcal{D}^{\geqslant 0})$ is a canonical t-structure on $\mathcal{D}(\mathbf{\Gamma})$. Similarly, we define $\overline{\mathcal{D}}^{\leqslant 0}$ and $\overline{\mathcal{D}}^{\geqslant 0}$. And the pair $(\overline{\mathcal{D}}^{\leqslant 0}, \overline{\mathcal{D}}^{\geqslant 0})$ is a canonical t-structure on $\overline{\mathcal{D}}$.

On $\overline{\mathcal{D}}$, we take the canonical t-structure $(\overline{\mathcal{D}}^{\leqslant 0}, \overline{\mathcal{D}}^{\geqslant 0})$ with heart $\heartsuit = \mathrm{Mod}(J(\overline{Q}, \overline{W}))$ and on $\mathcal{D}(e\mathbf{\Gamma}e)$, we take the trivial t-structure whose left aisle is $\mathcal{D}(e\mathbf{\Gamma}e)$.

We denote by $(i^*, i_*, i^!)$ the usual triple of adjoint functors between $\mathcal{D}(e\mathbf{\Gamma}e)$ and $\mathcal{D}(\mathbf{\Gamma})$ induced by the dg inclusion $i\colon e\mathbf{\Gamma}e \hookrightarrow \mathbf{\Gamma}$.

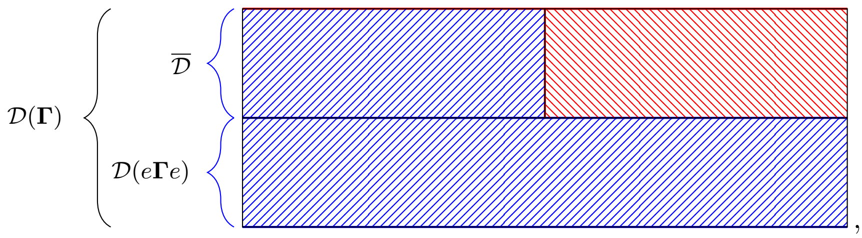
Proposition 3.5. [43, Corollary 4.4] *There is a t-structure in $\mathcal{D}(\mathbf{\Gamma})$ obtained by gluing the canonical t-structure on $\mathcal{D}(\overline{\mathbf{\Gamma}})$ with the trivial t-structure on $\mathcal{D}(e\mathbf{\Gamma}e)$ through a recollement diagram.*

We denote by $(\mathcal{D}^{\leqslant n}_{rel}, \mathcal{D}^{\geqslant n}_{rel})$ the glued t-structure on $\mathcal{D}(\mathbf{\Gamma})$. For each $n \in \mathbb{Z}$,

$$\mathcal{D}^{\leqslant n}_{rel} = \{X \in \mathcal{D}(\mathbf{\Gamma}) | H^l(p^*X) = 0, \forall l > n\},$$

$$\mathcal{D}^{\geqslant n}_{rel} = \{X \in \mathcal{D}(\mathbf{\Gamma}) | i_*(X) = 0, H^l(p^!X) \cong H^l(X) = 0, \forall l < n\}$$

and the heart $\heartsuit^{rel}$ of this glued t-structure is equivalent to $\mathrm{Mod}_e H^0(\mathbf{\Gamma})$*. We will call $(\mathcal{D}^{\leqslant n}_{rel}, \mathcal{D}^{\geqslant n}_{rel})$ the* relative t-structure *on $\mathcal{D}(\mathbf{\Gamma})$. We illustrate this glued t-structure in the following picture*



where the blue region represents the subcategory $\mathcal{D}^{\leqslant 0}_{rel}$ and the red region represents the subcategory $\mathcal{D}^{\geqslant 0}_{rel}$.

Definition 3.6. We define the *relative truncation functor* $\tau^{rel}_{>n}$ to be the following composition

$$\tau^{rel}_{>n} \colon \mathcal{D}(\mathbf{\Gamma}) \xrightarrow{p^*} \mathcal{D}(\overline{\mathbf{\Gamma}}) \xrightarrow{\tau_{>n}} \mathcal{D}(\overline{\mathbf{\Gamma}}) \xrightarrow{p_*} \mathcal{D}(\mathbf{\Gamma}).$$

Thus, for any $X \in \mathcal{D}(\mathbf{\Gamma})$, we have a canonical triangle in $\mathcal{D}(\mathbf{\Gamma})$

$$\tau^{rel}_{\leqslant n} X \to X \to \tau^{rel}_{>n} \to \Sigma \tau^{rel}_{\leqslant n} X$$

such that $\tau^{rel}_{\leqslant n} X$ belongs to $\mathcal{D}^{\leqslant n}_{rel}$ and $\tau^{rel}_{>n}(X) = p_*(\tau_{>n}(p^*X))$ belongs to $\mathcal{D}^{\geqslant n+1}_{rel}$.

Corollary 3.7. *If $(\overline{Q}, \overline{W})$ is Jacobi-finite, the relative t-structure on $\mathcal{D}(\mathbf{\Gamma})$ restricts to the perfect derived category* $\mathrm{per}\mathbf{\Gamma}$.

Proof. Let X be an object in $\mathrm{per}\mathbf{\Gamma}$. Consider the canonical triangle with respect to the relative t-structure on $\mathcal{D}(\Gamma)$

$$\tau^{rel}_{\leqslant 0} X \to X \to \tau^{rel}_{>0} X \to \Sigma \tau^{rel}_{\leqslant 0} X.$$

Since $(\overline{Q}, \overline{W})$ is Jacobi-finite, the perfect derived category $\mathrm{per}\overline{\mathbf{\Gamma}}$ is Home-finite (see [27, Proposition 2.5]). Thus, the space

$$H^l(\tau^{rel}_{>0} X) = \mathrm{Hom}_{\mathcal{D}\overline{\mathbf{\Gamma}}}(\overline{\mathbf{\Gamma}}, \Sigma^l \tau_{>0} p^* X)$$

equals zero or $H^l(\tau_{>0} p^*(X))$ which is finite-dimensional and vanishes for all $|l| \geqslant 0$. Thus, the object $\tau^{rel}_{>0} X$ is in $\mathrm{pvd}(\mathbf{\Gamma})$ and so in $\mathrm{per}\mathbf{\Gamma}$. This shows that the relative t-structure on $\mathcal{D}(\mathbf{\Gamma})$ restricts to $\mathrm{per}\mathbf{\Gamma}$. √

3.1. **Silting reduction.** Recall that a full subcategory $\mathcal{P}$ of a triangulated category $\mathcal{T}$ is *presilting* if $\mathrm{Hom}_{\mathcal{T}}(\mathcal{P}, \Sigma^i \mathcal{P}) = 0$ for any $i > 0$. It is *silting* if in addition $\mathcal{T} = \mathrm{thick}\mathcal{P}$. It is clear that $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$ is a presilting subcategory of $\mathrm{per}\mathbf{\Gamma}$.

Let $\mathcal{Z}$ be the following subcategory of $\mathrm{per}\mathbf{\Gamma}$

$$\mathcal{Z} = {}^{\perp}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp}.$$

Proposition 3.8. [22, Lemma 3.4] *The composition $\mathcal{Z} \subset \mathrm{per}\mathbf{\Gamma} \xrightarrow{p^*} \mathrm{per}\overline{\mathbf{\Gamma}}$ induces a fully faithful embedding*

$$p^* \colon \mathcal{Z}/[\mathcal{P}] \hookrightarrow \mathrm{per}\mathbf{\Gamma}/\mathrm{thick}(\mathcal{P}) \xrightarrow{\sim} \mathrm{per}\overline{\mathbf{\Gamma}}.$$

Similarly, the category $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma}) \subset \mathcal{C} = \mathcal{C}(Q, F, W)$ is a presilting subcategory of $\mathcal{C}$. Let $\mathcal{Y}$ be the following subcategory of $\mathcal{C}$

$$\mathcal{Y} = {}^{\perp}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp}.$$

Proposition 3.9. [22, Lemma 3.4] *The composition $\mathcal{Y} \subset \mathcal{C} \xrightarrow{p^*} \mathcal{C}(\overline{Q}, \overline{W})$ induces a fully faithful embedding*

$$p^* \colon \mathcal{Y}/[\mathcal{P}] \hookrightarrow \mathcal{C}/\mathrm{thick}(\mathcal{P}) \xrightarrow{\sim} \mathcal{C}(\overline{Q}, \overline{W}).$$

Remark 3.10. Since $\mathcal{P}$ doesn't satisfy the condition $(P1)$ in [22, Section 3.1], the functor p^* in Proposition 3.8 and Proposition 3.9 may not be dense.

3.2. **SMC reduction.** Let $\mathcal{S}$ be the subcategory of per$\mathbf{\Gamma}$ formed by the modules S_i associated with unfrozen vertices $i \in Q_0 \setminus F_0$. Then $\mathcal{S}$ is a simple-minded collection ([25, Definition 2.4]) of $\mathrm{pvd}_e(\mathbf{\Gamma})$ and a pre-simple-minded collection of $\mathrm{per}(\mathbf{\Gamma})$.

Consider the following subcategory of per$\mathbf{\Gamma}$

$$\mathcal{W} = (\Sigma^{\geqslant 0}\mathcal{S})^{\perp} \cap {}^{\perp}(\Sigma^{\leqslant 0}\mathcal{S}).$$

Proposition 3.11. [25, Theorem 3.1] *The composition* $\mathcal{W} \subseteq \mathrm{per}(\mathbf{\Gamma}) \xrightarrow{\pi^{rel}} \mathcal{C}(Q, F, W)$ *induces a fully faithful embedding*

$$\pi^{rel}_{\mathcal{W}} \colon \mathcal{W} \hookrightarrow \mathcal{C}(Q, F, W).$$

Remark 3.12. Since $\mathcal{S}$ doesn't satisfy the condition $(R1)$ in [25, Theorem 3.1], the functor $\pi^{rel}_{\mathcal{W}}$ above may not be dense.

Lemma 3.13. *If* $(\overline{Q}, \overline{W})$ *is Jacobi-finite, then* $\mathcal{S}$ *is functorially finite in* per$\mathbf{\Gamma}$.

Proof. Let $\overline{\mathcal{S}}$ be the the subcategory of per$\overline{\mathbf{\Gamma}}$ formed by all the simple $\overline{\mathbf{\Gamma}}$-modules. Since $(\overline{Q}, \overline{W})$ is Jacobi-finite, by a similar argument in [43, Lemma 4.18], $\overline{\mathcal{S}}$ is functorially finite in per$\overline{\mathbf{\Gamma}}$. The triple of adjoint functors

$$\mathcal{D}(\mathbf{\Gamma}) \underset{p^!}{\overset{p^*}{\underset{\longleftarrow p_* -}{\longrightarrow}}} \mathcal{D}(\overline{\mathbf{\Gamma}})$$

induced by $p : \mathbf{\Gamma} \to \overline{\mathbf{\Gamma}}$ induces a k-linear equivalence $p^* : \mathcal{S} \xrightarrow{\sim} \overline{\mathcal{S}}$.

Let X be an object of per$\mathbf{\Gamma}$. Since $\overline{\mathcal{S}}$ is covariantly finite in per$\overline{\mathbf{\Gamma}}$, there exists a left $\overline{\mathcal{S}}$-morphism $f' : p^*(X) \to S'$ in per$\overline{\mathbf{\Gamma}}$. Let f be the following composition

$$f \colon X \xrightarrow{\epsilon_X} p_*p^*X \xrightarrow{p_*(f')} p_*S',$$

where ϵ_X is the unite of the adjunction (p^*, p_*). It is easy to check that f is a left $\mathcal{S}$-approximation of X in per$\mathbf{\Gamma}$. Thus, $\mathcal{S}$ is covariantly finite in per$\mathbf{\Gamma}$.

Dually, we show that $\mathcal{S}$ is contravariantly finite in per$\mathbf{\Gamma}$.

$\checkmark$

By the above Lemma, if $(\overline{Q}, \overline{W})$ is Jacobi-finite, the category $\mathcal{S}$ also satisfies the condition $(R1)$ in [25, Theorem 3.1].

Proposition 3.14. *If* $(\overline{Q}, \overline{W})$ *is Jacobi-finite, the fully faithful embedding functor*

$$\pi^{rel}_{\mathcal{W}} \colon \mathcal{W} \hookrightarrow \mathcal{C}(Q, F, W).$$

is dense. Therefore we get a k*-linear equivalence*

$$\pi^{rel}_{\mathcal{W}} \colon \mathcal{W} \xrightarrow{\sim} \mathcal{C}(Q, F, Q).$$

3.3. **Higgs categories.** Let $\mathcal{T}$ be any triangulated category. Let $\mathcal{T}'$ be a full subcategory of $\mathcal{T}$. We denote by $\mathrm{pr}_{\mathcal{T}}\mathcal{T}'$ the full subcategory of $\mathcal{T}$ whose objects are cones of morphisms in add$\mathcal{T}'$. Similarly, we denote by $\mathrm{copr}_{\mathcal{T}}\mathcal{T}'$ the full subcategory of $\mathcal{T}$ whose objects are those X such that ΣX is in $\mathrm{pr}_{\mathcal{T}}\mathcal{T}'$. If $\mathcal{T}' = \mathrm{add}T$ for some object $T \in \mathcal{T}$, the categories $\mathrm{pr}_{\mathcal{T}}\mathcal{T}'$ and $\mathrm{copr}_{\mathcal{T}}\mathcal{T}'$ will be simply denoted by $\mathrm{pr}_{\mathcal{T}}T$ and $\mathrm{copr}_{\mathcal{T}}T$ respectively.

Lemma 3.15. [36, Lemma 2.11] *We have*

$$\mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma} = \mathcal{D}^{\leqslant 0} \cap {}^{\perp}\mathcal{D}^{\leqslant -2} \cap \mathrm{per}\mathbf{\Gamma}.$$

Thus, the category $\mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma}$ *is an extension closed subcategory of* per$\mathbf{\Gamma}$.

Proposition 3.16. *The quotient functor* $\pi^{rel} \colon \mathrm{per}\mathbf{\Gamma} \to \mathcal{C}$ *restricts to an equivalence of* k*-linear categories* $\mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma} \xrightarrow{\sim} \mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$.

Proof. Firstly, we show that $\pi^{rel}\colon \mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma} \to \mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$ is fully faithful.

Let X and Y be two objects in $\mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma}$. Hence X and Y lie in $\mathcal{D}^{\leqslant 0}(\mathbf{\Gamma})$. Suppose that a morphism $f\colon X \to Y$ is sent to zero in $\mathcal{C}$, i.e. f factors as

$$X \xrightarrow{g} M \xrightarrow{h} Y$$

with M in $\mathrm{pvd}_e(\mathbf{\Gamma})$. Since $X = \tau_{\leqslant 1}X$, g factors through $\tau_{\leqslant 1}M$, which is still in $\mathrm{pvd}_e(\mathbf{\Gamma})$. By Proposition 2.12, we have an isomorphism

$$D\mathrm{Hom}_{\mathbf{\Gamma}}(\tau_{\leqslant 1}M, Y) \cong \mathrm{Hom}_{\mathcal{D}(\mathbf{\Gamma})}(Y, \Sigma^3\tau_{\leqslant 1}M).$$

The space $\mathrm{Hom}_{\mathcal{D}(\mathbf{\Gamma})}(Y, \Sigma^3\tau_{\leqslant 1}M)$ vanishes, since Y belongs to ${}^{\perp}\mathcal{D}^{\leqslant -2}$. Thus, the morphism f is zero. This shows that $\pi^{rel}\colon \mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma} \to \mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$ is faithful. Let $f'\colon X \to Y$ be a morphism in $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$. Suppose that it can be represented by the following fraction

$$X \xrightarrow{f} Y' \xleftarrow{s} Y,$$

where the cone of s is an object N of $\mathrm{pvd}_e\mathbf{\Gamma}$. Consider the following diagram

$$\begin{array}{ccccc}
 & & Y & = & Y \\
 & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} \\
X & \xrightarrow{f} & Y' & \xrightarrow{g} & Y'' \\
 & & \downarrow & & \downarrow \\
\tau_{\leqslant 0}N & \longrightarrow & N & \longrightarrow & \tau_{\geqslant 1}N \\
 & & \downarrow & & \downarrow{\scriptstyle h} \\
 & & \Sigma Y & = & \Sigma Y
\end{array}$$

Since N is in $\mathrm{pvd}_e\mathbf{\Gamma}$, $\tau_{\leqslant 0}N$ is also in $\mathrm{pvd}_e\mathbf{\Gamma}$. Then the space $\mathrm{Hom}_{\mathcal{D}(\mathbf{\Gamma})}(\tau_{\leqslant 0}N, \Sigma Y)$ is isomorphic to $D\mathrm{Hom}_{\mathcal{D}(\mathbf{\Gamma})}(Y, \Sigma^2\tau_{\leqslant 0}N)$ because of Proposition 2.12. And this space vanishes since $\mathrm{Hom}_{\mathcal{D}(\mathbf{\Gamma})}(Y, \mathcal{D}^{\leqslant -2})$ vanishes.

Thus there exists a morphism $h\colon \tau_{\geqslant 1}N \to \Sigma Y$ such that the lower right square of the above diagram commutes. We extend h into a distinguished triangle which is the rightmost column of the diagram. Thus we have a fraction

$$X \xrightarrow{gf} Y'' \xleftarrow{t} Y$$

which is equal to

$$X \xrightarrow{f} Y' \xleftarrow{s} Y.$$

But the space $\mathrm{Hom}_{\mathcal{D}(\mathbf{\Gamma})}(X, \tau_{\geqslant 1}N)$ is zero since X is in $\mathcal{D}^{\leqslant 0}$ and $\tau_{\geqslant 1}N$ is in $\mathcal{D}^{\geqslant 1}$. Therefore, there exists a morphism $l\colon X \to Y$ such that $gf = tl$. It is easy to see that the fraction

$$X \xrightarrow{gf} Y'' \xleftarrow{t} Y$$

is the image of $l\colon X \to Y$ under the functor $\pi^{rel}\colon \mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma} \to \mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$. Thus, we have shown that $\pi^{rel}|_{\mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma}}$ is fully faithful.

It remains to be shown that it is dense. Let Z be an object of $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$. Then Z admits an $\mathrm{add}\mathbf{\Gamma}$-presentation

$$T_1 \xrightarrow{f'} T_0 \to Z \to \Sigma T_1$$

in $\mathcal{C}$. Since the functor $\pi^{rel}\colon \mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma} \to \mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$ is fully faithful, we can lift the morphism $f'\colon T_1 \to T_0$ to a morphism $f\colon P_1 \to P_0$ in $\mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma}$ with P_1 and P_0 in $\mathrm{add}\mathbf{\Gamma}$. Its cone is sent to Z in $\mathcal{C}$. This finishes the proof. $\surd$

An object X is in $\mathrm{copr}_{\mathcal{D}}\mathbf{\Gamma}$ if and only if ΣX is in $\mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma}$. Therefore we have the following dual statement of Proposition 3.16.

Corollary 3.17. *The quotient functor $\pi^{rel}\colon \mathrm{per}\mathbf{\Gamma} \to \mathcal{C}$ restricts to an equivalence of k-linear categories $\mathrm{copr}_{\mathcal{D}}\mathbf{\Gamma} \xrightarrow{\sim} \mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$.*

Let $\mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma}$ be the following subcategory of $\mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma}$

$$\{\mathrm{Cone}(X_1 \xrightarrow{f} X_0) \,|\, X_i \in \mathrm{add}(\mathbf{\Gamma}) \text{ and } \mathrm{Hom}_{\mathcal{D}}(f, I) \text{ is surjective for any object } I \in \mathcal{P}\}.$$

Clearly, we have $\mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma} = \mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma} \cap \mathcal{Z} = \mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma} \cap {}^{\perp}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp}$, where $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$.

Dually, we define $\mathrm{copr}^F_{\mathcal{D}}\mathbf{\Gamma}$ as the following subcategory of $\mathrm{copr}_{\mathcal{D}}\mathbf{\Gamma}$

$$\{\Sigma^{-1}\mathrm{Cone}(X_0 \xrightarrow{f} X_1) \,|\, X_i \in \mathrm{add}(\mathbf{\Gamma}) \text{ and } \mathrm{Hom}_{\mathcal{D}}(P, f) \text{ is surjective for any object } P \in \mathcal{P}\}.$$

And we have $\mathrm{copr}^F_{\mathcal{D}}\mathbf{\Gamma} = \mathrm{copr}_{\mathcal{D}}\mathbf{\Gamma} \cap \mathcal{Z}$.

Similarly, we define subcategories

$$\mathrm{pr}^F_{\mathcal{C}}\mathbf{\Gamma} = \mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma} \cap \mathcal{Y}$$

and

$$\mathrm{copr}^F_{\mathcal{C}}\mathbf{\Gamma} = \mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma} \cap \mathcal{Y}$$

of $\mathcal{C}$, where

$$\mathcal{Y} = {}^{\perp}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp} \subseteq \mathcal{C}.$$

Remark 3.18. It is easy to see that $\mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma}$ is a full subcategory of $\Sigma^{-1}\mathcal{W}$, where $\mathcal{W} = (\Sigma^{\geqslant 0}\mathcal{S})^{\perp} \cap {}^{\perp}(\Sigma^{\leqslant 0}\mathcal{S})$ and $\mathcal{S} = \mathrm{thick}(S_i \,|\, i \in Q_0 \setminus F_0)$.

Proposition 3.19. *The quotient functor* $\pi^{rel}\colon \mathrm{per}\mathbf{\Gamma} \to \mathcal{C}$ *induces equivalences of* k*-linear categories* $\mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma} \xrightarrow{\sim} \mathrm{pr}^F_{\mathcal{C}}\mathbf{\Gamma}$ *and* $\mathrm{copr}^F_{\mathcal{D}}\mathbf{\Gamma} \xrightarrow{\sim} \mathrm{copr}^F_{\mathcal{C}}\mathbf{\Gamma}$.

Proof. Let X be an object of $\mathrm{pr}^F_{\mathcal{C}}\mathbf{\Gamma} \subseteq \mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$. By Proposition 3.16, there is an object $X' \in \mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma}$ such that $\pi^{rel}(X') \cong X$. Since $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{<0}\mathcal{P}, X) \simeq \mathrm{Hom}_{\mathcal{D}}(\Sigma^{<0}\mathcal{P}, X')$ and $\mathrm{Hom}_{\mathcal{C}}(X, \Sigma^{>0}\mathcal{P}) \simeq \mathrm{Hom}_{\mathcal{D}}(X', \Sigma^{>0}\mathcal{P})$, we see that X' is in $\mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma}$. Thus, π^{rel} induces equivalences of k-linear categories $\mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma} \xrightarrow{\sim} \mathrm{pr}^F_{\mathcal{C}}\mathbf{\Gamma}$ and $\mathrm{copr}^F_{\mathcal{D}}\mathbf{\Gamma} \xrightarrow{\sim} \mathrm{copr}^F_{\mathcal{C}}\mathbf{\Gamma}$.

$\surd$

The following result relates morphisms in the relative cluster category and in the derived category.

Proposition 3.20. [36, Proposition 2.19] *Let* X *and* Y *be objects of* $\mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma}$ *such that* $\mathrm{Hom}_{\mathcal{D}}(X, \Sigma Y)$ *is finite-dimensional. Then there is an exact sequence of vector spaces*

$$0 \to \mathrm{Hom}_{\mathcal{D}}(X, \Sigma Y) \to \mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y) \to D\mathrm{Hom}_{\mathcal{D}}(Y, \Sigma X) \to 0.$$

Proof. The proof follows the lines of that [36, Proposition 2.19]. $\surd$

Definition 3.21. [43] We define the *Higgs category* $\mathcal{H}(Q, F, W)$ (or $\mathcal{H}$) as the full subcategory of $\mathrm{pr}^F_{\mathcal{C}}\mathbf{\Gamma} \cap \mathrm{copr}^F_{\mathcal{C}}\mathbf{\Gamma}$ whose objects are those X such that $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\mathbf{\Gamma}, X)$ is finite-dimensional.

Recall that the dg quotient functor $p\colon \mathbf{\Gamma}(Q, F, W) \to \mathbf{\Gamma}(\overline{Q}, \overline{W})$ induces the following Verdier quotient

$$p^*\colon \mathcal{C}(Q, F, W) \twoheadrightarrow \mathcal{C}(\overline{Q}, \overline{W}).$$

Definition 3.22. We define the category $\mathcal{D}(Q, F, W) \subseteq \mathcal{C}(Q, F, W)$ as the full subcategory of $\mathcal{C}(Q, F, W)$ whose objects are those objects M of $\mathcal{C}(Q, F, W)$ such that $p^*(M)$ lies in Plamondon's category $\mathcal{D}(\overline{Q}, \overline{W})$ (see Subsection 4.1).

By Proposition 4.5, we see that $\mathcal{D}(Q, F, W)$ is an extension closed subcategory of $\mathcal{C}(Q, F, W)$. Hence it has an extriangulated structure in the sense of Nakaoka-Palu [32], cf. also [35].

Lemma 3.23. *The Higgs category* $\mathcal{H}(Q, F, W)$ *is idempotent complete.*

Proof. The Higgs category is a full subcategory of $\mathrm{pr}^F_{\mathcal{C}}\mathbf{\Gamma} \cap \mathrm{copr}^F_{\mathcal{C}}\mathbf{\Gamma} = \mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma} \cap \mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma} \cap \mathcal{Y}$, where $\mathcal{Y} = {}^{\perp}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp} \subseteq \mathcal{C}$. By definition, the relative cluster category $\mathcal{C}$ is idempotent complete. Then $\mathcal{Y}$ is also idempotent complete.

By Lemma 3.15 and Proposition 3.16, $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$ is idempotent complete. Hence so is $\mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$. This shows that $\mathrm{pr}^F_{\mathcal{C}}\mathbf{\Gamma} \cap \mathrm{copr}^F_{\mathcal{C}}\mathbf{\Gamma}$ is idempotent complete. Thus, the Higgs category $\mathcal{H}(Q, F, W)$ is idempotent complete. $\surd$

3.4. **Modules.** Consider the functors

$$R = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\mathbf{\Gamma}, ?)\colon \mathcal{C} \to \mathrm{Mod}J_{rel}$$

and

$$G = \mathrm{Hom}_{\mathcal{C}}(?, \Sigma\mathbf{\Gamma})\colon (\mathcal{C})^{op} \to \mathrm{Mod}J_{rel}^{op},$$

where $\mathrm{Mod}J_{rel}$ is the category of right J_{rel}-modules.

Proposition 3.24. [36, Lemma 3.2] *Let X and Y be objects in $\mathcal{C}$.*

1) *If X lies in $\mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$, then R induces an isomorphism*
$$\mathrm{Hom}_{\mathcal{C}}(X, Y)/[\mathbf{\Gamma}] \to \mathrm{Hom}_{J_{rel}}(RX, RY).$$
2) *If Y lies in $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$, then G induces an isomorphism*
$$\mathrm{Hom}_{\mathcal{C}}(X, Y)/[\mathbf{\Gamma}] \to \mathrm{Hom}_{J_{rel}^{op}}(GY, GX).$$
3) *R induces an equivalence of categories*
$$\mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}/[\mathbf{\Gamma}] \to \mathrm{mod}J_{rel},$$
where $\mathrm{mod}J_{rel}$ denotes the category of finitely presented J_{rel}-modules.
4) *Any finite-dimensional J_{rel}-module can be lifted through R to an object in $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma} \cap \mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$. Any short exact sequence of finite-dimensional J_{rel}-modules can be lifted through R to a triangle of $\mathcal{C}$, whose three terms are in $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma} \cap \mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$.*

Proof. The proof follows the lines of that of [36, Lemma 3.2]. We give the proof of (4). It is easy to see that we can lift the simple modules at each vertex to an object of $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma} \cap \mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$. Let M be a finite dimensional J_{rel}-module. Then M is nilpotent and it can be obtained from the simple modules by repeated extensions. Thus, it is enough to show this property is preserved under extensions in $\mathrm{mod}J_{rel}$.

Let $0 \to L \to M \to N \to 0$ be a short exact sequence where L and N are finite-dimensional. Suppose that L and M admit lifts $\overline{L}$ and $\overline{N}$ in $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma} \cap \mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$, respectively. Let

$$P_1^L \to P_0^L \to L \to 0 \quad \text{and} \quad P_1^N \to P_0^N \to N \to 0$$

be projective presentations of L and N , respectively. Then we have the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1^L & \longrightarrow & P_1^L \oplus P_1^N & \longrightarrow & P_1^N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0^L & \longrightarrow & P_0^L \oplus P_0^N & \longrightarrow & P_0^N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0, & &
\end{array}$$

where the upper two rows are split. By part (3), we lift the upper left square to a commutative diagram in $\mathrm{add}(\Sigma^{-1}\mathbf{\Gamma}) \subseteq \mathcal{C}$

$$\begin{array}{ccc}
T_1^L & \longrightarrow & T_1^L \oplus T_1^N \\
\downarrow & & \downarrow \\
T_0^L & \longrightarrow & T_0^L \oplus T_0^N.
\end{array}$$

The above diagram embeds in a nine-diagram in $\mathcal{C}$ as follows

$$\begin{array}{ccccccc} T_1^L & \longrightarrow & T_1^L \oplus T_1^N & \longrightarrow & T_1^N & \longrightarrow & \Sigma T_1^L \\ \downarrow & & \downarrow & & \downarrow & & \\ T_0^L & \longrightarrow & T_0^L \oplus T_0^N & \longrightarrow & T_0^N & \longrightarrow & \Sigma T_0^L \\ \downarrow & & \downarrow & & \downarrow & & \\ \overline{L} & \longrightarrow & \overline{M} & \longrightarrow & \overline{N} & \longrightarrow & \Sigma\overline{L} \\ \downarrow & & \downarrow & & \downarrow & & \\ \Sigma T_1^L & \longrightarrow & \Sigma T_1^L \oplus \Sigma T_1^N & \longrightarrow & \Sigma T_1^N. & & \end{array}$$

Thus, $\overline{M}$ is a lift of M in $\mathrm{pr}_{\mathcal{C}}(\Sigma^{-1}\mathbf{\Gamma}) = \mathrm{copr}_{\mathcal{C}}(\mathbf{\Gamma})$. Since $\overline{N}$ is in $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$, $\Sigma^{-1}\overline{N}$ lies in $\mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$. By part (1), the morphism $\Sigma^{-1}\overline{N} \to \overline{L}$ is in $[\mathbf{\Gamma}]$. According to Lemma 3.25 below, $\overline{M}$ is also in $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$. $\surd$

Lemma 3.25. [36, Lemma 3.4] *Let $X \to Y \to Z \xrightarrow{\epsilon} \Sigma X$ be a triangle in $\mathcal{C}$ such that ϵ lies in $[\mathbf{\Gamma}]$. If two of X, Y and Z lie in $\mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$, so does the third one.*

Proof. The proof of [36, Lemma 3.4] also works for our situation. $\surd$

Proposition 3.26. *The Higgs category $\mathcal{H}(Q, F, W)$ is an extension closed subcategory of $\mathcal{C}$. Thus, it becomes an extriangulated category in the sense of Nakaoka-Palu* [32]*, cf. also* [35].

Proof. Let $X \to Y \to Z \to \Sigma X$ be a triangle in $\mathcal{C}$ such that $X, Z \in \mathcal{H}(Q, F, W)$. We need to show that Y also lies in $\mathcal{H}(Q, F, W)$.

Applying the functor $R = \mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}\mathbf{\Gamma}, ?) \colon \mathcal{C} \to \mathrm{Mod}J_{rel}$, we get a long exact sequence

$$\cdots \to R(X) \to R(Y) \to R(Z) \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{\Gamma}, \Sigma^2 X) = 0 \to \cdots .$$

Since X lies in $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$, we see that $\mathrm{Hom}_{\mathcal{C}}(\mathbf{\Gamma}, \Sigma^2 X)$ vanishes. By the definition of the Higgs category, the vector spaces $R(X)$ and $R(Z)$ are finite dimensional. Thus, $R(Y)$ is also finite dimensional. Then by 4) of Proposition 3.24, there exists an object $Y' \in \mathcal{H}(Q, F, W)$ such that $R(Y') \cong R(Y)$.

We next show that Y also lies in $\mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma} = \mathrm{pr}_{\mathcal{C}}(\Sigma^{-1}\mathbf{\Gamma})$. Let $\Sigma^{-1}T_1^X \to \Sigma^{-1}T_0^X \to X \to \Sigma T_1^X$ and $\Sigma^{-1}T_1^Z \to \Sigma^{-1}T_0^Z \to Z \to \Sigma T_1^Z$ be $\mathrm{add}(\Sigma^{-1}\mathbf{\Gamma})$-presentations of X and Z, respectively. Since $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}T_0^Z, \Sigma X) = 0$, the composition $\Sigma^{-1}T_0^Z \to Z \to \Sigma X$ factors through Y. This induces a commutative square

$$\begin{array}{ccc} \Sigma^{-1}T_0^X \oplus \Sigma^{-1}T_0^Z & \longrightarrow & \Sigma^{-1}T_0^Z \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z. \end{array}$$

It can be completed into a nine-diagram

$$\begin{array}{ccccccc} \Sigma^{-1}T_1^X & \longrightarrow & \Sigma^{-1}T_1^X \oplus \Sigma^{-1}T_1^Z & \longrightarrow & \Sigma^{-1}T_1^Z & \longrightarrow & T_1^X \\ \downarrow & & \downarrow & & \downarrow & & \\ \Sigma^{-1}T_0^X & \longrightarrow & \Sigma^{-1}T_0^X \oplus \Sigma^{-1}T_0^Z & \longrightarrow & \Sigma^{-1}T_0^Z & \longrightarrow & T_0^X \\ \downarrow & & \downarrow & & \downarrow & & \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ \downarrow & & \downarrow & & \downarrow & & \\ T_1^X & & T_1^X \oplus T_1^Z & & T_1^Z. & & \end{array}$$

This shows that Y is in $\mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$. By 3) of Proposition 3.24 and the fact that $R(Y) \cong R(Y')$, there exist objects T and T' in $\mathrm{add}(\mathbf{\Gamma})$ such that $Y \oplus T \cong Y' \oplus T'$ in $\mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$. By Lemma 3.23, the object Y lies in the Higgs category $\mathcal{H}(Q, F, W)$.

$\surd$

3.5. **Mutations induce equivalences.** Let (Q, F, W) be an ice quiver with potential. Let $\mathbf{\Gamma}'$ be the complete relative Ginzburg dg algebra of $\mu_v(Q, F, W) = (Q', F', W')$. For any vertex j of Q, let $\mathbf{\Gamma}_i = e_i\mathbf{\Gamma}$ and $\mathbf{\Gamma}'_i = e_i\mathbf{\Gamma}'$.

3.5.1. *Mutation at unfrozen vertices.*

Let v be an unfrozen vertex of Q not involved in any oriented cycle of length 2. As seen in Subsection 2.2, one can mutate (Q, F, W) at the vertex v. We assume that v is the source of at least one arrow.

Theorem 3.27. [42, Theorem 5.3]

1) *There is a triangle equivalence Φ_+ from $\mathcal{D}(\mathbf{\Gamma}')$ to $\mathcal{D}(\mathbf{\Gamma})$ sending $\mathbf{\Gamma}'_i$ to $\mathbf{\Gamma}_i$ if $i \neq v$ and to the cone $\mathbf{\Gamma}^*_v$ of the morphism*
$$\mathbf{\Gamma}_v \longrightarrow \bigoplus_{\alpha\in Q_1, s(\alpha)=v} \mathbf{\Gamma}_{t(\alpha)}$$
whose components are given by left multiplication by α if $i = v$. The functor Φ_+ restricts to triangle equivalences from $\mathrm{per}\mathbf{\Gamma}'$ to $\mathrm{per}\mathbf{\Gamma}$ and from $\mathrm{pvd}(\mathbf{\Gamma}')$ to $\mathrm{pvd}(\mathbf{\Gamma})$.
2) *The following diagram commutes*
$$\begin{array}{ccc} & & \mathcal{D}(\mathbf{\Gamma}'_{rel}) \\ & \nearrow^{(\boldsymbol{G}'_{rel})^*} & \\ \mathcal{D}(\mathbf{\Pi}_2(F)) & & \downarrow{\scriptstyle \Phi_+} \\ & \searrow_{(\boldsymbol{G}_{rel})^*} & \\ & & \mathcal{D}(\mathbf{\Gamma}). \end{array}$$

Remark 3.28. [42, Remark 5.5] If v is the target of at least one arrow, there is also a triangle equivalence $\Phi_- \colon \mathcal{D}(\mathbf{\Gamma}'_{rel}) \to \mathcal{D}(\mathbf{\Gamma})$ which, for $j \neq v$, sends the $\mathbf{\Gamma}'_j$ to $\mathbf{\Gamma}_j$ and for $j = v$, to the shifted cone
$$\Sigma^{-1}(\bigoplus_{\beta\in Q_1; t(\beta)=v} \mathbf{\Gamma}_{s(\beta)} \to \mathbf{\Gamma}_v),$$
where we have a summand $\mathbf{\Gamma}_{s(\beta)}$ for each arrow β of Q with target i and the corresponding component of the morphism is left multiplication by β. Moreover, the two equivalences Φ_+ and Φ_- are related by the twist functor t_{S_v} with respect to the 3-spherical object S_v, i.e. $\Phi_- = t_{S_v} \circ \Phi_+$. For each object X in $\mathcal{D}(\mathbf{\Gamma}_{rel})$, the object $t_{S_v}(X)$ is given by the following triangle
$$\mathbf{R}\mathrm{Hom}(S_v, X) \otimes_k S_v \to X \to t_{S_v}(X) \to \Sigma\mathbf{R}\mathrm{Hom}(S_v, X) \otimes_k S_v.$$

Proposition 3.29. *The functors*
$$\Phi_\pm \colon \mathrm{per}\mathbf{\Gamma}' \overset{\sim}{\underset{\sim}{\rightrightarrows}} \mathrm{per}\mathbf{\Gamma}$$
induce equivalences $\Phi_\pm \colon \mathrm{pvd}_e\mathbf{\Gamma}' \xrightarrow{\sim} \mathrm{pvd}_e\mathbf{\Gamma}$ and
$$\Phi_\pm \colon (\mathcal{C})' \coloneqq \mathcal{C}(Q', F', W') \overset{\sim}{\underset{\sim}{\rightrightarrows}} \mathcal{C}.$$
We have $\Phi_+ \simeq \Phi_- \coloneqq \Phi$ and Φ induces an equivalence $\mathcal{H}' \coloneqq \mathcal{H}(Q', F', W') \xrightarrow{\sim} \mathcal{H}(Q, F, W)$ such that the following diagram commutes
$$\begin{array}{ccccc} \mathrm{per}\mathbf{\Gamma}' & \twoheadrightarrow & (\mathcal{C})' & \supset & \mathcal{H}' \\ {\scriptstyle \Phi_+}\downarrow\downarrow{\scriptstyle \Phi_-} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ \mathrm{per}\mathbf{\Gamma} & \twoheadrightarrow & \mathcal{C} & \supset & \mathcal{H} \end{array}$$

Proof. We know that $\Phi_\pm$ induces triangle equivalences $\mathrm{pvd}(\mathbf{\Gamma}') \xrightarrow{\sim} \mathrm{pvd}(\mathbf{\Gamma})$. It is easy to see that we have identities
$$\mathrm{pvd}_e(\mathbf{\Gamma}) = \mathrm{pvd}(\mathbf{\Gamma}) \cap (\oplus_{i\in F_0}\mathbf{\Gamma}_i)^{\perp_{\mathrm{per}\mathbf{\Gamma}}}$$

and

$$\mathrm{pvd}_e(\mathbf{\Gamma}') = \mathrm{pvd}(\mathbf{\Gamma}') \cap (\oplus_{i\in F_0'}\mathbf{\Gamma}_i')^{\perp_{\mathrm{per}\mathbf{\Gamma}'}}.$$

Thus, $\Phi_\pm$ induces equivalences $\Phi_\pm\colon \mathrm{pvd}_e\mathbf{\Gamma}' \xrightarrow{\sim} \mathrm{pvd}_e\mathbf{\Gamma}$ and

$$\Phi_\pm\colon (\mathcal{C})' \coloneqq \mathcal{C}(\mu_v(Q,F,W)) \overset{\sim}{\underset{\sim}{\rightrightarrows}} \mathcal{C}.$$

On the level of objects, when $j \neq v$, it is clear that $\Phi_+(\mathbf{\Gamma}_j') = \Phi_-(\mathbf{\Gamma}_j) = \mathbf{\Gamma}_j$. When $j = v$, then $\Phi_+(\mathbf{\Gamma}_j')$ is the cone $\mathbf{\Gamma}_v^*$ of the morphism

$$\mathbf{\Gamma}_v \longrightarrow \bigoplus_{\alpha\in Q_1, s(\alpha)=v} \mathbf{\Gamma}_{t(\alpha)}$$

whose components are given by left multiplication by α. Let X_v be this mapping cone. It is easy to see that $\mathbf{R}\mathrm{Hom}(S_v, X_v)\otimes_k S_v$ belongs to $\mathrm{pvd}_e(\mathbf{\Gamma})$. Thus, for any vertex $j\in Q_0$, we have an isomorphism

$$\Phi_+(\mathbf{\Gamma}_j') \cong \Phi_-(\mathbf{\Gamma}_j')$$

in the relative cluster category $\mathcal{C}$. This shows that $\Phi_+ \simeq \Phi_- \coloneqq \Phi\colon (\mathcal{C})' \to \mathcal{C}$.

By [36, Proposition 2.7], we know that

$$\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma} = \mathrm{pr}_{\mathcal{C}}(\mu_v(\mathbf{\Gamma})),$$

where $\mu_v(\mathbf{\Gamma})$ is the $\mathbf{\Gamma}'$-$\mathbf{\Gamma}$-bimodule $\mathbf{\Gamma}_v^* \oplus \oplus_{j\neq v}\mathbf{\Gamma}_j$. Then it is clear that Φ induces an equivalence

$$\Phi\colon \mathrm{pr}_{(\mathcal{C})'}\mathbf{\Gamma}' \xrightarrow{\sim} \mathrm{pr}_{\mathcal{C}}(\mu_v(\mathbf{\Gamma})).$$

Hence, we have an equivalence $\Phi\colon \mathrm{pr}_{(\mathcal{C})'}\mathbf{\Gamma}' \xrightarrow{\sim} \mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$. Similarly, we have an equivalence

$$\Phi\colon \mathrm{copr}_{(\mathcal{C})'}\mathbf{\Gamma}' \xrightarrow{\sim} \mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}.$$

Let $\mathcal{P}' = \mathrm{add}(e'\mathbf{\Gamma}')$ with $e' = \sum_{i\in F_0'} e_i$. Then we define $\mathcal{Y}'$ to be the following full subcategory of $(\mathcal{C})'$

$$\mathcal{Y}' = {}^{\perp}(\Sigma^{>0}\mathcal{P}') \cap (\Sigma^{<0}\mathcal{P}')^{\perp}.$$

Hence Φ induces an equivalence

$$\mathrm{pr}_{(\mathcal{C})'}(\mathbf{\Gamma}') \cap \mathrm{copr}_{(\mathcal{C})'}(\mathbf{\Gamma}') \cap \mathcal{Y}' \xrightarrow{\sim} \mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma} \cap \mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma} \cap \mathcal{Y}.$$

Thus, by the definition of the Higgs category, we have an equivalence

$$\Phi\colon \mathcal{H}' \xrightarrow{\sim} \mathcal{H}.$$

$\surd$

3.5.2. *Mutation at frozen vertices.*

Now let v be a frozen source. As seen in Subsection 2.2, one can mutate (Q,F,W) at v. Write $(Q',F',W') = \mu_v(Q,F,W)$. Let $\mathbf{\Gamma} = \mathbf{\Gamma}_{rel}(Q,F,W)$ and $\mathbf{\Gamma}' = \mathbf{\Gamma}_{rel}(Q',F',W')$ be the complete relative Ginzburg dg algebras associated to (Q,F,W) and (Q',F',W') respectively. For a vertex i, let $\mathbf{\Gamma}_i = e_i\mathbf{\Gamma}$ and $\mathbf{\Gamma}_i' = e_i\mathbf{\Gamma}'$.

Theorem 3.30. [42, Theorem 6.8] *We have a triangle equivalence*

$$\Psi_+\colon \mathcal{D}(\mathbf{\Gamma}') \to \mathcal{D}(\mathbf{\Gamma}),$$

which sends the $\mathbf{\Gamma}_i'$ to $\mathbf{\Gamma}_i$ for $i\neq v$ and $\mathbf{\Gamma}_v$ to the cone

$$\mathrm{Cone}(\mathbf{\Gamma}_v \to \bigoplus_{\alpha} \mathbf{\Gamma}_{t(\alpha)}),$$

where we have a summand $\mathbf{\Gamma}_{t(\alpha)}$ for each arrow α of F with source v and the corresponding component of the map is the left multiplication by α. The functor Ψ_+ restricts to triangle equivalences from $\mathrm{per}(\mathbf{\Gamma}')$

to $\mathrm{per}(\mathbf{\Gamma})$ *and from* $\mathrm{pvd}(\mathbf{\Gamma}')$ *to* $\mathrm{pvd}(\mathbf{\Gamma})$. *Moreover, the following square commutes up to isomorphism*

$$\begin{array}{ccc} \mathcal{D}(\mathbf{\Pi}_2(F')) & \xrightarrow{G'^*} & \mathcal{D}(\mathbf{\Gamma}') \\ {\scriptstyle \mathrm{can}}\downarrow & & \downarrow{\scriptstyle \Psi_+} \\ \mathcal{D}(\mathbf{\Pi}_2(F)) & & \mathcal{D}(\mathbf{\Gamma}) \\ {\scriptstyle t_{S_v}^{-1}}\downarrow & & \| \\ \mathcal{D}(\mathbf{\Pi}_2(F)) & \xrightarrow[G^*]{} & \mathcal{D}(\mathbf{\Gamma}), \end{array}$$

where can *is the canonical functor induced by an identification between* $\mathbf{\Pi}_2(F')$ *and* $\mathbf{\Pi}_2(F)$ *and* $t_{S_v}^{-1}$ *is the inverse twist functor with respect to the* 2*-spherical object* S_v*, which gives rise to a triangle*

$$t_{S_v}^{-1}(X) \to X \to \mathrm{Hom}_k(\mathbf{R}\mathrm{Hom}_{\mathbf{\Pi}_2(F)}(X, S_v), S_v) \to \Sigma t_{S_v}^{-1}(X)$$

for each object X *of* $\mathcal{D}(\mathbf{\Pi}_2(F))$.

Proposition 3.31. *The triangle equivalence* $\Psi_+ \colon \mathrm{per}\mathbf{\Gamma}' \to \mathrm{per}\mathbf{\Gamma}$ *induces an equivalence* $\mathrm{pvd}_e(\mathbf{\Gamma}') \to \mathrm{pvd}_e(\mathbf{\Gamma})$. *Moreover we have a commutative diagram*

$$\begin{array}{ccccc} \mathrm{per}\mathbf{\Pi}_2(F') & \longrightarrow & \mathrm{per}\mathbf{\Gamma}' & \longrightarrow & \mathcal{C}(Q', F', W') \\ {\scriptstyle t_{S_v}^{-1}\circ\mathrm{can}}\downarrow & & \downarrow{\scriptstyle \Psi_+} & & \downarrow{\scriptstyle \Psi_+} \\ \mathrm{per}\mathbf{\Pi}_2(F) & \longrightarrow & \mathrm{per}\mathbf{\Gamma} & \longrightarrow & \mathcal{C}(Q, F, W). \end{array}$$

The functor $\Psi_+ \colon \mathcal{C}(Q', F', W') \to \mathcal{C}(Q, F, W)$ *does not take* $\mathcal{H}'$ *to* $\mathcal{H}$. *But it takes* $\mathcal{D}(Q', F', W')$ *to* $\mathcal{D}(Q, F, W)$.

Proof. The equivalence $\Psi_+ : \mathcal{D}(\mathbf{\Gamma}') \to \mathcal{D}(\mathbf{\Gamma})$ is the derived tensor product (see [42, Theorem 6.8])

$$? \overset{\mathbf{L}}{\otimes}_{\mathbf{\Gamma}'} U,$$

where the $\mathbf{\Gamma}'$-$\mathbf{\Gamma}$-bimodule U is given by

$$U = \bigoplus_{j \neq v} \mathbf{\Gamma}_j \oplus U_v$$

with $U_v = \mathrm{Cone}(\mathbf{\Gamma}_v \xrightarrow{(\alpha)} \bigoplus_{\alpha \in F_1 : s(\alpha) = v} \mathbf{\Gamma}_{t(\alpha)})$. It is clear that Ψ_+ induces an equivalence $\Psi_+ \colon \mathrm{per}\mathbf{\Gamma}' \to \mathrm{per}\mathbf{\Gamma}$. By using a similar computation in [29, Lemma 3.12], the functor Ψ_+ induces an equivalence $\Psi_+ \colon \mathrm{pvd}_e(\mathbf{\Gamma}') \to \mathrm{pvd}_e(\mathbf{\Gamma})$. Similarly, the equivalence $t_{S_v}^{-1} \circ \mathrm{can}$ induces an equivalence

$$t_{S_v}^{-1} \circ \mathrm{can} \colon \mathrm{per}\mathbf{\Pi}_2(F') \to \mathrm{per}\mathbf{\Pi}_2(F).$$

Thus, we have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{per}\mathbf{\Pi}_2(F') & \longrightarrow & \mathrm{per}\mathbf{\Gamma}' & \longrightarrow & \mathcal{C}(Q', F', W') \\ {\scriptstyle t_{S_v}^{-1}\circ\mathrm{can}}\downarrow & & \downarrow{\scriptstyle \Psi_+} & & \downarrow{\scriptstyle \Psi_+} \\ \mathrm{per}\mathbf{\Pi}_2(F) & \longrightarrow & \mathrm{per}\mathbf{\Gamma} & \longrightarrow & \mathcal{C}(Q, F, W). \end{array}$$

The object $\mathbf{\Gamma}'_v$ lies in $\mathcal{H}' \subseteq \mathcal{C}(Q', F', W')$. The object $\Psi_+(\mathbf{\Gamma}'_v)$ is equal to $\mathrm{Cone}(\mathbf{\Gamma}_v \xrightarrow{(\alpha)} \bigoplus_{\alpha \in F_1 : s(\alpha) = v} \mathbf{\Gamma}_{t(\alpha)})$. Since v is a frozen source, the induce map

$$(\alpha)^* \colon \mathrm{Hom}_{\mathcal{C}}(\bigoplus_{\alpha \in F_1 : s(\alpha) = v} \mathbf{\Gamma}_{t(\alpha)}, \mathbf{\Gamma}_v) \to \mathrm{Hom}_{\mathcal{C}}(\mathbf{\Gamma}_v, \mathbf{\Gamma}_v)$$

is not surjective. This shows that $\Psi_+(\mathbf{\Gamma}'_v)$ doesn't lie in ${}^{\perp}(\Sigma^{>0}\mathcal{P})$. Thus, the functor Ψ_+ does not take $\mathcal{H}'$ to $\mathcal{H}$.

It isn't hard to see that mutation at v doesn't change the unfrozen part of (Q', F', W'). After deleting the frozen part, the quiver with potential $(\overline{Q'}, \overline{W'})$ is equal to $(\overline{Q}, \overline{W})$. The functor $p^* : \mathcal{C}(Q, F, W) \to$

$\mathcal{C}(\overline{Q},\overline{W})$ is induced by the derived tensor product $?\overset{\mathbf{L}}{\otimes}_{\mathbf{\Gamma}}\overline{\mathbf{\Gamma}}\colon \mathrm{per}\mathbf{\Gamma}\to\overline{\mathbf{\Gamma}}$. And $p'^{*}:\mathcal{C}(Q',F',W')\to\mathcal{C}(\overline{Q},\overline{W})$ is induced by $?\overset{\mathbf{L}}{\otimes}_{\mathbf{\Gamma}'}\overline{\mathbf{\Gamma}}$.

Then we have

$$\begin{aligned}(?\overset{\mathbf{L}}{\otimes}_{\mathbf{\Gamma}'}U)\overset{\mathbf{L}}{\otimes}_{\mathbf{\Gamma}}\overline{\mathbf{\Gamma}} &\cong ?\overset{\mathbf{L}}{\otimes}_{\mathbf{\Gamma}'}(U\overset{\mathbf{L}}{\otimes}_{\mathbf{\Gamma}}\overline{\mathbf{\Gamma}})\\ &\cong ?\overset{\mathbf{L}}{\otimes}_{\mathbf{\Gamma}'}\overline{\mathbf{\Gamma}}.\end{aligned}$$

This computation gives us the following commutative square

$$\begin{array}{ccc}\mathcal{C}(Q',F',W') & \xrightarrow{\Psi_+} & \mathcal{C}(Q,F,W)\\ \big\downarrow{\scriptstyle p'^*} & & \big\downarrow{\scriptstyle p^*}\\ \mathcal{C}(\overline{Q'},\overline{W'}) & = & \mathcal{C}(\overline{Q},\overline{W}).\end{array}$$

For each object M of $\mathcal{D}(Q',F',W')$, we have $p^*(\Psi_+(M))\cong p'^*(M)\in\mathcal{D}(\overline{Q},\overline{W})$. Thus, Ψ_+ takes $\mathcal{D}(Q',F',W')$ to $\mathcal{D}(Q,F,W)$.

$\surd$

Remark 3.32. Dually, let v be a frozen sink. One can mutate (Q,F,W) at v. Write $(Q',F',W')=\mu_v(Q,F,W)$. Let $\mathbf{\Gamma}=\mathbf{\Gamma}_{rel}(Q,F,W)$ and $\mathbf{\Gamma}'=\mathbf{\Gamma}_{rel}(Q',F',W')$ be the complete relative Ginzburg dg algebras associated to (Q,F,W) and (Q',F',W') respectively. We also have a commutative diagram (see [42, Theorem 6.9])

$$\begin{array}{ccccc}\mathrm{per}\mathbf{\Pi}_2(F') & \longrightarrow & \mathrm{per}\mathbf{\Gamma}' & \longrightarrow & \mathcal{C}(Q',F',W')\\ {\scriptstyle t_{S_v}\circ\mathrm{can}}\big\downarrow & & \big\downarrow{\scriptstyle\Psi_-} & & \big\downarrow{\scriptstyle\Psi_-}\\ \mathrm{per}\mathbf{\Pi}_2(F) & \longrightarrow & \mathrm{per}\mathbf{\Gamma} & \longrightarrow & \mathcal{C}(Q,F,W).\end{array}$$

The functor $\Psi_-\colon\mathcal{C}(Q',F',W')\to\mathcal{C}(Q,F,W)$ does not take $\mathcal{H}'$ to $\mathcal{H}$. But it takes $\mathcal{D}(Q',F',W')$ to $\mathcal{D}(Q,F,W)$.

3.6. **Iyama–Kalck–Wemyss–Yang's theorem without the Noetherian hypothesis.** Let $\mathcal{E}$ be a Frobenius category and $\mathcal{P}\subseteq\mathcal{E}$ the subcategory of projective-injective objects. For each $P\in\mathcal{P}$, we put $P^\wedge=\mathcal{P}(?,P)\colon\mathcal{P}^{op}\to\mathrm{Mod}k$. We denote by $\mathrm{Mod}\mathcal{P}$ the category of right $\mathcal{P}$-modules. Then we have a functor

$$H\colon\mathcal{E}\to\mathrm{Mod}\mathcal{P}$$

which maps X to $\mathcal{E}(?,X)|_{\mathcal{P}}$.

Lemma 3.33. 1) *For any object X in $\mathcal{E}$, the right $\mathcal{P}$-module $H(X)$ is pseudo-coherent.*
2) *For any objects $X\in\mathcal{E}$ and $P\in\mathcal{P}$, we have $\mathrm{Ext}^i_{\mathcal{P}}(H(X),P^\wedge)=0$, $\forall i>0$.*

Proof. 1) Choose conflations

$$\Omega^iX\rightarrowtail P_{i-1}\twoheadrightarrow\Omega^{i-1}X\quad\text{for } i\geqslant 1 \text{ with proj-inj } P_{i-1}.$$

Their images under $\mathrm{Hom}_{\mathcal{E}}(P,?)$ are exact, $\forall P\in\mathcal{P}$. So the following exact sequence

$$\to H(P_i)\to\cdots\to H(P_1)\to H(P_0)\to H(X)\to 0$$

is a resolution for $H(X)\in\mathrm{Mod}\mathcal{P}$.

2) Choose a resolution $\ldots\to P_2\to P_1\to P_0\to X\to 0$ as in 1). Its image under H is a resolution of $H(X)$ in $\mathrm{Mod}\mathcal{P}$. Since P is also injective, the homologies of

$$\begin{array}{ccccccc}\mathrm{Hom}(HP_0,HP) & \longrightarrow & \mathrm{Hom}(HP_1,HP) & \longrightarrow & \mathrm{Hom}(HP_2,HP) & \longrightarrow & \cdots\\ \big\downarrow\simeq & & \big\downarrow\simeq & & \big\downarrow\simeq & & \\ \mathrm{Hom}(P_0,P) & & \mathrm{Hom}(P_1,P) & & \mathrm{Hom}(P_2,P) & & \end{array}$$

vanish in all degrees $i>0$. $\surd$

Let $\mathcal{M} \subseteq \mathcal{E}$ be a full additive subcategory stable under direct factors such that $\mathcal{P} \subsetneqq \mathcal{M}$ and gldim$\mathcal{M} \leqslant n$. An object X of Mod$\mathcal{P}$ is called *$\mathcal{M}$-pseudo-coherent* if X is pseudo-coherent and $\widetilde{X} = X \otimes_{\mathcal{P}} \mathcal{M}$ admits a finitely generated projective resolution.

We say that $\mathcal{P}$ is fp$_\infty$-*Gorenstein of dimension at most n with respect to* $\mathcal{M}$ if $\mathrm{Ext}^i_{\mathcal{P}}(X, P^\wedge) = 0$ for all $i \geqslant n+1$ and all $\mathcal{M}$-pseudo-coherent $X \in \mathrm{Mod}\mathcal{P}$ and similarly for $\mathcal{P}^{op}$. Let $\mathrm{fp}_\infty\mathcal{P}$ be the full subcategory of Mod$\mathcal{P}$ whose objects are the pseudo-coherent $\mathcal{P}$-modules. If $\mathcal{P}$ is fp$_\infty$-Gorenstein of dimension at most n with respect to $\mathcal{M}$, we denote by

$$\mathrm{grp}_\infty(\mathcal{P}) = \{X \in \mathrm{fp}_\infty\mathcal{P} \,|\, \mathrm{Ext}^i_{\mathcal{P}}(X, P^\wedge) = 0 \text{ for any } i > 0 \text{ and } P \in \mathcal{P}\},$$

the category of fp$_\infty$-*Gorenstein projective modules X over $\mathcal{P}$* and by $\mathrm{grp}^{\leqslant n}_\infty(\mathcal{P})$ the full subcategory of $\mathrm{grp}_\infty(\mathcal{P})$ whose objects are the $\mathcal{M}$-pseudo-coherent $\mathcal{P}$-modules.

Theorem 3.34. *Let $\mathcal{E}$ be an idempotent complete Frobenius category with* $\mathrm{proj}\,\mathcal{E} = \mathcal{P}$*. Assume that there exists a full additive subcategory $\mathcal{M} \subseteq \mathcal{E}$ which is stable under direct factors such that $\mathcal{P} \subsetneqq \mathcal{M}$ and* gldim$\mathcal{M} \leqslant n$*. The category $\mathcal{P}$ is* fp$_\infty$*-Gorenstein of dimension at most n with respect to $\mathcal{M}$ and the functor $H\colon \mathcal{E} \to \mathrm{Mod}\mathcal{P}$ induces an equivalence*

$$\mathcal{E} \xrightarrow{\sim} \mathrm{grp}^{\leqslant n}_\infty(\mathcal{P}).$$

Proof. Let L be a $\mathcal{M}$-pseudo-coherent $\mathcal{P}$-module. Since the global dimension of $\mathcal{M}$ is at most n, there exists a a finitely generated projective resolution in Mod$\mathcal{M}$

$$0 \to L_n \to L_{n-1} \to \cdots \to L_0 \to \widetilde{L} \to 0.$$

We restrict this resolution to $\mathcal{P}$ and get a finitely generated projective resolution in Mod$\mathcal{P}$ of length n. This shows that $\mathrm{Ext}^i_{\mathcal{P}}(L, P^\wedge) = 0$ for all $i \geqslant n+1$. The dual argument shows the situation in Mod$\mathcal{P}^{op}$.

By Lemma 3.33 and the argument above, the subcategory $\mathcal{P}$ is fp$_\infty$-Gorenstein of dimension at most n. It is obvious that H is fully faithful. Let X be an object of $\mathrm{grp}^{\leqslant n}_\infty\mathcal{P}$. Choose a projective resolution with finitely generated terms

$$0 \to M_n^\wedge \to M_{n-1}^\wedge \to \cdots \to M_0^\wedge \to \widetilde{X} \to 0.$$

Then we have a complex

$$0 \to M_n \xrightarrow{f_n} \cdots \xrightarrow{f_0} M_0 \tag{1}$$

in $\mathcal{E}$ with $M_i \in \mathcal{M}$ such that

$$0 \to HM_n \xrightarrow{Hf_n} HM_{n-1} \xrightarrow{Hf_{n-1}} \cdots \to HM_0 \to X \to 0$$

is exact.

For any object P of $\mathcal{P}$, we have a commutative diagram

$$\begin{array}{ccccccccc}
(M_0, P) & \longrightarrow & (M_1, P) & \longrightarrow & \cdots & \longrightarrow & (M_n, P) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & & & \downarrow \simeq & & \\
(H(M_0), H(P)) & \longrightarrow & (H(M_1), H(P)) & \longrightarrow & \cdots & \longrightarrow & (H(M_n), H(P)) & \longrightarrow & 0,
\end{array}$$

where the lower sequence is exact since $X \in \mathrm{grp}^{\leqslant n}_\infty\mathcal{P}$. Thus the upper sequence is also exact.

Applying [26, Lemma 2.6] repeatedly to the complex (1), we get an $\mathcal{E}$-acyclic complex in $\mathcal{E}$ (i.e. obtained by splicing conflation of $\mathcal{E}$)

$$0 \to M_n \xrightarrow{f_n} \cdots \xrightarrow{f_0} M_0 \to X' \to 0$$

and that $H(X') \cong X$. It follows that H induces an equivalence

$$\mathcal{E} \xrightarrow{\sim} \mathrm{grp}^{\leqslant n}_\infty(\mathcal{P}).$$

$\surd$

4. Relationship with Plamondon's category $\mathcal{D}(\overline{Q},\overline{W})$

Let (Q,F,W) be an ice quiver with potential. From now on, we make the following technical assumption:

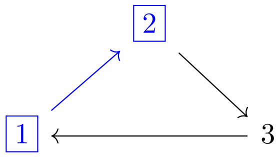
Assumption 1. *The additive subcategory* $\mathcal{P}=\mathrm{add}(e\mathbf{\Gamma})$ *is functorially finite in* $\mathrm{add}(\mathbf{\Gamma})$.

Remark 4.1. The above assumption is equivalent to the following conditions:

- $\mathrm{Hom}_{\mathrm{per}\mathbf{\Gamma}}(e\mathbf{\Gamma},e_i\mathbf{\Gamma})$ is a finitely generated right $\mathrm{End}_{\mathrm{per}\mathbf{\Gamma}}(e\mathbf{\Gamma})$-module, and
- $\mathrm{Hom}_{\mathrm{per}\mathbf{\Gamma}}(e_i\mathbf{\Gamma},e\mathbf{\Gamma})$ is a finitely generated left $\mathrm{End}_{\mathrm{per}\mathbf{\Gamma}}(e\mathbf{\Gamma})$-module for all vertices $i\in Q_0\setminus F_0$.

Remark 4.2. Suppose $\mathrm{add}(\mathbf{\Gamma})=\mathcal{T}$ is a cluster-titling subcategory in a stably 2-Calabi–Yau Frobenius category $\mathcal{E}$ whose subcategory of pro-injectives is $\mathcal{P}\subseteq\mathcal{T}$. Then the above assumption clearly holds. One of our aims is to construct such a Frobenius category $\mathcal{E}$ when $(\overline{Q},\overline{W})$ is Jacobi-finite.

Example 4.3. The assumption does not hold for the example below.



$W=0.$

4.1. **Plamondon's category $\mathcal{D}(Q,W)$.** In [36], Plamondon generalized Amiot's ([1]) construction of generalized cluster categories to the case of any quiver with potential. Let (Q,W) be any quiver with potential and $\mathbf{\Gamma}(Q,W)$ the associated Ginzburg algebra. The (generalized) cluster category (see [1]) of (Q,W) is defined as the idempotent completion of the triangulated quotient

$$\mathcal{C}(Q,W)=\mathrm{per}\mathbf{\Gamma}(Q,W)/\mathrm{pvd}(\mathbf{\Gamma}(Q,W)).$$

Definition 4.4. [36, Definition 3.9] The subcategory $\mathcal{D}(Q,W)$ is the full subcategory of $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}(Q,W)\cap\mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}(Q,W)$ whose objects are those X such that $\mathrm{Ext}^1_{\mathcal{C}(Q,W)}(\mathbf{\Gamma}(Q,W),X)$ is finite-dimensional.

In our situation, let $\overline{Q}$ be the quiver obtained from Q by deleting all vertices in F and all arrows incident with vertices in F. Let $\overline{W}$ be the potential on $\overline{Q}$ obtained by deleting all cycles passing through vertices of F in W. Let $\overline{\mathbf{\Gamma}}$ be the Ginzburg algebra of $(\overline{Q},\overline{W})$. Then we have

$$\mathcal{D}(\overline{Q},\overline{W})=\{X\in\mathrm{pr}_{\mathcal{C}}\overline{\mathbf{\Gamma}}\cap\mathrm{copr}_{\mathcal{C}}\overline{\mathbf{\Gamma}}\mid\mathrm{Ext}^1_{\mathcal{C}(\overline{Q},\overline{W})}(\overline{\mathbf{\Gamma}},X)\text{ is finite-dimensional}\}.$$

Proposition 4.5. *The category* $\mathcal{D}(\overline{Q},\overline{W})$ *is an extension closed subcategory of* $\mathcal{C}(\overline{Q},\overline{W})$*. Thus, it becomes an extriangulated category in the sense of Nakaoka-Palu* [32].

Proof. It follows from Proposition 3.26 for the case when F is empty. $\surd$

Recall that the functor $p^*\colon\mathrm{per}\mathbf{\Gamma}\to\mathrm{per}\overline{\mathbf{\Gamma}}$ is the extension of scalars along the dg quotient functor $p\colon\mathbf{\Gamma}\to\overline{\mathbf{\Gamma}}=\mathbf{\Gamma}(\overline{Q},\overline{W})$.

Proposition 4.6. *The functor* $p^*\colon\mathrm{per}\mathbf{\Gamma}\to\mathrm{per}\overline{\mathbf{\Gamma}}$ *induces an equivalence of k-linear categories*

$$\mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma}/[\mathcal{P}]\to\mathrm{pr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}},$$

where $\mathcal{D}=\mathcal{D}(\mathbf{\Gamma}(Q,F,W))$ *and* $\overline{\mathcal{D}}=\mathcal{D}(\mathbf{\Gamma}(\overline{Q},\overline{W}))$.

Proof. Let X be an object in $\mathrm{pr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}}\subseteq\mathrm{per}\overline{\mathbf{\Gamma}}$. By definition, X fits into the following triangle in $\mathrm{per}\overline{\mathbf{\Gamma}}$

$$P_1\xrightarrow{\alpha}P_0\to X\to\Sigma P_1$$

with $P_0,P_1\in\mathrm{add}\overline{\mathbf{\Gamma}}$. Since we have an equivalence of additive categories

$$p^*\colon\mathrm{add}\mathbf{\Gamma}/[\mathcal{P}]\xrightarrow{\sim}\mathrm{add}\overline{\mathbf{\Gamma}},$$

there exists a morphism $\beta'\colon M_1\to M_0'$ in $\mathrm{add}\mathbf{\Gamma}$ such that $p^*(M_1)=P_1$, $p^*(M_0')=P_0$ and $p^*(\beta')=\alpha$. Let $\gamma\colon M_1\to Q_0$ be a left $\mathcal{P}$-approximation of M_1. We define

$$(M_1\xrightarrow{\beta}M_0):=(M_1\xrightarrow{[\beta',\gamma]^t}M_0'\oplus Q_0).$$

Then we still have $p^*(\beta) = \alpha$. Moreover, $\mathrm{Hom}(\beta, I)$ is surjective for any object I in $\mathcal{P}$. Thus, the object $U := \mathrm{Cone}(\beta)$ is in $\mathcal{F}^{rel}$ and $p^*(U) \cong X$. Hence, the functor $p^*\colon \mathcal{F}^{rel} \to \mathcal{F}$ is dense.

Next we show that $p^*\colon \mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma} \to \mathrm{pr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}}$ is full. Let $f\colon X \to Y$ be a morphism in $\mathrm{pr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}}$. We have the following diagram in $\mathrm{per}\overline{\mathbf{\Gamma}}$

$$\begin{array}{ccccccc} P_1 & \xrightarrow{a} & P_0 & \xrightarrow{b} & X & \longrightarrow & \Sigma P_1 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle f} & & \\ Q_1 & \xrightarrow[c]{} & Q_0 & \xrightarrow[d]{} & Y & \longrightarrow & \Sigma Q_1 \end{array}$$

with $P_1, P_0, Q_1, Q_0 \in \mathrm{add}\overline{\mathbf{\Gamma}}$.

Since $\mathrm{Hom}_{\mathrm{per}\overline{\mathbf{\Gamma}}}(P_0, \Sigma Q_1) = 0$, there exists a morphism $e\colon P_0 \to Q_0$ such that $fb = de$. Then there exists a morphism $g\colon P_1 \to Q_1$ such that $ea = cg$. We lift the above commutative diagram to a commutative diagram in $\mathrm{per}\mathbf{\Gamma}$. Then we find a morphism β in $\mathcal{F}^{rel}$ such that $p^*(\beta) = f$.

It remains to show that the map $\mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma}/[\mathcal{P}](M,N) \to \mathrm{pr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}}(p^*(M), p^*(N))$ is injective for any $M, N \in \mathcal{F}^{rel}$. Assume that a morphism $\alpha \in \mathrm{pr}^F_{\mathcal{D}}(M,N)$ is zero in $\mathrm{per}\mathbf{\Gamma}/\mathrm{thick}(\mathcal{P}) \xrightarrow{\sim} \mathrm{per}\overline{\mathbf{\Gamma}}$. Then it factors through $\mathrm{thick}(\mathcal{P}) \cong \mathrm{per}(e\mathbf{\Gamma}e)$, that is, there exist $T \in \mathrm{thick}(\mathcal{P})$, $u \in \mathrm{Hom}_{\mathrm{per}\mathbf{\Gamma}}(M,T)$, and $v \in \mathrm{Hom}_{\mathrm{per}\mathbf{\Gamma}}(T,N)$ such that $\alpha = vu$.

Since $e\mathbf{\Gamma}$ is a silting object of $\mathrm{thick}(\mathcal{P})$, the pair $(\mathrm{thick}(\mathcal{P})_{\geqslant 0}, \mathrm{thick}(\mathcal{P})_{\leqslant 0})$ is a bounded co-t-structure (see [22, Proposition 2.8]) on $\mathrm{thick}(\mathcal{P})$, where

$$\mathrm{thick}(\mathcal{P})_{\geqslant l} = \mathrm{thick}(\mathcal{P})_{>l-1} := \bigcup_{n\geqslant 0} \Sigma^{-l-n}\mathcal{P} * \cdots * \Sigma^{-l-1}\mathcal{P} * \Sigma^{-l}\mathcal{P}$$

and

$$\mathrm{thick}(\mathcal{P})_{\leqslant l} = \mathrm{thick}(\mathcal{P})_{<l+1} := \bigcup_{n\geqslant 0} \Sigma^{-l}\mathcal{P} * \Sigma^{-l+1}\mathcal{P} * \cdots * \Sigma^{-l+n}\mathcal{P}.$$

Take a triangle

$$T_{>0} \xrightarrow{b} T \xrightarrow{c} T_{\leqslant 0} \to \Sigma T_{>0}$$

with $T_{>0} \in \mathrm{thick}(\mathcal{P})_{>0}$ and $T_{\leqslant 0} \in \mathrm{thick}(\mathcal{P})_{\leqslant 0}$. Since $\mathrm{Hom}_{\mathrm{per}\mathbf{\Gamma}}(T_{>0}, N) = 0$, we have $vb = 0$. Thus, there exists $d \in \mathrm{Hom}_{\mathrm{per}\mathbf{\Gamma}}(T_{\leqslant 0}, N)$ such that $v = dc$.

$$\begin{array}{ccccc} T_{>0} & \xrightarrow{b} & T & \xrightarrow{c} & T_{\leqslant 0} \\ & {\scriptstyle u}\nearrow & & \searrow{\scriptstyle v} & \downarrow{\scriptstyle d} \\ M & & \xrightarrow[\alpha]{} & & N \end{array}$$

Since $T_{\leqslant 0} \in \mathrm{thick}(\mathcal{P})_{\leqslant 0}$, we have triangle

$$P \to T_{\leqslant 0} \xrightarrow{e} T_{<0} \to \Sigma P$$

with $P \in \mathcal{P}$ and $T_{<0} \in \mathrm{thick}(\mathcal{P})_{<0}$. Then we have $ecu = 0$ by $M \in \mathrm{add}\mathbf{\Gamma}$ and $T_{<0} \in \mathrm{thick}(\mathcal{P})_{<0}$. Thus, cu factors through $\mathcal{P}$ and $\alpha = dcu = 0$ in $\mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma}/[\mathcal{P}]$.

$\surd$

Proposition 4.7. *The functor* $p^*\colon \mathrm{per}\mathbf{\Gamma} \to \mathrm{per}\overline{\mathbf{\Gamma}}$ *induces an equivalence of* k*-linear categories*

$$p^*\colon \frac{\mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma} \cap \mathrm{copr}^F_{\mathcal{D}}\mathbf{\Gamma}}{[\mathcal{P}]} \xrightarrow{\sim} \mathrm{pr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}} \cap \mathrm{copr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}}.$$

Proof. By Proposition 4.6, it is enough to show that the induced functor

$$p^*\colon \mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma} \cap \mathrm{copr}^F_{\mathcal{D}}\mathbf{\Gamma} \to \mathrm{pr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}} \cap \mathrm{copr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}}$$

is dense. Let M be an object of $\mathrm{pr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}} \cap \mathrm{copr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}}$. By Proposition 4.6, there exists an object $X \in \mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma}$ such that $p^*(X) \cong M$ in $\mathrm{per}\overline{\mathbf{\Gamma}}$.

Since M is also in $\mathrm{copr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}}$, we see that $\Sigma M \in \mathrm{pr}_{\overline{\mathcal{D}}}(\overline{\mathbf{\Gamma}})$. Again by Proposition 4.6, there exists an object $Y \in \mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma}$ such that $p^*(Y) \cong \Sigma M$ in $\mathrm{per}\overline{\mathbf{\Gamma}}$.

Thus, we have $p^*(X) \cong p^*(\Sigma^{-1}Y) \cong M$. Notice that $\mathrm{pr}^F_{\mathcal{D}}\mathbf{\Gamma}$ and $\mathrm{copr}^F_{\mathcal{D}}\mathbf{\Gamma}$ are subcategories of $\mathcal{Z} = {}^{\perp}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp}$. By Proposition 3.8, there exist objects P_0, P_1 in $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$ such that

$$X \oplus P_0 \cong \Sigma^{-1}Y \oplus P_1.$$

Then the object $X\oplus P_0$ belongs to $\mathrm{pr}_{\mathcal{D}}^F\mathbf{\Gamma}\cap\mathrm{copr}_{\mathcal{D}}^F\mathbf{\Gamma}$ and $p^*(X\oplus P_0)\cong M$. This shows that the functor

$$p^*\colon \mathrm{pr}_{\mathcal{D}}^F\mathbf{\Gamma}\cap\mathrm{copr}_{\mathcal{D}}^F\mathbf{\Gamma}\to\mathrm{pr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}}\cap\mathrm{copr}_{\overline{\mathcal{D}}}\overline{\mathbf{\Gamma}}$$

is dense. $\surd$

Similarly, we have the following propositions.

Proposition 4.8. *The functor $p^*\colon\mathcal{C}(Q,F,W)\to\mathcal{C}(\overline{Q},\overline{W})$ induces the following equivalences of k-linear categories*

$$p^*\colon \mathrm{pr}_{\mathcal{C}}^F\mathbf{\Gamma}/[\mathcal{P}]\to\mathrm{pr}_{\overline{\mathcal{C}}}\overline{\mathbf{\Gamma}}\,,\quad p^*\colon\frac{\mathrm{pr}_{\mathcal{C}}^F\mathbf{\Gamma}\cap\mathrm{copr}_{\mathcal{C}}^F\mathbf{\Gamma}}{[\mathcal{P}]}\xrightarrow{\sim}\mathrm{pr}_{\overline{\mathcal{C}}}\overline{\mathbf{\Gamma}}\cap\mathrm{copr}_{\overline{\mathcal{C}}}\overline{\mathbf{\Gamma}},$$

where $\mathcal{C}=\mathcal{C}(Q,F,W)$ and $\overline{\mathcal{C}}=\mathcal{C}(\overline{Q},\overline{W})$.

Corollary 4.9. *We have an equivalence of k-linear categories*

$$p^*\colon\mathcal{H}/[\mathcal{P}]\xrightarrow{\sim}\mathcal{D}(\overline{Q},\overline{W}).$$

Proof. It follows from the Lemma below and Proposition 4.8. $\surd$

Lemma 4.10. *Let Y be an object of $\mathrm{pr}_{\mathcal{C}}^F\mathbf{\Gamma}\cap\mathrm{copr}_{\mathcal{C}}^F\mathbf{\Gamma}$. For any object X in*

$$\mathcal{Y}={}^{\perp}(\Sigma^{>0}\mathcal{P})\cap(\Sigma^{<0}\mathcal{P})^{\perp}\subseteq\mathcal{C}(Q,F,W),$$

we have

$$\mathrm{Hom}_{\mathcal{C}}(X,\Sigma Y)\simeq\mathrm{Hom}_{\mathcal{C}(\overline{Q},\overline{W})}(p^*(X),\Sigma p^*(Y)).$$

Proof. The category $\mathcal{P}=e\mathbf{\Gamma}$ is functorially finite in $\mathrm{pr}_{\mathcal{C}}^F\mathbf{\Gamma}\cap\mathrm{copr}_{\mathcal{C}}^F\mathbf{\Gamma}$. We take a left $\mathcal{P}$-approximation $f\colon Y\to P$ of Y. Then we have a triangle in $\mathcal{C}$

$$Y\xrightarrow{f}P\to T\to\Sigma Y,$$

with $p^*(T)\cong\Sigma Y$. Since f is a left $\mathcal{P}$-approximation, we see that T is in $\mathcal{Y}$. Applying the functor $\mathrm{Hom}_{\mathcal{C}}(X,?)$ to the above triangle, we get a long exact sequence

$$\to\mathrm{Hom}_{\mathcal{C}}(X,P)\xrightarrow{\Phi}\mathrm{Hom}_{\mathcal{C}}(X,T)\to\mathrm{Hom}_{\mathcal{C}}(X,\Sigma Y)\to 0.$$

Thus, by Proposition 3.9, we have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(X,\Sigma Y)&\simeq\mathrm{Hom}_{\mathcal{C}}(X,T)/\mathrm{Im}(\Phi)\\&\simeq\mathcal{Y}/[\mathcal{P}](X,T)\\&\simeq\mathrm{Hom}_{\mathcal{C}(\overline{Q},\overline{W})}(p^*(X),p^*(T))\\&\simeq\mathrm{Hom}_{\mathcal{C}(\overline{Q},\overline{W})}(p^*(X),\Sigma p^*(Y)).\end{aligned}$$

$\surd$

Proposition 4.11. *We have bifunctorial isomorphisms $D\mathrm{Ext}^1_{\mathcal{H}}(X,Y)\simeq\mathrm{Ext}^1_{\mathcal{H}}(Y,X)$ for all $X,Y\in\mathcal{H}$.*

Proof. It follows from [36, Proposition 2.16] and Lemma 4.10. $\surd$

4.2. **The case where $(\overline{Q},\overline{W})$ is Jacobi-finite.** Let (Q,F,W) be an ice quiver with potential. In this subsection, we assume that $(\overline{Q},\overline{W})$ is Jacobi-finite.

By Corollary 3.7, the relative t-structure on $\mathcal{D}(\mathbf{\Gamma})$ restricts to the perfect derived category $\mathrm{per}\mathbf{\Gamma}$. For any object X of $\mathrm{per}\mathbf{\Gamma}$, the canonical triangle corresponding to the relative t-structure is given by

$$\tau^{rel}_{\leqslant n}X\to X\to\tau^{rel}_{>n}\to\Sigma\tau^{rel}_{\leqslant n}X$$

such that $\tau^{rel}_{\leqslant n}X\in\mathrm{per}\mathbf{\Gamma}$ belongs to $\mathcal{D}^{\leqslant n}_{rel}$ and $\tau^{rel}_{>n}(X)=p_*(\tau_{>n}(p^*X))$ belongs to $\mathcal{D}^{\geqslant n+1}_{rel}$.

Recall that the fundamental domain $\mathcal{F}$ (see [1, Lemma 2.10]) of $\mathrm{per}\overline{\mathbf{\Gamma}}$ is defined as

$$\mathcal{F}=\mathrm{add}\overline{\mathbf{\Gamma}}*\Sigma\mathrm{add}\overline{\mathbf{\Gamma}}=\mathrm{pr}\overline{\mathbf{\Gamma}}\subseteq\mathrm{per}\overline{\mathbf{\Gamma}}.$$

Proposition 4.12. [1, Proposition 2.9] *The projection functor* $\pi\colon \mathrm{per}\overline{\mathbf{\Gamma}} \to \mathcal{C}(\overline{Q},\overline{W})$ *induces a* k*-linear equivalence between* $\mathcal{F}$ *and* $\mathcal{C}(\overline{Q},\overline{W})$*.*

Let r be a non-negative integer. Let $\mathcal{H}\langle r\rangle$ be the full subcategory of $\mathcal{Y} = {}^{\perp}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp} \subseteq \mathcal{C}$ whose objects are those X such that $p^*(X)$ is isomorphic to $\Sigma^r Y$ in $\mathrm{per}\overline{\mathbf{\Gamma}}$ for some object $Y \in \mathcal{F}$. By Proposition 3.9 and Corollary 4.9, we see that $\mathcal{H} = \mathcal{H}\langle 0\rangle$.

Proposition 4.13. *Let* r *be a positive integer and* X *an object of* $\mathcal{H}\langle r\rangle$*. Then there exists an object* Y *in* $\mathcal{H}\langle r-1\rangle$ *such that* $\tau^{rel}_{\leqslant -r}Y \cong X$ *in* $\mathcal{C}$*.*

Proof. By definition, there exist an object Y of $\mathcal{F}$ such that $p^*(X) \cong \Sigma^r Y$ in $\mathrm{per}\overline{\mathbf{\Gamma}}$. We set $Z = \Sigma^{1-r}p^*(X) = \Sigma Y$. By [1, Proposition 2.9], there exists a triangle in $\mathrm{per}\overline{\mathbf{\Gamma}}$

$$\Sigma P_1 \to \Sigma P_0 \to Z \to \Sigma^2 P_1$$

with P_0 and P_1 in $\mathrm{add}\overline{\mathbf{\Gamma}}$. Denote by ν the Nakayama functor on the projectives of $\mathrm{mod}H^0(\overline{\mathbf{\Gamma}})$. Let M' be the kernel of the morphism $\nu H^0(P_1) \to \nu H^0(P_0)$. We define M to be $\Sigma^{r-1}p_*(M')$. It is easy to see that M lies in $\mathrm{pvd}_e(\mathbf{\Gamma}) \subseteq \mathcal{Y} = {}^{\perp}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp} \subseteq \mathcal{C}$.

By [1, Lemma 2.11], we have an isomorphism of functors

$$\mathrm{Hom}_{\mathrm{per}\overline{\mathbf{\Gamma}}}(?, \Sigma Z)|_{\mathrm{mod}H^0(\overline{\mathbf{\Gamma}})} \simeq \mathrm{Hom}_{\mathrm{mod}H^0(\overline{\mathbf{\Gamma}})}(?, M').$$

By the relative 3-Calabi–Yau property (see Proposition 2.12), we have an isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathrm{per}\mathbf{\Gamma}}(M,\Sigma X) &\simeq D\mathrm{Hom}_{\mathrm{per}\mathbf{\Gamma}}(\Sigma X, \Sigma^{r+2}p_*(M'))\\ &\simeq D\mathrm{Hom}_{\mathrm{per}\overline{\mathbf{\Gamma}}}(\Sigma p^* X, \Sigma^{r+2}M')\\ &\simeq \mathrm{Hom}_{\mathrm{per}\overline{\mathbf{\Gamma}}}(\Sigma^{r+2}M', \Sigma^4 p^*(X))\\ &= \mathrm{Hom}_{\mathrm{per}\overline{\mathbf{\Gamma}}}(p^*(M), \Sigma p^*(X)).\end{aligned}$$

Therefore we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{per}\mathbf{\Gamma}}(M,\Sigma X) &\simeq \mathrm{Hom}_{\mathrm{per}\overline{\mathbf{\Gamma}}}(p^*(M), \Sigma p^*(X))\\ &\simeq \mathrm{Hom}_{\mathrm{per}\overline{\mathbf{\Gamma}}}(\Sigma^{r-1}M', \Sigma p^*(X))\\ &\simeq \mathrm{Hom}_{\mathrm{per}\overline{\mathbf{\Gamma}}}(M', \Sigma^{2-r} p^*(X))\\ &\simeq \mathrm{Hom}_{\mathrm{per}\overline{\mathbf{\Gamma}}}(M', \Sigma Z).\end{aligned}$$

Let ϵ be the preimage of the identity map on M' under the isomorphism

$$\mathrm{Hom}_{\mathrm{per}\mathbf{\Gamma}}(M,\Sigma X) \simeq \mathrm{Hom}_{\mathrm{per}\overline{\mathbf{\Gamma}}}(M', \Sigma Z)|_{\mathrm{mod}H^0(\overline{\mathbf{\Gamma}})} \simeq \mathrm{Hom}_{\mathrm{mod}H^0(\overline{\mathbf{\Gamma}})}(M', M').$$

We form the corresponding triangle in $\mathrm{per}\mathbf{\Gamma}$

$$X \to L \to M \xrightarrow{\epsilon} \Sigma X.$$

Similarly, let ϵ' be the preimage of the identity map on M' under the isomorphism

$$\mathrm{Hom}_{\mathrm{per}\overline{\mathbf{\Gamma}}}(M', \Sigma^{2-r}p^*(X)) \simeq \mathrm{Hom}_{\mathrm{mod}H^0(\overline{\mathbf{\Gamma}})}(M', M').$$

Then we form the corresponding triangle in $\mathrm{per}\overline{\mathbf{\Gamma}}$

$$\Sigma^{1-r}p^*(X) \to L' \to M' \to \Sigma^{2-r}p^*(X).$$

We see that $p^*(L)$ is isomorphic to $\Sigma^{r-1}L'$.

By [1, Lemma 2.11], the object L' is in the fundamental domain $\mathcal{F} \subseteq \mathrm{per}\overline{\mathbf{\Gamma}}$. So L is an object of $\mathcal{H}\langle r-1\rangle$. Next, we will show that $\tau^{rel}_{\leqslant -r}L$ is isomorphic to X.

Since $X \in \mathcal{D}(\mathbf{\Gamma})^{\leqslant -r}_{rel}$ and $\tau^{rel}_{>-r}L \in \mathcal{D}(\mathbf{\Gamma})^{>-r}_{rel}$, the space $\mathrm{Hom}_{\mathcal{D}(\mathbf{\Gamma})}(X, \tau^{rel}_{>-r}L)$ vanishes. Hence, we obtain a commutative diagram of triangles

$$\begin{array}{ccccccc}\tau^{rel}_{\leqslant -r}L & \longrightarrow & L & \longrightarrow & \tau^{rel}_{>-r}L & \longrightarrow & \Sigma\tau^{rel}_{\leqslant -r}L\\ {\scriptstyle\delta_2}\uparrow & & \Vert & & {\scriptstyle\delta_1}\uparrow & & \uparrow\\ X & \longrightarrow & L & \longrightarrow & M & \longrightarrow & \Sigma X.\end{array}$$

By the octahedral axiom, we have the following commutative diagram

$$\begin{array}{ccccccc}
L & \longrightarrow & M & \longrightarrow & \Sigma X & \longrightarrow & \Sigma L \\
\| & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2[1]} & & \| \\
L & \longrightarrow & \tau^{rel}_{>-l}(L) & \longrightarrow & \Sigma\tau^{rel}_{\leqslant -l}L & \longrightarrow & \Sigma L \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Cone}(\delta_1) & \dashrightarrow & \Sigma\mathrm{Cone}(\delta_2) & \longrightarrow & \Sigma M \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma M & \longrightarrow & \Sigma^2 X & &
\end{array}$$

and the object $\mathrm{Cone}(\delta_1)$ is isomorphic to $\Sigma\mathrm{Cone}(\delta_2)$ in $\mathrm{per}\mathbf{\Gamma}$.

Since $\tau^{rel}_{\leqslant -r}L \in \mathcal{D}(\mathbf{\Gamma})^{\leqslant -r}_{rel}$ and $X \in \mathcal{D}(\mathbf{\Gamma})^{\leqslant -r}_{rel}$, the object $\mathrm{Cone}(\delta_2)$ is also in $\mathcal{D}(\mathbf{\Gamma})^{\leqslant -r}_{rel}$. Thus $\Sigma\mathrm{Cone}(\delta_2)$ is in $\mathcal{D}(\mathbf{\Gamma})^{\leqslant -r-1}_{rel}$. On the other hand, M and $\tau^{rel}_{>-r}(L)$ are in $\mathcal{D}^{\geqslant -r+1}_{rel}(\mathbf{\Gamma})$. Thus $\mathrm{Cone}(\delta_1)$ is in $\mathcal{D}^{\geqslant -r}_{rel}(\mathbf{\Gamma})$. Hence we can conclude that $\mathrm{Cone}(\delta_1) \cong \Sigma\mathrm{Cone}(\delta_2)$ is zero.

Thus, the relative truncation $\tau^{rel}_{\leqslant -r}L$ of L is isomorphic to X.

$\surd$

Lemma 4.14. *Let X be an object of $\mathcal{H}$. For any positive integer l, there exist objects an object $U \in \mathcal{H}$ and a triangle in $\mathcal{C}(Q,F,W)$*

$$P \xrightarrow{f} X \to \Sigma^l U \to \Sigma P$$

with f a right $(\mathcal{P} * \Sigma\mathcal{P} * \cdots \Sigma^{l-1}\mathcal{P})$-approximation, i.e. for each object P' in $\mathcal{P} * \Sigma\mathcal{P} * \cdots \Sigma^{l-1}\mathcal{P}$, the induced map $f_* : \mathrm{Hom}_{\mathcal{C}}(P',P) \to \mathrm{Hom}_{\mathcal{C}}(P',X)$ is surjective.

Dually, for any positive integer m, there exist an object $V \in \mathcal{H}$ and a triangle in $\mathcal{C}(Q,F,W)$

$$\Sigma^{-m-1}V \to X \xrightarrow{g} Q \to \Sigma^{-m}V$$

with g a left $(\Sigma^{-m}\mathcal{P} * \cdots \Sigma^{-1}\mathcal{P} * \mathcal{P})$-approximation, i.e. for each object Q' in $\Sigma^{-m}\mathcal{P} * \cdots \Sigma^{-1}\mathcal{P} * \mathcal{P}$, the induced map $g^* : \mathrm{Hom}_{\mathcal{C}}(Q,Q') \to \mathrm{Hom}_{\mathcal{C}}(X,Q')$ is surjective.

Proof. We only show the existences of the first statement since the second one can be shown dually. Let l be a positive integer. By Theorem 4.15 below, the Higgs category $\mathcal{H}$ is a Frobenius extriangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$. Thus, we have the following triangles in $\mathcal{C}(Q,F,W)$

$$\Omega X \to P_0 \to X \to \Sigma\Omega X,$$
$$\Omega^2 X \to P_1 \to \Omega X \to \Sigma\Omega^2 X,$$
$$\cdots$$
$$\Omega^l X \to P_{l-1} \to \Omega^{l-1}X \to \Sigma\Omega^l X,$$

where for each $0 \leqslant i \leqslant l-1$, the object P_i lies in $\mathcal{P}$ and $\Omega^{i+1}X$ lies in $\mathcal{H}$.

By the first two triangles and the octahedral axiom, we have

$$\begin{array}{ccccccc}
P_1 & \longrightarrow & \Omega X & \longrightarrow & \Sigma\Omega^2 X & \longrightarrow & \Sigma P_1 \\
\| & & \downarrow & & \downarrow & & \| \\
P_1 & \longrightarrow & P_0 & \longrightarrow & P_1' & \longrightarrow & \Sigma P_1 \\
 & & \downarrow & & \downarrow & & \\
 & & X & = & X & & \\
 & & \downarrow & & \downarrow & & \\
 & & \Sigma\Omega X & \longrightarrow & \Sigma^2\Omega X. & &
\end{array}$$

Then we get the following triangle in $\mathcal{C}(Q,F,W)$

$$P_1' \to X \to \Sigma^2\Omega X \to \Sigma P_1'$$

with P_1' in $\mathcal{P} * \Sigma\mathcal{P}$.

Repeating this process until the last triangle, we get a triangle in $\mathcal{C}(Q,F,W)$

$$P'_{l-1} \xrightarrow{f} X \to \Sigma^l \Omega^{l-1} X \to \Sigma P'_{l-1}$$

with P'_{l-1} in $\mathcal{P} * \Sigma\mathcal{P} * \cdots * \Sigma^{l-1}\mathcal{P}$. Since $\Omega^{l-1}X$ lies in $\mathcal{H}$, the space

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{P} * \Sigma\mathcal{P} * \cdots * \Sigma^{l-1}\mathcal{P}, \Sigma^l\mathcal{H})$$

vanishes. Thus, for each object P' in $\mathcal{P} * \Sigma\mathcal{P} * \cdots \Sigma^{l-1}\mathcal{P}$, the induced map $f_* : \mathrm{Hom}_{\mathcal{C}}(P', P'_{l-1}) \to \mathrm{Hom}_{\mathcal{C}}(P', X)$ is surjective.

$\surd$

Recall that an extriangulated category $\mathcal{E}$ is *Frobenius* if $\mathcal{E}$ has enough projectives and enough injectives and if moreover the projectives coincide with the injectives (see [30, Definition 3.2]).

Theorem 4.15. *The Higgs category $\mathcal{H}(Q,F,W)$ is a Frobenius extriangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$. In this case, we have equalities*

$$\mathcal{D}(\overline{Q},\overline{W}) = \mathcal{C}(\overline{Q},\overline{W})$$

and

$$\mathcal{D}(Q,F,W) = \mathcal{C}(Q,F,W).$$

Moreover, for any object X of $\mathcal{C}(Q,F,W)$, there exist $l \in \mathbb{Z}$, $U \in \mathcal{H}$ and $P \in \mathrm{thick}_{\mathcal{C}}\langle\mathcal{P}\rangle \simeq \mathrm{per}(e\mathbf{\Gamma}e)$ such that we have a triangle in $\mathcal{C}$

$$P \to X \to \Sigma^l U \to \Sigma P. \tag{2}$$

Dually, there exist $m \in \mathbb{Z}$, $V \in \mathcal{H}$ and $Q \in \mathrm{thick}_{\mathcal{C}}\langle\mathcal{P}\rangle \simeq \mathrm{per}(e\mathbf{\Gamma}e)$ such that we have a triangle in $\mathcal{C}$

$$\Sigma^m V \to X \xrightarrow{g} Q \to \Sigma^{m+1} V. \tag{3}$$

Proof.
Step 1. The Higgs category $\mathcal{H}(Q,F,W)$ is a Frobenius extriangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$.
Since $(\overline{Q},\overline{W})$ is Jacobi-finite, by [36, Remark 3.11], we have an equality

$$\mathcal{D}(\overline{Q},\overline{W}) = \mathcal{C}(\overline{Q},\overline{W}).$$

Let I be an object in $\mathcal{P}$. For any distinguished triangle in $\mathcal{H}$

$$X \to Y \to Z \overset{\delta}{-\to},$$

the space $\mathrm{Hom}_{\mathcal{C}}(\Sigma^{-1}Z, I) \cong \mathrm{Hom}_{\mathrm{per}\mathbf{\Gamma}}(\Sigma^{-1}Z, I)$ vanishes. Thus, we have an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(Y, I) \to \mathrm{Hom}_{\mathcal{C}}(X, I) \to 0.$$

This shows that any object in $\mathcal{P}$ is injective.
Let X be an object of $\mathcal{H}$. By definition, we have a triangle in $\mathcal{C}$

$$X \xrightarrow{f} T_0 \xrightarrow{g} T_1 \to \Sigma X$$

with $T_i \in \mathrm{add}\mathbf{\Gamma}$. Since the category $\mathcal{P}$ is functorially finite in $\mathrm{add}\mathbf{\Gamma}$, there exists a left $\mathcal{P}$-approximation $T_0 \xrightarrow{h} I_0$, i.e. $\mathrm{Hom}_{\mathcal{C}}(h, I)$ is surjective for any $I \in \mathcal{P}$. Thus, we get a triangle in $\mathcal{C}$

$$X \xrightarrow{hf} I_0 \to T'_1 \to \Sigma X$$

with $I_0 \in \mathcal{P}$ and $T'_1 \in \mathcal{Y} = ^{\perp}(\Sigma^{>0}\mathcal{P}) \cap (\Sigma^{<0}\mathcal{P})^{\perp}$. Thus T'_1 is an object of $\mathcal{H}\langle 1\rangle$.
By Proposition 4.13, there exists an object $I_1 \in \mathcal{H}$ such that $\tau^{rel}_{\leqslant -1} I_1$ is isomorphic to T'_1 in $\mathrm{per}\mathbf{\Gamma}$. Thus I_1 is isomorphic to T'_1 in $\mathcal{C}$. We get a triangle in $\mathcal{C}$

$$X \xrightarrow{hf} I_0 \to I_1 \to \Sigma X$$

with $I_0, I_1 \in \mathcal{H}$. Therefore, $\mathcal{H}$ has has enough injectives.
Dually, we show that any object in $\mathcal{P}$ is projective and $\mathcal{H}$ has enough projectives. Thus, the Higgs category $\mathcal{H}$ is a Frobenius extriangulated category with projective-injective objects $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$.
By the definition of $\mathcal{D}(Q,F,W)$ (see Definition 3.22), it is clear that we have an identity $\mathcal{D}(Q,F,W) = \mathcal{C}(Q,F,W)$.

Step 2. The existences of triangles (2) and (3).

We only show the existences of the first triangle since the second one can be shown dually.

By using the canonical co-t-structure on per$\mathbf{\Gamma}$, we see that per$\mathbf{\Gamma}$ = $\mathrm{tri}_{\mathrm{per}\mathbf{\Gamma}}(\mathrm{add}\mathbf{\Gamma})$, i.e. the smallest triangulated subcategory of per$\mathbf{\Gamma}$ containing add$\mathbf{\Gamma}$. Since $\pi^{rel}(\mathrm{add}\mathbf{\Gamma})$ lies in $\mathcal{H}$, we have

$$\mathcal{C}(Q,F,W) = \mathrm{tri}_{\mathcal{C}}(\mathcal{H}).$$

Let $\mathcal{K}$ be the full subcategory of $\mathcal{C}(Q,F,W)$ whose objects are those X which satisfy the following condition:

For any $l \gg 0$, there exist objects $P \in \mathrm{thick}_{\mathcal{C}}(\mathcal{P})$, $U \in \mathcal{H}$ and a triangle in $\mathcal{C}(Q,F,W)$

$$P \to X \to \Sigma^l U \to \Sigma P$$

such that P lies in $\Sigma^{k_1}\mathcal{P} * \Sigma^{k_2} * \cdots * \Sigma^{k_s}\mathcal{P}$ for some integers $k_1, k_2, \cdots, k_s$ less than l.

By Lemma 4.14, we have $\mathcal{H} \subseteq \mathcal{K}$. We next show that $\mathcal{K}$ is closed under shifts and extensions. It is easy to see that $\mathcal{K}$ is closed under shifts.

We next show that $\mathcal{K}$ is closed under extensions. Suppose we are give a triangle $X' \to X \to X'' \to \Sigma X'$ in $\mathcal{C}(Q,F,W)$ with X' and X'' in $\mathcal{K}$. For any $l \gg 0$, we have the following triangles in $\mathcal{C}(Q,F,W)$

$$P' \to X' \to \Sigma^l U' \to \Sigma P',$$
$$P'' \to X'' \to \Sigma^l U'' \to \Sigma P''$$

with U' and U'' in $\mathcal{H}$ and $P' \in \Sigma^{k'_1}\mathcal{P} * \Sigma^{k'_2} * \cdots * \Sigma^{k'_s}\mathcal{P}$, $P'' \in \Sigma^{k''_1}\mathcal{P} * \Sigma^{k''_2} * \cdots * \Sigma^{k''_r}\mathcal{P}$ for some integers $k'_1, k'_2, \cdots, k'_s, k''_1, k''_2, \cdots, k''_r$ less than l.

By using the fact that $\mathrm{Hom}_{\mathcal{C}}(P'', \Sigma^{l+1}U') = 0$ and Proposition 3.26, we get the following diagram

$$\begin{array}{ccccccc}
P' & \longrightarrow & X' & \longrightarrow & \Sigma^l U' & \longrightarrow & \Sigma P' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P & \longrightarrow & X & \longrightarrow & \Sigma^l U & \longrightarrow & \Sigma P \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P'' & \longrightarrow & X'' & \longrightarrow & \Sigma^l U'' & \longrightarrow & \Sigma P'' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma P' & \longrightarrow & \Sigma X' & \longrightarrow & \Sigma^{l+1} U' & \longrightarrow & \Sigma^2 P'.
\end{array}$$

This shows that X also lies in $\mathcal{K}$. Hence $\mathcal{K}$ is closed under extension. By the above arguments, we have $\mathrm{tri}_{\mathcal{C}}(\mathcal{H}) \subseteq \mathcal{K}$. Thus the category $\mathcal{K}$ is equal to $\mathcal{C}(Q,F,W)$. $\surd$

By Proposition 3.26, the Higgs category $\mathcal{H} = \mathcal{H}(Q,F,Q)$ is an extriangulated category. The extriangulated structure $(\mathcal{H}, \mathbb{E}, \mathfrak{s})$ can be described as follows:

(1) For any two objects X, Y of $\mathcal{H}$, the $\mathbb{E}$-extension space $\mathbb{E}(X,Y)$ is given by

$$\mathbb{E}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X, \Sigma Y).$$

(2) For any $\delta \in \mathbb{E}(X,Y)$, take a distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\delta} \Sigma X$$

and define $\mathfrak{s}(\delta) = [X \xrightarrow{f} Y \xrightarrow{g} Z]$. Then $\mathfrak{s}(\delta)$ does not depend on the choice of the distinguished triangle above.

Proposition 4.16. *The functor $p^* \colon \mathcal{C} \to \mathcal{C}(\overline{Q}, \overline{W})$ induces an equivalence of triangulated categories*

$$p^* \colon \mathcal{H}/[\mathcal{P}] \xrightarrow{\sim} \mathcal{D}(\overline{Q}, \overline{W}) = \mathcal{C}(\overline{Q}, \overline{W}).$$

Proof. Since $(\overline{Q}, \overline{W})$ is Jacobi-finite, the Higgs category $\mathcal{H}$ is a Frobenius extriangulated category. By [30, Lemma 3.12], the stable category $\mathcal{H}/[\mathcal{P}]$ also has an extriangulated structure where the distinguished triangles in $(\mathcal{H}/[\mathcal{P}], \overline{\mathbb{E}}, \overline{\mathfrak{s}})$ are the images of distinguished triangles in $(\mathcal{H}, \mathbb{E}, \mathfrak{s})$. By Corollary 4.9 and Lemma 4.10, we get this equivalence of triangulated categories. $\surd$

4.3. **Summary of results.** Let $\mathcal{E}$ be a Frobenius category and $\mathcal{M}$ a full subcategory of $\mathcal{E}$ which contains the full subcategory $\mathcal{P}$ of $\mathcal{E}$ formed by the projective-injective objects. We denote by $\mathcal{K}^b(\mathcal{E})$ and $\mathcal{D}^b(\mathcal{E})$ respectively the bounded homotopy category and the bounded derived category of $\mathcal{E}$.

We say that a complex $X\colon \cdots \to X^{i-1} \to X^i \to X^{i+1} \to \cdots$ in $\mathcal{K}^b(\mathcal{E})$ is *$\mathcal{E}$-acyclic* if there are conflations $Z^i \stackrel{l^i}{\rightarrowtail} X^i \stackrel{\pi^i}{\twoheadrightarrow} Z^{i+1}$ such that $d_X^i = l^{i+1}\circ\pi^i$ for each $i\in\mathbb{Z}$.

We also denote by $\mathcal{K}^b_{\mathcal{E}-ac}(\mathcal{E})$, $\mathcal{K}^b(\mathcal{P})$, $\mathcal{K}^b(\mathcal{M})$ and $\mathcal{K}^b_{\mathcal{E}-ac}(\mathcal{M})$ the full subcategory of $\mathcal{K}^b(\mathcal{E})$ whose objects are the $\mathcal{E}$-acyclic complexes, the complexes of projective objects in $\mathcal{E}$, the complexes of objects of $\mathcal{M}$ and the $\mathcal{E}$-acyclic complexes of objects of $\mathcal{M}$ respectively.

Combining Corollary 4.9, Theorem 4.15 and Theorem 3.34, we have the following result.

Theorem 4.17. *Let (Q,F,W) be an ice quiver with potential such that $\mathcal{P}=\mathrm{add}(e\mathbf{\Gamma})$ is functorially finite in $\mathrm{add}(\mathbf{\Gamma})$.*

1) *We have an equivalence of k-categories*
$$\mathcal{H}(Q,F,W)/[\mathcal{P}] \xrightarrow{\sim} \mathcal{D}(\overline{Q},\overline{W}).$$
2) *If $(\overline{Q},\overline{W})$ is Jacobi-finite, then $\mathcal{H}(Q,F,W)$ is a Frobenius extriangulated category with projective-injective objects $\mathcal{P}=\mathrm{add}(e\mathbf{\Gamma})$ and the equivalence in 1) preserves the extriangulated structure. Moreover, we have equalities*
$$\mathcal{D}(\overline{Q},\overline{W})=\mathcal{C}(\overline{Q},\overline{W})$$
and
$$\mathcal{D}(Q,F,W)=\mathcal{C}(Q,F,W).$$
3) *If moreover $\mathbf{\Gamma}$ is concentrated in degree 0, then the* boundary algebra $B = eJ_{rel}e$ *is* fp_∞*-Gorenstein of injective dimension at most $g\leqslant 3$ with respect to $\mathbf{\Gamma}$ and the Higgs category $\mathcal{H}$ is equivalent to the category $\mathrm{gpr}_\infty^{\leqslant 3}B$. Moreover, $\mathbf{\Gamma}$ is a canonical cluster-tilting object of $\mathcal{H}$ with endomorphism algebra $\mathrm{End}_{\mathcal{H}}(\mathbf{\Gamma})=J(Q,F,W)$.*
4) *Let $\mathcal{M}=\mathrm{add}(\mathbf{\Gamma})\subseteq\mathcal{H}$. Under the condition of 3), the exact sequence of triangulated categories*
$$0\to\mathrm{pvd}_e(\mathbf{\Gamma})\to\mathrm{per}\mathbf{\Gamma}\to\mathcal{C}(Q,F,W)\to 0$$
is equivalent to
$$0\to\mathcal{K}^b_{\mathcal{H}-ac}(\mathcal{M})\to\mathcal{K}^b(\mathcal{M})\to\mathcal{D}^b(\mathcal{H})\to 0.$$
In particular, the relative cluster category $\mathcal{C}(Q,F,W)$ is equivalent to the bounded derived category $\mathcal{D}^b(\mathcal{H})$ of $\mathcal{H}$.

Proof. 1), 2) and 3) follow from Corollary 4.9, Theorem 4.15 and Theorem 3.34, respectively.

Let $\mathrm{per}\mathcal{M}$ be the full subcategory of the derived category of modules over $\mathcal{M}$ generated by all representable functors and let $\mathrm{per}_{\underline{\mathcal{M}}}\mathcal{M}$ be its full subcategory consisting of complexes whose cohomologies are in $\mathrm{mod}\underline{\mathcal{M}}\cong\mathrm{mod}\overline{\mathbf{\Gamma}}$. Here $\underline{\mathcal{M}}$ is the additive quotient of $\mathcal{M}$ by $\mathcal{P}$. By [34, Lemma 2], we have the following exact sequence
$$0\to\mathcal{K}^b_{\mathcal{H}-ac}(\mathcal{M})\to\mathcal{K}^b(\mathcal{M})\to\mathcal{D}^b(\mathcal{H})\to 0.$$
By [34, Lemma 7], the Yoneda equivalence of triangulated categories $\mathcal{K}^b(\mathcal{M})\to\mathrm{per}\mathcal{M}\cong\mathrm{per}\mathbf{\Gamma}$ induces a triangle equivalence
$$\mathcal{K}^b_{\mathcal{H}-ac}(\mathcal{M})\to\mathrm{per}_{\underline{\mathcal{M}}}\mathcal{M}\cong\mathrm{pvd}_e(\mathbf{\Gamma}).$$
Thus, we finish the proof of 4). $\surd$

5. Cluster characters

Suppose $k=\mathbb{C}$. Let (Q,F,W) be an ice quiver with potential. Let $\mathbf{\Gamma}$ be the associated complete relative Ginzburg algebra. Let $Q_0=\{1,2,\ldots,n\}\supseteq F_0=\{r+1,\ldots,n\}$ for some integer $1\leqslant r\leqslant n$. We denote by $\mathbf{\Gamma}_i=e_i\mathbf{\Gamma}$ the indecomposable direct summand of $\mathbf{\Gamma}$ associated with the vertex i. Then $\mathcal{P}=\mathrm{add}(e\mathbf{\Gamma})$ is exactly the additive category $\mathrm{add}(\mathbf{\Gamma}_{r+1}\oplus\ldots\oplus\mathbf{\Gamma}_n)$. For $1\leqslant i\leqslant n$, let S_i be the simple J_{rel}-module associated with the vertex i. Let $e=\sum_{i\in F}e_i$ be the idempotent associated with the set of frozen vertices. We assume that $\mathcal{P}=\mathrm{add}(e\mathbf{\Gamma})$ is functorially finite in $\mathrm{add}(\mathbf{\Gamma})$. Let $\mathcal{H}$ be the Higgs category of (Q,F,W).

Definition 5.1. A *cluster character* on the Higgs category $\mathcal{H}$ with values in $\mathbb{Q}[x_{r+1},\dots,x_n][x_1^{\pm1},x_2^{\pm1},\dots,x_r^{\pm1}]$ is a map $X_?\colon \mathrm{obj}(\mathcal{H})\to\mathbb{Q}[x_{r+1},\dots,x_n][x_1^{\pm1},x_2^{\pm1},\dots,x_r^{\pm1}]$ such that

1) we have $X_L = X_{L'}$ if L and L' are isomorphic,
2) we have $X_{L\oplus M} = X_L X_M$ for all objects L and M and
3) (multiplication formula) if L and M are objects such that $\mathrm{Ext}^1_{\mathcal{H}}(L,M)$ is one-dimensional (hence $\mathrm{Ext}^1_{\mathcal{H}}(M,L)$ is one-dimensional) and
$$L\to E\to M\xrightarrow{+1}\quad\text{and}\quad M\to E'\to L\xrightarrow{+1}$$
are non-split triangles, then we have
$$X_L X_M = X_E + X_{E'}.$$

5.1. **Index.** Let X be an object of $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$. We define the *index with respect to* $\mathbf{\Gamma}$ of X as the element of $K_0(\mathrm{add}\mathbf{\Gamma})$ given by
$$\mathrm{ind}_{\mathbf{\Gamma}}X = [T_0^X]-[T_1^X],$$
where $T_1^X\to T_0^X\to X\to\Sigma T_1^X$ is an $(\mathrm{add}\mathbf{\Gamma})$-presentation of X. If $\widetilde{X}$ is the preimage of X under the k-linear equivalence
$$\mathrm{pr}_{\mathcal{D}}\mathbf{\Gamma}\to\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$$
induced by π^{rel} (cf. Proposition 3.16), then $\mathrm{ind}_{\mathbf{\Gamma}}(X)$ identifies with the class of $\widetilde{X}$ in
$$K_0(\mathrm{add}\mathbf{\Gamma})\xrightarrow{\sim}K_0(\mathrm{per}\mathbf{\Gamma}).$$
Thus, it is independent of the choice of presentation.

Lemma 5.2. *Let X be an object in $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}\cap\mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$ such that $R(X)=\mathrm{Ext}^1_{\mathcal{C}}(\mathbf{\Gamma},X)$ is finite-dimensional. Then the sum $\mathrm{ind}_{\mathbf{\Gamma}}X+\mathrm{ind}_{\mathbf{\Gamma}}\Sigma X$ only depends on the dimension vector of $F(X)$.*

Proof. The proof follows the lines of that of [36, Lemma 3.6]. We leave it to the reader. √

For a dimension vector e, we denote by $l(e)$ the sum $\mathrm{ind}_{\mathbf{\Gamma}}X+\mathrm{ind}_{\mathbf{\Gamma}}\Sigma X$, where $\underline{\dim}\,R(X)=e$. By the above lemma, this does not depend on the choice of such X.

The following Lemma will be very useful in the proof of our main result.

Lemma 5.3. *Let $X\xrightarrow{\alpha}Y\xrightarrow{\beta}Z\xrightarrow{\gamma}\Sigma X$ be a triangle in $\mathcal{C}$ with $X,Z\in\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$ such that $\mathrm{coker}(R(\beta))$ is finite-dimensional.*

a) *We have $Y\in\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}$.*
b) *Let C be an object of $\mathrm{pr}_{\mathcal{C}}\mathbf{\Gamma}\cap\mathrm{copr}_{\mathcal{C}}\mathbf{\Gamma}$ such that $R(C)=\mathrm{coker}(R(\beta))$. Then we have*
$$\mathrm{ind}_{\mathbf{\Gamma}}X+\mathrm{ind}_{\mathbf{\Gamma}}Z=\mathrm{ind}_{\mathbf{\Gamma}}C+l(C).$$

Proof. The proof in [36, Lemma 3.5] also works. We leave it to the reader. √

Now we define the map

$$X_?\colon \mathrm{obj}(\mathcal{H})\to\mathbb{Q}[x_{r+1},\dots,x_n][x_1^{\pm1},x_2^{\pm1},\dots,x_r^{\pm1}] \tag{4}$$

as follows: for any object M of $\mathcal{H}$, we put
$$X_M = x^{\mathrm{ind}_{\mathbf{\Gamma}}M}\sum_e \chi(\mathrm{Gr}_e(RM))x^{-l(e)},$$
where the sum ranges over all the elements of the Grothendieck group; for a J_{rel}-module L, the notation $\mathrm{Gr}_e(L)$ denotes the projective variety of submodules of L whose class in the Grothendieck group is e; for an algebraic variety V over $\mathbb{C}$, the notation $\chi(V)$ denotes the Euler characteristic.

Theorem 5.4. *The map $X_?$ defined above is a cluster character on $\mathcal{H}$.*

We prove this Theorem in the next subsection.

5.2. **Multiplication formula.** Let L and M be objects of $\mathcal{H}$ such that $\dim \mathrm{Ext}^1_{\mathcal{H}}(L, M) = 1$. By Proposition 4.11, we also have $\dim \mathrm{Ext}^1_{\mathcal{H}}(M, L) = 1$. Let

$$L \xrightarrow{i} E \xrightarrow{p} M \xrightarrow{\epsilon} \Sigma L$$

and

$$M \xrightarrow{i'} E' \xrightarrow{p'} L \xrightarrow{\epsilon'} \Sigma M$$

be non-split triangles in $\mathcal{C}$. Recall that R is the functor $R = \mathrm{Ext}^1_{\mathcal{C}}(\mathbf{\Gamma}, ?)\colon \mathcal{C} \to \mathrm{mod} J_{rel}$. For any submodules U of $R(L)$ and V of $R(M)$, define

$$G_{U,V} = \{W \in \bigcup_e \mathrm{Gr}_e(R(E)) \,|\, (Ri)^{-1}(W) = U,\ (Rp)(W) = V\}$$

and

$$G'_{U,V} = \{W \in \bigcup_e \mathrm{Gr}_e(R(E')) \,|\, (Ri')^{-1}(W) = V,\ (Rp')(W) = U\}.$$

Proposition 5.5. *Let U and V as above. Then exactly one of $G_{U,V}$ and $G'_{U,V}$ is non-empty.*

Proof. It follows from Lemma 4.10 and [36, Proposition 3.13]. √

For any dimension vectors e, f and g, define the following varieties:

$$G_{e,f} = \bigcup_{\substack{\underline{\dim} U = e \\ \underline{\dim} V = f}} G_{U,V}$$

$$G'_{e,f} = \bigcup_{\substack{\underline{\dim} U = e \\ \underline{\dim} V = f}} G'_{U,V}$$

$$G^g_{e,f} = G_{e,f} \cap \mathrm{Gr}_g(R(E))$$

$$G'^g_{e,f} = G'_{e,f} \cap \mathrm{Gr}_g(R(E')).$$

Lemma 5.6. [36, Lemma 3.17] *With the notations above, we have*

$$\chi(\mathrm{Gr}_e(R(L))) \cdot \chi(\mathrm{Gr}_f(R(M))) = \sum_g (\chi(\mathrm{Gr}^g_{e,f}) + \chi(\mathrm{Gr}'^g_{e,f})).$$

Lemma 5.7. [36, Lemma 3.18] *If $G^g_{e,f}$ is not empty, then we have*

$$\underline{\dim}(\mathrm{coker} R(\Sigma^{-1} p)) = e + f - g.$$

Proof of Theorem 5.4. It is easy to see that $X_?$ satisfies the first two conditions of Definition 5.1. It enough to show the multiplication formula holds.

Let L and M be objects of $\mathcal{H}$ such that $\mathrm{Ext}^1_{\mathcal{H}}(L, M)$ is one-dimensional. Then we have

$$\begin{aligned}
X_L \cdot X_M =& x^{\mathrm{ind}_{\mathbf{\Gamma}} L + \mathrm{ind}_{\mathbf{\Gamma}} M} \sum_{e,f} \chi(\mathrm{Gr}_e(RL)) \cdot \chi(\mathrm{Gr}_f(RM)) x^{-l(e+f)} \\
=& x^{\mathrm{ind}_{\mathbf{\Gamma}} L + \mathrm{ind}_{\mathbf{\Gamma}} M - l(e+f)} \sum_{e,f,g} (\chi(\mathrm{Gr}^g_{e,f}) + \chi(\mathrm{Gr}'^g_{e,f})) \\
=& x^{\mathrm{ind}_{\mathbf{\Gamma}} L + \mathrm{ind}_{\mathbf{\Gamma}} M - l(\underline{\dim}\,\mathrm{coker}(R(\Sigma^{-1} p)) - l(g)} \sum_{e,f,g} \chi(\mathrm{Gr}^g_{e,f}) + \\
& x^{\mathrm{ind}_{\mathbf{\Gamma}} L + \mathrm{ind}_{\mathbf{\Gamma}} M - l(\underline{\dim}\,\mathrm{coker}(R(\Sigma^{-1} p')) - l(g)} \sum_{e,f,g} \chi(\mathrm{Gr}'^g_{e,f}) \\
=& x^{\mathrm{ind}_{\mathbf{\Gamma}} L + \mathrm{ind}_{\mathbf{\Gamma}} M - l(\underline{\dim}\,\mathrm{coker}(R(\Sigma^{-1} p))} \sum_{e,f,g} \chi(\mathrm{Gr}^g_{e,f}) x^{-l(g)} + \\
& x^{\mathrm{ind}_{\mathbf{\Gamma}} L + \mathrm{ind}_{\mathbf{\Gamma}} M - l(\underline{\dim}\,\mathrm{coker}(R(\Sigma^{-1} p'))} \sum_{e,f,g} \chi(\mathrm{Gr}'^g_{e,f}) x^{-l(g)}
\end{aligned}$$

$$= x^{\mathrm{ind}_{\mathbf{\Gamma}} E} \sum_{g} \chi(\mathrm{Gr}_e(RE)) x^{-l(g)} + x^{\mathrm{ind}_{\mathbf{\Gamma}} E'} \sum_{g} \chi(\mathrm{Gr}_e(RE')) x^{-l(g)}$$
$$= X_E + X_{E'}.$$

The second equality is due to Lemma 5.6. The third one follows from Lemma 5.7. The fifth is a consequence of Lemma 5.3. This finishes the proof.

√

5.3. **Commutative diagram.** Recall that $\overline{\mathbf{\Gamma}}$ is the Ginzburg algebra associated with $(\overline{Q}, \overline{W})$ and $\mathcal{C}(\overline{Q}, \overline{W})$ is the corresponding cluster category. The functor $\mathcal{H} \to \mathcal{D}(\overline{Q}, \overline{W})$ induced by the quotient functor $p^* \colon \mathcal{C}(Q, F, W) \to \mathcal{C}(\overline{Q}, \overline{W})$ induces an equivalence

$$\underline{\mathcal{H}} \to \mathcal{D}(\overline{Q}, \overline{W}).$$

For an object X of $\mathrm{pr}_{\mathcal{C}(\overline{Q}, \overline{W})} \overline{\mathbf{\Gamma}} \subseteq \mathcal{D}(\overline{Q}, \overline{W})$, the index (see [36]) with respect to $\overline{\mathbf{\Gamma}}$ is given by

$$\mathrm{ind}_{\overline{\mathbf{\Gamma}}} X = [U_0^X] - [U_1^X] \in K_0(\mathrm{add}\overline{\mathbf{\Gamma}}),$$

where $U_1^X \to U_0^X \to X \to \Sigma U_1^X$ is an $\mathrm{add}\overline{\mathbf{\Gamma}}$-presentation of X. As in Subsection 5.1, we see that it does not depend on the choice of a presentation.

Let $\overline{R}$ be the functor

$$\mathrm{Ext}^1_{\mathcal{C}(\overline{Q}, \overline{W})}(\overline{\mathbf{\Gamma}}, ?) \colon \mathcal{D}(\overline{Q}, \overline{W}) \to \mathrm{mod} J(\overline{Q}, \overline{W}).$$

Let X be an object in $\mathcal{D}(\overline{Q}, \overline{W})$. For a dimension vector $\underline{\dim} \overline{R} X = e$, denote by $\bar{l}(e)$ the sum $\bar{l}(e) = \mathrm{ind}_{\overline{\mathbf{\Gamma}}} X + \mathrm{ind}_{\overline{\mathbf{\Gamma}}} \Sigma X$. By [36, Lemma 3.6], $\bar{l}(e)$ only depends on the dimension vector of $\overline{R} X$.

In [36, Theorem 3.12], Plamondon defined the canonical cluster character

$$\overline{CC} \colon \mathcal{D}(\overline{Q}, \overline{W}) \to \mathbb{Q}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$$

taking $\overline{\mathbf{\Gamma}_i} = e_i \mathbf{\Gamma}(\overline{Q}, \overline{W})$ to x_i. It is given as follows: for any object X of $\mathcal{D}(\overline{Q}, \overline{W})$, put

$$\overline{CC}(X) = x^{\mathrm{ind}_{\overline{\mathbf{\Gamma}}} X} \sum_{e} \chi(\mathrm{Gr}_e(\overline{R} X)) x^{-\bar{l}(e)},$$

where the sum ranges over all the elements of the Grothendieck group.

Let $\mathcal{S}$ be the subcategory of $\mathrm{per}\mathbf{\Gamma}$ formed by the modules S_i associated with unfrozen vertices $i \in Q_0 \setminus F_0$. Consider the following subcategory of $\mathrm{per}\mathbf{\Gamma}$

$$\mathcal{W} = (\Sigma^{\geqslant 0} \mathcal{S})^{\perp} \cap {}^{\perp}(\Sigma^{\leqslant 0} \mathcal{S}).$$

Since $(\overline{Q}, \overline{W})$ is Jacobi-finite, by Proposition 3.14, the following composition

$$\mathcal{W} \hookrightarrow \mathrm{per}\mathbf{\Gamma} \xrightarrow{\pi^{rel}} \mathcal{C}(Q, F, W)$$

induces a k-linear equivalence

$$\pi^{rel}_{\mathcal{W}} \colon \mathcal{W} \xrightarrow{\sim} \mathcal{C}(Q, F, W).$$

Remark 5.8. When the frozen part F is empty, the category $\mathcal{W}$ is equal to

$$\Sigma \mathcal{F} = \Sigma \mathrm{add}\, \mathbf{\Gamma}(Q, W) * \Sigma^2 \mathrm{add}\, \mathbf{\Gamma}(Q, W),$$

i.e. the shift of the fundamental domain, see [23, Theorem 2.12].

Definition 5.9. For any object X of $\mathcal{C}(Q, F, W)$, let X' be a pre-image of ΣX in $\mathcal{W}$ under $\pi^{rel}_{\mathcal{W}}$. We define $[X]^F$ to be the image of the class $-[X']$ of $K_0(\mathrm{add}\mathbf{\Gamma})$ under the projection onto $K_0(\mathrm{add}(e\mathbf{\Gamma}))$ along $K_0(\mathrm{add}(1-e)\mathbf{\Gamma})$.

Remark 5.10. By Definition 3.21, the Higgs category $\mathcal{H}$ is a full subcategory of $\mathrm{pr}_{\mathcal{C}} \mathbf{\Gamma} \cap \mathrm{copr}_{\mathcal{C}} \mathbf{\Gamma}$. By Proposition 3.16 and Corollary 3.17, let $\mathcal{H}'$ be the pre-image of $\mathcal{H}$ under the equivalence

$$\pi_{rel} : \mathrm{pr}_{\mathcal{D}} \mathbf{\Gamma} \cap \mathrm{copr}_{\mathcal{D}} \mathbf{\Gamma} \xrightarrow{\sim} \mathrm{pr}_{\mathcal{C}} \mathbf{\Gamma} \cap \mathrm{copr}_{\mathcal{C}} \mathbf{\Gamma}.$$

It is not hard to see that $\Sigma \mathcal{H}'$ is a subcategory of $\mathcal{W}$.

Let X be an object of $\mathcal{H}$. We denote by $\mathrm{ind}_{\mathbf{\Gamma}}^{F}(X)$ the image of the class $\mathrm{ind}_{\mathbf{\Gamma}}(X)$ of $K_0(\mathrm{add}\mathbf{\Gamma})$ under the projection onto $K_0(\mathrm{add}(e\mathbf{\Gamma}))$ along $K_0(\mathrm{add}(1-e)\mathbf{\Gamma})$. Then we see that $\mathrm{ind}_{\mathbf{\Gamma}}^{F}(X)$ is equal to $[X]^F$. Moreover, we have $[X]^F + \mathrm{ind}_{\overline{\mathbf{\Gamma}}}(p^*(X)) = \mathrm{ind}_{\mathbf{\Gamma}}(X)$.

Theorem 5.11. *Let (Q, F, W) be an ice quiver with potential such that $\mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$ is functorially finite in $\mathrm{add}(\mathbf{\Gamma})$. We assume that $(\overline{Q}, \overline{W})$ is Jacobi-finite. Consider the following diagram*

$$\begin{array}{ccccc} \mathcal{H} & \hookrightarrow & \mathcal{C}(Q,F,W) & \overset{CC_{loc}}{\dashrightarrow} & \mathbb{Q}[x_1^{\pm1},\ldots,x_r^{\pm1},x_{r+1}^{\pm1},\ldots,x_n^{\pm1}] \\ \big\downarrow{\scriptstyle p^*} & & & & \big\downarrow{\scriptstyle x_i\mapsto 1\ \forall i>r} \\ \underline{\mathcal{H}} & = & \mathcal{C}(\overline{Q},\overline{W}) & \xrightarrow{\overline{CC}} & \mathbb{Q}[x_1^{\pm1},\ldots,x_r^{\pm1}]. \end{array}$$

There is a unique map

$$CC_{loc}\colon \mathcal{C}(Q,F,W) \to \mathbb{Q}[x_1^{\pm1},\ldots,x_r^{\pm1},x_{r+1}^{\pm1},\ldots,x_n^{\pm1}]$$

such that the above diagram commutes and

1) *for each triangle in $\mathcal{C}(Q,F,W)$*

$$P \to X \to M \to \Sigma P$$

with $P \in \mathrm{thick}_{\mathcal{C}}(\mathcal{P})$, we have

$$CC_{loc}(X) = CC_{loc}(P) \cdot CC_{loc}(M).$$

2) *The restriction $CC_{loc}|_{\mathcal{H}}$ is the cluster character $X_?$ defined in (4).*
3) *For each object P in $\mathrm{thick}_{\mathcal{C}}(\mathcal{P})$, we have $CC_{loc}(P) = x^{[P]}$, where $[P] \in K_0(\mathrm{per}\mathbf{\Gamma}) \simeq \mathbb{Z}^n$.*

Proof. Let us show that CC_{loc} is unique if it exists. Let n be a positive integer and M an object of $\mathcal{H}$. By our assumption, $\mathcal{H}$ is a Frobenius extriangulated category. Therefore, we have the following triangles in $\mathcal{C}(Q,F,W)$

$$\begin{gathered} \Omega^1(M) \to P_1 \to M \to \Sigma\Omega^1(M), \\ \Omega^2(M) \to P_2 \to \Omega(M) \to \Sigma\Omega^2(M), \\ \cdots \\ \Omega^n(M) \to P_n \to \Omega^{n-1}(M) \to \Sigma\Omega^n(M), \end{gathered}$$

where $P_i \in \mathcal{P} = \mathrm{add}(e\mathbf{\Gamma})$ and $\Omega^i(M) \in \mathcal{H}$ for each $i \in \{1,\ldots,n\}$. By the property 1), we have the following formula

$$CC_{loc}(\Sigma^{-n}M) = CC_{loc}(\Sigma^{-n}P_1) \cdot CC_{loc}(\Sigma^{-n+1}P_2) \cdots CC_{loc}(\Sigma^{-1}P_n) \cdot CC_{loc}(\Omega^n(M)).$$

Dually, we have the following triangles in $\mathcal{C}(Q,F,W)$

$$\begin{gathered} M \to I_1 \to \Theta^1(M) \to \Sigma M, \\ \Theta^1(M) \to I_2 \to \Theta^2(M) \to \Sigma\Theta^1(M), \\ \cdots \\ \Theta^{n-1}(M) \to I_n \to \Theta^n(M) \to \Sigma\Theta^{n-1}(M), \end{gathered}$$

where $I_i \in \mathcal{P}$ and $\Theta^i(M) \in \mathcal{H}$ for each $i \in \{1,\ldots,n\}$. By the properties 1) and 3), we have the following formula

$$CC_{loc}(\Sigma^n M) = CC_{loc}(\Sigma^n I_1) \cdot CC_{loc}(\Sigma^{n-1} I_2) \cdots CC_{loc}(\Sigma I_n) \cdot CC_{loc}(\Theta^n(M)).$$

Now let X be an any object of $\mathcal{C}(Q,F,W)$. By Theorem 4.15, we have a triangle in $\mathcal{C}(Q,F,W)$

$$P \to X \to \Sigma^m M \to \Sigma P,$$

where m is an integer, $P \in \mathrm{thick}_{\mathcal{C}}(\mathcal{P})$ and $M \in \mathcal{H}$. Hence we have a formula

$$CC_{loc}(X) = CC_{loc}(P) \cdot CC_{loc}(\Sigma^m M).$$

Thus, this shows the uniqueness of CC_{loc}.

Now it remains to show the existence. For any object X of $\mathcal{C}(Q,F,W)$, we define

$$CC_{loc}(X) = x^{[X]^F} \cdot \overline{CC}(p^*(X)),$$

where $[X]^F$ is defined in Definition 5.9 and p^* is the quotient functor $\mathcal{C}(Q,F,W) \to \mathcal{C}(\overline{Q},\overline{W})$. It is clear that CC_{loc} satisfies condition 3). For a finite dimensional $J(\overline{Q},\overline{W})$-module with dimension vector e, we have $l(e) = \bar{l}(e)$. And by Remark 5.10, the map CC_{loc} also satisfies condition 2).

Let

$$P \xrightarrow{a} X \to M \to \Sigma P$$

be a triangle in $\mathcal{C}(Q,F,W)$ with $P \in \mathrm{thick}_{\mathcal{C}}(\mathcal{P})$. Suppose that X' is an object of $\mathcal{W}$ such that $\pi^{rel}_{\mathcal{W}}(X')$ is isomorphic to ΣX in $\mathcal{C}(Q,F,W)$.

Since we have an isomorphism

$$\mathrm{Hom}_{\mathcal{W}}(\Sigma P, X') \cong \mathrm{Hom}_{\mathcal{C}}(\Sigma P, \Sigma X),$$

there exists a morphism $a' \colon P \to X'$ in $\mathcal{W}$ such that $\pi^{rel}_{\mathcal{W}}(a') = \Sigma a$. We form a triangle

$$\Sigma P \xrightarrow{a'} X' \to M' \to \Sigma P$$

in per$\mathbf{\Gamma}$. It is easy to see that M' lies in $\mathcal{W}$ and $\pi^{rel}_{\mathcal{W}}(M') \cong \Sigma M$. Then it follows that

$$[X]^F = [P]^F + [M]^F$$

and

$$CC_{loc}(X) = CC_{loc}(P) \cdot CC_{loc}(M).$$

Hence CC_{loc} also satisfies condition 1).

$\surd$

6. Applications to quasi-cluster homomorphisms

In this section, we first recall the definition of quasi-cluster homomorphism defined by Fraser in [14]. Then our aim is to show that the decategorification of the equivalence associated with the mutation at a frozen source (or sink) is a quasi-cluster isomorphism.

Definition 6.1. [14, Definition 1.2] A *labeled r-regular tree*, $\mathbb{T}_r$, is an r-regular tree with edges labeled by integers so that the set of labels emanating from each vertex is $[1,r] = \{1,2,\ldots,r\}$. We write $t \xrightarrow{k} t'$ to indicate that vertices t, t' are joined by an edge with label k. An isomorphism $\mathbb{T}_r \to \overline{\mathbb{T}}_r$ of labeled trees sends vertices to vertices and edges to edges, preserving incidences of edges and the edge labels. Such an isomorphism is uniquely determined by its value at a single vertex $t \in \mathbb{T}_r$.

Let (Q,F) be a finite ice quiver, where Q has no oriented cycles of length $\leqslant 2$. We suppose that $Q_0 = \{1,\ldots,n\} \supseteq F_0 = \{r+1,\ldots,n\}$. We denote by $\mathbb{P}$ the tropical semifield $\mathrm{Trop}(x_{r+1},\ldots,x_n)$. Let $\mathcal{F}$ be the field of fractions of the ring of polynomials in r indeterminates with coefficients in $\mathbb{QP}$.

Definition 6.2. A *seed* is a pair $((Q,F),\mathbf{x})$, where (Q,F) is an ice quiver as above, and $\mathbf{x} = \{x_1,\ldots,x_n\}$ is a free generating set of the field $\mathcal{F}$.

Given a vertex i of $Q_0 \backslash F_0$, the *mutation* of the seed $((Q,F),\mathbf{x})$ at the vertex i is the pair $\mu_i((Q,F),\mathbf{x}) = ((Q',F'),\mathbf{x}')$, where

- (Q',F') is the mutated ice quiver $\mu_i(Q,F)$;
- $\mathbf{x}' = \mathbf{x} \setminus \{x_i\} \cup \{x_i'\}$, where x_i' is obtained from the *exchange relation*

$$x_i x_i' = \prod_{\alpha \in Q_1, s(\alpha)=i} x_{t(\alpha)} + \prod_{\alpha \in Q_1, t(\alpha)=i} x_{s(\alpha)}.$$

It is easy to see that the mutation at a fixed vertex is an involution.

Let $\mathbb{T}_r$ be the r-regular labeled tree with root t_0.

Definition 6.3. [14, Definition 1.3] A collection of seeds in $\mathcal{F}$, with one seed $\Sigma(t) = ((Q(t),F(t)),\mathbf{x}(t))$ for each $t \in \mathbb{T}_r$, is called a *seed pattern* if, for each edge $t \xrightarrow{k} t'$, the seeds $\Sigma(t)$ and $\Sigma(t')$ are related by a mutation in direction k. The seed $\Sigma(t_0)$ at the root t_0 is called the *initial seed* and the seed pattern is denoted by $\mathcal{E}$.

Fix an initial seed $((Q,F),\mathbf{x})$.

- The sets $\mathbf{x}'$ obtained by repeated mutation of the initial seed are the *clusters*.
- The elements of the clusters are the *cluster variables*.

- The corresponding *cluster algebra with invertible coefficients* $\mathcal{A}_Q$ is the $\mathbb{Z}\mathbb{P}$-subalgebra of $\mathcal{F}$ generated by all cluster variables.

As in the previous section, we denote by $\overline{Q}$ the full subquiver of Q on the non-frozen vertices. Then the associated *cluster algebra without coefficients* is denoted by $\mathcal{A}_{\overline{Q}}$. It is a quotient of $\mathcal{A}_Q/(m-1, \forall m \in \mathbb{P})$.

For any $t \in \mathbb{T}_r$, let $Q(t)$ be the mutated quiver associated with $t \in \mathbb{T}_r$. The associated exchange matrix $B(t) = (b_{ij}(t))$ is defined by

$$b_{ij}(t) = |\{i \to j \,\text{in}\, Q(t)\}| - |\{j \to i \,\text{in}\, Q(t)\}|$$

and the associated r-tuple of hatted variables $\{\widehat{y}_j(t) \,|\, j = 1, \ldots, r\}$ is given by

$$\widehat{y}_j(t) = \prod_{i \in Q_0} x_i^{b_{ij}(t)}.$$

Now let (Q', F') be another ice quiver, where Q' has no oriented cycles of length $\leqslant 2$, and $Q'_0 = \{1, \ldots, n\} \supseteq F'_0 = \{r+1, \ldots, n\}$. The corresponding seed pattern is $\mathcal{E}'$. It is built on a second copy of the r-regular tree, $\mathbb{T}'_r$.

Definition 6.4. [14, Definition 3.1] Let $\mathcal{A}_Q$ and $\mathcal{A}_{Q'}$ be the associated cluster algebras with invertible coefficients, respectively. A *quasi-cluster homomorphism* $\varphi\colon \mathcal{A}_Q \to \mathcal{A}_{Q'}$ is an algebra morphism such that

a) $\varphi(\mathbb{P}) \subseteq \mathbb{P}'$ and for each cluster variable x of $\mathcal{A}_Q$, there is a cluster variable x' of $\mathcal{A}_{Q'}$ such that $\varphi(x) \in \mathbb{P}' \cdot x'$;
b) the induced algebra morphism $\overline{\varphi}\colon \mathcal{A}_{\overline{Q}} \to \mathcal{A}_{\overline{Q'}}$ is an isomorphism of cluster algebras taking the initial seed $(\overline{Q}, \overline{x})$ to a seed $(\overline{Q'}, \overline{x'})$;
c) We have $\varphi(\widehat{y}_j(t_0)) = \widehat{y'}_{j'}(t')$, where j', t' satisfy

$$\overline{\varphi}(x_j(t_0)) = x'_{j'}(t'), \quad \forall 1 \leqslant j \leqslant r.$$

Remark 6.5. In *c*), we have used the fact that each cluster $\mathbf{x}(t) = \{x_j(t)) \,|\, 1 \leqslant j \leqslant r\}$ uniquely determines a seed $(\mathbf{x}(t), Q(t))$, cf. [21].

Let (Q, F, W) be an ice quiver with potential such that $\mathcal{P} = \text{add}(e\mathbf{\Gamma})$ is functorially finite in $\text{add}(\mathbf{\Gamma})$. We assume that $(\overline{Q}, \overline{W})$ is Jacobi-finite.

Proposition 6.6. *Let $v \in F_0$ be a frozen source and $(Q', F', W') = \mu_v(Q, F, W)$. Then there exists a unique isomorphism $\psi_+ \colon \mathbb{Q}[x_1'^{\pm 1}, \ldots, x_r'^{\pm 1}, x_{r+1}'^{\pm 1}, \ldots, x_n'^{\pm 1}] \to \mathbb{Q}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{C}(Q', F', W') & \xrightarrow{CC'_{loc}} & \mathbb{Q}[x_1'^{\pm 1}, \ldots, x_r'^{\pm 1}, x_{r+1}'^{\pm 1}, \ldots, x_n'^{\pm 1}] \\ \downarrow \Psi_+ & & \downarrow \psi_+ \\ \mathcal{C}(Q, F, W) & \xrightarrow{CC_{loc}} & \mathbb{Q}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}]. \end{array} \tag{5}$$

Proof. We define an algebra morphism

$$\psi_+ \colon \mathbb{Q}[x_1'^{\pm 1}, \ldots, x_r'^{\pm 1}, x_{r+1}'^{\pm 1}, \ldots, x_n'^{\pm 1}] \to \mathbb{Q}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$$

as follows:

$$\psi_+(x'_j) = \begin{cases} x_j & \text{if } j \neq v \\ \frac{\prod_{v \to k \in F_1} x_k}{x_v} & \text{if } j = v. \end{cases}$$

It is clear that ψ_+ is an isomorphism. Let X be an object of $\mathcal{C}(Q', F', W')$. By the proof of Proposition 3.31, we have $p^*(\Psi_+(X)) = p'^*(X)$. Hence we have the following identity

$$\overline{CC}(p'^*(X)) = \overline{CC}(p^*(\Psi_+(X))).$$

The equivalence $\Psi_+ : \text{per}\mathbf{\Gamma}' \to \text{per}\mathbf{\Gamma}$ induces an equivalence between their Grothendieck groups

$$[\Psi_+] : K_0(\text{add}\mathbf{\Gamma}') \to K_0(\text{add}\mathbf{\Gamma})$$

which maps $[\mathbf{\Gamma}'_i]$ to $\mathbf{\Gamma}_i$ for $i \neq v$ and $\mathbf{\Gamma}'_v$ to $\sum\limits_{\alpha \in F_1 : s(\alpha)=v} [\mathbf{\Gamma}_{t(\alpha)}] - [\mathbf{\Gamma}_v]$. This implies that

$$\psi_+(x'^{[X]^{F'}}) = x^{[\Psi_+(X)]^F}.$$

According to the proof of Theorem 5.11, we have

$$CC'_{loc}(X) = x'^{[X]^{F'}} \overline{CC}(p'^*(X))$$

and

$$CC_{loc}(\Psi_+(X)) = x^{[\Psi_+(X)]^F} \overline{CC}(p^*(\Psi_+(X))) = x^{[\Psi_+(X)]^F} \overline{CC}(p'^*(X)).$$

Therefore we obtain the equality $\psi_+(CC'_{loc}(X)) = CC_{loc}(\Psi_+(X))$. This shows the commutativety of diagram (5). By the equality $\varphi_+ \circ CC_{loc}(\mathbf{\Gamma}'_i) = CC_{loc} \circ \Psi_+(\mathbf{\Gamma}'_i)$, we see that such map φ_+ is unique.

$\surd$

Remark 6.7. Let $\mathcal{R}$ be the class of rigid indecomposables reachable (by left and right mutations) from $\mathbf{\Gamma}$. Denote by $\mathrm{rch}(Q, F, W)$ the full subcategory of $\mathcal{C}(Q, F, W)$ obtained as the closure of $\mathcal{R}$ under suspensions and desuspensions, extensions by objects in $\mathrm{thick}(\mathcal{P})$, finite direct sums and direct summands. Similarly, we define $\mathrm{rch}(Q', F', W') \subseteq \mathcal{C}(Q', F', W')$. The commutative diagram 5 induces the following commutative diagram

$$\begin{array}{ccc} \mathrm{rch}(Q', F', W') & \xrightarrow{CC'_{loc}} & \mathcal{A}_{Q'} \\ \downarrow{\scriptstyle \Psi_+} & & \downarrow{\scriptstyle \psi_+} \\ \mathrm{rch}(Q, F, W) & \xrightarrow{CC_{loc}} & \mathcal{A}_Q. \end{array}$$

Moreover, $\psi_+ \colon \mathcal{A}_{Q'} \to \mathcal{A}_Q$ is a quasi-cluster isomorphism.

We have the dual statement of Proposition 6.6 for mutation at a frozen sink.

Proposition 6.8. *Let $v \in F_0$ be a frozen sink and $(Q', F', W') = \mu_v(Q, F, W)$. Then there exists a unique isomorphism*

$$\begin{aligned} \psi_- \colon \mathbb{Q}[x_1'^{\pm 1}, \ldots, x_r'^{\pm 1}, x_{r+1}'^{\pm 1}, \ldots, x_n'^{\pm 1}] &\to \mathbb{Q}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}] \\ x'_i &\mapsto x_i, \text{ if } i \neq v; \\ x'_v &\mapsto \frac{\prod_{k \to v \in F_1} x_k}{x_v} \end{aligned}$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}(Q', F', W') & \xrightarrow{CC'_{loc}} & \mathbb{Q}[x_1'^{\pm 1}, \ldots, x_r'^{\pm 1}, x_{r+1}'^{\pm 1}, \ldots, x_n'^{\pm 1}] \\ \downarrow{\scriptstyle \Psi_-} & & \downarrow{\scriptstyle \psi_-} \\ \mathcal{C}(Q, F, W) & \xrightarrow{CC_{loc}} & \mathbb{Q}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}, x_{r+1}^{\pm 1}, \ldots, x_n^{\pm 1}]. \end{array} \tag{6}$$

Remark 6.9. The commutative diagram 6 induces the following commutative diagram

$$\begin{array}{ccc} \mathrm{rch}(Q', F', W') & \xrightarrow{CC'_{loc}} & \mathcal{A}_{Q'} \\ \downarrow{\scriptstyle \Psi_-} & & \downarrow{\scriptstyle \psi_-} \\ \mathrm{rch}(Q, F, W) & \xrightarrow{CC_{loc}} & \mathcal{A}_Q. \end{array}$$

And $\psi_- \colon \mathcal{A}_{Q'} \to \mathcal{A}_Q$ is a quasi-cluster isomorphism.

7. Examples from Postnikov diagrams

With each connected Postnikov diagram D in the disc, one can associate a canonical ice quiver with potential (Q_D, F_D, W_D). The corresponding relative Jacobian algebra $J_D = J(Q_D, F_D, W_D)$ is known as the *dimer algebra* of the diagram. Pressland has shown in [38] that it is internally 3-Calabi–Yau in the sense of his earlier work [37]. As a consequence, he obtains that the category of Gorenstein projective modules over the corresponding boundary algebra yields an additive categorification of the cluster algebra associated with the ice quiver (Q_D, F_D). Notice that, by a recent result of Galashin–Lam [16], after inverting the coefficients, this cluster algebra becomes isomorphic to the homogeneous coordinate algebra of the (open) positroid variety associated with D in the Grassmannian. In this section, we explain how this class of ice quivers with potential fits into the theory developed in this article. In particular, the categories of Gorenstein projective modules studied by Pressland turn out to be examples of Hom-infinite Higgs categories as constructed in section 3.

Let D be a connected Postnikov diagram in the disc (see [38, Definition 2.1]). With D, we can associate an ice quiver with potential (Q_D, F_D, W_D) (see [38, Definition 2.4]). By [38, Theorem 3.7] and [42, Lemma 5.11], the corresponding complete relative Ginzburg algebra $\mathbf{\Gamma}_D = \mathbf{\Gamma}(Q_D, F_D, W_D)$ is concentrated in degree 0. From [38, Proposition 4.4], we see that J_D is Noetherian and $(\overline{Q}_D, \overline{W}_D)$ is Jacobi-finite. Put $e = \sum_{v \in F_D} e_v$. The boundary algebra B_D is defined as eJ_De. It inherits the Noetherian property from J_D.

For each vertex v of Q_D, choose a path $t_v : v \to v$ representing a fundamental cycle (see [38, Definition 2.4]). Let $t = \sum_{v \in Q_D} t_v$. The element t is central in J_D so that J_D becomes an algebra over the power series algebra $Z = \mathbb{C}[[T]]$. By Proposition 2.11 of [38], for all vertices v, w of Q_D, the Z-module $e_w J_D e_v$ is free of rank 1. Thus, the algebra J_D is a finitely generated free Z-module (of rank equal to the square of the number of vertices of Q_D) and so is the boundary algebra (of rank equal to the square of the number of frozen vertices in Q_D). It follows that for each P in $\mathrm{add}(e\mathbf{\Gamma}_D)$ and each M in $\mathrm{add}(\mathbf{\Gamma}_D)$, the $\mathrm{End}(P)$-modules $\mathrm{Hom}(P, M)$ and $\mathrm{Hom}(M, P)$ are finitely generated so that the subcategory $\mathrm{add}(e\mathbf{\Gamma}_D)$ is functorially finite in $\mathrm{add}(\mathbf{\Gamma}_D)$.

Thus, the associated ice quiver with potential (Q_D, F_D, W_D) satisfies Assumption 1 of section 4. Then by Theorem 4.17, the corresponding Higgs category $\mathcal{H}$ is equivalent to $\mathrm{gpr}_{\infty}^{\leqslant 3}(B_D) = \mathrm{gpr}(B_D)$ which is exactly Pressland's category in [38, Theorem 4.5]. Under this equivalence, the canonical cluster-tilting object $\mathbf{\Gamma}_D$ of $\mathcal{H}$ corresponds to the canonical cluster-tilting object $T = J_De$ of $\mathrm{gpr}(B_D)$. Moreover, the relative cluster category $\mathcal{C}(Q_D, F_D, W_D)$ is equivalent to the bounded derived category $\mathcal{D}^b(\mathrm{gpr}(B_D))$ of $\mathrm{gpr}(B_D)$. In particular, by section 1.3 of [38], the category of Cohen-Macaulay modules introduced by Jensen–King–Su in [24] is equivalent to a Higgs category and the bounded derived category of their algebra A is equivalent to a relative cluster category.

Recall from Theorem 6.1 of [38] that the Frobenius category $\mathcal{H} \xrightarrow{\sim} \mathrm{gpr}(B_D)$ with the cluster-tilting object T is part of an additive categorification of the cluster algebra associated with the diagram D. The corresponding cluster character evaluated at an object $M \in \mathrm{gpr}(B_D)$ is given by Fu–Keller's [15] formula

$$CC(M) = x^{[FX]} \sum_{d} \chi(\mathrm{Gr}_d(\mathrm{Ext}^1_{\mathrm{gpr}(B_D)}(T, M)))x^{-[N]}. \tag{7}$$

Here we denote

a) by F be the functor

$$\mathrm{Hom}_{\mathrm{gpr}(B_D)}(T, ?) : \mathrm{gpr}(B_D) \to \mathrm{mod}(J_D) ;$$

b) by $[M]$ the class of a J_D-module M in the Grothendieck group $K_0(\mathrm{per}J_D) \cong K_0(\mathrm{proj}J_D)$, which we identify with $\mathbb{Z}^N$ via the choice of the basis formed by the classes $[P_i]$, $i \in (Q_D)_0 = \{1, \ldots, N\}$ of the indecomposable projective J_D-modules associated with the vertices of Q_D ;

c) by $[N]$ the class in $K_0(\mathrm{per}J_D)$ of any object N with dimension vector d (this class is independent of the choice of N as in the proof of [15, Proposition 3.2]).

Then it isn't hard to see that $\varphi(Ne)$ is equal to X_N (defined by formula (4)) for each object $N \in \mathcal{H}$ under the equivalence $\mathcal{H} \xrightarrow{\sim} \mathrm{grp}(B_D) : N \mapsto Ne$.

Appendix A. Frobenius categorification of quasi-cluster morphisms, by C. Fraser and B. Keller

A.1. **Quasi-cluster morphisms.** We recall the notion of quasi-cluster morphism from [14]. Let Q be an ice quiver with frozen subquiver $F \subseteq Q$. Let us assume that the set of non frozen vertices is formed by the integers $1, \ldots, r$ and the set of frozen vertices by $r+1, \ldots, n$. Let $\mathbb{T}_r$ be the r-regular tree with root t_0. Let $\mathcal{A}_Q$ be the cluster algebra *with invertible coefficients* associated with Q. Let us write x_1, $\ldots$, x_n for its initial cluster variables. Let $\mathbb{P}$ be its *coefficient group*, i.e. the group of Laurent monomials

$$x_{r+1}^{e_{r+1}} \cdots x_n^{e_n}, e_i \in \mathbb{Z} \ ,$$

in the frozen variables. Let $\overline{Q}$ be the full subquiver on the non frozen vertices of Q and $\mathcal{A}_{\overline{Q}}$ the associated cluster algebra without coefficients. For a vertex t of $\mathbb{T}_r$, we denote by $Q(t)$ denote the associated iterated mutation of Q, by $B(t) = (b_{ij}(t))$ the associated exchange matrix and by $x_j(t)$ the associated cluster variables for $1 \leq j \leq n$. Moreover, for each non frozen vertex j, we put

$$\widehat{y}_j(t) = \prod_{i \in Q_0} x_i^{b_{ij}(t)}.$$

Lemma A.1. *The specialization map*

$$\mathbb{Q}[x_1^{\pm}, \ldots, x_r^{\pm}, x_{r+1}^{\pm}, \ldots, x_n^{\pm}] \to \mathbb{Q}[x_1^{\pm}, \ldots, x_r^{\pm}]$$

taking the x_j, $j > r$, to 1 induces a cluster algebra isomorphism from the quotient $\mathcal{A}_Q/(m-1, m \in \mathbb{P})$ onto $\mathcal{A}_{\overline{Q}}$.

Let Q' be another ice quiver and $\mathbb{P}'$ the associated coefficient group. Slightly simplifying definition 3.1 of [14] we define a *quasi-cluster morphism* $\varphi : \mathcal{A}_Q \to \mathcal{A}_{Q'}$ to be a $\mathbb{Q}$-algebra morphism such that

a) we have $\varphi(\mathbb{P}) \subseteq \mathbb{P}'$ and for each cluster variable x of $\mathcal{A}_Q$, there is a cluster variable x' of $\mathcal{A}_{Q'}$ such that $\varphi(x) \subseteq \mathbb{P}' x'$;

b) the induced morphism $\overline{\varphi} : \mathcal{A}_{\overline{Q}} \to \mathcal{A}_{\overline{Q'}}$ is an isomorphism of cluster algebras taking the initial seed $(X, \overline{Q})$ to a seed $(X', \overline{Q})$ (with the same quiver $\overline{Q}$);

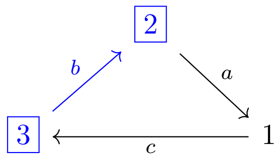
c) we have

$$\varphi(\widehat{y}_j(t_0)) = \widehat{y'}_{j'}(t') \ ,$$

where t' and j' are defined by the condition $\overline{\varphi}(x_j(t_0)) = x'_{j'}(t')$ for $1 \leq j \leq r$.

Notice that thanks to condition b), in condition c), there is a unique cluster $x'_{j'}(t')$, $1 \leq j' \leq r$, obtained as the image under $\overline{\varphi}$ of the cluster $x_j(t_0)$, $1 \leq j \leq r$, and that it determines the associated exchange matrix by Corollary 3.6 of [21]. Let us denote by $\mathcal{U}_Q$ the *upper cluster algebra with invertible coefficients* associated with Q. We define a *quasi-cluster morphism* $f : \mathcal{U}_Q \to \mathcal{U}_{Q'}$ to be a ring homomorphism inducing a quasi-cluster morphism $\mathcal{A}_Q \to \mathcal{A}_{Q'}$.

As a simple example, consider the ice quiver



Let us denote the initial cluster variables by x_1, $p_1 = x_2$ and $p_2 = x_3$. Then the only other cluster variable is

$$x_1' = \frac{p_1 + p_2}{x_1}.$$

The associated cluster algebra with invertible coefficients is

$$\mathcal{A}_Q = \mathbb{Q}[x_1, x_1', p_1^{\pm}, p_2^{\pm}]/(x_1 x_1' - p_1 - p_2)$$

and we have $\widehat{y}_1 = p_1/p_2$. The associated cluster algebra without coefficients is

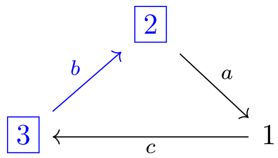
$$\mathcal{A}_{\overline{Q}} = \mathbb{Q}[x_1, 2/x_1].$$

Define the algebra automorphism $\sigma : \mathcal{A}_Q \to \mathcal{A}_Q$ to send p_i to $1/p_i$, x_1 to x_1'/p_2 and x_1' to x_1/p_1. We have

$$\widehat{y}_1' = \frac{\sigma(p_1)}{\sigma(p_2)} = \frac{1/p_1}{1/p_2} = \frac{p_2}{p_1} = \sigma(\widehat{y}_1)$$

so that σ is indeed a quasi-cluster automorphism.

A.2. **Frobenius categorification of cluster algebras with coefficients.** As before, let Q be an ice quiver. Denote by $\mathcal{A}_Q^+$ the cluster algebra with *non invertible* coefficients associated with Q. For example, for the quiver



we have

$$\mathcal{A}_Q^+ = \mathbb{Q}[x_1, x_1', p_1, p_2]/(x_1 x_1' - p_1 - p_2).$$

Now let k be an algebraically closed field and assume that $(\mathcal{E}, T)$ is a *Frobenius categorification* of the quiver Q. By definition, this means that

a) $\mathcal{E}$ is a k-linear Krull–Schmidt Frobenius category which is enriched over the monoidal category of pseudo-compact vector spaces, cf. section 4 of [4];
b) the stable category $\underline{\mathcal{E}}$ is Hom-finite and 2-Calabi–Yau, i.e. we have bifunctorial isomorphisms
$$D\mathrm{Hom}_{\underline{\mathcal{E}}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\underline{\mathcal{E}}}(Y, \Sigma^2 X)$$
for $X, Y \in \underline{\mathcal{E}}$, where D is the duality over the ground field k;
c) T is a basic cluster-tilting object of $\mathcal{E}$ and we are given an isomorphism between Q and the quiver of the endomorphism algebra A of T such that the frozen vertices correspond to the projective-injective indecomposable direct factors of T, the number of frozen arrows from i to j is
$$\dim \mathrm{Ext}^1_A(S_j, S_i) - \dim \mathrm{Ext}^2_A(S_i, S_j)$$
and the number of non frozen arrows from i to j is
$$\dim \mathrm{Ext}^2_A(S_i, S_j)$$
for all vertices i and j.
d) the basic cluster-tilting objects of $\mathcal{E}$ determine a cluster structure on $\mathcal{E}$ in the sense of section I.1 of [2].

By Theorem I.1.6 of [loc. cit.], if conditions a)–c) hold, then condition d) holds if no cluster-tilting object of $\mathcal{E}$ has loops or 2-cycles in the quiver of its endomorphism algebra. By Prop. 2.19 (v) of [19], this holds for many stably 2-CY categories occuring in Lie theory.

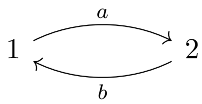
Let us assume that T is basic and that we have numbered its indecomposable direct factors T_i, $1 \le i \le n$, such that T_i is projective for $i > r$ and non projective for $i \le r$. Then, by iterated mutation, with each vertex t of the regular tree $\mathbb{T}_r$, we associate a basic cluster-tilting object $T(t)$ whose endomorphism algebra has the quiver $Q(t)$ obtained from $Q(t_0) = Q$ by iterated mutation.

Let us assume from now on the $k = \mathbb{C}$ is the field of complex numbers. Then with T, we have an associated cluster character $CC = CC_T : \mathcal{E} \to \mathcal{U}_Q$ with values in the upper cluster algebra $\mathcal{U}_Q$ and defined by the formula 7. The following theorem is a combination of results from [15] and [6].

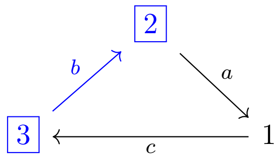
Theorem A.2. *The map $L \mapsto CC(L)$ induces bijections*

- *from the set of isoclasses of reachable rigid indecomposables of $\mathcal{E}$ to the set of cluster variables of $\mathcal{A}_Q^+$;*
- *from the set of isoclasses of reachable cluster-tilting objects of $\mathcal{E}$ to the set of clusters of $\mathcal{A}_Q^+$.*

As an example, let us consider the category $\mathcal{E} = \mathrm{mod}\Lambda$ of finite-dimensional modules over the preprojective algebra Λ of type A_2 given by the quiver



with the relations $ab = 0$ and $ba = 0$. Then $\mathcal{E}$ contains four isoclasses of indecomposable objects represented by the two simple modules S_1 and S_2 and the two indecomposable projective modules P_1 and P_2, where P_i is the projective cover of S_i, $i = 1, 2$. The object $T = S_1 \oplus P_1 \oplus P_2$ is cluster-tilting. Its endomorphism algebra is given by the quiver



with the relations $ab = 0$, $bc = 0$. Here the vertex 1 corresponds to S_1, the vertex 2 to P_1 and the vertex 3 to P_2. Notice that the endomorphism algebra is also the relative Jacobian algebra of the same quiver endowed with the potential $W = abc$. The cluster character associated with T takes the indecomposables respectively to x_1, x_2, p_1 and p_2. The space $\mathrm{Ext}^1_{\mathcal{E}}(S_1, S_2)$ is one-dimensional and we have the associated exchange conflations

$$S_1 \rightarrowtail P_2 \twoheadrightarrow S_2 \text{ and } S_2 \rightarrowtail P_1 \twoheadrightarrow S_1.$$

They decategorify to the exchange relation

$$CC(S_1)CC(S_2) = CC(P_1) + CC(P_2) \text{ respectively } x_1 x_2 = p_1 + p_2.$$

Our first aim in this appendix is to propose a categorical interpretation of expressions like $1/p_1$, $x_2/p_1, \ldots$ which have monomials in frozen variables as denominators. We will use objects of the derived category $\mathcal{D}^b(\mathcal{E})$ for this purpose, cf. Theorem A.4 below.

A.3. **On split Grothendieck groups of Frobenius categories.** Let $\mathcal{E}$ be a Krull–Schmidt Frobenius category and $\mathcal{P} \subseteq \mathcal{E}$ its full subcategory of projective-injectives. We denote by $K_0^{sp}(\mathcal{E})$ the *split Grothendieck group* of $\mathcal{E}$ which has a basis given by the isomorphism classes of the indecomposables in $\mathcal{E}$. Exceptionally, in this section, we will denote by $X \mapsto X[1]$ the suspension functor of the derived category $\mathcal{D}^b(\mathcal{E})$. We denote by $K_0^{sp}(\mathcal{D}^b(\mathcal{E}))$ the split Grothendieck group of $\mathcal{D}^b(\mathcal{E})$ and by $K_{\mathcal{E},\mathcal{P}}$ its quotient by the subgroup generated by all elements $[P] - [X] + [Y]$, where P, X and Y appear in a triangle

$$P \to X \to Y \to P[1]$$

of $\mathcal{D}^b(\mathcal{E})$ and P is isomorphic to a bounded complex of projectives of $\mathcal{E}$.

Proposition A.3. *The morphism $\varphi : K_0^{sp}(\mathcal{E}) \to K_{\mathcal{E},\mathcal{P}}$ induced by the canonical functor $\mathcal{E} \to \mathcal{D}^b(\mathcal{E})$ is bijective.*

Proof. We will construct the inverse ψ of φ. Let $\mathcal{C}^{-,b}(\mathcal{P})$ denote the category of right bounded complexes X with components in $\mathcal{P}$ which are acylic in all degrees $n \ll 0$, which means that we have conflations

$$Z^n(X) \rightarrowtail X^n \twoheadrightarrow Z^{n+1}(X).$$

The corresponding homotopy category $\mathcal{H}^{-,b}(\mathcal{P})$ is canonically equivalent to $\mathcal{D}^b(\mathcal{E})$. Thus it suffices to construct an element $\psi(X)$ in $K_0^{sp}(\mathcal{E})$ for each complex X in $C^{-,b}(\mathcal{P})$ such that

a) we have $\psi(X) = \psi(X')$ if X and X' are homotopy equivalent;
b) we have $\psi(P) + \psi(Y) = \psi(X)$ whenever there is a triangle

$$P \to X \to Y \to P[1]$$

of $\mathcal{H}^{-,b}(\mathcal{P})$, where P is bounded;
c) the induced map $\psi : K_{\mathcal{E},\mathcal{P}} \to K_0^{sp}(\mathcal{E})$ is inverse to φ.

1st step: For each $n \leq 0$, we define $\psi(X)$ for complexes X in $\mathcal{C}^{-,b}(\mathcal{E})$ which have non zero components only in degrees $\leq n$ and admit a conflation

$$Z^n(X) \rightarrowtail X^n \twoheadrightarrow Y$$

as well as conflations

$$Z^p(X) \rightarrowtail X^p \twoheadrightarrow Z^{p+1}(X)$$

for all $p \leq n-1$. In other words, such a complex is a projective resolution of $Y[n]$. For this, we choose an injective resolution

$$0 \to Y \to I^0 \to I^1 \to \cdots$$

From the conflation

$$Y \rightarrowtail I^0 \twoheadrightarrow \Sigma Y$$

we deduce that in $K_{\mathcal{E},\mathcal{P}}$, we have

$$[Y[1]] = [\Sigma Y] - [I^0]$$

and by induction that we have

$$[X] = [Y[n]] = [\Sigma^n Y] + (-1)^n \sum_{p=0}^{n-1} (-1)^p [I^p]$$

in $K_{\mathcal{E},\mathcal{P}}$. Therefore, we define the element $\psi(X)$ of $K_0^{sp}(\mathcal{E})$ by

$$\psi(X) = [\Sigma^n Y] + (-1)^n \sum_{p=0}^{n-1} (-1)^p [I^p].$$

It is clear that $\psi(X)$ does not change if we replace X with a homotopic complex X' concentrated in degrees $\leq n$ and that $\varphi(\psi(X)) = [X]$.

2nd step: We define $\psi(X)$ for bounded complexes X with projective components by

$$\psi(X) = \sum_{p \in \mathbb{Z}} (-1)^p [X^p].$$

3rd step: We define $\psi(X)$ for general X in $\mathcal{C}^{-,b}(\mathcal{P})$. For this, we choose $N \ll 0$ such that X is acyclic in degrees $n \leq N$. We then have a a triangle

$$\sigma_{>N}(X) \to X \to \sigma_{\leq N}(X) \to (\sigma_{>N}(X))[1]$$

and we define

$$\psi(X) = \psi(\sigma_{>N}(X)) + \psi(\sigma_{\leq N}(X))$$

using the first and the second step. One checks that this element of $K_0^{sp}(\mathcal{E})$ does not depend on the choice of N and that we have properties a), b) and c) above. √

A.4. **Extension of cluster characters to the derived category.** Let $\mathcal{E}$ be a Krull–Schmidt Frobenius category which is stably 2-Calabi–Yau. Suppose that $CC : \mathcal{E} \to R$ is a cluster character with values in a commutative domain R. Suppose that the isomorphism classes of the indecomposable projectives of $\mathcal{E}$ are represented by objects $P_{r+1}, \ldots, P_n$ and put $x_i = CC(P_i)$ for $r+1 \leq i \leq n$. Let $\iota : R \to R_{loc}$ be a ring homomorphism to a domain which makes $x_{r+1}, \ldots, x_n$ invertible. Let us also denote the canonical functor $\mathcal{E} \to \mathcal{D}^b(\mathcal{E})$ by ι. Let $\pi : R_{loc} \to \underline{R}$ be a ring homomorphism which sends the $\iota(x_i)$ to 1, $r+1 \leq i \leq n$. Let us also denote by $\pi : \mathcal{D}^b(\mathcal{E}) \to \underline{\mathcal{E}}$ the canonical triangle functor. Let $\underline{CC} : \underline{\mathcal{E}} \to \underline{R}$ be the cluster character induced by CC.

Theorem A.4. a) *There is a unique map $CC_{loc} : \mathcal{D}^b(\mathcal{E}) \to R_{loc}$ such that*

1) *the following diagram commutes*

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{CC} & R \\
\iota \downarrow & & \downarrow \iota \\
\mathcal{D}^b(\mathcal{E}) & \xrightarrow{CC_{loc}} & R_{loc} \\
\pi \downarrow & & \downarrow \pi \\
\underline{\mathcal{E}} & \xrightarrow{\underline{CC}} & \underline{R}
\end{array}$$

2) *whenever we have a triangle*

$$P \to X \to Y \to P[1]$$

of $\mathcal{D}^b(\mathcal{E})$*, where* P *is a bounded complex of projectives, we have*

$$CC_{loc}(X) = CC_{loc}(P) CC_{loc}(Y).$$

b) *Let* X *and* Y *be objects of* $\mathcal{D}^b(\mathcal{E})$ *such that the space* $\mathrm{Ext}^1_{\underline{\mathcal{E}}}(\pi X, \pi Y)$ *is one-dimensional. Let* E *and* E' *be the middle terms of triangles*

$$Y \to E \to X \to Y[1] \text{ and } X \to E' \to Y \to X[1]$$

of $\mathcal{D}^b(\mathcal{E})$ *whose images in* $\underline{\mathcal{E}}$ *are non split. Then we have*

$$CC_{loc}(X) CC_{loc}(Y) = CC_{loc}(E) + CC_{loc}(E').$$

Proof. a) We consider $\iota \circ CC$ as a map defined on $K_0^{sp}(\mathcal{E})$ with values in the multiplicative group of non zero elements of the fraction field of R_{loc}. By Proposition A.3, this map admits a unique extension CC_1 to $K_{\mathcal{E},\mathcal{P}}$ such that for each triangle

$$P \to X \to Y \to P[1]$$

of $\mathcal{D}^b(\mathcal{E})$, where P is a bounded complex of projectives, we have

$$CC_1(X) = CC_1(P) CC_1(Y).$$

An inspection of the proof of Proposition A.3 shows that CC_1 actually takes values in R_{loc}. Thus, if we define CC_{loc} to be the map $X \mapsto CC_1(X)$ from $\mathcal{D}^b(\mathcal{E})$ to R_{loc}, then the top square of the diagram commutes. Moreover, by the definition of $\underline{CC}$, the outer rectangle commutes. By the surjectivity of the map $K_0^{sp}(\mathcal{E}) \to K_{\mathcal{E},\mathcal{P}}$, the bottom square also commutes.

b) Suppose first that X and Y belong to $\mathcal{E} \subset \mathcal{D}^b(\mathcal{E})$. Then clearly, the claim holds for X and Y. Let us check it for $X[1]$ and $Y[1]$. Let us choose conflations

$$X \rightarrowtail IX \twoheadrightarrow \Sigma X \text{ and } Y \rightarrowtail IY \twoheadrightarrow \Sigma Y$$

with injective IX and IY. Then we can construct conflations

$$E \rightarrowtail IY \oplus IX \twoheadrightarrow \Sigma E \text{ and } E' \rightarrowtail IX \oplus IY \twoheadrightarrow \Sigma E'.$$

We have

$$CC_{loc}(X[1]) = CC(\Sigma X) CC(IX)^{-1}$$

and similarly for $CC_{loc}(U)$ for $U \in \{Y, E, E'\}$. This immediately implies that the formula holds for $X[1]$ and $Y[1]$. By induction, we obtain it for $X[n]$ and $Y[n]$ for $n \geq 0$. Now let X and Y be arbitrary objects of $\mathcal{D}^b(\mathcal{E})$. For some $N \ll 0$, we may assume that X^p is projective for all $p \geq N$ and $X^p = 0$ for all $p \leq N-2$ and similarly for Y, E and E'. Moreover, we may assume that we have $E^p \cong X^p \oplus Y^p \cong E'^p$ for $p \geq N$ and that we have conflations

$$X^{N-1} \rightarrowtail E^{N-1} \twoheadrightarrow Y^{N-1} \text{ and } Y^{N-1} \rightarrowtail E'^{N-1} \twoheadrightarrow X^{N-1}.$$

Notice that up to suspension, these give rise in the stable category $\underline{\mathcal{E}}$ to the images of the chosen triangles linking X and Y in $\mathcal{D}^b(\mathcal{E})$. By our assumption on the extension groups between πX and πY, these conflations are therefore exchange conflations between X^{N-1} and Y^{N-1} in $\mathcal{E}$. We have the componentwise split conflation of complexes

$$\sigma_{\geq N}(X) \rightarrowtail X \twoheadrightarrow X^{N-1}[1-N].$$

Since $\sigma_{\geq N}(X)$ is bounded with projective components, we have

$$CC_{loc}(X) = CC_{loc}(X^{N-1}[1-N]) CC_{loc}(\sigma_{\geq N}(X))$$

and similarly for Y, E and E'. The componentwise split conflations of complexes

$$\sigma_{\geq N}(Y) \rightarrowtail \sigma_{\geq N}(E) \twoheadrightarrow \sigma_{\geq N}(X) \text{ and } \sigma_{\geq N}(X) \rightarrowtail \sigma_{\geq N}(E') \twoheadrightarrow \sigma_{\geq N}(Y)$$

yield

$$CC_{loc}(\sigma_{\geq N}(E)) = CC_{loc}(\sigma_{\geq N}(X) CC_{loc}(\sigma_{\geq N}(Y)) = CC_{loc}(\sigma_{\geq N}(E')).$$

Together with the formula for $X^{N-1}[1-N]$ and $Y^{N-1}[1-N]$, this implies the formula for X and Y. $\surd$

Remark A.5. Let X and X' be objects of $\mathcal{D}^b(\mathcal{E})$ whose images in the stable category $\mathcal{D}^b(\mathcal{E})/\mathcal{H}^b(\mathcal{P}) \xrightarrow{\sim} \underline{\mathcal{E}}$ are isomorphic. By the calculus of fractions for this Verdier quotient, there are triangles of $\mathcal{D}^b(\mathcal{E})$

$$P' \to X'' \to X \to P'[1] \text{ and } P'' \to X'' \to X' \to P'[1]$$

such that P' and P'' belong to $\mathcal{H}^b(\mathcal{P})$. It follows that we have

$$CC_{loc}(X') = M \cdot CC_{loc}(X)$$

for a Laurent monomial M in the $CC_{loc}(P)$, $P \in \mathcal{P}$.

Example A.6. *Let us continue the example from the end of section A.2. With the notations used there, we clearly have $CC(P_1[1]) = 1/p_1$. Moreover, the triangle*

$$P_1 \to S_2 \to S_1[1] \to P_2[1]$$

shows that we have $CC(S_1[1]) = x_2/p_2$. In fact, it is not hard to check that we have a commutative square

$$\begin{array}{ccc} \mathcal{D}^b(\mathcal{E}) & \xrightarrow{CC} & \mathcal{A}_Q \\ {\scriptstyle [1]}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \mathcal{D}^b(\mathcal{E}) & \xrightarrow{CC} & \mathcal{A}_Q. \end{array}$$

This can be interpreted by saying that the suspension functor [1] *of $\mathcal{D}^b(\mathcal{E})$ categorifies the quasi-cluster automorphism σ of $\mathcal{A}_Q$.*

A.5. **Categorification of quasi-cluster morphisms.** Let $(\mathcal{E}, T)$ be a Frobenius categorification (cf. section A.2) of the ice quiver of Q of a cluster algebra $\mathcal{A}$ with invertible coefficients. Let $\mathcal{A}^+$ be the corresponding cluster algebra with non invertible coefficients and define $\mathcal{U}$ and $\mathcal{U}^+$ to be the corresponding upper cluster algebras. Let $\Phi : \mathcal{E} \to \mathcal{U}^+$ be the cluster character associated with T. Let $\mathcal{P} \subseteq \mathcal{E}$ be the full subcategory of the projective-injectives. Using part 1) of Theorem A.4 we extend $CC = \Phi$ to a map $CC_{loc} : \mathcal{D}^b(\mathcal{E}) \to \mathcal{U}$, which we will still denote by Φ. Let $(\mathcal{E}', T')$ be a Frobenius categorification of the ice quiver Q' of another cluster algebra $\mathcal{A}'$ with invertible coefficients. We define the notations $\mathcal{A}'^+$, $\mathcal{U}'$, $\mathcal{U}'^+$, Φ' and $\mathcal{P}'$ in the obvious way.

Theorem A.7. *A ring homomorphism $f : \mathcal{U} \to \mathcal{U}'$ is a quasi-cluster morphism if there is a triangle functor $F : \mathcal{D}^b(\mathcal{E}) \to \mathcal{D}^b(\mathcal{E}')$ such that*

1) *F takes $\mathcal{P}$ to $\mathcal{H}^b(\mathcal{P}') \subseteq \mathcal{D}^b(\mathcal{E}')$;*
2) *the induced functor $\underline{F} : \underline{\mathcal{E}} \to \underline{\mathcal{E}'}$ is a triangle equivalence taking T to a cluster tilting object T'' reachable from T';*
3) *there is a triangle functor $\widetilde{F} : \mathcal{H}^b(\mathrm{add}(T)) \to \mathcal{H}^b(\mathrm{add}(T'))$ making the following square commute (up to isomorphism)*

$$\begin{array}{ccc} \mathcal{H}^b(\mathrm{add}(T)) & \longrightarrow & \mathcal{D}^b(\mathcal{E}) \\ {\scriptstyle \widetilde{F}}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{H}^b(\mathrm{add}(T')) & \longrightarrow & \mathcal{D}^b(\mathcal{E}') \end{array} ,$$

   where the horizontal arrows are induced by the inclusions $\mathrm{add}(T) \subseteq \mathcal{E}$ *and* $\mathrm{add}(T') \subseteq \mathcal{E}'$.
4) *the square*

$$\begin{array}{ccc} \mathrm{add}(T) & \xrightarrow{\Phi} & \mathcal{U}^+ \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{D}^b(\mathcal{E}') & \xrightarrow[\Phi']{} & \mathcal{U}' \end{array}$$

   commutes.

Remark A.8. We thank Matthew Pressland for suggesting the weak hypothesis 4). In [40], he has recently applied this theorem to prove a conjecture by Muller–Speyer [31, Rem. 4.7] linking the two canonical cluster structures on a positroid variety by a quasi-cluster isomorphism.

Proof. Since $\mathcal{P}$ is contained in $\mathrm{add}(T)$, conditions 1) and 4) imply that f maps frozen variables of $\mathcal{A}^+$ to Laurent monomials in frozen variables of $\mathcal{A}'^+$. Let T_0 be an indecomposable summand of T and $x_0 \in \mathcal{A}^+$ the associated initial cluster variable. By condition 2), there is an indecomposable reachable rigid object U_0 which becomes isomorphic to FT_0 in $\underline{\mathcal{E}'}$. By Remark A.5, we have $\Phi'(FT_0) = M \cdot \Phi'(U_0)$ for a Laurent monomial M in the frozen variables of $\mathcal{A}'^+$. Thus, the image $f(x_0) = f(\Phi(T_0)) = \Phi'(FT_0)$, where we have used condition 4), is a product of the cluster variable $\Phi'(U_0)$ with a Laurent monomial in frozen variables. Thanks to part b) of Theorem A.4, condition 4) implies the commutativity of the square

$$\begin{array}{ccc} \mathrm{add}(T'') & \xrightarrow{\Phi} & \mathcal{U}^+ \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{D}^b(\mathcal{E}') & \xrightarrow[\Phi']{} & \mathcal{U}'^+ \end{array}$$

for any cluster-tilting object T'' of $\mathcal{E}$ reachable from T. We conclude that for any cluster variable x of $\mathcal{A}^+$, its image $f(x)$ is the product of a cluster variable of $\mathcal{A}'^+$ with a Laurent monomial in frozen variables.

Let $\underline{\mathcal{A}}$ and $\underline{\mathcal{A}'}$ denote the cluster algebras without coefficients associated with $\mathcal{A}^+$ and $\mathcal{A}'^+$. By Lemma A.1 and condition 2), the morphism $\underline{f}$ takes the initial seed of $\underline{\mathcal{A}}$ to a seed of $\underline{\mathcal{A}'}$ with the same quiver. Thus, the morphism $\underline{f}$ is a cluster algebra isomorphism.

It remains to check condition c) of section A.1. After composing f with a cluster isomorphism we may assume that $\underline{f}$ takes the initial seed of $\underline{\mathcal{A}}$ to the initial seed of $\underline{\mathcal{A}'}$. Let $B = (b_{ij})$ and B' denote the corresponding exchange matrices with $1 \leq i \leq n$ and $1 \leq j \leq r$, where n is the number of all and $r < n$ the number of non-frozen initial cluster variables. We have to show that $B = B'$. Thanks to the work [34] of Palu, we can deduce this from condition 3). Indeed, as shown in [loc. cit.], if we put $\mathcal{M} = \mathrm{add}(T)$, we have a short exact sequence of triangulated categories

$$0 \to \mathcal{H}^b_{ac}(\mathcal{M}) \to \mathcal{H}^b(\mathcal{M}) \to \mathcal{D}^b(\mathcal{E}) \to 0\ ,$$

where $\mathcal{H}^b_{ac}(\mathcal{M})$ denotes the kernel of the canonical functor $\mathcal{H}^b(\mathcal{M}) \to \mathcal{D}^b(\mathcal{E})$, i.e. the full subcategory whose objects are the complexes with components in $\mathcal{M}$ which are acyclic as complexes over $\mathcal{E}$. The triangulated category $\mathcal{H}^b_{ac}(\mathcal{M})$ admits a bounded non degenerate t-structure whose heart identifies with the category of finite-dimensional modules over the stable endomorphism algebra $\mathrm{End}_{\underline{\mathcal{E}}}(T)$. If T_j is an indecomposable non projective summand of T, the simple quotient S_j of the projective module $\mathrm{Hom}_{\underline{\mathcal{E}}}(T, T_j)$ corresponds to the acyclic complex

$$0 \to T_j \to E \to E' \to T_j \to 0$$

(with the last copy of T_j in degree 0) obtained by splicing the two exchange conflations

$$T_j^* \rightarrowtail E' \twoheadrightarrow T_j \text{ and } T_j \rightarrowtail E \twoheadrightarrow T_j^*.$$

Thus, the matrix of the morphism in the Grothendieck groups induced by the functor $\mathcal{H}^b_{ac}(\mathcal{M}) \to \mathcal{H}^b(\mathcal{M})$ in the bases given by the $[S_j]$ and the $[T_i]$ is (b_{ij}). The triangle functors $\widetilde{F}$ and F induce isomorphisms in the Grothendieck group which are compatible with the morphism induced by $\mathcal{H}^b_{ac}(\mathcal{M}) \to \mathcal{H}^b(\mathcal{M})$ and clearly preserve the bases. Thus, we have $B = B'$ as claimed. $\surd$

We keep the notations introduced at the beginning of this section. In the following variant of the above theorem, we make a slightly stronger assumption in condition 1) but do not suppose that we are given a ring homomorphism $\mathcal{A} \to \mathcal{A}'$ from the outset.

Theorem A.9. *Let $F : \mathcal{D}^b(\mathcal{E}) \to \mathcal{D}^b(\mathcal{E}')$ be a triangle functor such that*

1) *F takes $\mathcal{P}$ to $\mathcal{H}^b(\mathcal{P}') \subseteq \mathcal{D}^b(\mathcal{E}')$ and induces an isomorphism $K_0(\mathcal{P}) \xrightarrow{\sim} K_0(\mathcal{P}')$.*
2) *the induced functor $\underline{F} : \underline{\mathcal{E}} \to \underline{\mathcal{E}'}$ is a triangle equivalence taking T to a cluster tilting object T'' reachable from T';*

3) *there is a triangle functor* $\widetilde{F} : \mathcal{H}^b(\mathrm{add}(T)) \to \mathcal{H}^b(\mathrm{add}(T''))$ *making the following square commute (up to isomorphism)*

$$\begin{array}{ccc} \mathcal{H}^b(\mathrm{add}(T)) & \longrightarrow & \mathcal{D}^b(\mathcal{E}) \\ {\scriptstyle \widetilde{F}}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{H}^b(\mathrm{add}(T'')) & \longrightarrow & \mathcal{D}^b(\mathcal{E}') \end{array} ,$$

where the horizontal arrows are induced by the inclusions $\mathrm{add}(T) \subseteq \mathcal{E}$ *and* $\mathrm{add}(T'') \subseteq \mathcal{E}'$.

Then there is a unique quasi-cluster isomorphism $f : \mathcal{A} \xrightarrow{\sim} \mathcal{A}'$ *taking the initial variable* x_i *to* $\Phi'(FT_i)$, $1 \leq i \leq n$.

Proof. Let $(Q, (x_i))$ be the inital seed of $\mathcal{A}$ and $(Q', (x'_i))$ that of $\mathcal{A}'$. By condition 2), both have the same number r of non frozen initial cluster variables. Moreover, the isomorphism

$$K_0(F) : K_0(\mathcal{P}) \xrightarrow{\sim} K_0(\mathcal{P}')$$

of condition 1) yields an isomorphism between the groups of Laurent monomials in the frozen variables for Q and Q'. Put $u_i = \Phi'(FT_i)$. Let (x''_i) be the cluster of $\mathcal{A}'$ associated with the reachable cluster-tilting object of $\mathcal{E}'$ lifting $FT \in \underline{\mathcal{E}}'$. Since FT_i and T''_i are isomorphic in $\underline{\mathcal{E}'}$, by Remark A.5, we have $u_i = m_i x''_i$ for $1 \leq i \leq r$, where the m_i are Laurent polynomials in the frozen variables. Therefore, the u_i, $1 \leq i \leq r$, are algebraically independent over the Laurent polynomial algebra in the frozen variables of $\mathcal{A}'$. So there is a well-defined algebra morphism $f : \mathcal{U} \to \mathbb{Q}(x'_i, 1 \leq i \leq n)$ such that $f(x_i) = u_i$, $1 \leq i \leq r$, and f induces the isomorphism given by $K_0(F)$ in the coefficient groups. From part b) of Theorem A.4, we deduce by induction that the image under f of each cluster variable of $\mathcal{A}$ is the product of a cluster variable of $\mathcal{A}'$ with a Laurent monomial in the frozen variables. Thus, the image of each cluster of $\mathcal{A}$ lies in $\mathcal{A}'$ and $f(\mathcal{A}) \subseteq \mathcal{A}'$. We even have $f(\mathcal{A}) = \mathcal{A}'$ since T' is reachable from FT in $\underline{\mathcal{E}}$. Thus, we have a ring isomorphism $f : \mathcal{A} \to \mathcal{A}'$ taking the x_i to the u_i, $1 \leq i \leq n$. Clearly, it satisfies the assumptions of Theorem A.7 and thus is a quasi-cluster isomorphism. $\surd$

Example A.10. *Let us generalize Example A.6. We use the notations introduced at the beginning of this section. Suppose that the cluster-tilting object* ΣT *of* $\underline{\mathcal{E}}$ *is reachable. For each indecomposable summand* T_i *of* T, *we choose an inflation*

$$T_i \rightarrowtail I_i \twoheadrightarrow \Sigma T_i.$$

We define

$$u_i = CC_{loc}(T_i[1]) = CC(\Sigma T_i) CC(I_i)^{-1} \quad , 1 \leq i \leq n.$$

Clearly the shift functor $\mathcal{D}^b(\mathcal{E}) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{E})$ *satisfies the hypotheses of Theorem A.9. Thus, we have a unique quasi-cluster isomorphism* $DT : \mathcal{A} \xrightarrow{\sim} \mathcal{A}$ *taking* x_i *to* u_i, $1 \leq i \leq n$. *It is called the* twist *or the* Donaldson–Thomas transformation *of* $\mathcal{A}$.

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References

[1] Claire Amiot, *Cluster categories for algebras of global dimension 2 and quivers with potential*, Ann. Inst. Fourier (Grenoble) 59 (2009), no. 6, 2525–2590.

[2] Aslak Bakke Buan, Bethany Marsh, Markus Reineke, Idun Reiten, and Gordana Todorov, *Tilting theory and cluster combinatorics*, Advances in mathematics, 204(2):572–618, 2006.

[3] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. 126 (2005), no. 1, 1–52.

[4] Michel Van den Bergh, *Calabi-Yau algebras and superpotentials*, Sel. Math. New Ser. 21, 555–603 (2015).

[5] Philippe Caldero and Frédéric Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. 81 (2006), no. 3, pp. 595–616.

[6] Giovanni Cerulli Irelli, Bernhard Keller, Daniel Labardini-Fragoso, Pierre-Guy Plamondon, *Linear independence of cluster monomials for skew-symmetric cluster algebras*, Compos. Math. **149** (2013), 1753–1764.

[7] Philippe Caldero and Bernhard Keller, *From triangulated categories to cluster algebras*, Invent. math. 172, 169–211 (2008).

[8] Harm Derksen, Jerzy Weyman and Andrei Zelevinsky, *Quivers with potentials and their representations. I. Mutations* Selecta Math. (N.S.) 14 (2008), 59–119.

[9] Harm Derksen, Jerzy Weyman and Andrei Zelevinsky, *Quivers with potentials and their representations II: applications to cluster algebras* J. Amer. Math. Soc. 23 (2010), 749–790.

[10] Vladimir Drinfeld, *DG quotients of DG categories*, Journal of Algebra, 272(2):643–691, 2004.

[11] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras I: Foundations*, Journal of the American Mathematical Society, 15(2):497–529, 2002.

[12] Sergey Fomin, *Total positivity and cluster algebras*, Proceedings of the International Congress of Mathematicians., Volume II (New Delhi), Hindustan Book Agency, 2010, pp. 125–145.

[13] Sergey Fomin, Lauren Williams and Andrei Zelevinsky, *Introduction to cluster clgebras. Chapters 1-3*, arXiv:1608.05735 [math.CO].

[14] Chris Fraser, *Quasi-homomorphisms of cluster algebras*, Advances in Applied Mathematics, 81:40–77, 2016.

[15] Changjian Fu and Bernhard Keller, *On cluster algebras with coefficients and 2-Calabi-Yau categories*, Trans. Amer. Math. Soc. 362 (2010), 859-895.

[16] Pavel Galashin and Thomas Lam, *Positroid varieties and cluster algebras*, To appear in Ann. Sci. École Norm. Sup. 2019, arXiv:1906.03501v2.

[17] Christof Geiß, Bernard Leclerc, and Jan Schröer, *Rigid modules over preprojective algebras*, Invent. Math. 165 (2006), no. 3, 589–632.

[18] Christof Geiß, Bernard Leclerc, and Jan Schröer, *Cluster algebra structures and semicanonical bases for unipotent groups*, arXiv:math/0703039.

[19] Christof Geiß, Bernard Leclerc, and Jan Schröer, *Kac-Moody groups and cluster algebras*, Adv. Math. 228 (2011), no. 1, 329–433.

[20] Jan E. Grabowski and Matthew Pressland, *Electronic message to Bernhard Keller*, January 13, 2023.

[21] Giovanni Cerulli Irelli, Bernhard Keller, Daniel Labardini-Fragoso, and Pierre-Guy Plamondon, *Linear independence of cluster monomials for skew-symmetric cluster algebras*, Compos. Math. 149 (2013), no. 10, 1753–1764.

[22] Osamu Iyama and Dong Yang, *Silting reduction and Calabi–Yau reduction of triangulated categories*, Trans. Amer. Math. Soc., 370(11):7861–7898, 2018.

[23] Osamu Iyama and Dong Yang, *Quotients of triangulated categories and equivalences of Buchweitz, Orlov, and Amiot-Guo-Keller*, Amer. J. Math., 142(5), 1641-1659, 2020.

[24] Bernt Tore Jensen, Alastair D. King, and Xiuping Su, *A categorification of Grassmannian cluster algebras*, Proceedings of the London Mathematical Society, 113(2):185–212, 2016.

[25] Haibo Jin, *Reductions of triangulated categories and simple-minded collections*, arXiv:1907.05114 [math.RT].

[26] Martin Kalck, Osamu Iyama, Michael Wemyss and Dong Yang, *Frobenius categories, Gorenstein algebras and rational surface singularities*, Compositio Mathematica, 151(3), 502-534.

[27] Martin Kalck and Dong Yang, *Relative singularity categories I: Auslander resolutions*, Advances in Mathematics, 301:973–1021, 2016.

[28] Bernhard Keller, *Deformed Calabi–Yau completions*, with an appendix by Michel Van den Bergh. 2011(654):125–180, 2011.

[29] Bernhard Keller and Dong Yang, *Derived equivalences from mutations of quivers with potential*, Adv. Math. 226 (2011), no. 3, 2118–2168.

[30] Yu Liu and Panyue Zhou, *Frobenius n-exangulated categories*, arXiv:1909.13284 [math], 2020.

[31] Greg Muller, David E Speyer, *The twist for positroid varieties*, Proc. Lond. Math. Soc. (3) 115 (2017), pp. 1014–1071.

[32] Hiroyuki Nakaoka and Yann Palu, *Extriangulated categories, Hovey twin cotorsion pairs and model structures*, Cah. Topol. Géom. Différ. Catég, 60(2):117–193, 2019.

[33] Yann Palu, *Cluster characters for 2-Calabi–Yau triangulated categories*, Annales de l'Institut Fourier, 58(6):2221–2248, 2008.

[34] Yann Palu, *Grothendieck group and generalized mutation rule for 2-Calabi–Yau triangulated categories*, J.Pure Appl. Algebra 213 (2009), no. 7, 1438–1449.

[35] Yann Palu, *Some applications of extriangulated categories*, arXiv:2307.10019 [math.RT].

[36] Pierre-Guy Plamondon, *Cluster characters for cluster categories with infinite-dimensional morphism spaces*, Adv. Math. 227 (2011), no. 1, 1-39.

[37] Matthew Pressland, *Internally Calabi–Yau algebras and cluster-tilting objects*, Math. Z. 287.1-2 (2017), pp. 555–585.

[38] Matthew Pressland, *Calabi-Yau properties of Postnikov diagrams*, Forum Math. Sigma 10 (2022), Paper No. e56, 31 pp.

[39] Matthew Pressland, *Mutation of frozen Jacobian algebras*, Journal of Algebra, 546:236–273, 2020.

[40] Matthew Pressland, *Quasi-coincidence of cluster structures on positroid varieties*, in preparation.

[41] Joshua S. Scott, *Grassmannians and cluster algebras*, Proceedings of the London Mathematical Society, 92(2):345–380, 2006.

[42] Yilin Wu, *Categorification of ice quiver mutation*, arXiv:2109.04503 [math], 2021.

[43] Yilin Wu, *Relative cluster categories and Higgs categories*, arXiv:2109.03707 [math], 2021.

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